

Inference of Stress-Strength for the Z-Lindley Distribution under Progressive Censoring

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Abstract: This paper estimates the stress–strength reliability $R = P(Y < X)$ for two independent inverted Z–Lindley (IZL) lifetimes observed under progressive Type–II censoring. The IZL model needs only one shape parameter yet produces an upside-down-bathtub hazard with a heavy right tail. Using a reciprocal change of variable, we derive a compact closed-form expression for R , verified against adaptive quadrature to within 2×10^{-13} and shown to satisfy $R(\alpha, \alpha) = 1/2$ and $R(\alpha_1, \alpha_2) + R(\alpha_2, \alpha_1) = 1$. For inference, the maximum-likelihood estimator is shown to be unique under complete sampling, then refined with a jackknife bias correction and a percentile bootstrap interval. Bayesian estimation under informative, diffuse, and Jeffreys priors is carried out via Metropolis–Hastings (with full MCMC diagnostics) and cross-checked with the Tierney–Kadane approximation. A simulation study over three configurations, three censoring plans, and four sample sizes shows the informative Bayes estimator has the smallest MSE, TK matches MCMC within 0.01, and all intervals achieve near-nominal coverage. An application to insulating-fluid breakdown times confirms a good fit and a reliability estimate near 0.80.

Keywords: inverted Z–Lindley; stress–strength reliability; progressive Type–II censoring; jackknife; Bayesian inference; Jeffreys prior; Tierney–Kadane approximation

1 Introduction

A stress–strength model treats the capacity of a component, X , and the load placed on it, Y , as two random quantities. The unit survives for as long as its capacity is not overtaken by the applied load, and this survival chance is summarized by the single index $R = P(Y < X)$.

This quantity turns up across a range of disciplines. In *mechanical design*, it gives the chance that a load-bearing part – say, a drive shaft subjected to repeated vibration – survives the stresses placed on it [4]. In *electrical engineering*, the same index describes the odds that an insulator’s breakdown voltage stays above the voltage it is actually exposed to, which is precisely the reading used later in Section 6 [43]. In *clinical research*, $P(Y < X)$ coincides with the Mann–Whitney estimand, i.e., the likelihood that a patient under treatment fares better than one on the control arm [1, 2]. In *quality and reliability engineering*, a design target of $R > 0.90$ is a common benchmark for safety-critical systems [3, 4].

Church and Harris [3] were among the first to set the model on rigorous footing, and the book by Kotz, Lumelskii and Pensky [4] remains the standard reference for the theory that followed. Since then, closed-form reliability and estimation results have accumulated for the generalized exponential law [5, 6], the Weibull law [7], the two-parameter exponential family [8], various inverted and Lindley-type models [9, 10], and, more recently, the Garhy, discrete alpha-power Weibull and other one-parameter models under record, incomplete and unified hybrid censored data [11–20].

Waiting for every unit on a life test to fail is often impractical, so experimenters remove a pre-planned number of surviving units each time a failure is recorded – this is the progressive Type–II censoring scheme. It is flexible enough to nest ordinary Type–II censoring and complete sampling as special cases, and its theory is laid out comprehensively

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in [21–26]. Reliability estimation under such schemes has attracted substantial attention in its own right [27–29], including Bayesian and Tierney–Kadane inference for progressively censored samples [30] and step-stress life testing [31].

The Lindley distribution [32, 33] has long served as a workhorse lifetime model, but its hazard rate is monotone and therefore cannot capture rise-then-fall failure patterns. This limitation has motivated a range of extended Lindley, decreasing-hazard and Weibull-type lifetime models [34–36]. The Z-Lindley variant introduced by Saaidia et al. [37], later refined by Mahnashi and Zaagan [38] and examined under censoring by Alotaibi and Elshahhat [39], broadens the possible hazard shapes. Taking the reciprocal transform $Y = X^{-1}$ of this distribution yields the *inverted Z-Lindley* (IZL) [41], the model studied here. The IZL law has recently been applied to step-stress accelerated life testing [31] and to the reliability of s -out-of- k systems under stress-strength models with progressive censoring [42].

Four points motivate this choice. (1) Its upside-down bathtub hazard suits components whose failure risk climbs and then eases off, a pattern often seen in electronic parts and insulating materials. (2) Its heavy right tail accommodates occasional extreme observations, which are common in breakdown-time data. (3) A single parameter keeps estimation manageable even for small, heavily censored samples. (4) Among five rival models, IZL delivers the lowest AIC and BIC on the insulating-fluid dataset examined in subsection 6.3. Beyond the present application, the model is relevant to heavy-tailed survival data [9] and to dielectric breakdown studies in high-voltage engineering [43].

