Applied Mathematics & Information Sciences
An International Journal

A Scorza–Dragoni approach to second-order boundary value problems in abstract spaces

Jan Andres[1*], *Luisa Malaguti*[2] *and Martina Pavlačková*[1]

[1] Fac. of Science, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic
[2] Dept. of Engineering Sciences and Methods, University of Modena and Reggio Emilia, Via G. Amendola, 2 - pad. Morselli, I-42122 Reggio Emilia, Italy

Received: Jul 8, 2011; Revised Oct. 4, 2011; Accepted Jan. 26, 2012
Published online: 1 May 2012

Abstract: The existence and localization of strong (Carathéodory) solutions is proved for a second-order Floquet problem in a Banach space. The result is obtained by combining a continuation principle together with a bounding (Liapunov-like) functions approach. The application of the Scorza–Dragoni type technique allows us to use strictly localized transversality conditions.

Keywords: Second-order Floquet problem, Scorza–Dragoni type results, bounding functions, solutions in a given set, evolution equations, condensing multivalued operators.

1. Introduction

The main aim of this paper is to present a theorem concerning the existence and localization of solutions to second-order Floquet boundary value problems for upper-Carathéodory differential inclusions in Banach spaces. For some related references, see e.g. [6, 7] and those quoted in [3]. The novelty consists in the application of strictly localized Liapunov-type bounding functions guaranteeing the transversality behaviour of trajectories on bound sets, i.e. the fixed points free property required in the applied degree arguments.

The first-order problems were considered in [6, 7]. The same second-order problem was already studied by ourselves via a bound sets approach in [3]. The conditions concerning bounding functions were not however imposed directly on the boundaries of bound sets like here, but at some vicinity of them. On the other hand, such a strict localization, allowed by means of the Scorza–Dragoni type technique developed in [15], demands a higher regularity of applied bounding functions which brings here some obstructions. Nevertheless, our result is new even in a single-valued case of equations.

Hence, let E be a separable Banach space (with the norm $\|\cdot\|$) satisfying the Radon–Nikodym property (e.g. reflexivity) and let us consider the Floquet boundary value problem (b.v.p.)

$$\left.\begin{array}{l}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in F(t,x(t),\dot{x}(t)),\\ \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0),\ \dot{x}(T)=N\dot{x}(0).\end{array}\right\} \quad (1)$$

Throughout the paper, we assume (for the related definitions, see the next Section 2) that

(1_i) $A, B\colon [0,T] \to \mathcal{L}(E)$ are Bochner integrable, where $\mathcal{L}(E)$ stands for the Banach space of all linear, bounded transformations $L\colon E \to E$ endowed with the sup-norm,

(1_{ii}) $F\colon [0,T]\times E\times E \multimap E$ is an upper-Carathéodory multivalued mapping,

(1_{iii}) $M,\ N\in\mathcal{L}(E)$ with M non-singular.

Let us note that in the entire paper, all derivatives will be always understood in the sense of Fréchet, and by the measurability, we mean the one with respect to the Lebesque σ-algebra in $[0,T]$ and the Borel σ-algebra in E.

The notion of a solution will be understood in a strong (i.e. Carathéodory) sense. Namely, by a *solution* of problem (1), we mean a function $x\colon[0,T]\to E$ whose first derivative $\dot{x}(\cdot)$ is absolutely continuous and satisfies (1), for almost all $t\in[0,T]$.

* Corresponding author: e-mail: jan.andres@upol.cz

The solution of the b.v.p. (1) will be obtained as the limit of a sequence of solutions of approximating problems that we construct by means of a Scorza–Dragoni type result developed in [15]. The approximating problems will be treated by means of the continuation principle developed in [3].

For the main result (Theorem 1) in Section 3, we collect all necessary technicalities and applied tools in the next Section 2. Concluding remarks in Section 4 concern an illustrative example of the application of Theorem 1. Since the applied bounding function V takes the form $V(x) := \frac{1}{2}\left(\|x\|^2 - r\right)$ and since one condition in Theorem 1 deals with $V \in C^2(E,\mathbb{R})$, we only restrict ourselves there to Hilbert spaces, where $\ddot{V}(x) \equiv Id$. In particular, we take $E := L^2(\Omega)$, where Ω is a suitable nonempty, bounded domain in $\mathbb{R}^n$.

2. Preliminaries

Let E be a Banach space having the Radon-Nikodym property (see e.g. [13, pp. 694–695]) and $[0,T] \subset \mathbb{R}$ be a closed interval. By the symbol $L^1([0,T],E)$, we shall mean the set of all Bochner integrable functions $x\colon [0,T] \to E$. For the definition and properties, see e.g. [13, pp. 693–701]. The symbol $AC^1([0,T],E)$ will denote the set of functions $x\colon [0,T] \to E$ whose first derivative $\dot{x}(\cdot)$ is absolutely continuous. Then $\ddot{x} \in L^1([0,T],E)$ and the fundamental theorem of calculus (the Newton–Leibniz formula) holds (see e.g. [1, pp. 243–244], [13, pp. 695–696]). In the sequel, we shall always consider $AC^1([0,T],E)$ as a subspace of the Banach space $C^1([0,T],E)$.

Given $C \subset E$ and $\varepsilon > 0$, the symbol $B(C,\varepsilon)$ will denote, as usually, the set $C + \varepsilon B$, where B is the open unit ball in E, i.e. $B = \{x \in E \mid ||x|| < 1\}$. In what follows, the symbol μ will denote the Lebesque measure on $\mathbb{R}$.

For each $L \in \mathcal{L}(E \times E)$, there exist unique $L_{ij} \in \mathcal{L}(E)$, $i,j = 1,2$, such that

$$L(x,y) = (L_{11}x + L_{12}y,\, L_{21}x + L_{22}y)\,,$$

where $(x,y) \in E \times E$. For the sake of simplicity, we shall use the notation

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}.$$

Let E' be the Banach space dual to E and let us denote by $\langle \cdot,\cdot \rangle$ the pairing (the duality relation) between E and E', i.e., for all $\Phi \in E'$ and $x \in E$, we put $\Phi(x){:=}\langle \Phi, x \rangle$.

We shall also need the following definitions and notions from multivalued analysis. Let X, Y be two metric spaces. We say that F is a *multivalued mapping* from X to Y (written $F\colon X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is given. We associate with F its graph Γ_F, the subset of $X \times Y$, defined by $\Gamma_F := \{(x,y) \in X \times Y \mid y \in F(x)\}$.

A multivalued mapping $F\colon X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open subset $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X.

Let $J \subset \mathbf{R}$ be a compact interval. A mapping $F\colon J \multimap Y$ with closed values, where Y is a separable metric space, is called *measurable* if, for each open subset $U \subset Y$, the set $\{t \in J \mid F(t) \subset U\}$ belongs to a σ-algebra of subsets of J.

If $F\colon J \multimap Y$ is compact-valued and $Y = E$ is a separable Banach space, then the notion of measurability coincides with those of strong measurability (cf. e.g. [11, Theorem 1.3.1]) as well as of weak measurability (cf. e.g. [1, Proposition I.3.45.4]). For the definitions and more details, see e.g. [1,10,11].

A multivalued mapping $F\colon X \multimap Y$ is called *compact* if the set $F(X) = \bigcup_{x \in X} F(x)$ is contained in a compact subset of Y and it is called *quasi-compact* if it maps compact sets onto relatively compact sets.

The relationship between upper semicontinuous mappings and quasi-compact mappings with closed graphs is expressed by the following proposition (see, e.g., [11]).

Proposition 1. *Let X, Y be metric spaces and $F\colon X \multimap Y$ be a quasi-compact mapping with a closed graph. Then F is u.s.c.*

Let $J = [0,T]$ be a given compact interval. A multivalued mapping $F\colon J \times X \multimap Y$, where Y is a separable Banach space, is called an *upper-Carathéodory mapping* if the map $F(\cdot,x)\colon J \multimap Y$ is measurable, for all $x \in X$, the map $F(t,\cdot)\colon X \multimap Y$ is u.s.c., for almost all $t \in J$, and the set $F(t,x)$ is compact and convex, for all $(t,x) \in J \times X$.

The technique that will be used for proving the existence and localization result consists in constructing a sequence of approximating problems. This construction will be made on the basis of the Scorza–Dragoni type result in [15] (cf. [5]).

Definition 1. *An upper-Carathéodory mapping $F\colon [0,T] \times X \times X \multimap X$ is said to have the Scorza–Dragoni property if there exists a multivalued mapping $F_0\colon [0,T] \times X \times X \multimap X \cup \{\emptyset\}$ with compact, convex values having the following properties:*

(i) $F_0(t,x,y) \subset F(t,x,y)$, for all $(t,x,y) \in [0,T] \times X \times X$,

(ii) if $u,v\colon [0,T] \to X$ are measurable functions with $v(t) \in F(t,u(t),\dot{u}(t))$, for a.a. $t \in [0,T]$, then also $v(t) \in F_0(t,u(t),\dot{u}(t))$, for a.a. $t \in [0,T]$,

(iii) for every $\varepsilon > 0$, there exists a closed $I_\varepsilon \subset [0,T]$ such that $\mu([0,T] \setminus I_\varepsilon) < \varepsilon$, $F_0(t,x,y) \neq \emptyset$, for all $(t,x,y) \in I_\varepsilon \times X \times X$, and F_0 is u.s.c. on $I_\varepsilon \times X \times X$.

The following two propositions are crucial in our investigation. The first one is almost a direct consequence of the main result in [15] (cf. [5] and [7, Theorem 2.1]); precisely, the quoted results deal with a multivalued map $F\colon [0,T] \times X \multimap X$, but it is straightforward to see that they are still valid in this case, where F is defined on

$[0,T] \times X \times X$. The second one allows us to construct a sequence of approximating problems of (1).

Proposition 2. *Let X be a separable Banach space and $F\colon [0,T] \times X \times X \multimap X$ be an upper-Carathéodory mapping. If F is globally measurable or quasi-compact, then F has the Scorza–Dragoni property.*

Proposition 3. (cf. [7, Theorem 2.2]) *Let X be a Banach space and $K \subset X$ a nonempty, open, convex, bounded set such that $0 \in K$. Moreover, let $\varepsilon > 0$ and $V\colon X \to \mathbb{R}$ be a Fréchet differentiable function with $\dot V$ Lipschitzian in $\overline{B(\partial K, \varepsilon)}$ satisfying*

$(H1)$ $V|_{\partial K} = 0$,
$(H2)$ $V(x) \le 0$, *for all* $x \in \overline{K}$,
$(H3)$ $\|\dot V(x)\| \ge \delta$, *for all* $x \in \partial K$, *where* $\delta > 0$ *is given.*

Then there exists a bounded Lipschitzian function
$\phi\colon \overline{B(\partial K, \varepsilon)} \to X$
such that $\langle \dot V_x, \phi(x) \rangle = 1$, *for every* $x \in \overline{B(\partial K, \varepsilon)}$

Example 1. Let us note that the function $x \to \phi(x)||\dot V_x||$, where ϕ and $\dot V_x$ occur in Proposition 3, is Lipschitzian and bounded in $\overline{B(\partial K, \varepsilon)}$. The symbol $\dot V_x$ denotes as usually the first Fréchet derivative of V at x.

