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On Jain-Beta Linear Operators

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Abstract: Starting from a sequence of linear positive operators introduced by G.C. Jain, we present an integral version of it. Approximation properties and the rate of convergence are investigated. We use the concept of A-statistical convergence. An extension for smooth functions is also given.

Keywords: Linear positive operator, modulus of continuity, A-statistical convergence.

1. Introduction

Set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{R}_+ = [0, \infty)$. By using the Poisson-type distribution given by

$$w_\beta(k;\alpha) = \frac{\alpha}{k!}(\alpha + k\beta)^{k-1} e^{-(\alpha+k\beta)},\ k \in \mathbb{N}_0, \tag{1}$$

for $0 < \alpha < \infty$ and $|\beta| < 1$, G.C. Jain [7] introduced and studied the following class of positive linear operators

$$(P_n^{[\beta]} f)(x) = \sum_{k=0}^{\infty} w_\beta(k; nx) f\left(\frac{k}{n}\right),\ x \geq 0, \tag{2}$$

where $\beta \in [0, 1)$ and $f \in C(\mathbb{R}_+)$, the space of all real-valued continuous functions defined on $\mathbb{R}_+$. In the particular case $\beta = 0$, $P_n^{[0]}$, $n \in \mathbb{N}$, turn into well-known Szász-Mirakjan operators, see [11], [9] $(\mathrm{P}_n^{[0]} f)(x) \equiv (S_n f)(x)$ $= e^{-nx} \sum_{k=0}^{\infty} \dfrac{(nx)^k}{k!} f\left(\dfrac{k}{n}\right),\ x \geq 0.$

Due to their properties, the operators S_n have been intensively studied by many mathematicians. Thus, in our opinion, the class $(P_n^{[\beta]})$ should deeper investigate. This paper focuses on an integral variant of the discrete operators defined by (2). The construction is presented in Section 2. The approximation properties of our mixed summation-integral type operators are collected in Section 3.

We mention that a Kantorovich-type extension of $P_n^{[\beta]}$ was given in [12].

2. The operator $J_n^{[\beta]}$

Considering the weight function $\rho_\lambda : \mathbb{R}_+ \to [1, \infty)$, $\rho_\lambda(t) = 1 + x^{2+\lambda}$ $(\lambda \geq 0)$, we define the space

$$C_{\rho_\lambda}(\mathbb{R}_+) = \left\{ f \in C(\mathbb{R}_+) : \frac{f(x)}{\rho_\lambda(x)} \text{ is convergent as } x \to \infty \right\}$$

endowed with the usual norm $\|\cdot\|_{\rho_\lambda}$, $\|f\|_{\rho_\lambda} = \sup\limits_{x \geq 0} \dfrac{|f(x)|}{\rho_\lambda(x)}$.
Further on, we introduce a sequence of operators calling it Jain-Beta, as follows

$$(J_n^{[\beta]} f)(x)$$

$$= \sum_{k=1}^{\infty} \frac{w_\beta(k; nx)}{B(n+1, k)} \int_0^\infty f(t) \frac{t^{k-1}}{(1+t)^{n+k+1}} dt + e^{-nx} f(0), \tag{3}$$

$x \geq 0$, where $n \geq 2$, $f \in C_{\rho_0}(\mathbb{R}_+)$ and $w_\beta(k; nx)$ is given as in (1). They have Jain and Beta basis functions in summation and integration, respectively. One can see, for any $f \in C_{\rho_0}(\mathbb{R}_+)$ the integrals from (4) are well-defined. Indeed, if $f \in C_{\rho_0}(\mathbb{R}_+)$ then a positive constant M_f exists such that

$$|f(t)| \leq M_f(1 + t^2).$$

The convergence of the integrals

$$\int_0^\infty t^{k+1}(1+t)^{-n-k-1} dt,\ k \geq 1,$$

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guarantees that $\int_0^\infty f(t)t^{k-1}(1+t)^{-n-k-1}dt$, $k \geq 1$, are finite. The required condition is satisfied for $n > 1$, so we chose $n \geq 2$.

It is obvious that these operators are linear. Because any function $f \geq 0$ implies $J_n^{[\beta]} f \geq 0$, they are also positive. For $\beta = 0$, the operators $J_n^{[0]}$ reduce to the mixed Szász-Beta operators recently investigated by V. Gupta and M.A. Noor [6].