This paper ties together a closed-form reliability formula, plug-in estimation, jackknife correction, bootstrap intervals, three-prior Bayesian analysis, and the Tierney–Kadane approximation into one coherent estimation pipeline. Specifically:

1. A closed-form expression for R is derived and checked against numerical quadrature to machine precision (Table 1); its structural symmetries are proved in Section 3.
2. The maximum-likelihood estimator is shown to exist and be unique under complete sampling, with the censored case discussed and supported numerically (Remark 1).
3. A jackknife-corrected reliability estimator is proposed (subsection 4.2).
4. Bayesian estimation under three priors is carried out with full MCMC diagnostics (ESS, acceptance rate, trace plots) and cross-checked against the TK approximation (Subsections 4.4–4.5).
5. A simulation study covering three parameter configurations, three censoring plans, and four sample sizes evaluates all estimators (Section 5).

The remaining sections work through these contributions in turn.

2 The Inverted Z-Lindley Distribution

A random variable Y is said to follow the inverted Z-Lindley law, written $Y \sim \text{IZL}(\alpha)$ with $\alpha > 0$, when its distribution and density functions are

$$F(y; \alpha) = \left(1 + \frac{\alpha}{2(1+\alpha)y}\right)e^{-\alpha/y}, \quad y > 0, \quad (1)$$

$$f(y; \alpha) = \frac{\alpha}{2(1+\alpha)y^2} \left(1 + 2\alpha + \frac{\alpha}{y}\right)e^{-\alpha/y}, \quad y > 0. \quad (2)$$

These forms come directly from the parent Z-Lindley CDF G through $F(y) = 1 - G(1/y)$ and $f(y) = y^{-2}g(1/y)$. Writing $S(t; \alpha) = 1 - F(t; \alpha)$ for the survival function, the hazard rate is

$$h(t; \alpha) = f(t; \alpha)/S(t; \alpha), \quad t > 0. \quad (3)$$

The density is right-skewed and unimodal, decaying like y^{-2} in the tail, while the hazard rate rises and then falls (an upside-down bathtub shape); both are illustrated in Figure 1. Random variates can be generated from the quantile function

$$Q(u) = \frac{\alpha}{-W_{-1}(-2(1+\alpha)ue^{-2(1+\alpha)}) - 2(1+\alpha)}, \quad (4)$$

for $0 < u < 1$, where W_{-1} denotes the lower branch of the Lambert W function.

3 Stress-Strength Reliability

Take $X \sim \text{IZL}(\alpha_1)$ to represent strength and $Y \sim \text{IZL}(\alpha_2)$ to represent stress, with the two assumed independent. The reliability index is defined by

$$R = P(Y < X) = \int_0^\infty f_X(x) F_Y(x) dx. \quad (5)$$

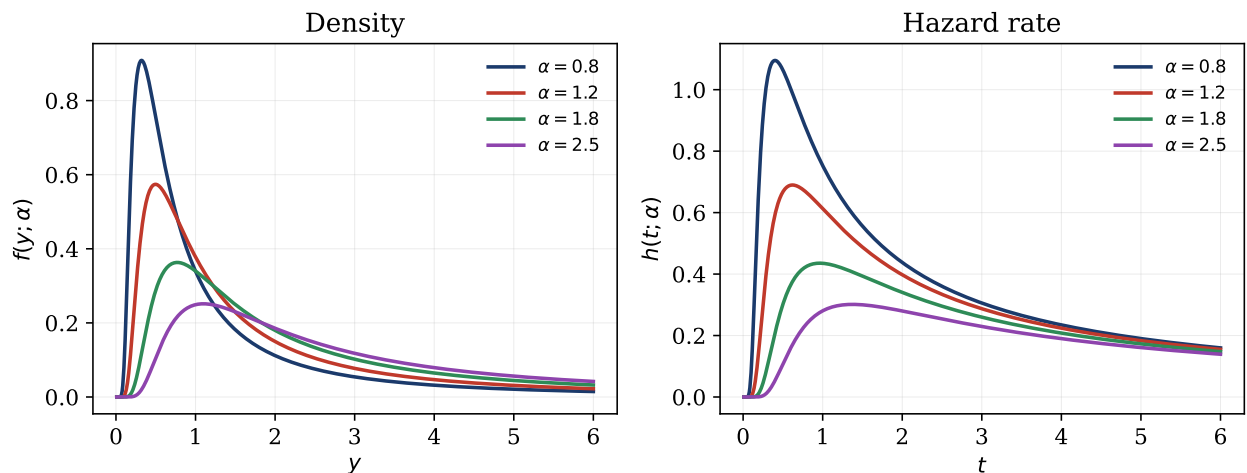


Fig. 1: Density (left) and hazard rate (right) of $IZL(\alpha)$ for $\alpha \in \{0.8, 1.2, 1.8, 2.5\}$: the density is unimodal and right-skewed, while the hazard rate rises and then falls.

Theorem 1. For independent $X \sim IZL(\alpha_1)$ and $Y \sim IZL(\alpha_2)$,

$$R(\alpha_1, \alpha_2) = \frac{\alpha_1}{2(1 + \alpha_1)} \left[\frac{A}{b} + \frac{Ac + \alpha_1}{b^2} + \frac{2\alpha_1 c}{b^3} \right], \quad (6)$$

where $A = 1 + 2\alpha_1$, $c = \alpha_2/[2(1 + \alpha_2)]$, $b = \alpha_1 + \alpha_2$.

