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Spectral Galerkin Method for Optimal Control Problems Governed by Integral and Integro- Differential Equations

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Abstract:: In this paper a Legendre integral method is proposed to solve integral and integro- differential problems and optimal control problems governed by integral and integro- differential equations. Galerkin method is used to reformulate the problem as constrained optimization problem. The resulting constrained optimization problem is solved by Hybrid penalty partial quadratic interpolation technique. Numerical results are included to confirm the efficiency and accuracy of the method.

Keywords: Spectral methods- Legendre polynomials – Hybrid penalty partial quadratic interpolation technique- Integral equations – Integro differential equations - Optimal control Problems.

1. Introduction

Spectral methods using expansion in orthogonal polynomials such as Chebyshev or ultraspherical polynomials is successful in the numerical approximation of various boundary value problems, see for instance Ahues et al [1], Canuto et al [3]and Kopriva [5]. If these polynomials are used as basis functions, then the rate of decay of the expansion coefficients is determined by the smoothness properties of the function being expanded. This choice of trial functions is responsible for the superior approximation properties of spectral methods compared with finite difference and finite element methods.

El-Hawary et al. [4] derived some useful properties of the ultrashperical polynomials. They introduced an ultrashperical approximation for any continuous function and its finite integrals. They derived error estimates for this approximation. They introduced an algorithm that gives an optimal approximation of the integrals. Optimal control problems governed by integral and integro- differential equations arise in a variety of problems with memory effects, including economics, population dynamics, epidemiology. Belbas[2] presented and analyzed a novel method for solving optimal control problems for Volterra integral equations, based on approximating the controlled Volterra integral equations by a sequence of systems of controlled ordinary differential equations.

Peng et al.[6] considered a class of second-order nonlinear impulsive integro-differential equations of mixed type whose principle is time-varying generating operators with unbounded perturbation on Banach spaces. They discussed the perturbation of time-varying operator matrix and constructed corresponding the evolution system generated by operator matrix. Ningning[7] considered the numerical simulation for a class of constrained optimal control problems governed by integral equations of Fredholm type. He provided a superconvergence analysis for the Galerkin approximation to these control problems. Based on the results of the superconvergence analysis, he established a recovery type a posteriori error estimator, which can be used for adaptive mesh refinemen The purpose of this paper is to solve integral and integro- differential problems and optimal control problems governed by integral and integro- differential equations. We shall combine Legendre integral approximation with Galerkinmethodto reformulate the integral and integro- differential problem to unconstrained optimization problem. Optimal control problems governed by integral and integro- differential equations shall be reformulated to be constrained optimization problem.

The resulting unconstrained optimization problem is solved by penalty partial quadratic interpolation technique whereas the constrained optimization problem is treated by Hybrid penalty partial quadratic interpolation technique.

The error bound for the used approximation is discussed to ensure the efficiency of the proposed method. We include enough numerical examples and comparisons to confirm accuracy of the method.

This paper is organized as follows. In Section 2 we will introduce the Legendre spectral approximation. In Section 3 we will discuss error bound. In Section 4, we will state basis of Galerkin method. Setting of the problem is found In Section 5. In Section 6 we will describe the proposed method of solution which uses Galerkin method to reformulate the problem as constrained optimization problem. In Section 7 we will state Hybrid penalty partial quadratic interpolation technique to solve the resulting constrained optimization problem. In Section 8, we will give numerical results. Finally in Section 9 we will conclude this paper.

2. The Legendre Spectral Approximation

The Legendre polynomials are a sequence of polynomials $Q_j(x),\ j=0,1,2,...$ each respectively satisfying the following two recurrence relations

$$(j+1)Q_{j+1}(x)=(2j+1)Q_j(x)-jQ_{j-1}, \qquad j\geq 1 \tag{2.1a}$$

$$\frac{j+1}{(2j+!)(j+1)}D^m Q_{j+1}(x)=\frac{1}{(2j+!)}D^m Q_{j-1}(x)+D^{m-1}Q_j \tag{2.1b}$$

The recurrence formula (2.1a) can be used to generate the Legendre polynomials starting with $Q_0(x)=1$ and $Q_1(x)=x$ (see Szegö [8]). The m order derivative of the Legendre polynomials can be generated by the recurrence formula (2.1b). where $m\geq 1$ is the order of differentiation.

