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Two-Mode Ring Lasers: Stable, Instable and Irregular Behaviors in Coupled Lasers Logistic Equations

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Abstract: We have analyzed two-mode ring coupled laser equations, after transforming them to the corresponding two dimensional coupled logistic equations. In our numerical experiment we find that for selected range of the three controlling parameters, chaotic pattern may emerge out of an infinite sequence of period-doubling bifurcations. However, there exists large regions of controlling parameter space where the approach to irregular behavior appears to be consistent with the Newhouse-Ruelle-Takens scenario, as evident by the appearance of two characteristics frequencies. As the ratio of controlling frequencies is varied, by changing the controlling parameters, phase locking periodic and quasi-periodic motions are observed on the way to chaos..

Keywords: Ring Lasers, controlling parameters, phase locking.

1. INTRODUCTION

One-dimensional maps are useful models not only for the description of specific population evolution but also as a kind of stroboscopic representation of the continuous solutions of nonlinear differential equations [1]. The competition between two species has already been discussed in the literature in term of coupled first order equations of form similar to that governing the single species growth [4]. In this paper we explain the two-mode ring lasers as a function of coupling strength, as the given equations reduce to coupled first order equations [5,6]. In coupled logistic maps, we found self-similarity stripe structure of basins, distortion of torus and transition to chaos [7]. V. A. Markelov in 1975, discussed fluctuations of two-mode emission from a ring laser, and concluded that in the self-locking of the difference frequencies, reduced the level of their fluctuations [8].

M. M-Tehrani and L. Mandel in August 1977, found that the weaker mode intensity did not grow with increase in pump parameter above threshold and that its relative intensity fluctuations did not die out, as in a conventional laser, but became thermal instead [9], and then in 1978, they explained intensity fluctuations. Shiqum Zhu in 1994, studied and explained the saturation effect in two mode ring lasers cavity [10,11]. The logistic map is natural a dynamical system capable of orderly and chaotic behavior depending on its parameters [12]. Coupling multiple logistic maps can result in multidimensional multidynamical behavior which is a powerful modeling tool in many fields of science.

In this paper the coupled laser equations have been transformed into coupled logistic equations. In 1983, Jian-Min Yuan and Lorenzo M. Narducci worked on coupled logistic maps and found that on changing controlling parameters, phase locking and quasi-periodic motion are observed on the way to chaos [13]. The layout of the paper is as follows. In the II section of the paper, we discuss the model and its corresponding mathematical equations, which are later transformed into coupled discrete laser equations. In III section we obtained bilinear coupled laser equations, we discuss analytically and numerically the obtain results. In last section we present conclusions.

2. Model

We consider a two-mode ring cavity in which waves are propagated in clockwise and anti-clockwise direction around

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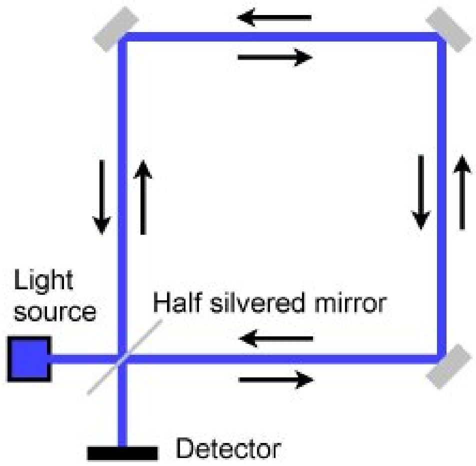

Figure 1 The possible diagram of ring cavity in which two waves are propagating in clockwise and anti-clockwise direction.

the ring [14,15], as shown in the Fig. 2.1. Mathematical formulation and simplification of these two-mode traveling waves in the cavity has been calculated [10,9]. These equations, written for electric field E_1 and E_2 are also called nonlinear coupled laser dynamical equation,