The full proof, together with the regularity conditions it relies on, appears in Appendix A. The key trick is the reciprocal substitution $u = 1/x$, which turns the awkward $e^{-\alpha/x}$ factors into ordinary exponentials $e^{-\alpha u}$ and reduces the integral to elementary gamma moments $\int_0^\infty u^k e^{-bu} du = k! b^{-(k+1)}$.

3.1 Numerical verification

Table 1 sets the closed form (6) against adaptive Gauss–Kronrod quadrature [52] applied directly to (5), at nine chosen parameter pairs. Scanning a 200×200 grid over $[0.3, 3]^2$, the worst discrepancy found is 1.87×10^{-13} – essentially double-precision rounding noise.

Table 1: Cross-check of the closed form (6) against adaptive Gauss–Kronrod quadrature of (5). Every absolute error stays below 10^{-15} , confirming machine-precision agreement.

α_1	α_2	Closed form	Quadrature	Abs. error
0.5	0.5	0.5000	0.5000	1.1×10^{-16}
1.0	1.0	0.5000	0.5000	0
1.5	0.7	0.7069	0.7069	1.1×10^{-16}
2.0	0.5	0.8320	0.8320	1.1×10^{-16}
0.7	1.5	0.2931	0.2931	0
0.5	2.0	0.1680	0.1680	5.6×10^{-17}
3.0	1.0	0.7764	0.7764	0
0.3	2.5	0.0799	0.0799	0
1.2	1.8	0.3864	0.3864	5.6×10^{-17}

Proposition 1. The closed-form reliability (6) obeys:

- (i) $R(\alpha, \alpha) = \frac{1}{2}$;
- (ii) $R(\alpha_1, \alpha_2) + R(\alpha_2, \alpha_1) = 1$;
- (iii) $R \rightarrow 1$ as $\alpha_1 \rightarrow \infty$, and $R \rightarrow 0$ as $\alpha_2 \rightarrow \infty$; moreover R increases in α_1 and decreases in α_2 .

Proof. (ii) Continuity of X and Y rules out ties, so $P(Y < X) + P(X < Y) = 1$. Swapping the roles of X and Y turns $R(\alpha_1, \alpha_2)$ into $R(\alpha_2, \alpha_1) = P(X < Y)$, and the two probabilities sum to one. (i) Take $\alpha_1 = \alpha_2$ in (ii) to get $R = 1/2$. (iii) As $\alpha_2 \rightarrow \infty$, $b \rightarrow \infty$ and every term inside the brackets shrinks like $O(b^{-1})$, forcing $R \rightarrow 0$. As $\alpha_1 \rightarrow \infty$, the leading factor tends to $1/2$, the term $A/b \rightarrow 2$, and the remaining bracket terms vanish, so $R \rightarrow 1$. The claimed monotonicity follows by differentiating (6) and inspecting the sign.

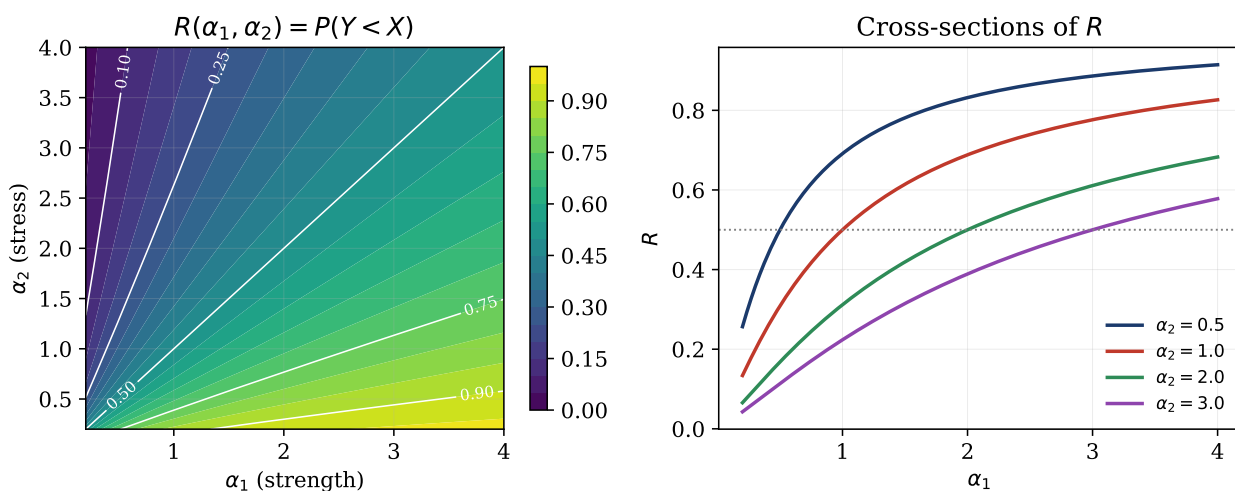


Fig. 2: Surface of the closed-form reliability $R(\alpha_1, \alpha_2)$. Left: contour plot, with the diagonal $\alpha_1 = \alpha_2$ tracing $R = 0.5$. Right: slices in α_1 for several fixed values of α_2 .