We have from (2.1)

$$\begin{aligned} Q_{j+1}(x) &= \frac{2j+1}{j+1}x\left(\frac{2f-1}{j}Q_{j-1}-......\right)-\frac{j}{j+1}Q_{j-1} \\ &= \theta_{\mathrm{j+1}}X^{j+1}+\text{Polynomial of flower order} \end{aligned} \tag{2.2}$$

where,

$$\theta_j=\frac{\Gamma(2j)}{2^j\,\Gamma(j+1)\Gamma(j)}$$

The Legendre polynomials satisfy the orthogonal property

$$\int_{-1}^{1}Q_j(x)Q_k(x)dx=\lambda_j\delta_{jk},$$

where $\lambda_j=\dfrac{2}{2j+1},$ (2.3)

and $\delta_{jk}=\begin{cases}0 & if\ \ j\neq k\\ 1 & if\ \ j=k\end{cases}$

We present here the Legendre approximation of the integration of any function $f(x)=C^{\infty}[-1,1]$ at equally spaced points

$$S=\left\{x_k=-1+\frac{2k}{N},\ k=0,1,...,N\right\}.$$

Let f(x) be approximated at Sby Legendre polynomials, namely

$$f(x)=\sum_{j=0}^{N}a_j Q_j(x)\ ,\ \ x\in S$$

where

(2.4a)

$$a_{\mathrm{j}}=\left(\lambda_{\mathrm{j}}\right)^{-1}\int_{-1}^{1}Q_{\mathrm{j}}(x)f(x)dx$$

(*)

We can write trapezoidal rule for approximating this integral

$$\int_a^b g(x)dx=h\sum_{j=0}^{N}{}^{"}g(x_{\mathrm{j}})-\frac{(b-a)h^2}{12}g^{(2)}(\xi),\ h=\frac{(b-a)}{N},$$

so equation (*) will be:

$$a_{\mathrm{j}}=\tfrac{2}{N}\left(\lambda_{\mathrm{j}}\right)^{-1}\sum_{k=0}^{N}{}^{"}Q_{\mathrm{j}}(x_k)f(x_k)-\tfrac{2}{3\mathrm{N}^2}\left(\lambda_{\mathrm{j}}\right)^{-1}\left[Q_j(\xi)f(\xi)\right]^{(2)}$$

(2.4b)

where the double prime on the summation denotes that the first and last term halved.

From Eqs. (2.2) -(2.4), we can define the elements of the matrices

$D^{(n)}=\left[d_{ij}^{(n)}\right],\ i,\ j=0(1)N,\ n=1,2,...$, n=1, 2,…. as follows:

$$D^{(n)}f(x)\Big|_{x=x_i}=\sum_{k=0}^{N}d_{ik}^{(n)}f(x_k),\ x\in S,$$

$i=0(1)N$, (2.5a)

where

$$d_{ik}^{(n)}=\sum_{j=0}^{N}\frac{\beta_k}{N}(\lambda_j)^{-1}Q_j(x_k)D^{(n)}Q_j(x)\Big|_{x=x_i},$$

(2.5b)

with

$$\beta_k=\begin{cases}1, & k=0,N\\ 2 & o.w\end{cases},\qquad x_k\in S \quad (2.6)$$

We approximate the m-times integral of a function f at $x_i\in S$ by the following relation:

$$\underbrace{\int_{-1}^{x_i}\int_{-1}^{x_i}\cdots\int_{-1}^{x_i}f(x)dxdx\ldots dx}_{m-times}=\sum_{k=0}^{N}q_{ik}^{(m)}f(x_k),$$

$x_i, x_k\in S, i=0(1)N$ (2.7a)

where

$$q_{ij}^{[m]}=\frac{(x_i-x_j)^{m-1}}{(m-1)!}q_{ij}$$

$i, j=0(1)N$,

(2.7b)

and

$$q_{ik}=\sum_{r=0}^{N}(\lambda_r)^{-1}\varpi_k Q_r(x_k)\int_{-1}^{x_i}Q_r(x)dx$$

3. Error Bound

Let $f(x)=C^{\infty}[-1,1]$ be approximated by (2.4a). We define the function:

