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Benard Convection in a Horizontal Porous Layer Permeated by a Non-Linear Magnetic Fluid under the Influence of Both Magnetic Field and Coriolis Forces

F. M. Allehiany and A. A. Abdullah
Department of Mathematical Sciences, Faculty of Applied Sciences,
Umm Al-Qura University, Makkah, Saudi Arabia
Email Address: abdullah@uqu.edu.sa

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This work examines the Benard convection of an infinite horizontal layer occupied by a porous medium permeated by an incompressible, thermally and electrically conducting viscous fluid heated from below when subjected to both uniform vertical magnetic field and Coriolis forces. A model proposed by P. H. Roberts (1981) in the context of neutron stars is used. We show that the nonlinearity in this model has no effect on the development of instabilities through the mechanism of stationary convection which is the preferred process in terrestrial applications. However, in non-terrestrial applications the non-linearity influences the onset of overstable convection and overstability is the preferred mechanism. Some numerical results are presented for the overstability case when both boundaries are free and rigid.

Keywords: Benard convection, linear stability, porous medium, stationary instability, overstable convection.

1 Introduction

Thermal instability theory has attracted considerable interest and has been recognized as a problem of fundamental importance in many fields of fluid dynamics. The earliest experiments to demonstrate the onset of thermal instability in fluids are attributed to Benard (1900, 1901). Rayleigh (1916) provided a theoretical basis for Benard's experimental results.

Thermal instability theory has been enlarged by the interest in hydrodynamic flows of electrically conducting fluids in the presence of magnetic field. The presence of such fields in an electrically conducting fluid usually has the effect of inhibiting the development of instabilities. Benard convection of a magnetohydrodynamic fluid has been examined by Thompson (1951), Chandrasekhar (1952) and others. The instability of a layer of fluid heated from below in the presence of Coriolis forces has been studied by Chandrasekhar (1953,1955) for stationary and overstability cases. He showed that the presence of these forces usually has the effect of inhibiting the onset of thermal convection. The simultaneous effect of both magnetic field and Coriolis forces on the thermal instability of a layer of fluid heated from below has been studied theoretically by Chandrasekhar (1954,1956) and experimentally by Nakagawa (1957). The analysis of this problem showed that under the influence of rotation and magnetic field, fluid motion can show unexpected pattern of behaviour.

A non-linear relation between the magnetic field H and the magnetic induction B has been suggested by Roberts (1981) and Muzikar & Pethick (1981) in the context of neutron stars. This non-linear relation has been used by Abdullah & Lindsy (1990, 1991) to discuss the Benard convection in the presence of vertical and non-vertical magnetic field. They showed that this relationship has no effect on the development of instabilities through the mechanism of stationary convection which is the preferred process in terrestial applications. However, in non-terrestial applications the non-linearity influences the onset of overstable convection and overstability is the preferred mechanism. This work has been extended by Abdullah (1992, 1994) to include the effect of Coriolis forces in the presence of vertical and non-vertical magnetic field. The strength of the non-linearity is measured by a non-dimensional parameter. Appropriate values of this parameter have been obtained by Abdullah (1990) using real data.

Rayleigh instability of a layer of fluid subject to a vertical temperature gradient in a porous medium has been discussed by Horton & Rogers (1945) and Lapwood (1948) using Darcy's law. They obtained the criterion for the formation of convection currents in the presence of porous medium. This criterion is compared with the equivalent conditions developed by Rayleigh (1916) for the formation of convection currents in a simpler fluid.

Brinkman (1947, 1947) suggested a modification of Darcy's law by assuming that the viscosity term in the Navier-stokes equation should be included in the equation of motion. Yamamoto & Iwamura (1976) and Rudraiah *et al.* (1980) showed that the Brinkman model is valid up to magnitude of $\varkappa_1/d^2 \cong 10^{-4}$ or 10^{-3} where $\varkappa_1$ is the permeability of porous medium.

A numerical study of buoyancy-driven two-dimensional convection in a fluid saturated horizontal porous layer confined between two impermeable walls and heated isothermally from below has been studied by Georgiadis & Catton (1986). Kladias & Prasad (1990) studied numerically thermoconductive instabilities in horizontal porous layers heated from below using the Brinkman-Forchheimer-Darcy model. They showed that there are four flow regimes in the case of free convection: conduction, stable convection, periodic oscillatory and randomly oscillatory convection. Jan & Abdullah (2000) used Brinkman model to investigate the convective instability of a horizontal porous layer permeated by a conducting magnetic field.

The effect of the earth's magnetic field on the stability of a layer of porous medium is of interest in geophysics particularly in the study of the earth's core where the earth's mantle, which consists of conducting fluid, behaves like a porous medium which can become convectively unstable as a result of diffusion. Another application of the results of flow through a porous medium in the presence of a magnetic field lies in the study of the stability of a convective flow in the geothermal region.

This paper studies convective instability in a horizontal porous layer permeated by an incompressible, thermally and electrically conducting fluid using Brinkman model in the presence of a uniform vertical magnetic field and a uniform vertical rotation when the relationship between the magnetic induction B and the magnetic field H is non-linear. It is an extension work of Abdullah (2000). Analytical solutions were obtained when both boundaries are free and numerical results were presented for the cases of free and rigid boundaries. The numerical computations were performed using expansions of Chebyshev polynomials.

2 Mathematical Formulation

Consider an infinite horizontal porous layer permeated by incompressible, thermally and electrically conducting viscous fluid of density ρ under the influence of both magnetic field and Coriolis forces when the relation between the magnetic field $\underline{H}$ and the magnetic induction $\underline{B}$ is non-linear. Constitutive relationship between magnetic field, magnetic induction and density has form (see Roberts (1981)).

$$H_i = \rho \frac{\partial \psi^*}{\partial B_i}, \tag{2.1}$$

where $\psi^* = \psi(\rho,\underline{B})$ is the internal energy function. Since ψ^* must be invariant then the dependence of ψ^*on B is reduced to $\psi^* = \psi(\rho, B)$. Thus from (2.1) $H_i = \rho\phi B_i$ where

$$\phi = \frac{1}{B}\frac{\partial \psi}{\partial B} \tag{2.2}$$

is the susceptibility. Thus the equation of motion is

$$\rho\frac{\partial \upsilon_i}{\partial t} = -P_{,i} + \rho\nu\nabla^2\upsilon_i + \rho g_i + (\rho\phi B_i)_{,k} B_k - \frac{\mu'}{k_1}\upsilon_i + 2\rho(\underline{\upsilon}\times\underline{\Omega})_i,$$

where

$$P = P + \frac{\rho}{2}\left(B^2\phi + \int B^2\phi_B dB\right) - \frac{1}{2}\rho_0|\underline{\Omega}\times\underline{r}|^2.$$