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} = \left[A - C_1 - B(|E_1|^2 + \xi|E_2|^2)\right] E_1 \tag{1}$$

$$\frac{\mathrm{d}E_2}{\mathrm{d}t} = \left[A - C_2 - B(|E_2|^2 + \xi|E_1|^2)\right] E_2 \tag{2}$$

where A, B and C_1, C_2 are respectively, Einstein gain coefficient, nonlinear saturation, and losses in the system. Moreover ξ is given as,

$$\xi = \frac{1}{[1 + (\Delta\omega \mathrm{T}_1)^2]}, \tag{3}$$

Here T_1 be natural life time of laser transitions, $\Delta\omega$ is detuning [9]. Multiplying $2E_1^*$ and $2E_2^*$ to Eqs. (3) and (4), respectively, and simplifying provides,

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} = \left[A - C_1 - B(|E_1|^2 + \xi|E_2|^2)\right] E_1 \tag{4}$$

$$\frac{\mathrm{d}E_2}{\mathrm{d}t} = \left[A - C_2 - B(|E_2|^2 + \xi|E_1|^2)\right] E_2 \tag{5}$$

$$\dot{I}_1 = 2(A - C_1)\left[1 - \frac{\mathrm{B}}{(\mathrm{A} - \mathrm{C}_1)}(I_1 + \xi I_2)\right] I_1, \tag{6}$$

$$\dot{I}_2 = 2(A - C_2)\left[1 - \frac{\mathrm{B}}{(\mathrm{A} - \mathrm{C}_2)}(I_2 + \xi I_1)\right] I_2, \tag{7}$$

where for real E_i, $I_i = |E_i|^2$, and $i = 1, 2$. Since intensity is proportional to the number of photons therefore, our equations will take the form

$$\dot{x} = \dot{n_1} = 2(A - C_1)\left[1 - (n_1 + \xi n_2)\right] n_1, \tag{8}$$

$$\dot{y} = \dot{n_2} = 2(A - C_2)\left[1 - (n_2 + \xi n_1)\right] n_2. \tag{9}$$

Since $\left[\frac{B}{(A-C_i)}\right]^{-1}$ = are the steady state photon number in the absence of coupling and too small value nearly equal to zero, we can get more simple equations, since $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x_{j+1} - x_j}{\Delta t}$.

If $(A - C_i)\Delta t = \lambda_i$, controlling parameter and Δt be the cavity round trip time then t^+ and t^- are the times of the waves which are moving clockwise and anticlockwise directions respectively. Therefore, in general form,

$$x_{n+1} = x_n + 2\lambda_1 x_n \left[1 - (x_n + \xi y_n)\right], \tag{10}$$

$$y_{n+1} = y_n + 2\lambda_2 y_n \left[1 - (y_n + \xi x_n)\right]. \tag{11}$$

It is our required solution in form of coupled logistic maps, on the application of chaos theory and will be discussed in detail in next section, in which value of ξ is complex.

3. Bilinear Coupled Laser Logistic Equations

The purpose of this article is to study the instability and chaotic behavior of two-mode ring laser system governed by the difference equations [16]. The symmetric two dimensional coupled laser logistic equations are,

$$x_{n+1} = F(x_n, y_n) = x_n + 2\lambda_1 x_n(1 - x_n) + \gamma y_n x_n, \tag{12}$$

$$y_{n+1} = F(x_n, y_n) = y_n + 2\lambda_2 y_n(1 - y_n) + \gamma x_n y_n, \tag{13}$$

where $\gamma = 2\lambda_i\xi$, λ_1 and λ_2 are controlling parameter and γ is coupling constant, therefore, our system depends on three parameters. We will see how our system show chaotic behavior corresponding to change the parameters.