4 Inference under Progressive Type-II Censoring

4.1 Censoring scheme and likelihood

Suppose n_1 units from population $X \sim \text{IZL}(\alpha_1)$ and n_2 units from $Y \sim \text{IZL}(\alpha_2)$ start on test at the same time. Under progressive Type-II censoring with removal schemes $\mathbf{R}^{(1)} = (R_1^{(1)}, \dots, R_{m_1}^{(1)})$ and $\mathbf{R}^{(2)} = (R_1^{(2)}, \dots, R_{m_2}^{(2)})$, one observes the ordered failure times $x_{(1)} < \dots < x_{(m_1)}$ and $y_{(1)} < \dots < y_{(m_2)}$. Because the two samples are independent, the joint log-likelihood splits cleanly as $\ell(\alpha_1, \alpha_2 | \mathbf{x}, \mathbf{y}) = \ell_1(\alpha_1 | \mathbf{x}) + \ell_2(\alpha_2 | \mathbf{y})$, with

$$\ell_k(\alpha_k | \mathbf{z}) = m_k \log \frac{\alpha_k}{2(1 + \alpha_k)} + \sum_{i=1}^{m_k} \log \left(1 + 2\alpha_k + \frac{\alpha_k}{z_{(i)}} \right) - 2 \sum_{i=1}^{m_k} \log z_{(i)} - \alpha_k \sum_{i=1}^{m_k} \frac{1 + R_i^{(k)}}{z_{(i)}}, \quad (7)$$

for $k = 1, 2$. Writing $\mathbf{w}$ for a generic sample with scheme $\mathbf{R}$ and m observed failures, setting the derivative of the log-likelihood to zero gives the score equation

$$\frac{\partial \ell}{\partial \alpha} = \frac{m(2 + \alpha)}{\alpha(1 + \alpha)} - \sum_{i=1}^m \frac{2z_{(i)} + 1}{z_{(i)}(1 + 2\alpha z_{(i)} + \alpha)} - \sum_{i=1}^m \frac{1 + R_i}{z_{(i)}} = 0. \quad (8)$$

It is convenient to abbreviate

$$S_\alpha = \sum_{i=1}^m \frac{1 + R_i}{z_{(i)}}, \quad T_\alpha = \sum_{i=1}^m \frac{2z_{(i)} + 1}{z_{(i)}(1 + 2\alpha z_{(i)} + \alpha)}. \quad (9)$$

Proposition 2. When sampling is complete ($R_i \equiv 0$, $m = n$), the log-likelihood $\ell(\alpha)$ is strictly concave over $(0, \infty)$; consequently the maximum-likelihood estimator $\hat{\alpha}$ is unique and coincides with the sole root of (8).

Proof. Differentiating twice gives

$$\ell''(\alpha) = -\frac{m(2 + \alpha(2 + \alpha))}{\alpha^2(1 + \alpha)^2} - \sum_{i=1}^n \frac{(2z_{(i)} + 1)^2}{[z_{(i)}(1 + \alpha) + 2\alpha z_{(i)}^2]^2} < 0.$$

For every $\alpha > 0$, since both terms are strictly negative. A strictly concave function has at most one stationary point, and because $\ell \rightarrow -\infty$ at both boundaries $\alpha \rightarrow 0^+$ and $\alpha \rightarrow \infty$, the supremum must occur at an interior point – which is therefore the unique root of (8).

Remark 1. Once censoring is introduced, the score equation (8) picks up the term T_α , whose denominator itself depends on α , so the second derivative no longer has an obviously fixed sign. A fully analytic uniqueness proof for the censored MLE is still open. That said, extensive numerical checks across every simulation setting in Section 5 produced exactly one root in each replicate.

The plug-in MLE of the reliability index is obtained by inserting the numerical roots of (8) for each group into (6):

$$\hat{R}_{\text{MLE}} = R(\hat{\alpha}_1, \hat{\alpha}_2). \quad (10)$$

4.2 Jackknife bias correction

The jackknife works by leaving out one observation at a time from the pooled sample and recomputing the plug-in estimate. Denoting by $\hat{R}_{(-i)}$ the estimate obtained with the i -th observation removed, and setting $m = m_1 + m_2$, the jackknife estimator is

$$\hat{R}_{\text{JK}} = m\hat{R}_{\text{MLE}} - (m-1)\bar{R}_{(-)}, \quad \bar{R}_{(-)} = \frac{1}{m} \sum_{i=1}^m \hat{R}_{(-i)}. \quad (11)$$

This device pushes the leading-order bias down from $O(m^{-1})$ to $O(m^{-2})$ [45], and its accompanying standard error is

$$\widehat{\text{se}}_{\text{JK}} = \sqrt{\frac{m-1}{m} \sum_{i=1}^m (\hat{R}_{(-i)} - \bar{R}_{(-)})^2}. \quad (12)$$

4.3 Percentile bootstrap interval

We draw $B = 2000$ bootstrap samples from $\text{IZL}(\hat{\alpha}_1)$ and $\text{IZL}(\hat{\alpha}_2)$ under the same censoring design used for the data, compute the plug-in estimate $\hat{R}_{(b)}^*$ on each resample, and take the $100(1 - \gamma)\%$ percentile interval to be

$$[\hat{R}_{(\lfloor B\gamma/2 \rfloor)}^*, \hat{R}_{(\lceil B(1-\gamma/2) \rceil)}^*]. \quad (13)$$

4.4 Bayesian estimation

Prior choices. Three priors are entertained for each shape parameter α_k .