If we now make the Boussinesq approximation then the governing field equations become

$$\upsilon_{i,i} = 0$$

$$\frac{Dv_i}{Dt} = -\left(\frac{P}{\rho_0}\right)_i + v\nabla^2 v_i - g(1-\alpha\theta)\delta_{i3} + (\phi B_i)_k B_k - \frac{v}{k_1}v_i + 2e_{ijk}v_j\Omega_k, \quad (2.3)$$
$$\frac{D\theta}{Dt} = k\nabla^2\theta,$$
$$\frac{\partial B_i}{\partial t} = B_j v_{i,j} - v_j B_{i,j} - \eta e_{ijk} J_{k,j},$$

together with the Maxwell equations

$$div\underline{B} = B_{i,i} = 0,$$
$$curl\underline{H} = e_{ijk}H_{k,j} = J_i,$$
$$curl\underline{E} = e_{ijk}E_{k,j} = -\frac{\partial B_i}{\partial t}, \quad (2.4)$$

where g_i is the acceleration of gravity, v is the kinematic viscosity, μ' is the kinematic viscosity, k_1 is the permeability of porous medium, D/Dt is the convected derivative and the terms $(1/2)|\underline{\Omega} \times \underline{r}|^2_{,i}$ and $2(\underline{v} \times \underline{\Omega})$ represent respectively the centrifugal force and the Coriolis acceleration. We observe that equations (2.3) and (2.4) have a steady state solution in which

$$\begin{aligned} \underline{v} &= 0, \\ \theta &= \theta(x_3) = T_0 - \beta x_3, \\ P &= P(x_3), \\ \underline{B} &= (0,0,B), \quad B = \text{ constant} \\ \underline{J} &= 0, \\ \phi &= \phi(B), \end{aligned} \quad (2.5)$$

where the temperatures on the planes $x_3 = 0$, and $x_3 = d$ are respectively T_0 and $T_0 - \widetilde{T}$ so that $\beta = \widetilde{T}/d$.

3 The Perturbation Equations

Suppose that the initial state described by equations (2.5) is slightly perturbed so that

$$\underline{v} = \underline{0} + \varepsilon^*\underline{v}^*, \qquad \theta = T_0 - \beta x_3 + \varepsilon^*\theta^*, \qquad P = P + \varepsilon^* P^*,$$
$$B = (0,0,B) + \varepsilon^*\underline{b}^*, \qquad \underline{J} = \underline{0} + \varepsilon^*\underline{J}^*, \qquad \phi = \phi + \varepsilon^* b_3^*\phi_B,$$

where ε^* is the perturbation parameter and $\underline{v}^*, \theta^*, P^*, \underline{b}^*, \underline{J}^*$ are respectively the linear perturbation of velocity, temperature, pressure, magnetic induction and current density about their values described in (2.5). The linear perturbation of ϕ about its value can be obtained in the following way:

$$B = \sqrt{B_i B_i}$$

$$
\begin{aligned}
&= \sqrt{[(0,0,B)+\varepsilon^* \underline{b}^*] \,.\, [(0,0,B)+\varepsilon^* \underline{b}^*]} \\
&\simeq \sqrt{B^2 + 2\varepsilon^*(0,0,B).\underline{b}^*} \\
&\simeq B + \frac{\varepsilon^*(0,0,B).\underline{b}^*}{B}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\phi &= \phi\left(B + \frac{\varepsilon^*(0,0,B).\underline{b}^*}{B}\right) \\
&\simeq \phi(B) + \varepsilon^* b_3^* \phi_B(B), \quad \underline{b}^* = (b_1^*, b_2^*, b_3^*).
\end{aligned}
$$

We may verify that the linearized versions of equations (2.3) and (2.4) are

$$
\begin{aligned}
\frac{\partial v_i^*}{\partial t} &= -\left(\frac{P^*}{\rho_0}\right)_{,i} + v\nabla^2 v_i^* - B\phi b_{i,3} + B^2 \phi_B b_{3,3}^* \delta_{i3} - \frac{v}{k_1} v_i^* + 2e_{ijk} v_j^* \Omega_k, \\
v_{i,i}^* &= 0, \\
\frac{\partial \theta^*}{\partial t} &= \beta v_3^* + k\nabla^2 \theta^*, \\
b_{i,i}^* &= 0, \\
\frac{\partial b_{i,i}^*}{\partial t} &= B_j v_{i,j}^* - \eta e_{ijk} J_{k,j}^*, \\
J^* &= e_{ijk}(\rho\phi b_k^* + \rho B \phi_B b_3^* \delta_{k3})_j.
\end{aligned} \tag{3.1}
$$

Now we introduce dimensionless variables $\widehat{x}_i, \widehat{v}_i, \widehat{t}, \widehat{j}_i, \widehat{\theta}, \widehat{P}$ and $\widehat{b}_i$ such that

$$
\begin{aligned}
&x_i^* = d\widehat{x}_i, \quad v_i^* = \frac{k}{d}\widehat{v}_i, \quad t = \frac{d^2}{\mathrm{v}}\widehat{t}, \quad J_i^* = \frac{kv\rho_0}{Bd^3}\widehat{J}_i \\
&\theta^* = \frac{k}{d}\sqrt{\frac{|\mathrm{v}|\beta}{k\alpha g}}\widehat{\theta}, \quad P^* = \frac{k\mathrm{v}\rho_0}{d^2}\widehat{P} \quad b_i^* = \frac{k\mathrm{v}}{B\phi d^2}\widehat{b_i},
\end{aligned}
$$