$u(x)=f(x)-P_N(x)-\rho Q_{N+1}(x)$,

and choose ρ such that $U(x)$ has a root $\bar{x}$ satisfying $Q_{N+1}(\bar{x})\neq 0$, then

$f(\bar{x})-P_N(\bar{x})-\rho Q_{N+1}(\bar{x})=0$,

$$\rho=\frac{f(\bar{x})-P_N(\bar{x})}{Q_{N+1}(\bar{x})} \quad (3.1)$$

Since

$f(x)\in C^{\infty}[-1,1]$, $P_N(x)\in C^{N}[-1,1]$, $Q_{N+1}(x)\in C^{N+1}[-1,1]$

it follows that u(x) has more than N+1 roots. This ensure that $u^{(N+1)}(x)$ has at least one root, then there exists $\xi=\xi(x)$ in [-1,1] at which

$$u^{(N+1)}(\xi)=f^{(N+1)}(\xi)-P_N^{(N+1)}(\xi)-\rho Q_{N+1}^{(N+1)}(\xi)=0 \quad (3.2)$$

Since $P_N^{(N+1)}(\xi)=0$ and by (2.2)

$Q_{N+1}^{(N+1)}(\xi)=(N+1)!\theta_{N+1}$,

$$\rho=\frac{f^{(N+1)}(\xi)}{(N+1)!\theta_{N+1}}$$

Then, f rom (3.1) and (3.2) we have:

$$\frac{f^{(N+1)}(\xi)}{(N+1)!\theta_{N+1}}=\frac{f(x)-P_N(x)}{Q_{N+1}(x)}$$, hence

$$f(\bar{x})-P_N(\bar{x})=\frac{f^{(N+1)}(\xi)}{(N+1)!\theta_{N+1}}Q_{N+1}(\bar{x}).$$

This error is bounded since

$$\|R_N(X,\xi)\|=\|f(\bar{X})-P_N(\bar{X})\|=\frac{1}{(N+1)!\theta_{N+1}}\|f^{(N+1)}(\xi)\|\|Q_{N+1}(\bar{x})\|$$

Hence

$$\|R_N(X,\xi)\|\leq \max_{-1\leq\xi\leq 1}\frac{1}{(N+1)!\theta_{N+1}}\|f^{(N+1)}(\xi)\| \quad (3.3)$$

Now Let $f(x)$ be approximated (2.4a), then by using (2.4b) and (3.3)

$$f(x)=\sum_{j=0}^{N}\left(\frac{\lambda}{N}\right)(\lambda_J)^{-1}\sum_{k=0}^{N}Q_j(x_k)f(x_k)-\frac{2}{3N}(\lambda_j)^{-1}\left[Q_j(\xi)f(\xi)\right]^{(2)}Q_j(x)+\frac{f^{(N+1)}(\xi)}{(N+1)}Q_{N+1}(x)$$

so

$$f(x)=\sum_{j=0}^{N}\left(\frac{\lambda}{N}\right)(\lambda_J)^{-1}\sum_{k=0}^{N}Q_j(x_k)f(x_k)-\frac{2}{3N^2}\sum(\lambda_j)^{-1}\left[Q_j(\xi)f(\xi)\right]^{(2)}Q_j(x)+\frac{f^{(N+1)}(\xi)}{(N+1)}Q_{N+1}(x)$$

by differentiation m - times, we get

$$D_x^m f(x)\big|_{x=x_i} = \sum_{j=0}^{N} \frac{2}{N} (\lambda_j)^{-1} \sum_{k=0}^{N} Q_j(x_k) f(x_k) D_x^m Q_j(x)\big|_{x=x_i}$$
$$- \frac{2}{3N^2} \sum_{j=0}^{N} (\lambda_j)^{-1} \left[Q_j(\xi) f(\xi)\right]^{(2)} D_x^m Q_j(x)\big|_{x=x_i}$$
$$+ \frac{f^{(N+1)}(\xi)}{(N+1)!\theta_{N+1}} D_x^m Q_{N+1}(x)\Big|_{x=x_i}$$
$$= \sum_{k=0}^{N} d_{ik}^m f(x_k) + E_m(x_i,\xi),$$