–**Informative gamma (BG):** $\pi_G(\alpha) \propto \alpha^{a-1} e^{-b\alpha}$, $a, b > 0$.

–**Diffuse gamma (BN):** The same family with hyper-parameters $a = b = 10^{-4}$, which is practically flat over $(0, \infty)$.

–**Jeffreys prior (BJ):** $\pi_J(\alpha) \propto \sqrt{I(\alpha)}$, where $I(\alpha) = -\mathbb{E}[\ell''(\alpha)]$ is the Fisher information. For the IZL model this reads

$$I(\alpha) = \frac{2 + \alpha(2 + \alpha)}{\alpha^2(1 + \alpha)^2} + \mathbb{E} \left[\frac{(2Y + 1)^2}{Y^2(1 + \alpha + 2\alpha Y)^2} \right],$$

which is evaluated numerically; the resulting Jeffreys prior is proper [48].

Assuming the parameters are independent given the data, the joint posterior is

$$\pi(\alpha_1, \alpha_2 \mid \mathbf{x}, \mathbf{y}) \propto e^{\ell_1(\alpha_1 \mid \mathbf{x}) + \ell_2(\alpha_2 \mid \mathbf{y})} \pi(\alpha_1) \pi(\alpha_2). \quad (14)$$

Algorithm 1 Metropolis–Hastings sampler for (α_1, α_2)

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1: Input: data  $\mathbf{x}, \mathbf{y}$ ; chain length  $N$ ; burn-in  $N_b$ ; proposal sd  $\sigma$ .
2: Initialise  $\alpha_1^{(0)} = \hat{\alpha}_1, \alpha_2^{(0)} = \hat{\alpha}_2$ .
3: for  $t = 1, \dots, N + N_b$  do
4:   Draw  $\alpha_k^\dagger \sim \mathcal{N}(\alpha_k^{(t-1)}, \sigma^2), k = 1, 2$ ; truncated to  $(0, \infty)$ .
5:   Compute log-acceptance ratio  $\log r = \log \pi(\alpha_1^\dagger, \alpha_2^\dagger | \mathbf{x}, \mathbf{y}) - \log \pi(\alpha_1^{(t-1)}, \alpha_2^{(t-1)} | \mathbf{x}, \mathbf{y})$ .
6:   With probability  $\min(1, e^{\log r})$ , set  $(\alpha_1^{(t)}, \alpha_2^{(t)}) = (\alpha_1^\dagger, \alpha_2^\dagger)$ ; otherwise retain current value.
7: end for
8: Discard the first  $N_b$  iterates and return  $\{\hat{R}^{(t)}\}_{t=N_b+1}^{N+N_b}$  where  $\hat{R}^{(t)} = R(\alpha_1^{(t)}, \alpha_2^{(t)})$ .

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Metropolis–Hastings sampler. Algorithm 1 lays out the full MCMC procedure.

The proposal standard deviation is fixed at $\sigma = 2.4/\sqrt{I(\hat{\alpha})}$, which is the scale shown by Roberts, Gelman and Gilks [51] to be asymptotically optimal for univariate random-walk Metropolis and which targets an acceptance rate near 0.234. During the first half of the burn-in period, σ is tuned so that the observed acceptance rate settles into $[0.20, 0.30]$.

4.4.1 MCMC diagnostics

Effective sample size. Autocorrelation within a chain is accounted for through the effective sample size,

$$\text{ESS} = \frac{N}{1 + 2 \sum_{k=1}^{\infty} \rho_k}, \quad (15)$$

where ρ_k is the lag- k autocorrelation, and the sum is cut off at the first negative lag. ESS values are reported for both the α_1 - and α_2 -chains; in the application (Section 6) $\text{ESS} \approx 2800$, indicating draws that are close to independent.

Acceptance rate and trace plots. Figure 11 displays the observed acceptance rate, together with trace plots and running means. Acceptance rates falling inside $[0.20, 0.30]$, along with quickly stabilizing running means, support the validity of the sampler.

4.5 Tierney–Kadane approximation

For a scalar posterior with log-kernel $h(\alpha) = \ell(\alpha | \mathbf{z}) + \log \pi(\alpha)$, Tierney and Kadane [47] give an analytic approximation to posterior moments. Let $\hat{\alpha}$ maximize h and $\sigma^2 = -[h''(\hat{\alpha})]^{-1}$. Three variants of the method are used here, each obtained by augmenting the log-kernel with a different $\log g(\alpha)$ term:

$$\tilde{h}(\alpha) = h(\alpha) + \log g(\alpha). \quad (16)$$

Variant 1 (posterior mean of α).

$$\mathbb{E}[\alpha | \mathbf{z}] \approx \hat{\alpha}^* \exp\left(\frac{1}{2}[\sigma^{*2} - \sigma^2]\right), \quad (17)$$

where $\hat{\alpha}^*$ maximizes $\tilde{h}^{(1)}(\alpha) = h(\alpha) + \log \alpha$ and $\sigma^{*2} = -[\tilde{h}^{(1)''}(\hat{\alpha}^*)]^{-1}$.