After this non-dimensionalization, equations (3.1) is simplified to

$$
\begin{aligned}
v_{i,i} &= 0, \\
\frac{\partial v_i}{\partial t} &= -P_{,i} + \nabla^2 v_i + \sqrt{R}\theta^* \delta_{i3} + b_{i,3} + \varepsilon b_{3,3}\delta_{i3} - \frac{1}{N} v_i + \sqrt{T} e_{ijk} v_j \delta_{k3}, \\
P_r \frac{\partial \theta}{\partial t} &= -M\sqrt{R} v_3 + \nabla^2 \theta, \\
b_{i,i} &= 0, \\
P_m \frac{\partial b_i}{\partial t} &= Q v_{i,j} - e_{ijk}\varepsilon b_{3,3}\delta_{k3}, \\
J_i &= e_{ijk} b_{k,j} + e_{ijk} b_3^* \delta_{i3}.
\end{aligned} \tag{3.2}
$$

where the (^) superscript has been dropped and the non-dimensional $R, N, P_r, P_m, \varepsilon, Q$ and T are given by

$$\begin{aligned} R &= \frac{\alpha g|\beta|}{\mathrm{v}k}, \quad N = \frac{k_1}{d^2}, \ P_r = \frac{\mathrm{v}}{k}, \quad P_m = \frac{\mathrm{v}\mu}{\eta}, \\ \varepsilon &= \frac{B}{\phi}\phi_B, \quad Q = \frac{B^2 d^2}{\rho_0 \mathrm{v}\eta}, \quad T = \frac{4\Omega^2 d^4}{\mathrm{v}^2}. \end{aligned} \tag{3.3}$$

and

$$M = -\frac{\beta}{|\beta|}\begin{cases} +1 & \text{when heating from above} \\ -1 & \text{when heating from below} \end{cases}$$

Equations $(3.1)_5$ and $(3.1)_6$ can be combined to obtain

$$Pm\frac{\partial b_i}{\partial t} = Qv_{i,3} + \nabla^2 b_i + \varepsilon\nabla^2 b_3\delta_{i3} - \varepsilon b_{3,i3}. \tag{3.4}$$

4 The Boundary Conditions

The fluid is confined between the planes $x_3 = 0$ and $x_3 = 1$ and on these planes we need to specify mechanical, thermal and electromagnetic conditions. Suitable mechanical conditions assume either a rigid or free boundary, suitable thermal conditions assume either a perfectly conducting or an insulating boundary and suitable electromagnetic boundary conditions assume either an electrically insulating or a perfectly conducting boundary.

4.1 Mechanical conditions

For a free surface, the conditions are

$$v_3 = 0, \quad \frac{\partial^2 v_3}{\partial v_3^2} = 0, \quad \frac{\partial \zeta_3}{\partial x_3} = 0, \tag{4.1}$$

where ζ_3 is the third component of the vorticity.

For a rigid surface, the conditions are

$$v_3 = 0, \quad \frac{\partial v_3}{\partial v_3} = 0, \quad \zeta_3 = 0. \tag{4.2}$$

4.2 Thermal conditions

At a perfectly conducting boundary, the temperatures of the boundary and impinging fluid match whereas on a perfectly insulating boundary, no heat transfer can take place between the fluid and the surroundings and thus the normal derivative of temperature is zero. In mathematical terms, the possible thermal conditions are

$$\theta = \theta_{ext} \qquad \text{on a conducting boundary,} \tag{4.3}$$

$$\frac{\partial \theta}{\partial x_3} = 0 \qquad \text{on an insulated boundary,} \tag{4.4}$$

where θ_{ext} is the temperature of the region exterior to the fluid boundary.

4.3 Electromagnetic conditions

On a perfectly insulating electromagnetic boundary, no current can flow to the exterior region and so $J_3 = 0$ and the magnetic field is continous across the boundary with the external magnetic field being derived from a scalar potential since $curl\underline{H} = 0$ in the exterior region. On a stationary perfectly conducting boundary $b_3 = 0$ and there can be no surface components of electric field. It is common practice to associate mechanically rigid and electrically perfectly conducting stationary boundaries. Also the surface components of the current density are zero and since $div\underline{J} = 0$ then

$$\frac{\partial J_3}{\partial x_3} = 0.$$

5 The Eigenvalue Problem

We aim to investigate the non-linear stability of the convection solution (2.5) and, with this aim in mined, we construct the related eigenvalue problem from equations (3.2). Many of the tedious algebraic details are suppressed so that the essential direction of the argument is clear.

As in many convection problems, vector components parallel to the direction of gravity (i.e. the x3 direction) play a central role and so it is convenient to introduce the variables w, b, J, ζ and z by the definitions $w = v_3, b = b_3, J = J_3, \zeta = \zeta_3, z = x_3$. When we take the curl of equations $(3.2)_2$ and (3.4) we obtain

$$\begin{aligned} \frac{\partial \zeta_i}{\partial t} &= \nabla^2 \zeta_i + \sqrt{R} e_{ijk} \theta_{,j} \delta_{k3} + e_{ijk} b_{k,3j} + \varepsilon e_{ijk} b_{3,3j} \delta_{k3} - \frac{1}{N} \zeta_i + \sqrt{T} v_{i,3}, \\ P_m \frac{\partial J_i}{\partial t} &= \varepsilon P_m e_{ijk} \frac{\partial b_{3,j}}{\partial t} \delta_{k3} + Q \zeta_{i,3} + J_{i,jj}. \end{aligned} \tag{5.1}$$

Taking the curl of equation $(5.1)_1$ once again, we obtain

$$\begin{aligned} \frac{\partial \nabla^2 v_i}{\partial t^2} = \nabla^4 v_i - \sqrt{R}\left(\frac{\partial^2 \theta}{\partial x_3 \partial x_i} - \frac{\partial^2 \theta}{\partial x_j^2} \delta_{i3}\right) + \nabla^2 b_{i,3} \\ + \epsilon \nabla^2 b_{3,3} \delta_{i3} - \epsilon b_{3,33i} - \frac{1}{N} \nabla^2 v_i - \sqrt{T} \xi_{i,3}. \end{aligned} \tag{5.2}$$