For more details concerning multivalued analysis, see e.g. [1,10,11].

Definition 2. *Let N be a partially ordered set, E be a Banach space and let $P(E)$ denote the family of all subsets of E. A function $\beta\colon P(E) \to N$ is called a measure of non-compactness (m.n.c.) in E if $\beta(\overline{co\ \Omega}) = \beta(\Omega)$, for all $\Omega \in P(E)$, where $\overline{co\ \Omega}$ denotes the closed convex hull of Ω.*

A m.n.c. β is called:

(i) monotone if $\beta(\Omega_1) \le \beta(\Omega_2)$, for all $\Omega_1 \subset \Omega_2 \subset E$,
(ii) nonsingular if $\beta(\{x\} \cup \Omega) = \beta(\Omega)$, for all $x \in E$ and $\Omega \subset E$,
(iii) invariant with respect to the union with compact sets if $\beta(K \cup \Omega) = \beta(\Omega)$, for every relatively compact $K \subset E$ and every $\Omega \subset E$,
(iv) regular when $\beta(\Omega) = 0$ if and only if Ω is relatively compact.

It is obvious that the m.n.c. which is invariant with respect to the union with compact sets is also nonsingular.

The typical example of an m.n.c. is the *Hausdorff measure of noncompactness* γ defined, for all $\Omega \subset E$ by

$$\gamma(\Omega) :=$$

$$\inf\{\varepsilon > 0 \mid \exists x_1, \dots, x_n \in E\colon \Omega \subset \cup_{i=1}^n B(\{x_i\}, \varepsilon)\}.$$

The Hausdorff m.n.c. is monotone, invariant with respect to the union with compact sets and regular. Moreover, if $L \in \mathcal{L}(E)$ and $\Omega \subset E$, then (see, e.g., [11])

$$\gamma(L\Omega) \le \|L\|_{\mathcal{L}(E)} \gamma(\Omega). \tag{2}$$

Let $\{f_n\} \subset L([0,T], E)$ be such that $||f_n(t)|| \le \alpha(t)$, $\gamma(\{f_n(t)\}) \le c(t)$, for a.a. $t \in [0,T]$, all $n \in \mathbb{N}$ and suitable $\alpha, c \in L([0,T], \mathbb{R})$, then (cf. [11])

$$\gamma\left(\left\{\int_0^T f_n(t)\, dt\right\}\right) \le \int_0^T c(t)\, dt. \tag{3}$$

Moreover, for all subsets Ω of E (see e.g. [4]),

$$\gamma(\cup_{\lambda \in [0,1]} \lambda\Omega) = \gamma(\Omega).$$

Let us now introduce the function

$$\mu(\Omega) := \max_{\{w_n\}_n \subset \Omega} \Big(\sup_{t \in [0,T]} [\gamma(\{w_n(t)\}_n) + \gamma(\{\dot w_n(t)\}_n)], \\ \mathrm{mod}_C(\{w_n\}_n) + \mathrm{mod}_C(\{\dot w_n\}_n) \Big), \tag{4}$$

defined on the bounded $\Omega \subset C^1([0,T], E)$, where the ordering is induced by the positive cone in $\mathbb{R}^2$ and where $\mathrm{mod}_C(\Omega)$ denotes the modulus of continuity of a subset $\Omega \subset C([0,T], E)$.[1] It was proved in [3] that the function μ given by (4) is an m.n.c. in $C^1([0,T], E)$ that is monotone, invariant with respect to the union with compact sets and regular.

Definition 3. *Let E be a Banach space and $X \subset E$. A multivalued mapping $F\colon X \multimap E$ with compact values is called condensing with respect to an m.n.c. β (shortly, β-condensing) if, for every $\Omega \subset X$ such that $\beta(F(\Omega)) \ge \beta(\Omega)$, it holds that Ω is relatively compact.*

A family of mappings $G\colon X \times [0,1] \multimap E$ with compact values is called β-condensing if, for every $\Omega \subset X$ such that $\beta(G(\Omega \times [0,1])) \ge \beta(\Omega)$, it holds that Ω is relatively compact.

It will be also convenient to recall some basic facts concerning evolution equations. For a suitable introduction and more details, we refer, e.g., to [8,12,16].

Hence, let $C\colon [0,T] \to \mathcal{L}(E)$ be Bochner integrable and let $f \in L([0,T], E)$. Given $x_0 \in E$, consider the linear initial value problem

$$\dot x(t) = C(t)x(t) + f(t), \quad x(0) = x_0. \tag{5}$$

It is well-known (see, e.g., [8]) that, for the uniquely solvable problem (5), there exists the evolution operator

$$\{U(t,s)\}_{(t,s) \in \Delta},$$

where $\Delta := \{(t,s)\colon 0 \le s \le t \le T\}$, such that

$$U(t,s) \in \mathcal{L}(E) \quad \text{and} \quad \|U(t,s)\| \le \mathrm{e}^{\int_s^t \|C(\tau)\|\, d\tau},$$

$$\text{for all } (t,s) \in \Delta; \tag{6}$$

in addition, the unique solution $x(\cdot)$ of (5) is given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s) f(s)\, ds, \qquad t \in [0,T].$$

[1] The m.n.c. $\mathrm{mod}_C(\Omega)$ is a monotone, nonsingular and algebraically subadditive on $C([0,T], E)$ (cf. e.g. [11]) and it is equal to zero if and only if all the elements $x \in \Omega$ are equi-continuos.

Given $D \in \mathcal{L}(E)$, the linear Floquet b.v.p.

$$\left.\begin{aligned} \dot{x}(t) &= C(t)x(t) + f(t), \\ x(T) &= Dx(0), \end{aligned}\right\} \tag{7}$$

associated with the equation in (5), satisfies the following property.

Lemma 1. (cf. [4]) *If the linear operator $D - U(T,0)$ is invertible, then (7) admits a unique solution given, for all $t \in [0,T]$, by*

$$x(t) = U(t,0)\Big[D - U(T,0)\Big]^{-1} \int_0^T U(T,\tau)f(\tau)\,d\tau + \int_0^t U(t,\tau)f(\tau)\,d\tau. \tag{8}$$

Example 2. Denoting

$$\Lambda := \mathrm{e}^{\int_0^T \|C(s)\|\,ds}, \quad \Gamma := \|[D - U(T,0)]^{-1}\|,$$

we obtain, in view of (6), (8) and the growth estimate imposed on $C(t)$, the following inequality for the solution $x(\cdot)$ of (7):

$$\|x(t)\| \le \Lambda\,(\Lambda\Gamma + 1) \int_0^T \|f(s)\|\,ds. \tag{9}$$

Now, consider the second-order linear Floquet b.v.p.

$$\left.\begin{aligned} &\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = f(t), \\ &\text{for a.a. } t \in [0,T], \\ &x(T) = Mx(0),\ \dot{x}(T) = N\dot{x}(0), \end{aligned}\right\}, \tag{10}$$

where A, B are Bochner integrable and $f \in L^1([0,T],E)$, and let

$\|(x,y)\|_{E\times E} := \sqrt{\|x\|^2 + \|y\|^2}$, for all $x, y \in E$.

Problem (10) is equivalent to the following first-order linear one

$$\left.\begin{aligned} &\dot{\xi}(t) + C(t)\xi(t) = h(t), \\ &\text{for a.a. } t \in [0,T], \\ &\xi(T) = \tilde{D}\xi(0), \end{aligned}\right\} \tag{11}$$

where

$$\xi = (x,y) = (x,\dot{x}), \tag{12}$$

$$h(t) = (0, f(t)), \tag{13}$$

$$C(t)\colon E\times E \to E\times E, (x,y) \longmapsto (-y, B(t)x + A(t)y) \tag{14}$$

and

$$\tilde{D}\colon E \times E \to E \times E, \quad (x,y) \longmapsto (Mx, Ny)\,. \tag{15}$$

Let us denote, for all $(t,s) \in [0,T] \times [0,T]$, by

$$U(t,s) := \begin{pmatrix} U_{11}(t,s) & U_{12}(t,s) \\ U_{21}(t,s) & U_{22}(t,s) \end{pmatrix}$$

the evolution operator associated with

$$\left.\begin{aligned} &\dot{\xi}(t) + C(t)\xi(t) = h(t), \quad \text{for a.a. } t \in [0,T], \\ &\xi(0) = \xi_0, \end{aligned}\right\} \tag{16}$$

where ξ, h and C are defined by relations (12), (13) and (14), respectively, and $\xi_0 \in E \times E$. It is easy to see that $\|C(t)\| \le 1 + \|A(t)\| + \|B(t)\|$ and, according to (6), we obtain

$$\|U(t,s)\| \le \mathrm{e}^{\int_0^T (1+\|A(t)\|+\|B(t)\|)\,dt}, \quad \text{for all } (t,s) \in \Delta.$$

Consequently, for all $i,j = 1,2$,

$$\|U_{ij}(t,s)\| \le \mathrm{e}^{\int_0^T (1+\|A(t)\|+\|B(t)\|)\,dt}, \quad \text{for all } (t,s) \in \Delta. \tag{17}$$

Moreover, if we assume that $\tilde{D} - U(T,0)$ is invertible, denote

$$[\tilde{D} - U(T,0)]^{-1} := \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

and put

$$k := \|[\tilde{D} - U(T,0)]^{-1}\|, \tag{18}$$

then $\|K_{ij}\| \le k$, for $i,j = 1,2$, and the solution $x(\cdot)$ of (10) and its derivative $\dot{x}(\cdot)$ take, for all $t \in [0,T]$, the forms

$$\begin{aligned} x(t) = A_1(t) &\int_0^T U_{12}(T,\tau)f(\tau)\,d\tau \\ +A_2(t) &\int_0^T U_{22}(T,\tau)f(\tau)\,d\tau \\ + &\int_0^t U_{12}(t,\tau)f(\tau)\,d\tau, \end{aligned} \tag{19}$$

and

$$\begin{aligned} \dot{x}(t) = A_3(t) &\int_0^T U_{12}(T,\tau)f(\tau)\,d\tau \\ +A_4(t) &\int_0^T U_{22}(T,\tau)f(\tau)\,d\tau \\ + &\int_0^t U_{22}(t,\tau)f(\tau)\,d\tau, \end{aligned} \tag{20}$$

where

$$\begin{aligned} A_1(t) &:= U_{11}(t,0)K_{11} + U_{12}(t,0)K_{21}, \\ A_2(t) &:= U_{11}(t,0)K_{12} + U_{12}(t,0)K_{22}, \\ A_3(t) &:= U_{21}(t,0)K_{11} + U_{22}(t,0)K_{21}, \\ A_4(t) &:= U_{21}(t,0)K_{12} + U_{22}(t,0)K_{22}, \end{aligned}$$

for all $t \in [0,T]$. It holds that

$$\|A_i(t)\| \le 2k\mathrm{e}^{\int_0^T (1+\|A(t)\|+\|B(t)\|)\,dt}, \quad \text{for } i = 1,2,3,4 \text{ and } t \in [0,T]. \tag{21}$$

If there exists $\alpha \in L^1([0,T],[0,\infty))$ such that $\|f(t)\| \le \alpha(t)$, for a.a. $t \in [0,T]$, then it immediately follows from Remark 2 that the following estimates hold for each solution $x(\cdot)$ of (10) and its derivative $\dot{x}(\cdot)$:

$$\|x(t)\| \le Z\,(4Zk + 1) \int_0^T \alpha(s)\,ds$$

and

$$\|\dot{x}(t)\| \le Z\,(4Zk+1)\int_0^T \alpha(s)\,ds,$$

where

$$Z := \mathrm{e}^{\int_0^T (\|A(s)\|+\|B(s)\|+1)\,ds} \tag{22}$$

with k defined in (18).