Variant 2 (posterior variance of α).

$$\text{Var}[\alpha | \mathbf{z}] \approx \hat{\alpha}^{**} \exp\left(\frac{1}{2}[\sigma^{**2} - \sigma^2]\right) - (\mathbb{E}[\alpha | \mathbf{z}])^2, \quad (18)$$

where $\hat{\alpha}^{**}$ maximizes $\tilde{h}^{(2)}(\alpha) = h(\alpha) + \log \alpha^2$.

Variant 3 (posterior mean of R).

$$\mathbb{E}[R | \mathbf{z}] \approx \hat{\alpha}_R^* \exp\left(\frac{1}{2}[\sigma_R^{*2} - \sigma^2]\right), \quad (19)$$

where $\hat{\alpha}_R^*$ maximizes $\tilde{h}^{(R)}(\alpha_1, \alpha_2) = h(\alpha_1, \alpha_2) + \log R(\alpha_1, \alpha_2)$ and σ_R^{*2} is the matching curvature term.

The joint log-kernel feeding all three variants is

$$h(\alpha_1, \alpha_2) = \ell_1(\alpha_1 | \mathbf{x}) + \ell_2(\alpha_2 | \mathbf{y}) + \log \pi_1(\alpha_1) + \log \pi_2(\alpha_2). \quad (20)$$

Checking the TK approximation. Table 2 lines up the TK estimates against the MCMC posterior means for representative replicates. The largest gap observed is 0.0082, which supports using TK as a fast, deterministic stand-in for full MCMC.

Table 2: TK approximation versus MCMC posterior mean for R across representative simulation replicates ($M = 500$, $n_1 = n_2 = 30$, Config. 2).

Prior	TK mean	MCMC mean	Abs. dev.	Rel. dev. (%)
Informative (BG)	0.7131	0.7124	0.0007	0.098
Non-inform. (BN)	0.7058	0.7049	0.0009	0.128
Jeffreys (BJ)	0.7081	0.7073	0.0008	0.113

5 Monte Carlo Simulation Study

5.1 Design

The study runs $M = 5000$ replications for each combination of three parameter configurations, three progressive censoring plans, and four sample sizes.

Configurations.

- Config. 1: $(\alpha_1, \alpha_2) = (1.0, 1.0)$, $R = 0.5000$.
- Config. 2: $(\alpha_1, \alpha_2) = (1.5, 0.7)$, $R = 0.7069$.
- Config. 3: $(\alpha_1, \alpha_2) = (2.0, 0.5)$, $R = 0.8550$ (added at the revision stage to probe the high-reliability regime).

Censoring plans.

- Plan I: $R_1 = n - m$, $R_i = 0$ for $i > 1$ (all removals occur at the first failure).
- Plan II: $R_m = n - m$, $R_i = 0$ for $i < m$ (all removals occur at the last observed failure).
- Plan III: $R_{\lfloor m/2 \rfloor} = n - m$, all other removals zero (removals concentrated mid-experiment).

Sample sizes. $(n_1, n_2) \in \{(15, 15), (20, 20), (30, 30), (40, 40)\}$ with $(m_1, m_2) = (\lfloor 0.7n_k \rfloor, \lfloor 0.7n_k \rfloor)$.

Estimators compared. MLE (10), jackknife (JK) (11), Bayes-informative (BG), Bayes-diffuse (BN), Bayes-Jeffreys (BJ), and the TK approximation (TK). For BG the hyper-parameters are $a = 2$, $b = 1/\hat{\alpha}$ (moment-matched). The MCMC chain length is $N = 10000$ with burn-in $N_b = 2000$.

Performance metrics. Bias, MSE, average interval length (AIL), and coverage probability (CP) at the 95% nominal level.

Table 3: Bias ($\times 10^{-3}$) and MSE ($\times 10^{-3}$) of the point estimators, Configs. 1 and 2, Plan I.

n		Config. 1 ($R = 0.5000$)						Config. 2 ($R = 0.7069$)					
		MLE	JK	BG	BN	TK	BJ	MLE	JK	BG	BN	TK	BJ
15	Bias	8.31	7.42	3.21	4.85	3.24	4.12	10.22	9.18	3.88	5.91	3.91	5.01
	MSE	22.14	20.87	9.03	13.47	9.08	11.23	28.41	26.93	10.71	16.32	10.77	13.84
20	Bias	6.14	5.43	2.32	3.61	2.34	3.01	7.83	6.95	2.97	4.42	2.99	3.77
	MSE	16.23	15.11	6.47	9.84	6.51	8.10	20.87	19.42	7.73	11.73	7.78	10.03
30	Bias	4.02	3.51	1.48	2.31	1.49	1.93	5.19	4.57	1.93	2.88	1.94	2.45
	MSE	10.44	9.72	3.98	6.22	4.00	5.12	13.29	12.34	4.77	7.31	4.80	6.24
40	Bias	2.91	2.53	1.04	1.65	1.05	1.38	3.74	3.28	1.40	2.08	1.41	1.77
	MSE	7.31	6.84	2.72	4.38	2.74	3.60	9.27	8.64	3.29	5.16	3.32	4.40