The third components of equations (5.1), (5.2), $(3.2)_3$ and (3.4) yield

$$\begin{aligned} \frac{\partial \zeta}{\partial t} &= \nabla^2 \zeta + \frac{\partial J}{\partial z} - \frac{1}{N} \zeta + \sqrt{T} \frac{\partial W}{\partial z}, \\ P_m \frac{\partial J}{\partial t} &= Q \frac{\partial \zeta}{\partial z} + \nabla^2 J, \end{aligned}$$

$$
\begin{aligned}
\frac{\partial \nabla^2 w}{\partial t} &= \nabla^{\,4} w + \sqrt{R}\left(\frac{\partial^2 \theta}{\partial x_1^2} + \frac{\partial^2 \theta}{\partial x_2^2}\right) + \nabla^{\,2}\left(\frac{\partial b}{\partial z}\right) \\
&\quad + \epsilon \frac{\partial}{\partial z}\left(\frac{\partial^2 b}{\partial x_1^2} + \frac{\partial^2 b}{\partial x_2^2}\right) - \frac{1}{N}\nabla^{\,2} w - \sqrt{T}\frac{\partial \xi}{\partial z} \qquad (5.3)\\
P_r \frac{\partial Q}{\partial t} &= -M\sqrt{R} v_3 + \nabla^2 \theta, \\
P_m \frac{\partial b}{\partial t} &= Q\frac{\partial W}{\partial z} + \nabla^2 b + \varepsilon\left(\frac{\partial^2 b}{\partial x_1^2} + \frac{\partial^2 b}{\partial x_2^2}\right)
\end{aligned}
$$

Now we look for a solution of the form $\Phi = \Phi(z)\exp\left(i(nx+my)+\sigma t\right)$. Thus equations (5.3) becomes

$$
\begin{aligned}
\sigma \xi &= L\xi + DJ - \frac{1}{N}\xi + \sqrt{T} Dw, \\
\sigma P_m J &= QD\xi + LJ, \\
\sigma L w &= L^2 w - a^2\sqrt{R}\theta + L(Db) - \varepsilon a^2 (Db) - \frac{1}{N} Lw - \sqrt{T} D\xi, \qquad (5.4)\\
\sigma P_m b &= QDw - + b - \varepsilon a^2 b \\
\sigma P_r \theta &= L\theta - M\sqrt{R} w.
\end{aligned}
$$

Eliminating J from equation $(5.4)_1$ using equation $(5.4)_2$, we find that

$$
[(L-\sigma P_m)(L-\sigma-\frac{1}{N}) - QD^2]\xi = -\sqrt{T}(L - P_m\sigma) Dw. \qquad (5.5)
$$

We may eliminate b, θ and ξ from equation $(5.4)_3$ by applying the operator

$$
(L-\sigma P_m - \epsilon a^2)(L-\sigma P_r)[(L-\sigma P_m)(L-\sigma-\frac{1}{N}) - QD^2]
$$

to equations $(5.4)_3$ and we use equation (5.5) to obtain a twelfth order ordinary differential equation to be satisfied by *w*.

$$
\begin{aligned}
&\sigma^5 P_m P_r L w - \sigma^4 \left\{ P_m\left[P_m + 2P_r(1+P_m)\right] L^2 w - \left[\frac{2P_r P_m^2}{N} + 2P_r P_m \varepsilon a^2\right] Lw \right\} \\
&+ \sigma^3 \Big\{ \left[P_r(P_m+1)2 + 2P_m(P_m+P_r+1)\right] L^3 w \\
&- \left[\frac{2P_m}{N}\big(P_m + P_r(2+P_m)\big) + \varepsilon a^2 (P_m + P_r + 2P_m P_r)\right] L^2 w \\
&- P_m P_r\Big(2QD^2 - \frac{P_m}{N^2} - \frac{2\varepsilon a^2}{N}\Big) Lw - a^2 RM P_m^2 w + T P_r P_m^2 D^2 w \Big\} \\
&- \sigma^2 \Big\{ \left[(P_m+1)^2 + 2(P_m + P_r + P_m P_r)\right] L^4 w \\
&- \frac{1}{N}\Big[4P_m(1+P_r) + 2(P_r + P_m^2) - \varepsilon a^2\big(1 + 2(P_m+P_r) + P_m P_r\big)\Big] L^3 w \\
&+ \Big[\frac{P_m^2}{N^2}\Big(1 + \frac{2P_r}{P_m}\Big) - 2QD^2(P_m + P_r + P_m P_r) + \frac{2\varepsilon a^2}{N}(P_m + P_r + P_m P_r)\Big] L^2 w
\end{aligned}
$$

$$+ P_mP_r\Big[\Big(\frac{2Q}{N} + T(2+\frac{P_m}{P_r})\Big)D^2 - a^2R\frac{M}{P_r}(2+P_m) + \varepsilon a^2\Big(Q(2P_r+P_m+P_rP_m)D^2$$
$$-\frac{P_rP_m}{N}\Big)\Big]Lw + \Big[a^2\frac{R}{N}MP_m^2 - \varepsilon a^2P_mP_r\Big(\big(\frac{Q}{N}+T\big)D^2 - \frac{a^2}{P_r}RM\Big)\Big]w\Big\}$$
$$+ \sigma\Big\{(2P_m+P_r+2)L^5w - \Big[\frac{2}{N}(P_r+2P_m+1) + \varepsilon a^2(P_r+P_m+2)\Big]L^4w$$
$$+ \Big[\Big(-2QD^2(P_r+P_m+1) - \frac{1}{N^2}(2P_m+P_r)\Big) + \frac{2\varepsilon a^2}{N}(1+P_m+P_r)\Big]L^3w$$
$$+ \Big[-a^2RM(1+2P_m) + \Big(\frac{2Q}{N}(P_r+P_m) + T(P_r+2P_m)\Big)D^2$$
$$+ \varepsilon a^2\Big(QD^2(1+P_m+2P_r) - \frac{1}{N^2}(P_r+P_m)\Big)\Big]L^2w$$
$$+ \Big[\Big(2a^2\frac{R}{N}P_mM + Q^2D^4P_r\Big) + \varepsilon a^2\Big(a^2RM(1+P_m)$$
$$- \Big(\frac{Q}{N}(P_r+P_m) - T(P_r+P_m)\Big)D^2\Big]Lw$$
$$+ \Big[a^2RQMP_mD^2 - \varepsilon a^2\big(a^2P_m\frac{R}{N}M + P_rQ^2D^4\big)\Big]w\Big\} - L^6w + \Big(\frac{2}{N} + \varepsilon a^2\Big)L^5w$$
$$+ \Big(2QD^2 - \frac{1}{N^2} - \frac{2\varepsilon a^2}{N}\Big)L^4w + \Big[A^2RM - \big(2\frac{Q}{N}+T\big)D^2 + \varepsilon a^2\big(\frac{1}{N} - 2QD^2\big)\Big]L^3w$$
$$+ \Big[-a^2\frac{R}{N}M - Q^2D^4 + \varepsilon a^2\big(\frac{2Q}{N}+T\big)D^2 - \varepsilon a^4RM\Big]L^2w$$
$$+ \Big[\varepsilon a^2\big(Q^2D^4 + \frac{a^2R}{N}M\big) - a^2RQMD^2\Big]Lw + \varepsilon a^4RMQD^2 = 0. \tag{5.6}$$