The proof of the main result (cf. Theorem 1 below) will be based on the following slight modification of the continuation principle developed in [3]. Since the proof of this modified version differs from the one in [3] only slightly in technical details, we omit it here.

Proposition 4. *Let us consider the b.v.p.*

$$\left.\begin{array}{l}\ddot{x}(t) \in \varphi(t, x(t), \dot{x}(t)), \text{ for a.a. } t \in [0,T],\\ x \in S,\end{array}\right\} \tag{23}$$

where $\varphi\colon [0,T]\times E\times E \multimap E$ is an upper-Carathéodory mapping and $S \subset AC^1([0,T],E)$. Let $H\colon [0,T]\times E\times E\times E\times E\times[0,1] \multimap E$ be an upper-Carathéodory mapping such that

$$H(t,c,d,c,d,1) \subset \varphi(t,c,d), \text{ for all } (t,c,d) \in [0,T]\times E\times E.$$

Moreover, assume that the following conditions hold:

(i) There exist a closed set $S_1 \subset S$ and a closed, convex set $Q \subset C^1([0,T],E)$ with a non-empty interior $Int\,Q$ such that each associated problem

$$\left.\begin{array}{l}\ddot{x}(t) \in H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda),\\ \text{for a.a. } t \in [0,T],\\ x \in S_1,\end{array}\right\}$$

where $q \in Q$ and $\lambda \in [0,1]$, has a non-empty, convex set of solutions (denoted by $\mathfrak{T}(q,\lambda)$).

(ii) For every non-empty, bounded set $\Omega \subset E\times E\times E\times E$, there exists $\nu_\Omega \in L^1([0,T],[0,\infty))$ such that

$$\|H(t,x,y,u,v,\lambda)\| \le \nu_\Omega(t),$$

for a.a. $t \in [0,T]$ and all $(x,y,u,v) \in \Omega$ and $\lambda \in [0,1]$.

(iii) The solution mapping $\mathfrak{T}$ is quasi-compact and μ-condensing with respect to a monotone and nonsingular m.n.c. μ defined on $C^1([0,T],E)$.

(iv) For each $q \in Q$, the set of solutions of the problem $P(q,0)$ is a subset of $Int\,Q$, i.e. $\mathfrak{T}(q,0) \subset Int\,Q$, for all $q \in Q$.

(v) For each $\lambda \in (0,1)$, the solution mapping $\mathfrak{T}(\cdot,\lambda)$ has no fixed points on the boundary ∂Q of Q.

Then the b.v.p. (23) has a solution in Q.

3. Main result

Combining the foregoing continuation principle with the Scorza–Dragoni type technique (cf. Proposition 2), we are ready to state the main result of the paper concerning the solvability and localization of a solution of the multivalued Floquet problem (1).

For this purpose, let us consider again the single-valued Floquet b.v.p. (10) which is equivalent to the first-order Floquet b.v.p. (11), provided ξ, $h(\cdot)$, $C(\cdot)$ and $\tilde{D}$ are defined by relations (12)–(15). Moreover, let $U(t,s)$ be the evolution operator associated with (16).

Theorem 1. *Consider the Floquet b.v.p. (1), under conditions (1_i)–(1_{iii}), and suppose that F has the Scorza–Dragoni property. Assume that an open, convex, bounded set $K \subset E$ containing 0 exists such that $M\partial K = \partial K$. Furthermore, let the following conditions (2_i)–(2_{iv}) be satisfied:*

(2_i) *$\tilde{D} - U(T,0)$ is invertible.*

(2_{ii}) *$\gamma\left(F(t, \Omega_1\times\Omega_2)\right) \le g(t)\left(\gamma(\Omega_1)+\gamma(\Omega_2)\right)$, for a.a. $t \in [0,T]$ and each bounded $\Omega_1, \Omega_2 \subset E$, where $g \in L^1([0,T],[0,\infty))$ and γ is the Hausdorff m.n.c. in E.*

(2_{iii}) *For every non-empty, bounded $\Omega \subset E$, there exists $\nu_\Omega \in L^1([0,T],[0,\infty))$ such that*

$$\|F(t,x,y)\| \le \nu_\Omega(t), \tag{24}$$

for a.a. $t \in [0,T]$ and all $(x,y) \in \Omega\times E$.

(2_{iv}) *The inequality*

$$2e^{\int_0^T (1+\|A(t)\|+\|B(t)\|)\,dt} \times \left(4ke^{\int_0^T (1+\|A(t)\|+\|B(t)\|)\,dt}+1\right) \times \|g\|_{L^1([0,T],[0,\infty))} < 1$$

holds, where k is defined in (18). $P(q,\lambda)$

Furthermore, let there exist $\varepsilon > 0$ and a function $V \in C^2(E,\mathbb{R})$, i.e. a twice continuously differentiable function in the sense of Fréchet, satisfying (H1)–(H3) with Fréchet derivative $\dot{V}$ Lipschitzian in $\overline{B(\partial K,\varepsilon)}$.[2] *Moreover, let there exist $h > 0$ such that*

$$\left\langle \ddot{V}_x(v), v\right\rangle \ge 0, forall x \in B(\partial K,h),\ v \in E, \tag{25}$$

where $\ddot{V}_x(v)$ denotes the second Fréchet derivative of V at x in the direction $(v,v) \in E\times E$. Finally, let

$$\langle \dot{V}_x, w\rangle > 0, \tag{26}$$

and

$$\left\langle \dot{V}_{Mx}, Nv\right\rangle \cdot \left\langle \dot{V}_x, v\right\rangle > 0,$$

or

$$\left\langle \dot{V}_{Mx}, Nv\right\rangle = \left\langle \dot{V}_x, v\right\rangle = 0, \tag{27}$$

and for all $x \in \partial K$, $t \in (0,T)$, $v \in E$, $\lambda \in (0,1)$ and $w \in \lambda F(t,x,v) - A(t)v - B(t)x$.

Then the Floquet b.v.p. (1) admits a solution whose values are located in $\overline{K}$.

[2] Since a C^2-function V has only a locally Lipschitzian Fréchet derivative $\dot{V}$ (cf. e.g. [13]), we had to assume explicitly the global Lipschitzianity of $\dot{V}$ in a noncompact set $\overline{B(\partial K,\varepsilon)}$.

Proof. Since the proof of this result is rather technical, it will be divided into several steps. At first, let us define the sequence of approximating problems. For this purpose, let us consider a continuous function $\tau : E \to [0,1]$ such that $\tau(x) = 0$, for all $x \in E \setminus B(\partial K, \varepsilon)$, and $\tau(x) = 1$, for all $x \in \overline{B(\partial K, \frac{\varepsilon}{2})}$. According to Proposition 3 (see also Remark 1), the function $\hat{\phi}: E \to E$, where

$$\hat{\phi}(x) = \begin{cases} \tau(x) \cdot \phi(x) \cdot \|\dot{V}_x\|, & \text{for all } x \in \overline{B(\partial K, \varepsilon)}, \\ 0, & \text{for all } x \in E \setminus \overline{B(\partial K, \varepsilon)}, \end{cases}$$

is well-defined, continuous and bounded. Since $(t,y) \to A(t)y$ and $(t,x) \to B(t)x$ are Carathéodory maps, on $[0,T]\times E$, they are also almost-continuous (cf. [14]). Therefore, the mapping $(t,x,y) \multimap -A(t)y - B(t)x + F(t,x,y)$ has the Scorza–Dragoni property. So, we are able to find a decreasing sequence $\{J_m\}$ of subsets of $[0,T]$ and a mapping $F_0 : [0,T] \times E \times E \multimap E \cup \{\emptyset\}$ such that, for all $m \in \mathbb{N}$,

– $\mu(J_m) < \frac{1}{m}$,
– $[0,T] \setminus J_m$ is closed,
– $(t,x,y) \multimap -A(t)y - B(t)x + F_0(t,x,y)$ is u.s.c. on $[0,T] \setminus J_m \times E \times E$,
– $\nu_{\overline{K}}(t)$ is continuous in $[0,T] \setminus J_m$.

If we put $J = \cap_{m=1}^{\infty} J_m$, then $\mu(J) = 0$, $F_0(t,x,y) \neq \emptyset$, for all $t \in [0,T]\setminus J$ and the mapping $(t,x,y) \multimap -A(t)y - B(t)x + F_0(t,x,y)$ is u.s.c. on $[0,T] \setminus J \times E \times E$.

For each $m \in \mathbb{N}$, let us define the mapping $F_m : [0,T] \times E \times E \multimap E$ with compact, convex values by the formula

$$F_m(t,x,y) := \begin{cases} F_0(t,x,y) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(x), \\ \quad \text{for all } (t,x,y) \in [0,T] \setminus J \times E \times E, \\ \\ -p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(x), \\ \quad \text{for all } (t,x,y) \in J \times E \times E, \end{cases}$$

where

$$\begin{aligned} p(t) &= : -\nu_{\overline{K}}(t) - \|A(t)\| Z(4Zk+1) \|\nu_{\overline{K}}\|_{L^1([0,T],[0,\infty))} \\ &\quad -\|B(t)\| \left(\|\partial K\| + \frac{\varepsilon}{2}\right). \end{aligned} \tag{28}$$

with k and Z defined by (18) and (22), respectively.

Let us consider the b.v.p.

$$\left.\begin{array}{l} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F_m(t, x(t), \dot{x}(t)), \\ \quad \text{for a.a. } t \in [0,T], \\ x(T) = Mx(0),\ \dot{x}(T) = N\dot{x}(0). \end{array}\right\}$$

Now, let us verify the solvability of problems (P_m). Let $m \in \mathbb{N}$ be fixed. Since F_0 is globally u.s.c. on $[0,T] \setminus J \times E \times E$, $F_m(\cdot, x, y)$ is measurable, for each $(x,y) \in E \times E$, and, due to the continuity of $\hat{\phi}$, $F_m(t,\cdot,\cdot)$ is u.s.c., for all $t \in [0,T] \setminus J$. Therefore, F_m is an upper-Carathéodory mapping. Moreover, let us define the upper-Carathéodory mapping $H_m: [0,T] \times E \times E \times E \times E \times [0,1] \multimap E$ by the formula

$$H_m(t,x,y,u,v,\lambda) \equiv H_m(t,u,v,\lambda)$$
$$:= \begin{cases} \lambda F_0(t,u,v) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(u), \\ \quad \text{for all } (t,x,y,u,v,\lambda) \in [0,T] \setminus J \times E^4 \times [0,1], \\ \\ -p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(u), \\ \quad \text{for all } (t,x,y,u,v,\lambda) \in J \times E^4 \times [0,1]. \end{cases}$$

Let us show that, when $m \in \mathbb{N}$ is sufficiently large, all assumptions of Proposition 4 (for $\varphi(t,x,\dot{x}) := F_m(t,x,\dot{x}) - A(t)\dot{x} - B(t)x$) are satisfied.