Where $E_m(x_i,\xi)$ is defined by

$$E_m(x_i,\xi) = \frac{f^{(N+1)}(\xi)}{(N+1)\theta_{N+1}} D_x^m Q_{N+1}(x) - \frac{2}{3N} \sum_{j-0}^{N} (\lambda_i)^{-1} \left[Q_j(\xi) f(\xi)^2\right] D_x^m Q_j(x) \tag{3.4}$$

The error bound of the integral is obtained from

$$\int_{-1}^{x_i}\int_{-1}^{x_i}\ldots\ldots\int_{-1}^{x_i} f(x)dxdx\ldots.dx = \sum_{k=0}^{N} q_{ik}^{(m)} f(x_k) + \left[R_m(x,\xi)\right],\;\; x_k \in S,\, i = 0,1,\ldots\ldots N \tag{3.5}$$

where

$$R_m(x,\xi) = \frac{f^{(N+1)}(\xi)}{(N+1)\theta_{N+1}} \underbrace{\int_{-1}^{x_i}\int_{-1}^{x_i}\ldots\int_{-1}^{x_i}}_{m-times} Q_{N+1}(x)dxdx\ldots dx$$
$$-\frac{2}{3N}\sum_{j=0}^{N}(\lambda_j)^{-1}\left[Q_j(\xi)f(\xi)\right]^{(2)} \underbrace{\int_{-1}^{x_i}\int_{-1}^{x_i}\ldots\int_{-1}^{x_i}}_{m-times} Q_j(x)dxdx\ldots dx \tag{3.6}$$

and $\xi = \xi(x)$ in [0,1]. and $Q_j(x),\;\; j = 0,1,2,\ldots$ are the Legendre polynomials

4. Galerkinmethod

Consider the spectral approximation $u_N(x)$ is used to solve the integro- differential problem $L(u) = f$ with orthogonal base function $\psi_k(x), k = 1,2,\ldots N$. The Galerkin idea is to require the residual ([12])

$$R_N(x) = L(u_N(x)) - f(x) \tag{4.1}$$

To be orthogonal to each of the base functions. That is the inner product

$$\left\langle R_N, \psi_j \right\rangle = 0,\;\; j = 1,2,\ldots,N \tag{4.2}$$

using the inner product definition [17]

$$\int_{-1}^{1} w(x) R_N(x) \psi_j(x) dx = 0,\;\; j = 1,2,\ldots,N \tag{4.3}$$

With w is the weight function corresponds to the orthogonal base functions $\psi_k(x), k = 1,2,\ldots N$.

5. Setting of the problem

We are dealing with the optimal control problem governed by integral or integro- differential equations. The problems can be stated as:

Find the control $u(t)$ which minimizes the functional

$$I = \int_{\Gamma} f\left(x(t), u(t), t\right) dt \,, \tag{5.1}$$

subject to an integro- differential equation

$$L\left(x\left(t\right),u\left(t\right),t\right)=g\left(t\right), \qquad (5.2a)$$

where

$$L(x(t),u(t),t)=x^{(m)}(t)+\sum_{k=0}^{m-1}a_k x^{(k)}(t)+\int_{\Gamma}K\left(t,s,x(s).u(s)\right)ds \qquad (5.2b)$$

$$x^{(r)}(0)=A_r\,,r=0,1,....,m-1, \qquad (5.2c)$$

Where $x^r(t)$ denotes the rth- ordinary derivative of $x(t)$ and the coefficients $\alpha_r, r=0,1,....,m-1$, may be functions in t and $\Gamma[-1,1]$.

6. Description of the solution

We approximate the derivatives in equation (5.2b) via differentiation spectral matrix (2.5), i. e.

$$x^{(n)}(t_i)=\sum_{k=0}^{N}d_{ik}^{(n)}x(t_k),\ n=1,2,...,m \qquad (6.1)$$

We use integration matrix (2.7) to approximate the integral in (5.1) and (5.2b). Furthermore, We shall use approximating for the control variable,

$$u\left(t_i\right)=\omega_0+\sum_{k=1}^{M}\omega_k\int_{\Gamma}Q_k\left(t\right)dt \qquad (6.2)$$

With ω_j, $j=0(1)M$ are unknown coefficients.