Now we shall consider both boundaries to be free but later on we shall present results for the corresponding rigid boundary value problems. For the free boundary value problems

$$w = D^2, \quad w = 0 \text{ on } z = 0, 1.$$

Thus equation (5.6) has eigenfunctions $w = A\sin(l\pi z)$ where A is constant and l is an integer. Consequently $Lw = -\lambda w$ where $\lambda = l^2\pi^2 + a^2$ and σ satisfies the fifth order equation

$$\sigma^5P_mP_r + \sigma^4\Big\{P_m\Big[P_m + 2P_r(1+P_m)\Big]\lambda - \Big[\frac{2P_rP_m^2}{N} + 2P_rP_m\varepsilon a^2\Big]\Big\}$$
$$+ \sigma^3\Big\{\Big[P_r(P_m+1)^2 + 2P_m(P_m+P_r+1)\Big]\lambda^2$$
$$+ \Big[\frac{2P_m}{N}(P_m + P_r(2+P_m)) + \varepsilon a^2(P_m+P_r+2P_mP_r)\Big]\lambda$$
$$+ P_mP_r(2QD^2l^2\pi^2 + \frac{P_m}{N^2} + \frac{2\varepsilon a^2}{N}) + p^2(a^2RM + TP_rP_m^2l^2\pi^2)\lambda^{-1}\Big\}$$
$$+ \sigma^2\Big\{\Big[(P_m+1)^2 + 2(P_m+P_r+P_mP_r)\Big]\lambda^3$$

$$
\begin{aligned}
&+\frac{1}{N}\Big[4P_m(1+P_r)+2(P_r+P_m^2)+\varepsilon a^2\big(1+2(P_m+P_r)+P_mP_r\big)\Big]\lambda^2\\
&+\Big[\frac{P_m^2}{N^2}(1+\frac{2P_r}{P_m})+2Ql^2\pi^2(P_m+P_r+P_mP_r)+\frac{2\varepsilon a^2}{N}(P_m+P_r+P_mP_r)\Big]\lambda\\
&+P_mP_r\Big[(\frac{2Q}{N}+T(2+\frac{P_m}{P_r}))l^2\pi^2+a^2R\frac{M}{P_r}(2+P_m)+\varepsilon a^2Q(2P_r+P_m+P_rP_m)l^2\pi^2\\
&-\frac{\varepsilon a^2P_rP_m}{N}\Big]+\Big[a^2\frac{R}{N}MP_m^2+\varepsilon a^2P_mP_r\big((\frac{2Q}{N}+T)l^2\pi^2+\frac{a^2}{P_r}RM\big)\Big]\lambda^{-1}\Big\}\\
&+\sigma\Big\{(2P_m+P_r+2)\lambda^4+\Big[\frac{2}{N}(P_r+2P_m+1)+\varepsilon a^2(P_r+P_m+2)\Big]\lambda^3\\
&+\Big[2Ql^2\pi^2(P_r+P_m+1)+\frac{1}{N^2}(2P_m+P_r)+\frac{2\varepsilon a^2}{N}(1+P_m+P_r)\Big]\lambda^2\\
&+\Big[a^2RM(1+2P_m)+\big(\frac{2Q}{N}(P_r+P_m)+T(P_r+2P_m)\big)l^2\pi^2\\
&+\varepsilon a^2\big(Ql^2\pi^2(2+P_m+2P_r)-\frac{1}{N^2}(P_r+P_m)\big)\Big]\lambda\\
&+2a^2\frac{R}{N}P_mM+Q^2l^4\pi^4P_r+\varepsilon a^4RM(1+P_m)+\varepsilon a^2\big[\frac{Q}{N}+T\big](P_m+P_r)l^2\pi^2\\
&+\Big[a^2RQMP_ml^2\pi^2+\varepsilon a^2\big(a^2P_m\frac{R}{N}M+P_rQ^2l^4\pi^4\big)\Big]\lambda^{-1}\Big\}\\
&+(\lambda+\varepsilon a^2)\Big[\lambda^4+\frac{2}{N}\lambda^3+\big(Ql^2\pi^2+\frac{1}{N}\big)\lambda^2+\big[a^2RM+(\frac{2Q}{N}+T)l^2\pi^2\big]\lambda\\
&+a^2\frac{R}{N}M+Q^2l^4\pi^4+a^2RMQl^2\pi^2\lambda^{-1}\Big]=0.
\end{aligned}
\tag{5.7}
$$

Since the coefficients of this polynomial are real, its solutions satisfy one of the following conditions:

1. All solutions are real.
2. Three solutions are real and two are complex conjugate pair solutions.
3. One solution is real and four are complex conjugate solutions.