For this purpose, let us define the closed set $S = S_1$ by

$$S := \{x \in AC^1([0,T],E): x(T) = Mx(0),\ \dot{x}(T) = N\dot{x}(0)\}$$

and let the set Q of candidate solutions be defined as $Q := C^1([0,T], \overline{K})$. Because of the convexity of K, the set Q is closed and convex.

For all $q \in Q$ and $\lambda \in [0,1]$, consider still the associated fully linearized problem

$$\left.\begin{array}{l} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in H_m(t, q(t), \dot{q}(t), \lambda), \\ \text{for a.a. } t \in [0,T], \\ x(T) = Mx(0),\ \dot{x}(T) = N\dot{x}(0), \end{array}\right\}$$

and denote by $\mathfrak{T}_m$ the solution mapping which assigns to each $(q,\lambda) \in Q \times [0,1]$ the set of solutions of $P_m(q,\lambda)$.

ad (i) In order to verify condition (i) in Proposition 4, we need to show that, for each $(q,\lambda) \in Q \times [0,1]$, the problem $P_m(q,\lambda)$ is solvable with a convex set of solutions. So, let $(q,\lambda) \in Q \times [0,1]$ be arbitrary and let $f_q(\cdot)$ be a measurable selection of $H_m(\cdot, q(\cdot), \dot{q}(\cdot), \lambda)$. Then, according to (2_i), Lemma 1 and the equivalence, stated in Section 2, between the b.v.p. (10) and (11), the single-valued Floquet problem

$$\left.\begin{array}{l} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = f_q(t), \\ \quad \text{for a.a. } t \in [0,T], \\ x(T) = Mx(0),\ \dot{x}(T) = N\dot{x}(0) \end{array}\right\}$$

admits a unique solution which is one of solutions of $P_m(q,\lambda)$. Thus, the set of solutions of $P_m(q,\lambda)$ is nonempty. The convexity of the solution sets follows immediately from the definition of H_m and the fact that problems $P_m(q,\lambda)$ are fully linearized.

ad (ii) Let $\Omega \subset E \times E \times E \times E$ be bounded. Then, there exists a bounded $\Omega_1 \subset E$ such that $\Omega \subset \Omega_1 \times \Omega_1 \times \Omega_1 \times \Omega_1$ and, according to (2_{iii}) and the definition of H_m, there exists $\hat{J} \subset [0,T]$ with $\mu\left(\hat{J}\right) = 0$ such that, for all $t \in [0,T] \setminus \left(J \cup \hat{J}\right)$, $(x,y,u,v) \in \Omega$ and $\lambda \in [0,1]$, (P_m)

$$\|H_m(t,u,v,\lambda) - A(t)y - B(t)x\| \leq \nu_{\Omega_1}(t)$$
$$+2p(t) \cdot \max_{x \in \overline{B(\partial K, \varepsilon)}} \|\hat{\phi}(x)\| + \|A(t)\| \cdot \|y\| + \|B(t)\| \cdot \|x\|.$$

Therefore, the mapping $H_m(t,q(t),\dot{q}(t),\lambda) - A(t)\dot{x}(t) - B(t)x(t)$ satisfies condition (ii) from Proposition 4.

ad (iii) Since the verification of condition (iii) in Proposition 4 is technically the most complicated, it will be split into two parts: (iii_1) the quasi-compactness of the solution operator $\mathfrak{T}_m$, (iii_2) the condensity of $\mathfrak{T}_m$ w.r.t. the monotone and non-singular m.n.c. μ defined by (4).

ad (iii_1) Let us firstly prove that the solution mapping $\mathfrak{T}_m$ is quasi-compact. Since $C^1([0,T],E)$ is a complete metric space, it is sufficient to prove the sequential quasi-compactness of $\mathfrak{T}_m$. Hence, let us consider the sequences $\{q_n\},\{\lambda_n\}, q_n \in Q, \lambda_n \in [0,1]$, for all $n \in \mathbb{N}$, such that $q_n \to q$ in $C^1([0,T],E)$ and $\lambda_n \to \lambda$. Moreover, let $x_n \in \mathfrak{T}_m(q_n,\lambda_n)$, for all $n \in \mathbb{N}$. Then there exists, for all $n \in \mathbb{N}$, $k_n(\cdot) \in F_0(\cdot, q_n(\cdot), \dot{q}_n(\cdot))$ such that

$$\ddot{x}_n(t) + A(t)\dot{x}_n(t) + B(t)x_n(t) = \lambda_n k_n(t) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(q_n(t)), \quad \text{for a.a. } t \in [0,T], \tag{29}$$

and that $x_n(T) = Mx_n(0),\ \dot{x}_n(T) = N\dot{x}_n(0)$.

According to condition (2_{iii}) and the definition of Q, $\|k_n(t)\| \le \nu_{\overline{K}}(t)$, for every $n \in \mathbb{N}$ and a.a. $t \in [0,T]$. According to formula (19),

$$x_n(t) = A_1(t)\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau + A_2(t)\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau + \int_0^t U_{12}(t,\tau)f_n(\tau)\,d\tau, \tag{30}$$

where

$$f_n(t) = \lambda_n k_n(t) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(q_n(t)). \tag{31}$$

Therefore, for all $t \in [0,T]$ and $n \in \mathbb{N}$,

$$\|x_n(t)\| \le Z(4Zk+1)\hat{C},$$

where k, Z are defined by relations (18), (22) and

$$\hat{C} := \left[\|\nu_{\overline{K}}\|_{L^1([0,T],[0,\infty))} + 2\cdot \max_{x\in \overline{B(\partial K,\varepsilon)}} \|\hat{\phi}(x)\| \cdot \|p\|_{L^1([0,T],[0,\infty))}\right]. \tag{32}$$

This implies that the sequence $\{x_n\}$ is bounded.

Moreover, since

$$\dot{x}_n(t) = A_3(t)\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau + A_4(t)\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau + \int_0^t U_{22}(t,\tau)f_n(\tau)\,d\tau, \tag{33}$$

where $f_n(t)$ is defined by formula (31), we can obtain, by the similar arguments, that $\|\dot{x}_n(t)\| \le Z(4Zk+1)\hat{C}$ for all $t \in [0,T]$ and $n \in \mathbb{N}$.

Consequently, for a.a. $t \in [0,T]$, we have

$$\|\ddot{x}_n(t)\| \le \|A(t)\|\cdot\|\dot{x}_n(t)\| + \|B(t)\|\cdot\|x_n(t)\| + \|f_n(t)\|$$

$$\le (\|A(t)\| + \|B(t)\|)\cdot Z(4Zk+1)\hat{C} + \nu_{\overline{K}}(t) + 2\cdot \max_{x\in \overline{B(\partial K,\varepsilon)}} \|\hat{\phi}(x)\|\cdot p(t).$$

Thus, $\{\ddot{x}_n\}$ is uniformly integrable.

For each $t \in [0,T]$, the properties of the Hausdorff m.n.c. yield

$$\gamma(\{f_n(t)\}_n) \le \gamma(\{\lambda_n k_n(t)\}_n) + p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\gamma\left(\{\hat{\phi}(q_n(t))\}_n\right)$$

$$\le \gamma\left(\cup_{\lambda\in[0,1]}\{\lambda k_n(t)\}_n\right) + p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\gamma\left(\{\phi(q_n(t))\|\dot{V}_{q_n(t)}\| : q_n(t) \in \overline{B(\partial K,\varepsilon)}\}\right)$$

$$= \gamma\left(\{k_n(t)\}_n\right) + p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\gamma\left(\{\phi(q_n(t))\|\dot{V}_{q_n(t)}\| : q_n(t) \in \overline{B(\partial K,\varepsilon)}\}\right).$$

Therefore, according to condition (2_{ii}), for a.a. $t \in [0,T]$,

$$\gamma(\{f_n(t)\}_n) \le g(t)\left(\gamma\left(\{q_n(t)\}_n\right) + \gamma\left(\{\dot{q}_n(t)\}_n\right)\right) + p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\gamma\left(\{\phi(q_n(t))\|\dot{V}_{q_n(t)}\| : q_n(t) \in \overline{B(\partial K,\varepsilon)}\}\right)$$

$$\leq g(t) \sup_{t\in[0,T]} \left(\gamma\left(\{q_n(t)\}_n\right) + \gamma\left(\{\dot{q}_n(t)\}_n\right)\right) + p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\gamma\left(\{\phi(q_n(t))\|\dot{V}_{q_n(t)}\| : q_n(t) \in \overline{B(\partial K,\varepsilon)}\}\right).$$

Since the function $x \to \phi(x)\|\dot{V}_x\|$ is Lipschitzian on $\overline{B(\partial K,\varepsilon)}$ with some Lipschitz constant $\hat{L} > 0$ (see Remark 1), we get that

$$\gamma(\{f_n(t)\}_n) \leq \left(g(t) + \hat{L}p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\right) \sup_{t\in[0,T]} \left(\gamma\left(\{q_n(t)\}_n\right) + \gamma\left(\{\dot{q}_n(t)\}_n\right)\right).$$

Since $q_n \to q$ and $\dot{q}_n \to \dot{q}$ in $C([0,T],E)$, we get that, for a.a. $t \in [0,T]$, $\gamma\left(\{q_n(t)\}_n\right) = \gamma\left(\{\dot{q}_n(t)\}_n\right) = 0$, which implies that $\gamma(\{f_n(t)\}_n) = 0$, for a.a. $t \in [0,T]$.

For a given $t \in (0,T]$, the sequences $\{U_{ij}(t,s)f_n(s)\}$, $i,j \in \{1,2\}$, are relatively compact as well, for a.a. $s \in [0,t]$, because, according to (2),

$$\gamma(\{U_{ij}(t,s)f_n(s)\}_n) \leq \|U_{ij}(t,s)\|\gamma(\{f_n(s)\}_n) = 0, \tag{34}$$

for all $i,j \in \{1,2\}$.

Moreover, according to (17) and (22),

$$\|U_{ij}(t,s)f_n(s)\| \leq Z\left(\nu_{\overline{K}}(s) + 2\cdot \max_{x\in\overline{B(\partial K,\varepsilon)}} \|\hat{\phi}(x)\| \cdot p(s)\right), \tag{35}$$

for a.a. $s \in [0,t]$ and all $n \in \mathbb{N}$.