Let e_j, $j \in \mathbb{N}_0$, be the j-th monomial, $e_j(t) = t^j$. It is known (see, e.g., [2; *Proposition 4.2.5*]) that $\{e_0, e_1, e_2\}$ is a strict Korovkin set in $C_{\rho_0}(\mathbb{R}_+)$. So, our first concern is to determine values of these test functions. To do this, we recall the following identities established in [7; *Eqs. (2.12)-(2.14)*]

$$\begin{cases} (P_n^{[\beta]}e_0)(x) = 1,\ (P_n^{[\beta]}e_1)(x) = \dfrac{x}{1-\beta}, \\ (P_n^{[\beta]}e_2)(x) = \dfrac{x^2}{(1-\beta)^2} + \dfrac{x}{n(1-\beta)^3},\ x \geq 0. \end{cases} \tag{4}$$

Lemma 1. *The operators $J_n^{[\beta]}$, $n \geq 2$, defined by (4) satisfy the following relations*

$$\begin{cases} J_n^{[\beta]} e_0 = e_0,\ J_n^{[\beta]} e_1 = \dfrac{e_1}{1-\beta}, \\ J_n^{[\beta]} e_2 = \dfrac{n}{(n-1)(1-\beta)^2}\left(e_2 + \dfrac{1+(1-\beta)^2}{n(1-\beta)} e_1\right). \end{cases} \tag{5}$$

Proof. By simple computation we get

$$(J_n^{[\beta]} e_0)(x) = \sum_{k=1}^\infty w_\beta(k; nx) + e^{-nx} = (P_n^{[\beta]} e_0)(x).$$

$$(J_n^{[\beta]} e_1)(x) = \sum_{k=1}^\infty w_\beta(k; nx) \frac{B(n, k+1)}{B(n+1, k)}$$

$$= \sum_{k=0}^\infty w_\beta(k; nx) \frac{k}{n} = (P_n^{[\beta]} e_1)(x).$$

$$(J_n^{[\beta]} e_2)(x) = \sum_{k=1}^\infty w_\beta(k; nx) \frac{B(n-1, k+2)}{B(n+1, k)}$$

$$= \frac{n}{n-1} \sum_{k=1}^\infty w_\beta(k; nx) \frac{k^2}{n^2} + \frac{1}{n-1} \sum_{k=1}^\infty w_\beta(k; nx) \frac{k}{n}$$

$$= \frac{n}{n-1} (P_n^{[\beta]} e_2)(x) + \frac{1}{n-1} (P_n^{[\beta]} e_1)(x).$$

Taking into account (5) we easily obtain (6) and the proof is completed. □

We also introduce the s-th order central moment of the operator $J_n^{[\beta]}$, that is $J_n^{[\beta]} \varphi_x^s$, where $\varphi_x(t) = t - x$, $(x, t) \in \mathbb{R}_+ \times \mathbb{R}_+$. On the basis of (6), by a straightforward calculation, we obtain

Lemma 2. *The first and the second central moment of $J_n^{[\beta]}$, $n \geq 2$, operators are given by*

$$(J_n^{[\beta]} \varphi_x)(x) = \frac{\beta}{1-\beta} x,$$

$$(J_n^{[\beta]} \varphi_x^2)(x) = \left(\frac{n}{(n-1)(1-\beta)^2} - \frac{1+\beta}{1-\beta}\right) x^2$$

$$+ \frac{1+(1-\beta)^2}{(n-1)(1-\beta)^3} x, \tag{6}$$

respectively.

Since $\max\{x, x^2\} \leq x + x^2$, $(1-\beta)^2 \leq 1$ and

$$(1-\beta)^{-2} \leq (1-\beta)^{-3},$$

relation (7) implies

$$(J_n^{[\beta]} \varphi_x^2)(x) \leq \delta_{n,\beta}(x^2 + x), \tag{7}$$

where

$$\delta_{n,\beta} = \frac{n+2}{(n-1)(1-\beta)^3} - \frac{1}{1-\beta}. \tag{8}$$

3. Approximation properties

We establish the rate of convergence of the sequence $(J_n^{[\beta]} f)$ to f in terms of the rate of convergence of the test functions. The modulus of continuity $\omega_{[0,a]}(f; \cdot)$ is also involved, where

$$\omega_{[0,a]}(f; \delta) = \sup\{|f(x') - f(x'')| : x'x,'' \in [0, a],$$

$$|x' - x''| \leq \delta\},\ \delta \geq 0,$$

f continuous on the interval $[0, a]$.

Theorem 1. *Let $J_n^{[\beta]}$, $n \geq 2$, be defined by (4). For any function $f \in C_{\rho_0}(\mathbb{R}_+)$ one has*

$$|(J_n^{[\beta]} f)(x) - f(x)|$$

$$\leq \left(1 + \sqrt{x(x+1)}\right) \omega_{[0,a]} f \left(f; \sqrt{\delta_{n,\beta}}\right),\ x \in [0, a],$$

where $\delta_{n,\beta}$ is defined at (9).