The stationary instability happens if any real solution is positive while overstability happens if any real part of the complex conjugate solutions is positive. Solutions of (5.7) are functions of $P_r, P_m, N, \varepsilon, Q, T$ and R and we have to examine how the nature of these solutions depends on $P_r, P_m, N, \varepsilon, Q, T$ and R in the context of heating the fluid layer from below. Let us assume that $|\underline{B}|$ is an increasing function of $|\underline{H}|$ so that $dB/dH > 0$. Consequently

$$
\begin{aligned}
\frac{d(\phi B)}{dB} &> 0 \Rightarrow \phi + B\phi_B > 0,\\
1+\frac{B\phi_B}{\phi} &> 0 \Rightarrow 1+\varepsilon > 0,
\end{aligned}
$$

and so $\varepsilon > -1$, which implies that

$$\lambda + \varepsilon a^2 > 0. \tag{5.8}$$

5.1 Stationary convection case

To find the critical Rayleigh number for the onset of stationary convection we set $\sigma = 0$ in equation (5.7). Thus

$$(\lambda + \varepsilon a^2)\left[\frac{\lambda}{a^2 C}\left(C^2 + T\lambda l^2 \pi^2\right) - R\right] = 0,$$

where $C = \lambda^2 + \lambda/N + l^2\pi^2 Q$, i.e.

$$R = \frac{\lambda}{a^2 C}\left(C^2 + T\lambda l^2 \pi^2\right). \tag{5.9}$$

Since this equation does not contain P_r, P_m or ε the critical Rayleigh number for stationary convection is independent of P_r, P_m or ε, which means that the non-linear relation between $\underline{B}$ and $\underline{H}$ has no effect on the development of stationary instability. From equation (5.9) we find that

$$\begin{aligned} \frac{dR}{dQ} &= \frac{\lambda l^2 \pi^2}{a^2}\left[1 - \frac{T\lambda l^2 \pi^2}{C^2}\right], \\ \frac{dR}{dN} &= -\frac{\lambda^2}{a^2 N^2}\left[1 - \frac{T\lambda l^2 \pi^2}{C^2}\right], \\ \frac{dR}{dT} &= \frac{\lambda^2 l^2 \pi^2}{a^2 C^2}. \end{aligned} \tag{5.10}$$

It is clear from equation $(5.10)_1$ that the magnetic field has a stabilizing effect on the system in the absence of rotation. Also it has a stabilizing effect on the system in the presence of rotation provided that $T < C^2/(l^2\pi^2\lambda)$. From equation $(5.10)_2$ we find that the permeability of porous medium has a destabilizing effect on the system in the absence of rotation. Also it has a destabilizing effect on the system in the presence of rotation provided that $T < C^2/(l^2\pi^2\lambda)$. From equation $(5.10)_3$ it is clear that the rotation has a stabilizing effect on the system.

5.2 The overstability case

Since the equations are so complicated, we were unable to obtain analytical solution for the overstability case but we have produced numerical solutions for the corresponding problem.

6 Numerical Discussion

The eigenvalue problem (5.4) together with the boundary conditions are solved using expansion of Chebyshev polynomials. The non-linear relationship between the magnetic field $\underline{H}$ and the magnetic induction $\underline{B}$ has no effects on the relation between the critical

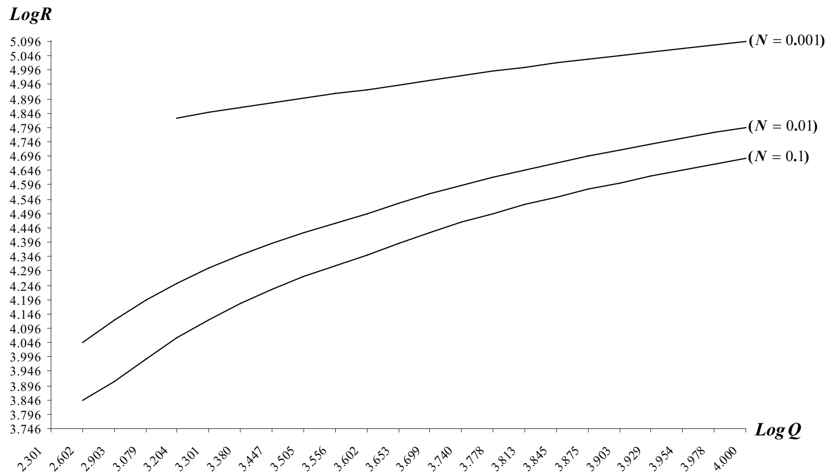

Figure 6.1: The relation between R and Q for the overstability case when both boundaries are free for $T = 10000$ and $\varepsilon = 1$.

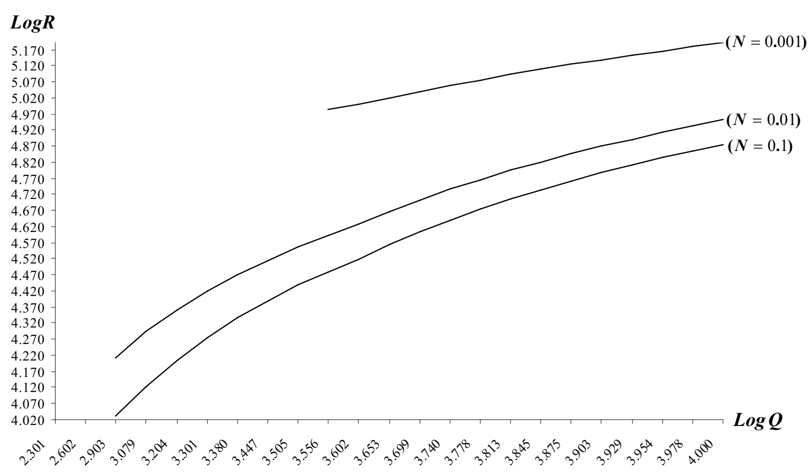

Figure 6.2: The relation between R and Q for the overstability case when both boundaries are free for $T = 10000$ and $\varepsilon = 2$.

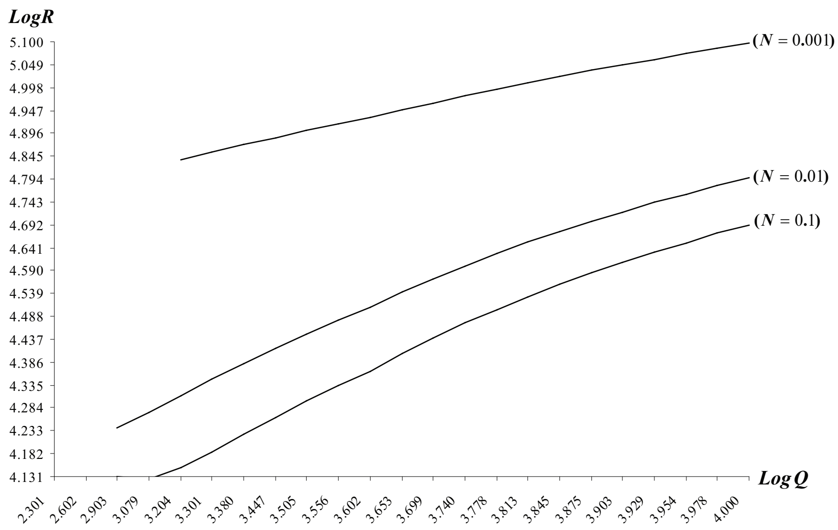

Figure 6.3: The relation between R and Q for the overstability case when both boundaries are free for $T = 50000$ and $\varepsilon = 1$.