By virtue of (2), (3), (34), (35) and the sub-additivity of γ, we finally arrive at

$$\gamma(\{x_n(t)\}_n) \leq \gamma\left(\left\{\int_0^t U_{12}(t,\tau)f_n(\tau)\,d\tau\right\}_n\right) + \|A_1(t)\|\cdot\gamma\left(\left\{\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau\right\}_n\right)$$

$$+\|A_2(t)\|\cdot\gamma\left(\left\{\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau\right\}_n\right) = 0.$$

By similar reasonings, when using (20) instead of (19), we also get

$$\gamma(\{\dot{x}_n(t)\}_n) = 0$$

by which $\{x_n(t)\}$, $\{\dot{x}_n(t)\}$ are relatively compact, for a.a. $t \in [0,T]$. Moreover, since x_n satisfies for all $n \in \mathbb{N}$ equation (29), $\{\ddot{x}_n(t)\}$ is relatively compact, for a.a. $t \in [0,T]$. Thus, according to [1, Lemma III.1.30], there exist a subsequence of $\{\dot{x}_n\}$, for the sake of simplicity denoted in the same way as the sequence, and $x \in C^1([0,T],E)$ such that $\{\dot{x}_n\}$ converges to $\dot{x}$ in $C([0,T],E)$ and $\{\ddot{x}_n\}$ converges weakly to $\ddot{x}$ in $L^1([0,T],E)$. According to the classical closure results (cf. e.g. [11, Lemma 5.1.1]), $x \in \mathfrak{T}_m(q,\lambda)$, which implies the quasi-compactness of $\mathfrak{T}_m$.

ad (iii_2) In order to show that, for $m \in \mathbb{N}$ sufficiently large, $\mathfrak{T}_m$ is μ-condensing with respect to the m.n.c. μ defined by (4), let us consider a bounded subset $\Theta \subset Q$ such that $\mu\left(\mathfrak{T}_m(\Theta\times[0,1])\right) \geq \mu(\Theta)$. Let $\{x_n\} \subset \mathfrak{T}_m\left(\Theta\times[0,1]\right)$ be a sequence such that

$$\mu\left(\mathfrak{T}_m\left(\Theta\times[0,1]\right)\right) = \left(\sup_{t\in[0,T]}\left[\gamma(\{x_n(t)\}_n) + \gamma(\{\dot{x}_n(t)\}_n)\right], \mathrm{mod}_C\left(\{x_n\}_n\right) + \mathrm{mod}_C\left(\{\dot{x}_n\}_n\right)\right).$$

According to (19) and (20), we can find $\{q_n\} \subset \Theta$, $\{\lambda_n\} \subset [0,1]$ and $\{k_n\}$ satisfying $k_n(t) \in F_0(t,q_n(t),\dot{q}_n(t))$, for a.a. $t \in [0,T]$, such that, for all $t \in [0,T]$, $x_n(t)$ and $\dot{x}_n(t)$ are defined by formulas (30) and (33), respectively, where $f_n(t)$ is defined by formula (31).

By the similar reasonings as in the part **ad** (iii_1), we can obtain that

$$\gamma\left(\{f_n(t)\}_n\right) \leq \left(g(t) + \hat{L}p(t)(\chi_{J_m}(t) + \frac{1}{m})\right) \sup_{t\in[0,T]} \left(\gamma\left(\{q_n(t)\}_n\right) + \gamma\left(\{\dot{q}_n(t)\}_n\right)\right),$$

for a.a. $t \in [0,T]$.

Let us put

$$\mathcal{S} := \sup_{t\in[0,T]} \left(\gamma\left(\{q_n(t)\}_n\right) + \gamma\left(\{\dot{q}_n(t)\}_n\right)\right),$$

fix $\tau \in [0,T]$ and let $i,j=1,2$. Then, according to (17) and (22), we have that, for all $n \in \mathbb{N}$,

$$\|U_{ij}(\tau,t)f_n(t)\| \leq \|U_{ij}(\tau,t)\| \cdot \|f_n(t)\| \leq Z\left(\|k_n(t)\| + 2 \cdot \max_{x \in \overline{B(\partial K,\varepsilon)}} \|\hat{\phi}(x)\| \cdot p(t)\right), \quad fora.a.t \in [0,\tau].$$

Since $k_n(t) \in F_0(t, q_n(t), \dot{q}_n(t))$, for a.a. $t \in [0,T]$, and $q_n \in \Theta$, for all $n \in \mathbb{N}$, where Θ is a bounded subset of $C^1([0,T],E)$, there exists $\Omega \subset \overline{K}$ such that $q_n(t) \in \Omega$, for all $n \in \mathbb{N}$ and $t \in [0,T]$. Hence, it follows from condition (2_{iii}) that

$$\|U_{ij}(\tau,t)f_n(t)\| \leq Z\left(\nu_\Omega(t) + 2 \cdot p(t) \cdot \max_{x \in \overline{B(\partial K,\varepsilon)}} \|\hat{\phi}(x)\|\right), \quad fora.a.t \in [0,\tau].$$

As a consequence of (17), (22) and property (2), we also have that

$$\gamma\left(\{U_{ij}(\tau,t)f_n(t)\}_n\right) \leq Z\gamma\left(\{f_n(t)\}_n\right), \quad fora.a.t \in [0,\tau].$$

Therefore, we can use (3) in order to show that

$$\gamma\left(\left\{\int_0^T U_{ij}(T,t)f_n(t)\,dt\right\}_n\right) \leq Z\mathcal{S}\int_0^T \left(g(t) + \hat{L}p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\right)dt, \quad ij=1,2,$$

and also

$$\gamma\left(\left\{\int_0^t U_{i2}(t,\tau)f_n(\tau)\,d\tau\right\}_n\right) \leq Z\mathcal{S}\int_0^t \left(g(\tau) + \hat{L}p(\tau)\left(\chi_{J_m}(\tau) + \frac{1}{m}\right)\right)d\tau, \quad i=1,2.$$

Consequently, according to (2), (21), (30) and the subadditivity of γ, we have that, for a.a. $t \in [0,T]$,

$$\gamma\left(\{x_n(t)\}_n\right) \leq Z\mathcal{S}\left(\|A_1(t)\| + \|A_2(t)\| + 1\right)\int_0^T \left(g(t) + \hat{L}p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\right)dt$$

$$\leq Z\mathcal{S}\left(4kZ+1\right)\left(\|g\|_{L^1([0,T],[0,\infty))} + \hat{L}\left(\|p\|_{L^1(J_m)} + \frac{1}{m}\|p\|_{L^1([0,T],[0,\infty))}\right)\right).$$

The same estimate can be obtained for $\gamma\left(\{\dot{x}_n(t)\}_n\right)$, when starting from condition (33). Subsequently,

$$\gamma\left(\{x_n(t)\}_n\right) + \gamma\left(\{\dot{x}_n(t)\}_n\right) \leq 2Z\mathcal{S}\left(4kZ+1\right)\left(\|g\|_{L^1([0,T],[0,\infty))} + \hat{L}\left(\|p\|_{L^1(J_m)} + \frac{1}{m}\|p\|_{L^1([0,T],[0,\infty))}\right)\right).$$

Since we assume that $\mu\left(\mathfrak{T}_m(\Theta \times [0,1])\right) \geq \mu(\Theta)$ and $\{q_n\}_n \subset \Theta$, we get

$$\mathcal{S} = \sup_{t \in [0,T]}\left(\gamma\left(\{q_n(t)\}_n\right) + \gamma\left(\{\dot{q}_n(t)\}_n\right)\right) \leq \sup_{t \in [0,T]}\left(\gamma\left(\{x_n(t)\}_n\right) + \gamma\left(\{\dot{x}_n(t)\}_n\right)\right)$$

$$\leq 2Z(4Zk+1)\left(\|g\|_{L^1([0,T],[0,\infty))} + \hat{L}\left(\|p\|_{L^1(J_m)} + \frac{1}{m}\|p\|_{L^1([0,T],[0,\infty))}\right)\right)\mathcal{S}.$$

Since we have, according to (2_{iv}), that $2Z(4kZ+1)\|g\|_{L^1([0,T],[0,\infty))} < 1$, we can choose $m_0 \in \mathbb{N}$ such that, for all $m \in \mathbb{N}$, $m \geq m_0$, it holds that

$$2Z(4kZ+1)\left(\|g\|_{L^1([0,T],[0,\infty))} + \hat{L}\left(\|p\|_{L^1(J_m)} + \frac{1}{m}\|p\|_{L^1([0,T],[0,\infty))}\right)\right) < 1.$$

Therefore, we get, for sufficiently large $m \in \mathbb{N}$, the contradiction $\mathcal{S} < \mathcal{S}$ which ensures the validity of condition (iii) in Proposition 4.

ad (iv) For all $q \in Q$, the set $\mathfrak{T}_m(q,0)$ coincides with the unique solution x_m of the linear system

$$\left.\begin{array}{l} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = -p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(q(t)), \text{ for a.a. } t \in [0,T], \\ x(T) = Mx(0),\ \dot{x}(T) = N\dot{x}(0). \end{array}\right\}$$

According to (19) and (20), for all $t \in [0,T]$,

$$x_m(t) = A_1(t)\int_0^T U_{12}(T,\tau)\varphi_m(\tau)\,d\tau + A_2(t)\int_0^T U_{22}(T,\tau)\varphi_m(\tau)\,d\tau + \int_0^t U_{12}(t,\tau)\varphi_m(\tau)\,d\tau,$$

and

$$\dot{x}_m(t) = A_3(t)\int_0^T U_{12}(T,\tau)\varphi_m(\tau)\,d\tau + A_4(t)\int_0^T U_{22}(T,\tau)\varphi_m(\tau)\,d\tau + \int_0^t U_{22}(t,\tau)\varphi_m(\tau)\,d\tau,$$

where $\varphi_m(t) := -p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(q_m(t))$.

Since

$$\|\varphi_m\|_{L^1([0,T],[0,\infty))} \le \max_{x\in B(\partial K,\varepsilon)} \|\hat{\phi}(x)\| \cdot \left(\|p\|_{L^1(J_m,[0,\infty))} + \frac{\|p\|_{L^1([0,T],[0,\infty))}}{m}\right),$$

we have that, for all $t \in [0,T]$,

$$\|x_m(t)\| \le Z\cdot(4Zk+1)\cdot \max_{x\in B(\partial K,\varepsilon)} \|\hat{\phi}(x)\| \cdot \left(\|p\|_{L^1(J_m,[0,\infty))} + \frac{\|p\|_{L^1([0,T],[0,\infty))}}{m}\right), \tag{36}$$

where k, Z are defined by relations (18), (22).

Let us now consider $r > 0$ such that $rB \subset K$. Then, it follows from (36) that we are able to find $m_0 \in \mathbb{N}$ such that, for all $m \in \mathbb{N},\ m \ge m_0$, and $t \in [0,T]$, $\|x_m\| \le r$. Therefore, for all $m \in \mathbb{N},\ m \ge m_0$, $\mathfrak{T}_m(q,0) \subset Int\ Q$, for all $q \in Q$, which ensures the validity of condition (iv) in Proposition 4.

ad (v) Let $m \in \mathbb{N}$ be fixed and let us show that each (P_m) satisfies the transversality condition (v) in Proposition 4. We reason by a contradiction, and assume the existence of $\lambda \in (0,1)$ and $q \in \partial Q$ such that $q \in \mathfrak{T}_m(q,\lambda)$. According to the definition of the solution operator $\mathfrak{T}_m$, there is $f_0 \in L^1([0,T],E)$ with $f_0(t) \in F_0(t,q(t),\dot{q}(t))$, for a.a. $t \in [0,T]\setminus J$, satisfying

$$\ddot{q}(t) + A(t)\dot{q}(t) + B(t)q(t) = \lambda f_0(t) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(q(t)), \quad fora.a.t \in [0,T]\setminus J. \tag{37}$$

Since, moreover, $\mu(J) = 0$, condition (37) is indeed valid for a.a. $t \in [0,T]$.