Stability of Fixed Points: It is not difficult to understand the behavior of coupled functions for λ_i and γ. The key to the structure of $f_i(\lambda_i, \gamma)$ is careful analysis of fixed point of the mapping functions (1 and 2) as well as that iterate. Since we have two coupled function and symmetrical so we will discuss only one function. Therefore we have two fixed points [6]. The Eq. (3.1) has four fixed points but each is same. Therefore we calculate,

$$x_1^{(1)} = 0,\ and\ x_2^{(1)} = \frac{2\lambda_2(2\lambda_1 + \gamma)}{(4\lambda_1\lambda_2 - \gamma^2)}, \tag{14}$$

If $\lambda_1 = \lambda_2$ then fixed points are [6],

$$x_1^{(1)} = 0,\ and\ x_2^{(1)} = \frac{1}{(1 - \frac{\gamma}{\lambda})}. \tag{15}$$

A point is said to be a stable fixed point, if

$$\|(\frac{dx_{n+1}}{dx_n} - \frac{dx_{n+1}}{dy_n})\| < 1 \tag{16}$$

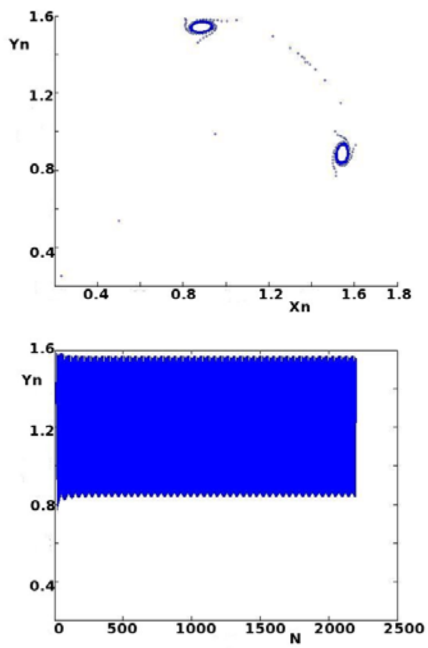

Figure 2 Phase space plots (x_n, y_n) of Eq. (3.1), corresponding to $\lambda_1 = \lambda_2 = 0.7$; $\gamma = 0.379$. Isolated points in the phase space are part of transient evolution; These maps have been constructed for few thousand iteration and the plot on right side shows the behavior of our system, which is unstable. Horizontal axis labels the number of iterations. Vertical axis is the y axis of the phase space, for the same parameter values given above.

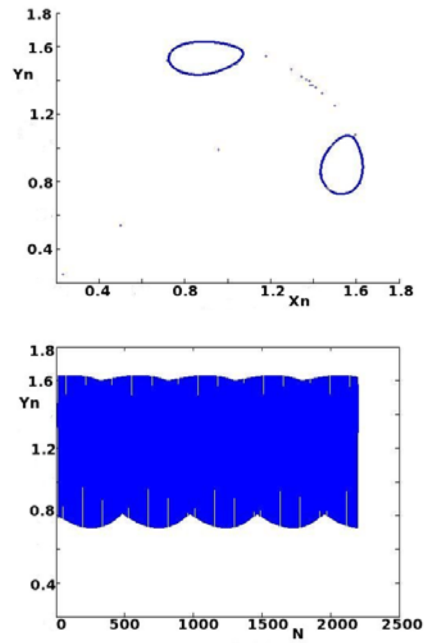

Figure 3 Phase space plots (x_n, y_n) of Eq. (3.1), corresponding to $\lambda_1 = \lambda_2 = 0.7$; $\gamma = 0.389$. The plot on right side shows the behavior of our system, which is stable. Horizontal axis labels the number of iterations. Vertical axis is the y-axis of the phase space, for the same parameter values given above.