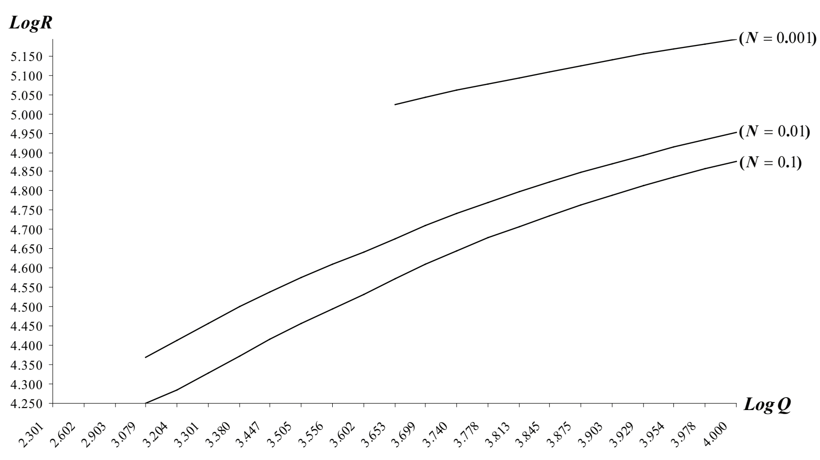

Figure 6.4: The relation between R and Q for the overstability case when both boundaries are free for $T = 50000$ and $\varepsilon = 2$.

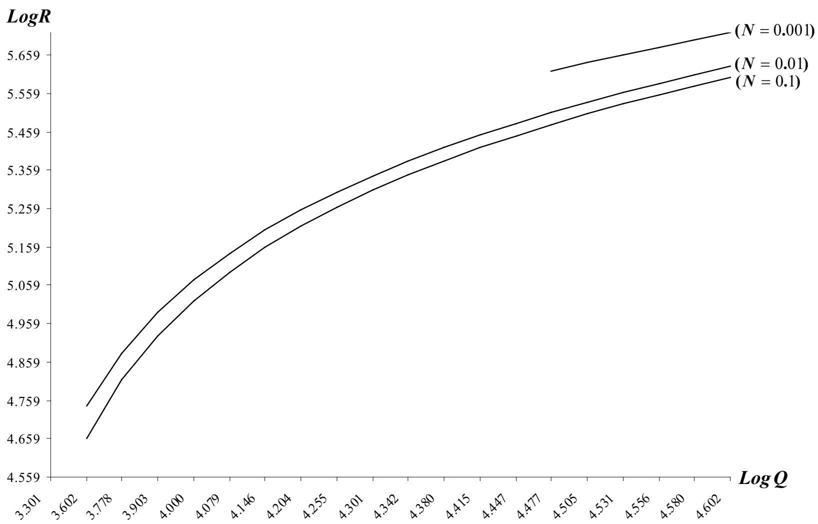

Figure 6.5: The relation between R and Q for the overstability case when both boundaries are free for $T = 50000$ and $\varepsilon = 3$.

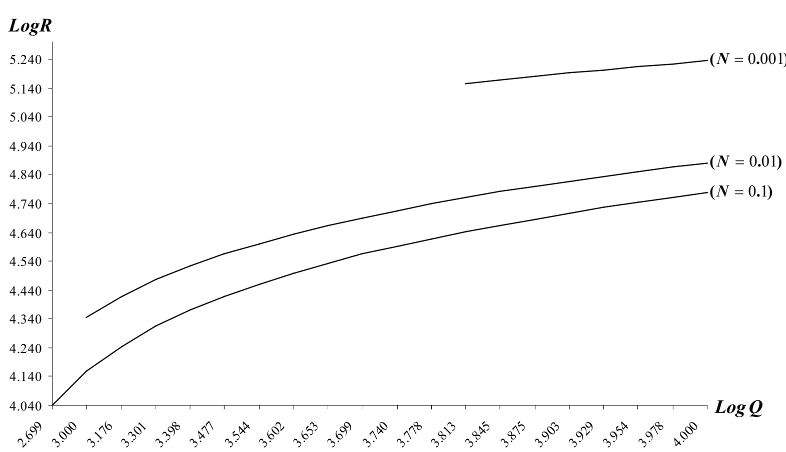

Figure 6.6: The relation between R and Q for the overstability case when both boundaries are rigid $T = 10000$ and $\varepsilon = 0.25$.

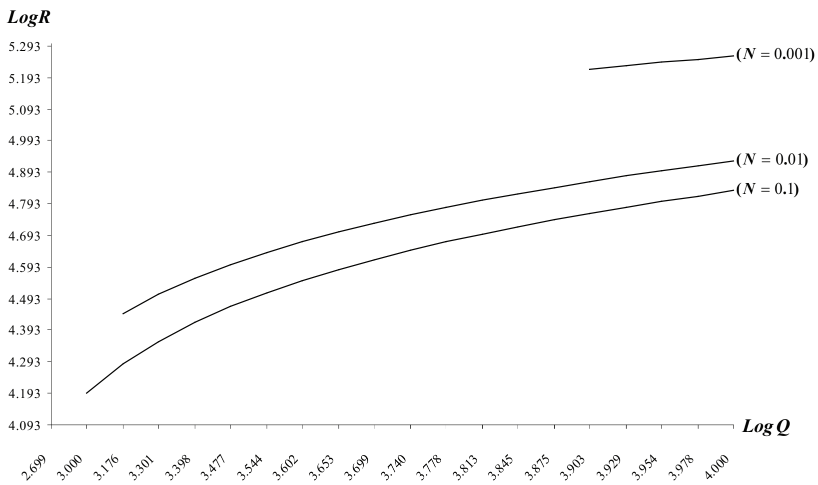

Figure 6.7: The relation between R and Q for the overstability case when both boundaries are rigid $T = 10000$ and $\varepsilon = 0.5$.