Since $q \in \partial Q$, there exists $t_0 \in [0,T]$ satisfying $q(t_0) \in \partial K$. If we further assume that $t_0 = 0$, then $q(T) = Mq(0) \in M\partial K = \partial K$. With no loss of generality we can then take $t_0 \in (0,T]$. According to condition (H3), $\|\dot{V}_{q(t_0)}\| \ge \delta$. Furthermore, since $t \longmapsto \|\dot{V}_{q(t)}\|$ is continuous, there is $h_0 > 0$ such that $q(t) \in B\left(\partial K, \min\{h, \frac{\varepsilon}{2}\}\right)$ and $\|\dot{V}_{q(t)}\| \ge \frac{\delta}{2}$, for all $t \in [t_0 - h_0, t_0]$. Since J_m is open in $[0,T]$, if, in addition, $t_0 \in J_m$, we can take h_0 in such a way that $[t_0 - h_0, t_0] \subset J_m$.

Consider now the function $g\colon [0,T] \to \mathbb{R}$ defined by $g(t) = V(q(t))$.

According to the regularity conditions imposed on V and q, we have that $g \in C^1([0,T],\mathbb{R})$ and $\dot{g}(t) = \langle \dot{V}_{q(t)},\ \dot{q}(t)\rangle$, for all $t \in [0,T]$. Since, moreover, $V \in C^2(E,\mathbb{R})$ and $\dot{q}$ is absolutely continuous on $[0,T]$, we obtain that also $\dot{g}$ is absolutely continuous, implying that $\ddot{g}(t)$ exists, for a.a. $t \in [t_0 - h_0, t_0]$.

Since $g(t) \le 0$, for all $t \in [0,T]$ with $g(t_0) = 0$, t_0 is a local maximum point. Hence, $\dot{g}(t_0) \ge 0$ and $\dot{g}(t_0) = 0$, whenever $t_0 \in (0,T)$. Consider now the special case when $t_0 = T$. Since $q(0) = M^{-1}q(T)$, according to the properties of M, we have that $q(0) \in \partial K$, and thus $\dot{g}(0) = \langle \dot{V}_{q(0)},\ \dot{q}(0)\rangle \le 0$. Note, moreover, that $\dot{q}(T) = N\dot{q}(0)$. Consequently, we have that $\langle \dot{V}_{Mq(0)}, N\dot{q}(0)\rangle \cdot \langle \dot{V}_{q(0)}, \dot{q}(0)\rangle = \dot{g}(T)\cdot\dot{g}(0) \le 0$ and according to (27) we obtain that

$$\dot{g}(0) = \left\langle \dot{V}_{q(0)}, \dot{q}(0)\right\rangle = \dot{g}(T) = \left\langle \dot{V}_{q(T)}, \dot{q}(T)\right\rangle = 0.$$

Let $t \in [t_0 - h_0, t_0]$ be such that both $\ddot{q}(t)$ and $\ddot{x}(t)$ exist. Then

$$\ddot{g}(t) = \lim_{h\to 0} \frac{\dot{g}(t+h) - \dot{g}(t)}{h} = \lim_{h\to 0} \frac{\langle \dot{V}_{q(t+h)}, \dot{q}(t+h)\rangle - \langle \dot{V}_{q(t)}, \dot{q}(t)\rangle}{h}.$$

According to the regularity of q, there exist two functions $a(h)$ and $b(h)$ from $[-t, T-t]$ to E with $a(h) \to 0$ and $b(h) \to 0$ when $h \to 0$ such that

$$\dot{q}(t+h) = \dot{q}(t) + h\left[\ddot{q}(t) + a(h)\right], \quad q(t+h) = q(t) + h\left[\dot{q}(t) + b(h)\right].$$

Consequently,

$$\ddot{g}(t) = \lim_{h\to 0} \frac{\langle \dot{V}_{q(t+h)}, \dot{q}(t) + h\left[\ddot{q}(t) + a(h)\right]\rangle - \langle \dot{V}_{q(t)}, \dot{q}(t)\rangle}{h}$$

$$= \lim_{h\to 0} \frac{\langle \dot{V}_{q(t+h)}, \dot{q}(t)\rangle - \langle \dot{V}_{q(t)}, \dot{q}(t)\rangle}{h} + \frac{\langle \dot{V}_{q(t+h)}, h\left[a(h)\right]\rangle}{h} + \langle \dot{V}_{q(t+h)}, \ddot{q}(t)\rangle.$$

Since $h \longmapsto \|\dot{V}_{q(t+h)}\|$ is continuous, it is bounded, for $t \in [-t, T-t]$, and therefore

$$\left| \frac{\langle \dot{V}_{q(t+h)}, h\,[a(h)] \rangle}{h} \right| \leq \|\dot{V}_{q(t+h)}\| \|a(h)\| \to 0, \quad h \to 0.$$

Thus, we obtain that

$$\ddot{g}(t) = \lim_{h\to 0} \frac{\langle \dot{V}_{q(t+h)}, \dot{q}(t)\rangle - \langle \dot{V}_{q(t)}, \dot{q}(t)\rangle}{h} + \langle \dot{V}_{q(t+h)}, \ddot{q}(t)\rangle$$

$$= \lim_{h\to 0} \frac{\langle \dot{V}_{q(t)+h[\dot{q}(t)+b(h)]}, \dot{q}(t)\rangle - \langle \dot{V}_{q(t)}, \dot{q}(t)\rangle}{h} + \langle \dot{V}_{q(t+h)}, \ddot{q}(t)\rangle.$$

According to the regularity condition imposed on V, there exists $O(h) \in E'$ with

$$\frac{\|O(h)\|}{h} \to 0 \quad forh \to 0$$

such that

$$\dot{V}_{q(t)+h[\dot{q}(t)+b(h)]} = \dot{V}_{q(t)} + \ddot{V}_{q(t)}\,(h\dot{q}(t) + hb(h)) + O(h)$$

implying

$$\frac{\langle \dot{V}_{q(t)+h[\dot{q}(t)+b(h)]}, \dot{q}(t)\rangle - \langle \dot{V}_{q(t)}, \dot{q}(t)\rangle}{h} = \frac{\langle \ddot{V}_{q(t)}(\dot{h}q(t)), \dot{q}(t)\rangle}{h} + \frac{\langle \ddot{V}_{q(t)}(hb(h)), \dot{q}(t)\rangle}{h} + \frac{\langle O(h), \dot{q}(t)\rangle}{h}$$

$$= \langle \ddot{V}_{q(t)}(\dot{q}(t)), \dot{q}(t)\rangle + \langle \ddot{V}_{q(t)}(b(h)), \dot{q}(t)\rangle + \frac{\langle O(h), \dot{q}(t)\rangle}{h}.$$

Therefore,

$$\ddot{g}(t) = \lim_{h\to 0} \langle \ddot{V}_{q(t)}(\dot{q}(t)), \dot{q}(t)\rangle + \langle \ddot{V}_{q(t)}(b(h)), \dot{q}(t)\rangle + \langle \dot{V}_{q(t+h)}, \ddot{q}(t)\rangle + \frac{\langle O(h), \dot{q}(t)\rangle}{h}$$

$$= \langle \ddot{V}_{q(t)}(\dot{q}(t)), \dot{q}(t)\rangle + \langle \dot{V}_{q(t)}, \ddot{q}(t)\rangle. \tag{38}$$

Let us now consider the case when $t_0 \in J_m$. According to the properties of g, it is possible to find $\hat{t}_0 \in (t_0 - h_0, t_0)$ such that $\dot{g}(\hat{t}_0) \geq 0$. Therefore, we obtain that

$$0 \geq -\dot{g}(\hat{t}_0) = \dot{g}(t_0) - \dot{g}(\hat{t}_0) = \int_{\hat{t}_0}^{t_0} \ddot{g}(t)\,dt.$$

According to (25) and (38), we have that

$$0 \geq -\dot{g}(\hat{t}_0) = \int_{\hat{t}_0}^{t_0} \ddot{g}(t)\,dt = \int_{\hat{t}_0}^{t_0} \langle \ddot{V}_{q(t)}(\dot{q}(t)), \dot{q}(t)\rangle + \langle \dot{V}_{q(t)}, \ddot{q}(t)\rangle\,dt \geq \int_{\hat{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \ddot{q}(t)\rangle\,dt$$

$$= \int_{\hat{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - \left(1 + \frac{1}{m}\right) p(t)\hat{\phi}(q(t))\rangle\,dt$$

$$= \int_{\hat{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - \left(1 + \frac{1}{m}\right) p(t)\tau(q(t))\|\dot{V}_{q(t)}\|\phi(q(t))\rangle\,dt.$$

Since $q(t) \in B(\partial K, \frac{\epsilon}{2})$, for all $t \in [\hat{t}_0, t_0]$, $\tau(q(t)) = 1$ and, according to Proposition 3, $\langle \dot{V}_{q(t)}, \phi(q(t))\rangle = 1$. Therefore, we obtain that

$$0 \geq -\dot{g}(\hat{t}_0) \geq \int_{\hat{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - \left(1 + \frac{1}{m}\right) p(t)\tau(q(t))\|\dot{V}_{q(t)}\|\phi(q(t))\rangle\,dt$$

$$= \int_{\hat{t}_0}^{t_0} \left(\langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t)\rangle - \left(1 + \frac{1}{m}\right) p(t)\|\dot{V}_{q(t)}\| \right)\,dt$$

$$\geq \int_{\hat{t}_0}^{t_0} \|\dot{V}_{q(t)}\| \left(\kappa(t) - \left(1 + \frac{1}{m}\right) p(t)\right)\, dt,$$

where

$$\kappa(t) := -\nu_{\overline{K}}(t) - \|A(t)\| Z(4Zk+1)\|\nu_{\overline{K}}\|_{L^1([0,T],[0,\infty))} - \|B(t)\| \left(\|\partial K\| + \frac{\varepsilon}{2}\right).$$

According to the definition of p, we have that the last integral is strictly positive, so we get the contradictory conclusion $0 \geq -\dot{g}(\hat{t}_0) > 0$. It implies that $t_0 \notin J_m$.

Therefore, let us study the case when $t_0 \in [0,T] \setminus J_m$. If we are able to get a contradiction also when $t_0 \in [0,T] \setminus J_m$, then $q \in \mathfrak{T}_m(\lambda, q)$ with $q \in \partial Q$ is not possible, and so problem (P_m) satisfies the required tranversality condition.