The objective (5.1) can be approximated by:

$$I=\sum_{k=0}^{N}q_{ik}f\left(x\left(t_k\right),u\left(t_k\right),t_k\right), \qquad (6.3)$$

The constraints (5.2a) in view of Galerkin method become as follows

$$\bar{L}_j=\int_a^b L\left(x\left(t\right),u\left(t\right),t\right)Q_j(t)dt=\int_a^b g(t)Q_j(t)dt,$$

$$j=0,1,2,...,N \qquad (6.4)$$

Eq (6.4) in view of (5.2b) become

$$\bar{L}_j=\int_a^b\left[x^{(m)}(t)+\sum_{k=0}^{m-1}\alpha_k x^{(k)}(t)+\int_{\Gamma}K\left(t,s,x\left(s\right),u\left(s\right)\right)ds\right]Q_j(t)dt=\int_a^b g(t)Q_j(t)dt,\ j=0,1,2,...,N$$

Which can be rewritten making use of (2.5) and (2.7) as follows

$$\bar{L}_j=\sum_{l=0}^{N}q_{Nl}Q_j(t_l)\left[\sum_{\nu=0}^{N}d_{l\nu}^{(m)}x(t_\nu)+\sum_{k=0}^{N}\alpha_k\sum d_{l\nu}^{(m)}x(t_\nu)+\sum_{\nu=0}^{N}q_{N\nu}K\left(t_l,s_\nu,x(s_\nu),u(s_\nu)\right)\right]$$

$$=\sum_{\ell=0}^{N}q_{N\ell}\,g(t_\ell)Q_j(t_\ell),\ j=0,1,2,...,N \qquad (6.5)$$

Also (5.2c) become

$$\sum_{k=0}^{N}d_{0k}^{(r)}x(t_0)=A_r\,,r=0,1,....,m-1, \qquad (6.6)$$

Hence the problem (5.1)- (5.3) can be approximated by:

Find the control $u(t)$ which minimizes the functional(6.3) subject to (6.5)-(6.6).

We can reformulate this optimal control problem as the constrained optimization proble

Minimize $j = I(\varpi, x)$, (6.7-a)

subject to $F(\omega, x) = \sum_{j=0}^{N} \bar{L}_j^2$, (6.7-b)

The problem (6.7) has (N+M+2) unknown variables. It will be solved via the following technique.

7. Hybrid penalty partial quadratic interpolation technique

The problem of solving optimal control problems by means of our Legendre approximation method reduced to a constrained optimization problem. We apply the Hybrid penalty partial quadratic interpolation technique (HPPQI), see Salim et al. [9] for the constrained optimization problem.

The constrained minimization problem can be stated as:

Minimize $f(T),\ T = \{t, t, ..., t\},$ (7.1)

subject to $g(T) = 0, for\ all\ i = 1(1)M$ (7.2)

where the functions f and $g_i, i = 1(1)M$ are assumed to be twice continuously differentiable function of T.

The Hybrid penalty partial quadratic interpolation technique (HPPQI) to solve the constrained optimization problem (7.1) by means of a sequential minimization of the Penalty function

$\Phi(T, \mu) = f(T) + \mu \sum g(T)$ (7.3)

Let T_k is an initial feasible or infeasible value, and $\mu_k > 0$. We solve (7.3) as unconstrained optimization problem by means of HPPQI algorithm [see Salim et al. [9].

8. Numerical results:

Now, we consider two examples to show the effectiveness of our technique in case of optimal control problems governed by Volterra equation or integro- differential equation.

Example 8.1:

Minimize $I = \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right]dt,$ (8.1)

Subject to $x(t) = t + \int_0^t \sqrt{ts}\left[x(s) + u(s)\right]ds \quad ,0 \le t \le 1 \cdot$

(8.2)

Approximating $x(t)$ and $u(t)$ using (6.1) and (6.2) respectively, making use of (2.7) for approximation integrals and (4.3) for treating the condition(8.2), the given problem is transformed to:

$$J = \frac{1}{2}\sum_{k=0}^{N} q_{Nk}\left(x^2(t_k) + u^2(t_k)\right)$$

$$\bar{L}_j = \sum_{\ell=0}^{N} q_{N\ell} Q_j(t_\ell)\left[x(t_j) - t_j - \sum_{\ell=0}^{N} q_{j\ell}(t_j + t_\ell)\left[x(t_\ell) + u(s_\ell)\right]\right], \quad F = \sum_{j=0}^{N} \bar{L}_j^2$$

The optimal cost ***J*** and maximum absolute error in the constraints ***F***obtained for this problem is presented in Table 1. These results are compared to those obtained by Salim [11] and El-Kady[10]. In figure 1 we plot the state and control variables obtained from present method.