Rayleigh number *R* and the magnetic parameter Q for the stationary convection case for different boundary conditions. However it has a great effect in the development of instabilities through overstability case. The relation between the critical Rayleigh number R and the magnetic parameter Q for the overstability case for different values of the porous medium permeability when both boundaries are free is displayed in figures 6.1 and 6.2 when $T = 10^4$ for $\varepsilon = 1, 2$ respectively and in figures 6.3, 6.4 and 6.5 when $T = 5 \times 10^4$ for $\varepsilon = 1, 2, 3$ respectively. For rigid boundary conditions the relation is displayed in figures 6.6 and 6.7 when $T = 10^4$ for $\varepsilon = 0.25, 0.5$ respectively and in figures 6.8 and 6.9 when $T = 5 \times 10^4$ for $\varepsilon = 0.25, 0.5$ respectively. The figures show that as *Q* increases *R* increases which means that the magnetic field has a stabilizing effect. Also it appears from the figures that as the permeability of the porous medium, *N*, decreases *R* increases which means that as the fluid becomes less porous it becomes more stable. Moreover the critical value of the magnetic parameter *Q*, at which overstability becomes possible, increases as the porous medium permeability *N* decreases. Figure 6.10 shows a comparison between the free boundary conditions and the rigid boundary conditions for $T = 10^4$ when $\varepsilon = 0.5$. In fact, it appears that as the value of the parameter ε increases, the critical Rayleigh number increases which means that the non-linearity has a stabilizing effect for the overstability case.

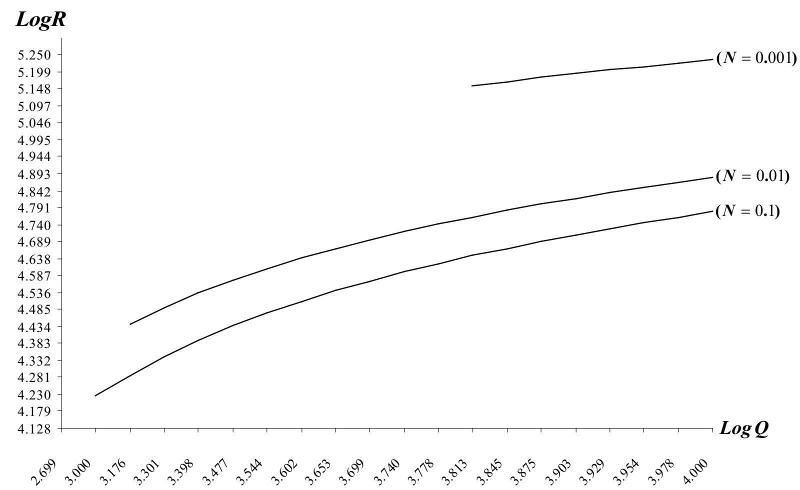

Figure 6.8: The relation between R and Q for the overstability case when both boundaries are rigid $T = 50000$ and $\varepsilon = 0.25$.

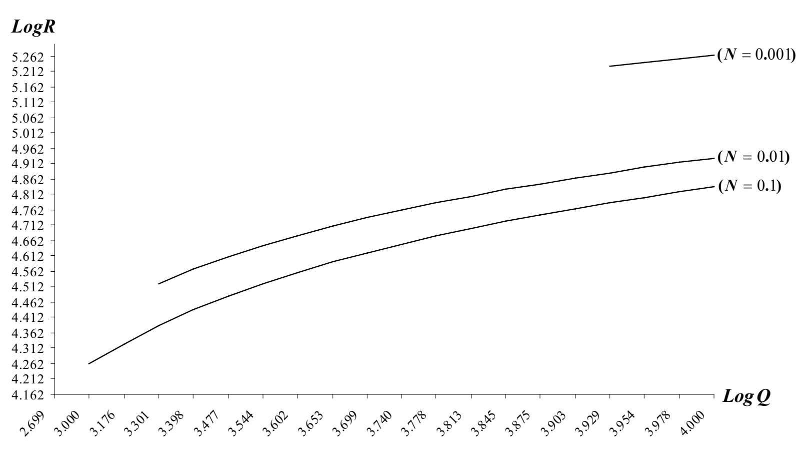

Figure 6.9: The relation between R and Q for the overstability case when both boundaries are rigid $T = 50000$ and $\varepsilon = 0.5$.

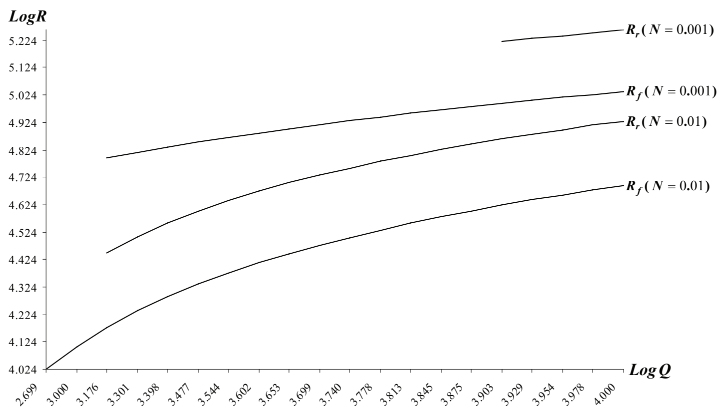

Figure 6.10: A comparison between the free boundary conditions and the rigid boundary conditions for the overstability case when $T = 1000$ and $\varepsilon = 0.5$.

7 Conclusion

Benard convection instability of an infinite horizontal layer occupied by a conducting viscous fluid using Brinkmann model when the relation between $\underline{H}$ and $\underline{B}$ is non-linear is investigated. The non-linearity has no effect on the development of instabilities through the mechanism of stationary convection, which is preferred process from the viewpoint of terrestrial applications. However it has great effect in the development of instabilities through overstability case. The presence of porous medium increases the stability of fluid. Moreover the critical value of the magnetic parameter Q, at which overstability becomes possible, increases as the porous medium permeability *N* decreases.

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