Let $w_0 \in F(t_0, q(t_0), \dot{q}(t_0))$. According to Proposition 3, and since $t_0 \notin J_m$, we have that

$$\langle \dot{V}_{q(t_0)}, \lambda w_0 - A(t_0)\dot{q}(t_0) - B(t_0)q(t_0) - p(t_0)(\chi_{J_m}(t_0) + \frac{1}{m})\hat{\phi}(q(t_0))\rangle$$

$$= \langle \dot{V}_{q(t_0)}, \lambda w_0 - A(t_0)\dot{q}(t_0) - B(t_0)q(t_0) - \frac{p(t_0)}{m}\hat{\phi}(q(t_0))\rangle$$

$$= \langle \dot{V}_{q(t_0)}, \lambda w_0 - A(t_0)\dot{q}(t_0) - B(t_0)q(t_0)\rangle - \frac{p(t_0)}{m}\|\dot{V}_{q(t_0)}\|.$$

Therefore, as a consequence of (26), the negativity of p and condition $(H3)$, we have that

$$\left\langle \dot{V}_{q(t_0)}, \lambda w_0 - A(t_0)\dot{q}(t_0) - B(t_0)q(t_0) - \frac{p(t_0)}{m}\hat{\phi}(q(t_0))\right\rangle \geq -\frac{p(t_0)}{m}\|\dot{V}_{q(t_0)}\| \geq -\frac{\delta p(t_0)}{m} > 0,$$

for all $w_0 \in F(t_0, q(t_0), \dot{q}(t_0))$. The multivalued map F is compact-valued and the map $\dot{V}_{q(t_0)}\colon E \to \mathbb{R}$ is continuous. Thus, we can find $\sigma > 0$ such that

$$\left\langle \dot{V}_{q(t_0)}, \lambda w_0 - A(t_0)\dot{q}(t_0) - B(t_0)q(t_0) - \frac{p(t_0)}{m}\hat{\phi}(q(t_0))\right\rangle \geq 2\sigma,$$

for all $w_0 \in F(t_0, q(t_0), \dot{q}(t_0))$.

In $[0,T] \setminus J_m$, the multivalued map

$$t \multimap \lambda F_0(t, q(t), \dot{q}(t)) - A(t)\dot{q}(t) - B(t)q(t) - \frac{p(t)}{m}\hat{\phi}(q(t))$$

is u.s.c. and, therefore, Φ: $[0,T] \setminus J_m \multimap \mathbb{R}$ defined by

$$t \multimap \left\{\langle \dot{V}_{q(t)}, \lambda w - A(t)\dot{q}(t) - B(t)q(t) - \frac{p(t)}{m}\hat{\phi}(q(t))\rangle, :\ w \in F_0(t, q(t), \dot{q}(t))\right\}$$

is u.s.c. Thus, we can find $\tilde{h}_0 \leq h_0$ such that $\Phi(t) \in [\sigma, +\infty)$, for all $t \in [t_0 - \tilde{h}_0, t_0] \setminus J_m$.

Since $g(t_0 - \tilde{h}_0) \leq 0$, also in $[t_0 - \tilde{h}_0, t_0]$, we can find $\tilde{t}_0$ with $\dot{g}(\tilde{t}_0) \geq 0$. Now, we reason as before and get

$$0 \geq -\dot{g}(\tilde{t}_0) = \dot{g}(t_0) - \dot{g}(\tilde{t}_0) = \int_{\tilde{t}_0}^{t_0} \ddot{g}(t)\, dt$$

$$= \int_{\tilde{t}_0}^{t_0} \langle \ddot{V}_{q(t)}(\dot{q}(t)), \dot{q}(t)\rangle\, dt + \int_{\tilde{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \ddot{q}(t)\rangle\, dt \geq \int_{\hat{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \ddot{q}(t)\rangle\, dt$$

$$= \int_{\tilde{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - p(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(q(t))\rangle\, dt$$

$$= \int_{[\tilde{t}_0,t_0]\setminus J_m} \langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - \frac{p(t)}{m}\hat{\phi}(q(t))\rangle\, dt$$

$$+ \int_{[\tilde{t}_0,t_0]\cap J_m} \langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - p(t)(1 + \frac{1}{m})\hat{\phi}(q(t))\rangle\, dt.$$

Since the multivalued map $\Phi(t)$ is u.s.c. and since $t_0 \notin J_m$, we have that

$$\int_{[\tilde{t}_0,t_0]\setminus J_m} \left\langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - \frac{p(t)}{m}\hat{\phi}(q(t)) \right\rangle dt \geq \sigma \int_{[\tilde{t}_0,t_0]\setminus J_m} > 0.$$

Otherwise, from the definition of p and by a similar reasoning as before, we obtain that

$$\int_{[\tilde{t}_0,t_0]\cap J_m} \left\langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - p(t)\left(1+\frac{1}{m}\right)\hat{\phi}(q(t)) \right\rangle dt$$

$$= \int_{[\tilde{t}_0,t_0]\cap J_m} \left\langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - p(t)\left(1+\frac{1}{m}\right)\|\dot{V}_{q(t)}\|\phi(q(t)) \right\rangle dt$$

$$= \int_{[\tilde{t}_0,t_0]\cap J_m} \left(\left\langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) \right\rangle - p(t)\left(1+\frac{1}{m}\right)\|\dot{V}_{q(t)}\| \right) dt$$

$$\geq \int_{[\tilde{t}_0,t_0]\cap J_m} \|\dot{V}_{q(t)}\| \left(-\nu_{\overline{K}}(t) - \|A(t)\|Z(4Zk+1)\|\nu_{\overline{K}}\|_{L^1([0,T],[0,\infty))}\right) dt$$

$$-\int_{[\tilde{t}_0,t_0]\cap J_m} \|\dot{V}_{q(t)}\| \left(\|B(t)\|\left(\|\partial K\| + \frac{\varepsilon}{2}\right) + \left(1+\frac{1}{m}\right)p(t)\,dt \right) > 0$$

In the case when $t_0 \in [0,T]\setminus J_m$, we obtain the contradictory conclusion $0 \geq -\dot{g}(\tilde{t}_0) > 0$ as well, and the tranversality condition (v) in Proposition 4 is so verified.

Summing up, we have proved that there exists $m_0 \in \mathbb{N}$ such that every problem (P_m), where $m \geq m_0$, satisfies all the assumptions of Proposition 4. This implies that every such (P_m) admits a solution, denoted by x_m, with $x_m(t) \in \overline{K}$, for all $t \in [0,T]$. Consequently, there exists a sequence $\{k_m\}_m$ in $L^1([0,T],E)$ satisfying

$$\ddot{x}_m(t) + A(t)\dot{x}_m(t) + B(t)x_m(t) = k_m(t) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(x_m(t)) \tag{39}$$

and also $k_m(t) \in F(t, x_m(t), \dot{x}_m(t))$, for a.a. $t \in [0,T]$ and every $m \geq m_0$. Moreover, according to (2_{ii}), we obtain that $\|k_m(t)\| \leq \nu_{\overline{K}}(t)$, for a.a. $t \in [0,T]$ and every $m \geq m_0$. Therefore, reasoning as in **ad** (iii_1), we have that $\|\dot{x}_m(t)\| \leq Z(4Zk+1)\hat{C}$ with $\hat{C}$ defined by (32). We can then apply (2_{ii}) and get

$$\gamma\left(\{k_m(t)\}_m\right) \leq g(t)\left[\gamma\left(\{x_m(t)\}_m\right) + \gamma\left(\{\dot{x}_m(t)\}_m\right)\right], \quad for a.a. t \in [0,T]. \tag{40}$$

Let us put $\hat{\mathcal{S}} := \gamma\left(\{x_m(t)\}_m\right) + \gamma\left(\{\dot{x}_m(t)\}_m\right)$ and let $\{f_m\} \subset L^1([0,T],E)$ be defined by $f_m(t) := k_m(t) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(x_m(t))$, for a.a. $t \in [0,T]$. When $t \notin J$, there is $\hat{m} = \hat{m}(t) \geq m_0$ such that $t \notin J_m$, for all $m \geq \hat{m}$. If we further apply the subadditivity of the Hausdorff m.n.c., we obtain

$$\gamma\left(\{f_m(t)\}_m\right) \leq \gamma\left(\{k_m(t)\}_m\right) + \gamma\left(\{-p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(x_m(t))\}_m\right)$$

$$\leq \gamma\left(\{k_m(t)\}_m\right) + \gamma\left(\{-p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(x_m(t)),\ m = m_0,\ldots,\hat{m}(t)-1\}_m\right)$$

$$+\gamma\left(\{-\frac{p(t)}{m}\hat{\phi}(x_m(t)),\, m \geq \hat{m}(t)\}_m\right) = \gamma\left(\{k_m(t)\}_m\right) + \gamma\left(\{-\frac{p(t)}{m}\hat{\phi}(x_m(t)),\, m \geq \hat{m}(t)\}_m\right).$$

Since $\hat{\phi}$ is bounded, we obtain that

$$\frac{p(t)}{m}\hat{\phi}(x_m(t)) \to 0, \quad m \to \infty$$

implying that $\gamma\left(\{f_m(t)\}_m\right) \leq \gamma\left(\{k_m(t)\}_m\right)$, for a.a. $t \in [0,T]$. According to (40), we have that $\gamma\left(\{f_m(t)\}_m\right) \leq \hat{\mathcal{S}}g(t)$, for a.a. $t \in [0,T]$. Reasoning as in **ad**(iii_1), it is also possible to show that

$$\gamma\left(\{x_m(t)\}_m\right) \leq Z(4Zk+1)\hat{\mathcal{S}}\|g\|_{L^1([0,T],[0,\infty))},$$

and the same estimate is valid for $\gamma(\{\dot{x}_m(t)\}_m)$. Consequently, according (2_{iii}), we obtain that

$$\hat{\mathcal{S}} = \gamma(\{x_m(t)\}_m) + \gamma(\{\dot{x}_m(t)\}_m)$$

$$\leq 2Z(4Zk+1)\hat{\mathcal{S}}\|g\|_{L^1([0,T],[0,\infty))} < \hat{\mathcal{S}},$$

implying that $\hat{\mathcal{S}} = 0$. Hence, $\gamma(\{x_m(t)\}_m) = \gamma(\{\dot{x}_m(t)\}_m) = 0$, for every $t \notin J$. Thus, also $\gamma(\{f_m(t)\}_m) = 0$. According to (39), we then obtain that $\gamma(\{\ddot{x}_m(t)\}_m) = 0$, for a.a. $t \in [0,T]$. Therefore, a classical convergence result (see e.g. [1, Lemma III.1.30])) assures the existence of a subsequence, denoted as the sequence, and of a function $x \in AC^1([0,T],E)$ such that $x_m \to x$ and $\dot{x}_m \to \dot{x}$ in $C([0,T],E)$ and also $\ddot{x}_m \rightharpoonup x$ in $L^1([0,T],E)$, when $m \to \infty$. Finally, a classical closure result (see e.g. [11, Lemma 5.1.1]) guarantees that x is a solution of (1) satisfying $x(t) \in \overline{K}$, for all $t \in [0,T]$, and the proof is so complete.

4. Concluding remarks

Observe that in a Hilbert space E, for $V(x) := \frac{1}{2}\left(\|x\|^2 - r\right)$, we have that (cf. [3], [13]) $\partial V(x) = \{\dot{V}(x)\} = x$, i.e. we obtain that $\ddot{V}(x) \equiv Id$. In particular $V \in C^2(E,\mathbb{R})$, as required in Theorem 1. On the other hand, if $\|\cdot\|^2$ (i.e. also $V(\cdot)$) is twice Fréchet differentiable at 0 in a Banach space $(E,\|\cdot\|)$, then E is isomorphic to a Hilbert space (see e.g. [9, p. 180]).