A mapping of bilinear and linear coupling term [17, **?**], have been shown to exhibit complicated dynamical behavior, including quasi periodicity, phase locking, intermittency, period adding, long-lived chaotic transitions and etc [19]. We have also found evidence for a boundary crises in it like or similar to found in Henon's maps by Grebogi, Ott and Yorke [17] as selecting $\gamma = 0.1$ and varied the λ_i. A boundary crises occurs in our case through the collision of a chaotic attractor with the basin boundaries that separate it from the several other coexistent periodic attractors (an other chaotic attractor) [19]. Upon an increase of λ_i beyond its critical value for the onset of crises, the chaotic attractors and its basin disappear while the basins of remaining attractors undergo a sudden expansion. This in turn, produces hysteresis effects [18], in the dynamical as λ_i decreases through it critical value. From the research on basins of attraction and Mandelbrot-Julia set of coupled logistic map, the following result indicate that: the boundary between periodic and quasi-periodic regions is fractal [20], (that indicates the impossibility to predict the moving result of points in phase plane); the structure of Mandelbrot-Julia sets are determined by control parameters, and their boundaries have fractal characteristic [20].

We will explain it by mean of two cases. In first case we fixed $\lambda_1 = \lambda_2 = 0.7$ and vary the $0.1 \leq \gamma \leq 0.48$ and giving $x = 0.1$ and $y = 0.11$ [13]. For $\gamma = 0.1$ to 0.24 the trajectory of (x, y) in phase space converges to a fixed point 1P. If, γ varies from 0.24 to 0.375 we get 2P character, asymptotic solution of our system. At the upper value of γ from this interval [0.34, 0.375], the transient solution displays a curious three wing pattern before settle down to asymptotic 2P state. When $\gamma = 0.379$ the two original stable points become unstable and bifurcate into two invariant orbits and behavior of the system, in phase space is quasi-periodic, as show in Fig. 3.2. The large value of γ lead the same result but trajectory grow.

At certain values of coupling parameter strange attractors are mirror images of each other but different at certain other value. Mirror image means they get synchronized frequency locking. At $\gamma = 0.391$ suddenly the trajectory converges to a periodic solution with period 6, as shown in Fig. 3.3. This is interpreted as a phase-locking with a quasi-periodic domain. The effect persists for a small, but measurable range of γ value. Quasi-periodicity again appeared at $\gamma = 0.392$, to replaced again by phase-locking with a period of 20, 26, 14, 22 and 8 iterations, at $\gamma = 0.4$, 0.407, 0.415, 0.419 and 0.425 respectively, among each interval except these values our dynamical lasing system show quasi-periodic and chaotic behaviors.

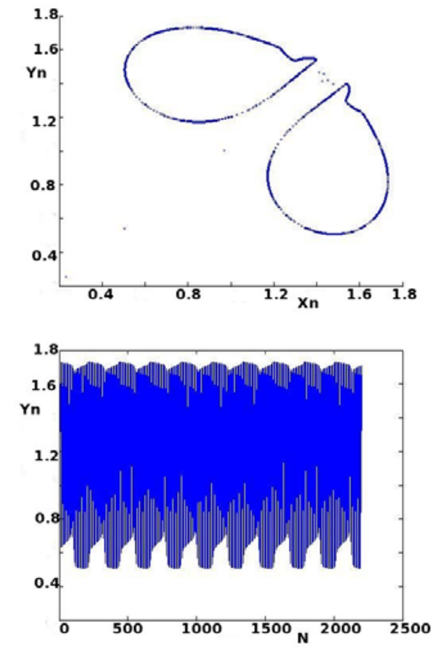

Figure 4 Phase space plots (x_n, y_n) of Eq. (3.1), corresponding to $\lambda_1 = \lambda_2 = 0.7$; $\gamma = 0.413$. The plot below side shows the behavior of our system, which is almost unstable. Horizontal axis labels the number of iterations. Vertical axis is the y-axis of the phase space, for the same parameters values given above.