Proof. We use the result of Shisha and Mond [10]. Considering the interval $[0, a]$, it says: if L is a linear positive operator defined on $C(I)$, $[0, a] \subseteq I$, then for every $x \in [0, a]$ and $\delta > 0$ one has

$$|(Lf)(x) - f(x)|$$

$$\leq |f(x)||(Le_0)(x) - 1|$$

$$+ \left((Le_0)(x) + \delta^{-1} \sqrt{(Le_0)(x)(L\varphi_x^2)(x)}\right) \omega_{[0,a]}(f; \delta).$$

Knowing that $J_n^{[\beta]} e_0 = e_0$ and relation (8), by choosing $\delta = \sqrt{\delta_{n,\beta}}$ the above inequality leads us to the desired result. □

Examining relation (6) and based on famous Korovkin theorem [8], it is clear that $(J_n^{[\beta]})_{n \geq 2}$ does not form an approximation process. The next step is to transform it for enjoying of this property. For each $n \geq 2$, the constant β will be replaced by a number $\beta_n \in [0, 1)$. If

$$\lim_{n\to\infty} \beta_n = 0, \tag{9}$$

then Lemma 1 ensures $\lim_n (J_n^{[\beta_n]} e_j)(x) = x^j$, $j = 0, 1, 2$, uniformly on any interval compact $K \subset \mathbb{R}_+$. Based on Korovkin criterion we can state

Theorem 2. *Let $J_n^{[\beta_n]}$, $n \geq 2$, be defined as in (4), where $(\beta_n)_{n\geq 2}$ satisfies (10). For any compact $K \subset \mathbb{R}_+$ and for each $f \in C_{\rho_0}(\mathbb{R}_+)$ one has*

$$\lim_n (J_n^{[\beta_n]} f)(x) = f(x), \ \textit{uniformly in } x \in K.$$

Our next concern is the study of statistical convergence of the sequence of operators. For the convenience of the reader, let recall the concept of this type of convergence.

The density of a set $S \subset \mathbb{N}$ is defined by

$$\delta(S) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_S(k),$$

provided the limit exists, where χ_S is the characteristic function of S. Following [4], a real sequence $x = (x_n)_{n\geq 1}$ is *statistically convergent* to a real number L if, for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0.$$

We write $st - \lim_n x_n = L$. It is known that any convergent sequence is statistically convergent, but not conversely. Closely related to this notion is A-statistical convergence where $A = (a_{n,k})$ is an infinite summability matrix. For a given sequence $x = (x_n)_{n\geq 1}$, the A-transform of x denoted by $Ax = (Ax)_{n\geq 1}$ is defined by

$$(Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k, \ n \in \mathbb{N},$$

provided the series converges for each n. Suppose that A is non-negative regular summability matrix, i.e., $a_{n,k} \geq 0$ and the matrix transformation of any convergent sequence preserves its limit.

The sequence $x = (x_n)_{n\geq 1}$ is *A-statistically convergent* to the real number L if, for every $\varepsilon > 0$, one has

$$\lim_{n\to\infty} \sum_{k\in I(\varepsilon)} a_{n,k} = 0,$$

where $I(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$.

We write $st_A - \lim_n x_n = l$, see e.g. [5].

Duman and Orhan [3; *Theorem 3*] proved the following weighted Korovkin-type theorem via A-statistical convergence.

Theorem 3. *Let $A = (a_{n,k})$ be a non-negative regular summability matrix and let ρ, ρ' weight functions such that*

$$\lim_{|x|\to\infty} \frac{\rho(x)}{\rho'(x)} = 0. \tag{10}$$

Assume that $(T_n)_{n\geq 1}$ is a sequence of positive linear operators from $C_\rho(\mathbb{R})$ into $C_{\rho'}(\mathbb{R})$. One has

$$st_A - \lim_n \|T_n f - f\|_{\rho'} = 0, \ f \in C_\rho(\mathbb{R}), \tag{11}$$

if and only if

$$st_A - \lim_n \|T_n F_k - F_k\|_\rho = 0, \ k = 0, 1, 2, \tag{12}$$

where $F_k(x) = x^k \rho(x)/(1 + x^2)$.

As regards to our sequence we prove the following result.