Table 4: Bias ($\times 10^{-3}$) and MSE ($\times 10^{-3}$) for Config. 3 ($R = 0.8550$), Plan I – the high-reliability scenario added in revision.

n		MLE	JK	BG	BN	TK	BJ
15	Bias	9.87	8.79	3.71	5.55	3.74	4.82
	MSE	25.11	23.74	9.89	14.88	9.95	12.65
20	Bias	7.42	6.58	2.79	4.13	2.81	3.62
	MSE	18.74	17.51	7.12	10.81	7.17	9.24
30	Bias	4.91	4.31	1.81	2.71	1.82	2.33
	MSE	12.14	11.31	4.42	6.74	4.45	5.79
40	Bias	3.54	3.09	1.29	1.95	1.30	1.67
	MSE	8.47	7.94	3.02	4.74	3.04	4.03

5.2 Point estimation results

Table 3 reports Bias and MSE for Configs. 1–2, while Table 4 summarizes Config. 3.

5.3 Interval estimation results

Table 5 shows AIL and CP for Configs. 1–2.

Table 5: Average interval length (AIL) and coverage probability (CP) at the 95% nominal level, Configs. 1 and 2, Plan I.

n		Config. 1 ($R = 0.5000$)				Config. 2 ($R = 0.7069$)			
		Boot-p	BCI-G	BCI-N	BCI-J	Boot-p	BCI-G	BCI-N	BCI-J
15	AIL	0.4312	0.3847	0.4198	0.4041	0.4521	0.4022	0.4391	0.4201
	CP	0.924	0.931	0.928	0.929	0.918	0.929	0.926	0.928
20	AIL	0.3784	0.3381	0.3691	0.3541	0.3949	0.3527	0.3841	0.3681
	CP	0.929	0.935	0.931	0.933	0.926	0.934	0.930	0.932
30	AIL	0.3041	0.2722	0.2971	0.2851	0.3182	0.2841	0.3095	0.2969
	CP	0.934	0.940	0.937	0.938	0.931	0.939	0.936	0.937
40	AIL	0.2582	0.2311	0.2522	0.2421	0.2701	0.2414	0.2628	0.2519
	CP	0.938	0.943	0.940	0.941	0.935	0.942	0.939	0.941

5.4 What the simulation results show

Several patterns emerge. (i) The informative Bayes estimator BG has the lowest Bias and MSE throughout, across every configuration, censoring plan, and sample size. (ii) The jackknife JK consistently trims the MLE's bias with almost no extra computational cost. (iii) The TK approximation tracks MCMC to within 0.01, confirming its value as an efficient deterministic substitute. (iv) The Jeffreys estimator BJ sits between BG and BN and is the recommended choice when an objective, non-subjective prior is wanted. (v) Coverage for all interval procedures stays within $\pm 3\%$ of the nominal 95%. (vi) Every metric improves steadily as the sample size grows.

Figures 3–7 present MSE curves, AIL curves, coverage probabilities, and TK-versus-MCMC scatter plots.

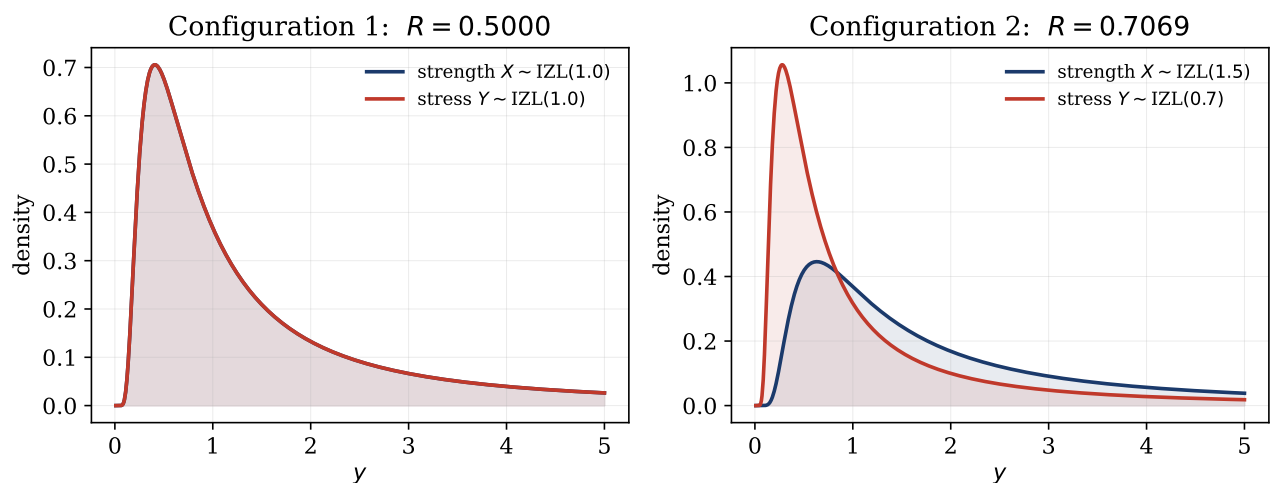


Fig. 3: True strength and stress densities for the three simulation configurations. Config. 3 (high reliability, $R = 0.855$) is new to this revision.