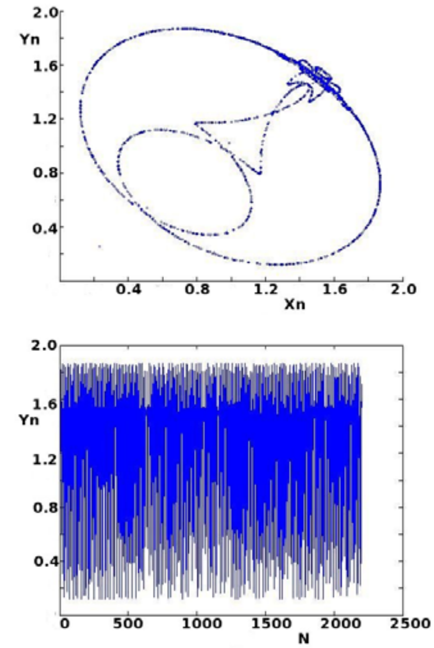

Figure 5 Orbital diagram in phase space of Eq. (3.1) for $\lambda_1 = \lambda_2 = 0.7$ and $\gamma = 0.47$. This orbital diagram is obtained due to boundary crises, the plot on right side shows chaotic behavior, for same parameter values given above

As coupling between two waves propagating in counter-clockwise direction in ring cavity is increased from $\gamma = 0.41$ to 0.424 the invariant orbit grow in size and became distorted in shape; eventually develop "ear" like shape, as shown in Fig. 3.4. At $\gamma = 0431$ the system is again phase-locked with a period of 18 iterations. As γ increased beyond 0.439, the two halves of the mapping grow and eventually overlap with each other (exterior crises occur between the boundary of two orbits) the orbital diagram, as shown in Fig. 3.5. When $\gamma = 0.475$, the system is again phase-locked with a period of 18P iterations. At the above value of $\gamma = 0.475$ our dynamical system display of iterated values of n Versus x_n appear to be quite irregular.

If one analyzes these two dimensional plots as Poincare surfaces of section for the discrete system, the sequence can be describe as: The 1P corresponds to a stable limit cycle [5]. As the γ increasing further, the limit cycle becomes unstable and bifurcates into a two-loop limit cycle and then evolve into a two-loop torus through a Hopf bifurcation [7]. The torus represents quasi-periodic behavior of our system and responsible for the two invariant orbit on the Poincare surface of section. The two intermittence periodic behavior is obtained when the two characteristic frequencies on the torus are in ratio of two small integers [13]. Higher bifurcation of the torus occurs as the system moves out of quasi-periodic region, by increasing γ. These bifurcations are characterized by the growing ear, folding of the boundary onto itself, the merging of separate part into a single one because of boundary crises, frequency locking and eventually breakup the torus [13, **?**].

Case 2: When λ_1 and λ_2 are not equal to each other then irregular behavior display by means of period doubling bifurcation. We have to study in this region for which, $\lambda_1 = 0.7$ and $\lambda_2 = 0.2$ and γ is varied from 0.2 to 0.4. For $\gamma = 0.37$, the asymptotic solution has 6P and y is running very small value. If γ is varied from 0.3 to 0.4, we detect period doubling bifurcation which leads to 64P solutions and then to chaos.

Thus from the above discussion, we can conclude that when $\lambda_1 = \lambda_2$ and variation exist in γ chaos emerges through quasi-periodicity. When $\lambda_1 \neq \lambda_2$, different values to each other, so chaos emerges by mean of period doubling bifurcation sequence. Since we have noted that oscillatory behavior at $\lambda_1 = 0.7$, $\lambda_2 = 0.2$ and in range of $\gamma \in [0.35, 0.376]$, from 2P to 6P through quasi-periodic state. The transition from 2P to quasi-periodic behavior is one (as a function of γ) and is accompanied, during the transient, by considerable noisier structure then observed in pervious scans; furthermore the duration of the irregular transient is a sensitive function of the position of the initial point [13], a feature which is reminiscent of the