As pointed out in [3], problems of type (1) can be related to those for *abstract nonlinear wave equations* in Hilbert spaces $E := L^2(\Omega)$. Hence, for $t \in [0,T]$ and $\xi \in \Omega$, where Ω is a nonempty, bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$, consider the functional evolution equation

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + bu(t,\cdot) + \mathcal{B}\|u(t,\cdot)\|^{p-2}u = \varphi(t,u), \quad (41)$$

where $u = u(t,\xi)$, subject to boundary conditions

$$u(T,\cdot) = Mu(0,\cdot), \qquad \frac{\partial u(T,\cdot)}{\partial t} = N\frac{\partial u(0,\cdot)}{\partial t}. \quad (42)$$

Assume that $a \geq 0, b < 0, \mathcal{B} \geq 0$, $p \in [3,\infty)$ are constants and that $\varphi\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ is sufficiently regular. The problem under consideration can be still restricted by a constraint $u(t,\cdot) \in \overline{K_1}$, where

$$K_1 := \{e \in L^2(\Omega) \mid \|e\| < 1\},\ t \in [0,T].$$

Taking $x(t) := u(t,\cdot)$ with $x \in AC^1([0,T],L^2(\Omega))$, $A(t) \equiv A := a$, $B(t) \equiv B := b$, $f\colon [0,T]\times L^2(\Omega) \to L^2(\Omega)$ defined by $(t,v) \to \varphi(t,v(\cdot))$, and $F(t,x,y) \equiv F(t,x) := -\mathcal{B}\|x\|^{p-2}x + f(t,x)$, the above problem can be rewritten into the form (1), possibly together with $x(t) \in \overline{K_1}, t \in [0,T]$.

In view of the above arguments, all illustrative examples in [3], related to $V(x) := \frac{1}{2}\|x\|^2 - R$ acting in Hilbert spaces, can be improved by means of Theorem 1 in the sense that all relations holding for $(t,x) \in (0,T)\times\overline{K_1} \cap B(\partial K_1,\varepsilon)$ can be strictly localized to $(0,T)\times\partial K_1$. More concretely, problem (41), (42), where $M = N = Id$ or $M = N = -Id$ together with $\varphi(t,-u) \equiv -\varphi(t,u)$, admits in this way a (strong) solution $x(t) := u(t,\cdot)$ such that $x(t) \in \overline{K}_1$, $t \in [0,T]$, provided (for more details, see [3])

(i) $a \geq 0,\ b < 0, 0 \leq \mathcal{B} < \frac{1}{p-1}$, where $p \in [3,\infty)$,

(ii) φ is Carathéodory (resp. continuous) and such that

$$|\varphi(t,\xi)| \leq \frac{c_0(t)}{\sqrt{|\Omega|+1}} + \frac{c_1(t)}{\sqrt{|\Omega|+1}}|\xi|^{2m},$$

$$t \in [0,T],\ \xi \in \Omega,$$

where c_0, c_1 are suitable integrable coefficients

($\Rightarrow f$ is Carathéodory and such that $\|f(t,x)\| \leq c_0(t) + c_1(t)\|x\|^m$, for all $x \in L^2(\Omega)$),

(iii) $\varphi(t,\xi)$ is Lipschitzian in ξ with a constant L (independent of t) such that (k will be specified below)

$$4\mathrm{e}^{T(1+a-b)}\left(4k\mathrm{e}^{T(1+a-b)} + 1\right)LT < 1 \quad (43)$$

($\Rightarrow f$ satisfies the γ−regularity condition, namely $\gamma\left(f(t,\tilde{\Omega})\right) \leq L\gamma(\tilde{\Omega})$, for a.a. $t \in [0,T]$ and each bounded $\tilde{\Omega} \subset E$, with $g(t) := L$ satisfying the inequality

$$4\mathrm{e}^{T(1+a-b)}\left(4k\mathrm{e}^{T(1+a-b)} + 1\right)\|g\|_{L^1([0,T],[0,\infty))} < 1),$$

(iv) condition $(d-\mathcal{B})\|x\|^2 + \langle x, f(t,x)\rangle \geq 0$, holds on the set $[0,T]\times\partial K_1$, where $d \geq 0$ is a suitable constant such that $a^2 \leq -4b(b+d)$.

It would be nice to express condition (iv), as conditions (i)–(iii), for function φ. For instance, the related equality $\sqrt{\int_\Omega x^2(\xi)\,d\xi} = r$ would then, however, lead to the inequality

$$z\varphi(t,z) \geq (\mathcal{B}-d)z^2$$

required, for all $(t,z) \in [0,T]\times\mathbb{R}$. In this way, the information concerning the localization of solutions would be lost.

The most technical requirement (in nontrivial situations) is so the inequality (43) in condition (iii). Nevertheless, the quotient in (43)

$$k := \|[\tilde{D} - U(T,0)]^{-1}\| = \|[\pm Id - \mathrm{e}^{CT}]^{-1}\|_{E\times E}$$

can be calculated as

$$k = k_0^{-1}$$

$$\left\|\begin{matrix} \pm 1 + \frac{\lambda_1\mathrm{e}^{\lambda_1 T} - \lambda_2\mathrm{e}^{\lambda_2 T}}{\lambda_2-\lambda_1}, & \frac{\mathrm{e}^{\lambda_2 T} - \mathrm{e}^{\lambda_1 T}}{\lambda_2-\lambda_1} \\ \frac{\lambda_1\lambda_2(\mathrm{e}^{\lambda_1 T} - \mathrm{e}^{\lambda_2 T})}{\lambda_2-\lambda_1}, & \pm 1 + \frac{\lambda_1\mathrm{e}^{\lambda_2 T} - \lambda_2\mathrm{e}^{\lambda_1 T}}{\lambda_2-\lambda_1} \end{matrix}\right\|_{\mathbb{R}^2\times\mathbb{R}^2},$$

where

$$k_0^{-1} = \left[1 \mp (\mathrm{e}^{\lambda_1 T} + \mathrm{e}^{\lambda_2 T}) + \mathrm{e}^{\lambda_1 T + \lambda_2 T}\right]^{-1},$$

$$\lambda_1 = \frac{-a - \sqrt{a^2 - 4b}}{2}, \quad \lambda_2 = \frac{-a + \sqrt{a^2 - 4b}}{2}.$$

For instance, for $a = 0, b = -1$, we get $k \leq \frac{1+\mathrm{e}^T}{2+\mathrm{e}^T+\mathrm{e}^{-T}} < 1$; condition (43) can be then satisfied, when e.g. $L \leq \frac{1}{T(16\mathrm{e}^{4T}+4\mathrm{e}^{2T})}$.

After all, since the usage of bounding function $V(x) := \frac{1}{2}\|x\|^2 - R$ is the most standard one, the illustrative example demonstrates that, in view of the above arguments, the practical application of Theorem 1 reduces to separable Hilbert spaces.

Acknowledgements. The first and the third author were supported by the Council of Czech Government (MSM 6198959214). The second author were supported by the national research project PRIN "Ordinary Differential Equations and Applications".

References

[1] J. Andres and L. Górniewicz, Topological Fixed Point Principles for Boundary Value Problems, Topological Fixed Point Theory and Its Applications, Vol. 1, Kluwer, Dordrecht, 2003.

[2] J. Andres, M. Kožušníková and L. Malaguti, *Bound sets approach to boundary value problems for vector second-order differential inclusions*. Nonlin. Anal. **71**, 1-2 (2009), 28–44.

[3] J. Andres, L. Malaguti and M. Pavlačková, *On second-order boundary value problems in Banach spaces: a bound sets approach*. Topol. Meth. Nonlin. Anal. **37**, 2 (2011), 303-341.

[4] J. Andres, L. Malaguti and V. Taddei, *On boundary value problems in Banach spaces*. Dynam. Syst. Appl. **18** (2009), 275–302.

[5] R. Bader and W. Kryszewski, *On the solution set of differential inclusions and the periodic problem in Banach spaces*. Nonlin. Anal. **54**, 4 (2003), 707–754.

[6] I. Benedetti, E. Panasenko and V. Taddei, *BVP for Carathéodory inclusions in Hilbert spaces: sharp existence conditions and applications*. J. Appl. Anal. **16**, 2 (2010), 237–258.

[7] S. Cecchini, L. Malaguti and V. Taddei, *Strictly localized bounding functions and Floquet boundary value problems*. Electr. J. Qual. Th. Diff. Eqns. **47** (2011), 1–18.

[8] Ju. L. Daleckiĭ and M.G. Kreĭn, *Stability of Solutions of Differential Equations in Banach Space*. Translation of Mathematical Monographs, American Mathematical Society, Providence, R. I., 1974.

[9] R. Deville, G. Godefroy and V. Zizler, Smoothness and Renorming in Banach Spaces, Longman Scientific and Technical, Harlow, 1993.

[10] S. Hu and N. S. Papageorgiou, Handbook of Multivalued Analysis, Vol. I: Theory, Kluwer, Dordrecht, 1997.

[11] M. I. Kamenskii, V. V. Obukhovskii and P. Zecca, Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces. W. de Gruyter, Berlin, 2001.

[12] J. L. Massera and J. J. Schäffer, Linear Differential Equations and Functional Spaces. Academic Press, New York, 1966.

[13] N. S. Papageorgiou and S. Th. Kyritsi-Yiallourou, Handbook of Applied Analysis. Springer, Berlin, 2009.

[14] B. Ricceri and A. Villani, *Separability and Scorza–Dragoni's property*. Le Matematiche (Catania) **37** (1982), 156–161.

[15] T. Rzezuchowski, *Scorza–Dragoni type theorem for upper semicontinuous multivalued functions*. Bull. Acad. Polon. Sci. Sr. Sci. Math. **28**, 1-2 (1980), 61–66.

[16] G. Sell and Y. You, Dynamics of Evolutionary Equations. Springer, Berlin, 2002.

[17] I. I. Vrabie, Compactness Methods for Nonlinear Evolutions, 2nd ed., Longman House, Burn Mill, Harlow, 1990.

Jan Andres is presently employed as a full Professor at Palacký University in Olomouc, Czech Republic. He is there a chairman of Department of Mathematical Analysis and Applications of Mathematics. He was also a Visiting Professor at two American and many European Universities; in the last years, especially at University of Paris 1 – Sorbonne and University of Roma 1 – La Sapienza. He received the highest scientific degrees in Czech Republic (DSc.) and in Poland (dr hab.). He is a member of 15 editorial boards of international scientific journals. He is also a coautor (jointly with Lech Górniewicz) of a reputed monograph published in 2003 at Kluwer (its second edition will be published in this year by Springer jointly with World Publ. Corp., Beijing), the author of three invited monographic chapters (two of them in handbooks of ordinary differential equations and a topological fixed point theory) and a (co)author of more than 160 scientific articles. His main research activities concern topological methods in nonlinear analysis, differential equations and inclusions, multivalued dynamical systems, fractals and their application to quantitative linguistics.

Luisa Malaguti is an Associate Professor at the University of Modena and Reggio Emilia, Italy. She obtained the degree in Mathematics with honors in 1982 at the University of Modena. In 1987 she received the Ph.D. in applied mathematics from the University of Trieste, Italy. Her research interests include: boundary value problems, reaction-diffusion processes, multivalued analysis and evolution equations.

Martina Pavlačková is presently employed as an Assistant Professor at Palacký University in Olomouc, Czech Republic. She obtained her PhD from Palacký University in 2008. Her research concerns mainly the existence and localization of solutions of the second-order multivalued boundary value problems. She is an active researcher coupled with the several years of teaching experience. She published several research articles in reputed international mathematical journals.