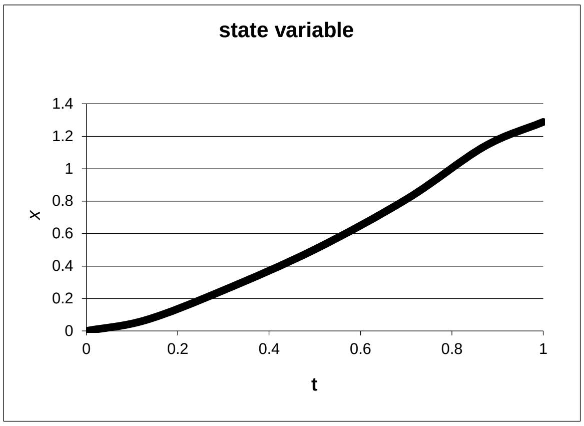
Example 8.2

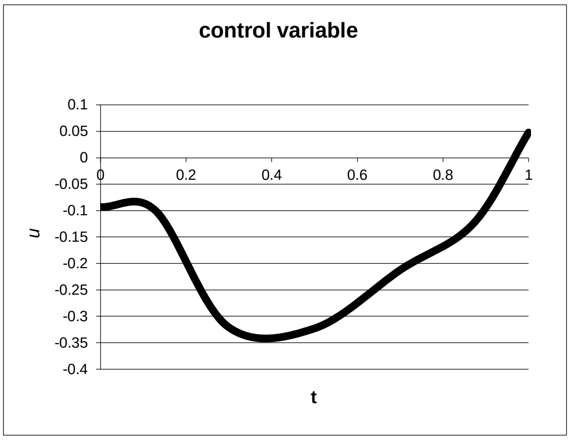
Minimize $I = \int_0^1 \left[x^2(t) + u^2(t)\right]dt,$

(8.3)

Table 1: Results of example 8.1, N=M=4

Method	***J***	***F***
Present method	0.2410745	4.63E-09
Salim [11]	0.241549	1.66E-07
El-Kady[10]	0.241437	4.21E-08





Figures 1: State and control variables

Subject to $x''(t)+\int_0^t\left[x(s)+u(s)\right]ds = 2-\text{Cos}(t) \quad ,0 \le t \le 1,$ (8.4)

with $x(0)=1$ and $x'(0)=-1.$

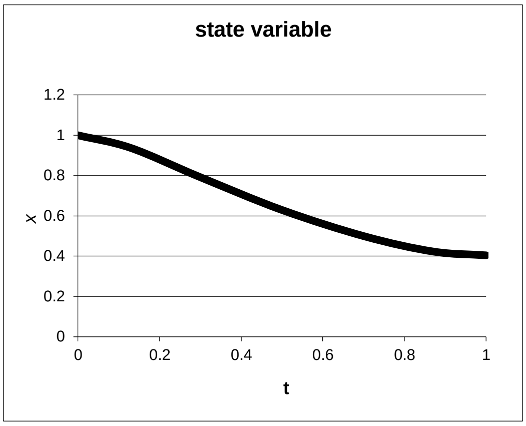
Approximating $x(t)$ and $u(t)$ using (6.1) and (6.2) respectively, making use of (2.7) for approximation integrals and (4.3) for treating the condition(8.2), the given problem is transformed to:

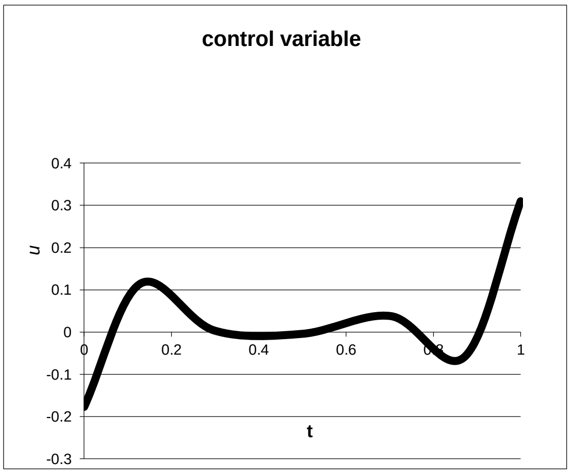
$$J = \sum_{k=0}^{N} q_{Nk}\left(x^2(t_k)+u^2(t_k)\right)$$

The optimal cost ***J*** and maximum absolute error in the constraints ***F***obtainedfor this problem is presented in Table 2. These results are compared to those obtained by El-Kady[10]. In figure 2 we plot the state and control variables obtained from present method.

Table 2: Results of example 8.2, N=M=4

Method	***J***	***F***
Present method	0.4387209	2.23E-09
El-Kady[10]	0.43874	4.02E -08





Figures 2: State and control variables

9. Conclusion

The basic idea of our present method is to combine Legendre integral approximation with Galerkinmethodto reformulate the integral and integro- differential problem to unconstrained optimization problem. Optimal control problems governed by integral and integro- differential equations shall be reformulated to be constrained optimization problem. Solving the resulting constrained optimization problem is easier than solving the original problem.

The convergence of the proposed method depends on the Legendre approximation method (El-Hawary et al [4]) and the Hybrid penalty partial quadratic interpolation technique ([9] Salim).

The results given previously show that the suggested technique is quite reliable. It can be successfully applied to both linear and nonlinear integral and integro- differential problems and optimal control problems governed by integral and integro- differential equations. The method produces an accurate solution at small number of nodes. The comparison of the maximum absolute error resulting from the proposed method and those obtained by Salim[11], El-Kady[10], show favorable agreement and is more accurate.

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References

[1] Ahues, M. ,Largillier, A. and Limaye, B.V. ,Spectral Computations or Bounded Operators, Chapman & Hall/CRC, 2001.

[2] Belbas, S.A.,A reduction method for optimal control of Volterra integral equations, Applied Mathematics and Computation 197 (2008) 880–890.

[3] Canuto, C., A. Quarteroni M. Y. Hussaini, and Zang, T. A. ,Spectral Methods Evolution to Complex Geometries and Applications to Fluid Dynamics, Springer-Verlag, Berlin Heidelberg, 2007

[4] El-Hawary, H. M., Salim, M. S and Hussien, H. S. (2000), An Optimal Ultraspherical Approximation of Integrals, Intern. J. of Computer Math., 76 219-237.

[5]Kopriva ,D. A. ,Implementing Spectral Methods for Partial Differential Equations, Algorithms for Scientists and Engineers, Springer Science + Business Media B.V. 2009.

[6] Peng, Y., Xiang, X. and Wei, W. ,Second-order nonlinear impulsive integro-differential equations of mixed type with time- varying generating operators and optimal controls on Banach spaces, Computers and Mathematics with Applications 57 (2009) 42-53.

[7] Ningning Y., B. ,Superconvergence Analysis And A Posteriori Error Estimation Of A Finite Element Method For An Optimal Control Problem Governed By Integral Equations, Applications Of Mathematics No. 54 (2009) 3, 267–283.

[8] Szegö, G. ,Orthogonal Polynomials , Am. Math. Soc. Colloq. Pub. , 23, 1985.

[9] Salim, M. S., El-Kady, M.M. and El-Sagheer ,A. M. , Hybrid penalty partial quadratic interpolation technique in the constrained optimization problems, Journal of the Institution of Mathematics and computer Science (computer science series) 9 (1998), 7-18.

[10] El-Kady, M. M., Numerical Studies for Optimal Control Problems, Ph.D. Thesis, Assuit University, 1994.

[11] Salim , M. S, Numerical Studied of Optimal Control Problems and its Applications , Ph.D. Thesis, Assiut University, Egypt ,1990.

[12] Maleknejad, K. Kajani, M. T. , Solving linear integro-differential equation system by Galerkin methods with hybrid functions, Applied Mathematics and Computation 159 (2004) 603–612.