Theorem 4. *Let $A = (a_{n,k})$ be a non-negative regular summability matrix and $\lambda > 0$ be fixed. Let $J_n^{[\beta_n]}$, $n \geq 2$, be defined as in (4), where $(\beta_n)_{n\geq 2}$, $0 \leq \beta_n < 1$, satisfies*

$$st_A - \lim_n \beta_n = 0. \tag{13}$$

One has

$$st_A - \lim_n \|J_n^{[\beta_n]} f - f\|_{\rho_\lambda} = 0, \ f \in C_{\rho_0}(\mathbb{R}_+). \tag{14}$$

Proof. We use Theorem 3 which is still valid if one replaces the domain $\mathbb{R}$ by $\mathbb{R}_+$. Also, we choose the weight functions $\rho := \rho_0$ and $\rho' := \rho_\lambda$. Since $\lambda > 0$, relation (11) is fulfilled and one has $C_{\rho_0}(\mathbb{R}_+) \subseteq C_{\rho_\lambda}(\mathbb{R}_+)$. The test function are $F_k = e_k$, $k = 0, 1, 2$. Taking in view Lemma 1 we have $\|J_n^{[\beta_n]} e_0 - e_0\|_{\rho_0} = 0$,
$\|J_n^{[\beta_n]} e_1 - e_1\|_{\rho_0} \leq \dfrac{\beta_n}{1-\beta_n}$,
$\|J_n^{[\beta_n]} e_2 - e_2\|_{\rho_0} \leq \dfrac{1}{(1-\beta_n)^2} + \dfrac{1}{n(1-\beta_n)^3}$. Hypothesis (14) and above relations imply

$$st_A - \lim_n \|J_n^{[\beta_n]} e_k - e_k\|_{\rho_0} = 0, \ k = 0, 1, 2.$$

Since (13) holds, on the basis of Theorem 3, identity (15) takes place and this ends the proof. □

To increase the rate of convergence we can replace $J_n^{[\beta]}$ by its generalization of the r-th order, see [1].

Let $f \in C^r(\mathbb{R}_+)$ such that $e_s f^{(s)} \in C_{\rho_0}(\mathbb{R}_+)$ for $s = 0, 1, \ldots, r$, and let $T_r f(x; \cdot)$ be the r-th degree Taylor polynomial associated to the function f at the point $x \in \mathbb{R}_+$. For $n \geq 2$ and any $x \geq 0$ we define the linear operators

$$(J_{n,r}^{[\beta_n]} f)(x) = J_n^{[\beta_n]}(T_r f; x)$$

$$= \sum_{k=1}^{\infty} \frac{w_{\beta_n}(k; nx)}{B(n+1,k)} \sum_{s=0}^{r} \frac{1}{s!} \int_0^\infty f^{(s)}(t) \frac{(x-t)^s t^{k-1}}{(1+t)^{n+k+1}} dt$$

$$+ e^{-nx} f(0). \tag{15}$$

Clearly, $J_{n,0}^{[\beta_n]} = J_n^{[\beta_n]}$, $n \geq 2$. These operators keep the linearity property but loose the positivity.

In what follows, for $\alpha \in (0, 1]$ and $M > 0$, $Lip_M \alpha$ stands for the the subset of all Hölder continuous functions f on $\mathbb{R}_+$ with exponent α and constant M, i.e.,

$$|f(x) - f(y)| \leq M|x - y|^\alpha, \ (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Applying [1; *Theorem 1*] we obtain

Theorem 5. *Let A be a non-negative regular summability matrix. Let $r \in \mathbb{N}$ be fixed, $\alpha \in (0, 1]$ and $M > 0$.*

Let the operators $J_n^{[\beta_n]}$ *and* $J_{n,r}^{[\beta_n]}$, $n \geq 2$, *be defined by (4) and (16), respectively.*

If $x \geq 0$ *and* $\varphi_x^{r+\alpha} \in C_{\rho_0}(\mathbb{R}_+)$ *such that*

$$st_A - \lim_n \beta_n = 0 \text{ and } st_A - \lim_n (J_{n,r}^{[\beta_n]} \varphi_x^{r+\alpha})(x) = 0,$$

then

$$st_A - \lim_n |(J_{n,r}^{[\beta_n]} f)(x) - f(x)| = 0$$

holds for any function $f \in C^r(\mathbb{R}_+) \cap C_{\rho_0}(\mathbb{R}_+)$ *with the properties* $e_s f^{(s)} \in C_{\rho_0}(\mathbb{R}_+)$, $s = 0, 1, \ldots, r$ *and* $f^{(r)} \in Lip_M \alpha$.

In other words, considering that the initial approximation process is A-statistically pointwise convergent, the result says that the property is inherited by the new sequence $J_{n,r}^{[\beta_n]}$ under additional conditions imposed on the smooth signal f.

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