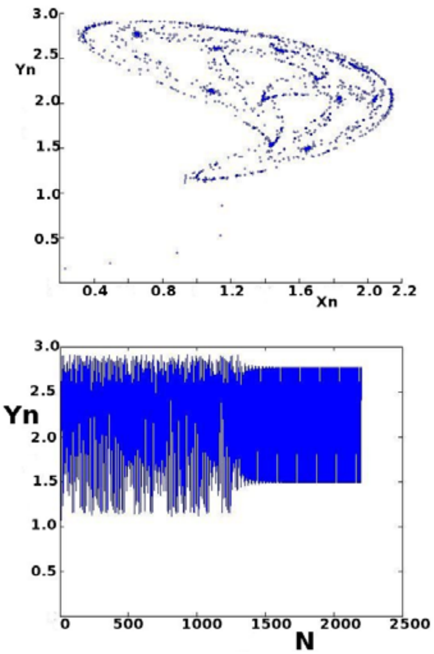

Figure 6 Phase space plots (x_n, y_n) of Eq. (3.1), corresponding to $\lambda_1 = 0.7,\ \lambda_2 = 0.2$; $\gamma = 0.379$. The plot on right side shows chaotic the behavior which is not denser. Horizontal axis labels the number of iterations. Vertical axis is the y-axis of the phase space, for the same parameters values given above.

metastable behavior observed in the Lorenz model for the certain range of coupling parameter. An example of the trajectories is shown in Fig. 3.6.

The general method to map out the different kind of dynamical behaviors existing in various regions of the three-dimensional control-parameter space is to fix the parameter γ and to scan the (λ_1, λ_2) plane. We see again that along and around the main diagonal $\lambda_1 = \lambda_2$ the transition from periodic behavior to chaos takes place through intermediary quasi-periodic motion, while away from the main diagonal, chaos results from a series of period-doubling bifurcations [13].

4. Conclusion

The coupling between the two laser logistic equations has a profound effect on the character of the solutions and on the approach to chaos that is observed for one dimensional logistic map. In general, one may find it only natural that, when complicated nonlinear system are coupled to each other, they became more complicated. On the other hand, at least locally, multidimensional systems are known to exhibit essentially one-dimensional character, but depending on the linearized eigenvalues. To calculate the result

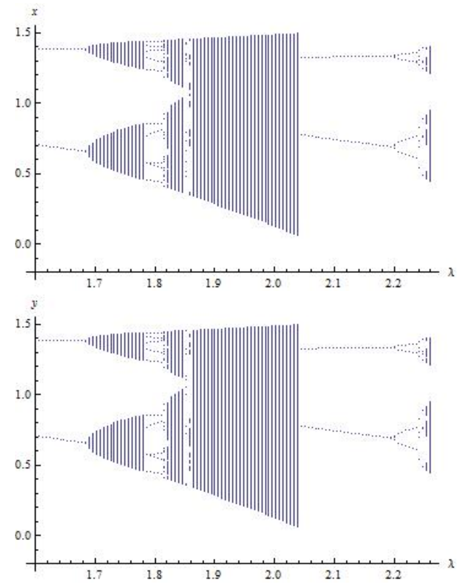

Figure 7 Bifurcation diagram for the coupled lasers Eqs. (3.1) with $\lambda_1 = \lambda_2 = 1.5$ range from 1.6 to 2.2 and $\gamma = 0.25$. For each value of λ_i we used the final point of the previous λ_i value and 1600 iterates are plotted. This shows the period-doubling sequence as well as Quasi-periodic and chaotic regions.

from the above discussion, there are regions of controlling-parameter space where indeed the qualitative behavior of solutions is some identical to that predicated by the one dimensional (laser) logistic equation. But here we have looked that our coupled laser logistic system remain stable which mean period oscillation from period 2 to period 6 through quasi-periodic state. The cause of not getting period 4 here is that, in phase space plots at $\lambda_1 = \lambda_2 = 0.7$ and $\gamma = 0.379$ each laser attractors, which are orthogonal, has three wing pattern so cause period 6, corresponding to change the frequency. So we can say that in laser coupled system there exist period tripling-furcation.

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