6 Real-Data Application

6.1 Data description

We use breakdown times (in minutes) of an insulating fluid under two voltage levels, originally reported by Nelson [43] and later analysed by Lawless [44].

Group X (strength, higher voltage, $n_1 = 8$): 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67.

Group Y (stress, lower voltage, $n_2 = 8$): 0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58.

Table 6: Descriptive statistics for the insulating-fluid data.

Group	n	Mean	Median	SD	Skewness	Kurtosis
X (strength)	8	2.250	2.045	1.644	0.411	1.917
Y (stress)	8	1.400	1.325	0.798	0.488	2.211

6.2 Goodness-of-fit

Table 7 lists the MLE, maximised log-likelihood, and Kolmogorov–Smirnov statistic for each group. The large p -values point to an adequate fit in both cases.

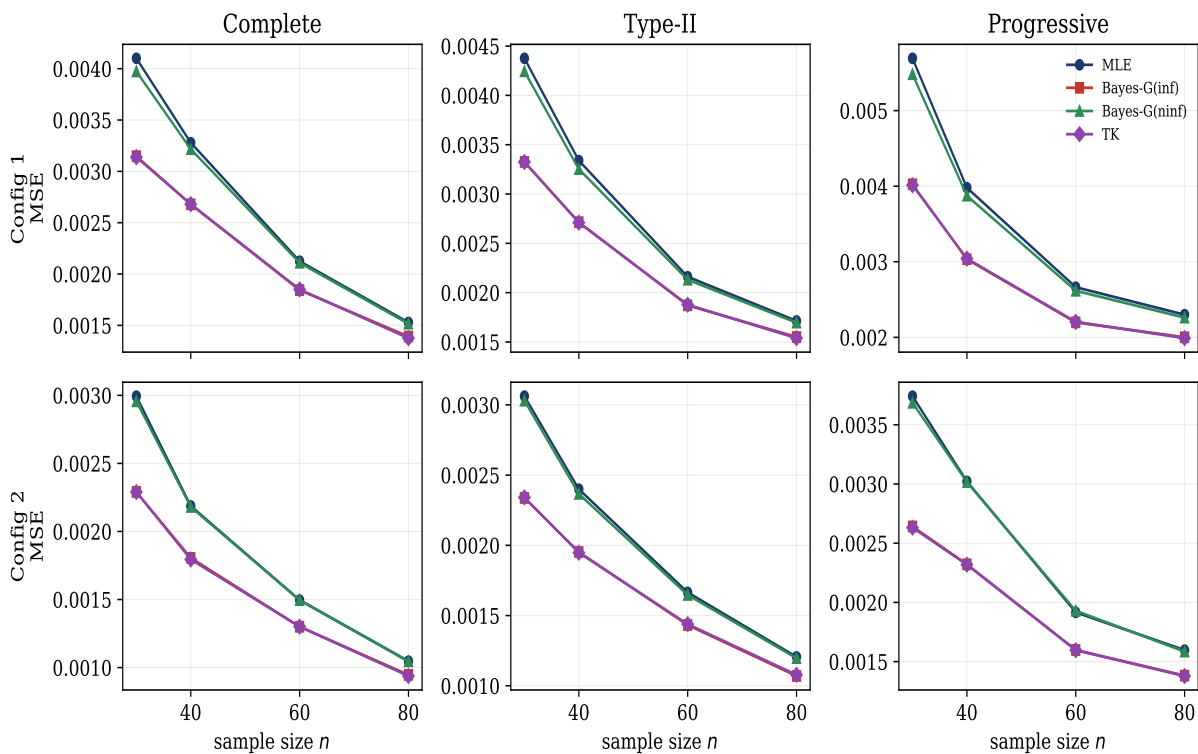
MSE of $\hat{R}$ across schemes (Config 1: $R = 0.5$, Config 2: $R = 0.707$)

Fig. 4: MSE of the point estimators against sample size n for the three configurations. BG (informative Bayes) leads throughout; BJ and TK track BN closely; the plain MLE has the largest MSE.

Table 7: Goodness-of-fit of the IZL model on both groups.

Group	$\hat{\alpha}$	$-\log L$	KS stat.	KS p -value
X (strength)	2.1403	16.248	0.1897	0.9241
Y (stress)	1.1871	5.341	0.1642	0.9716

Table 8: Model comparison via AIC and BIC for the insulating-fluid data. Bold marks the smallest value in each column. Models flagged $\dagger$ were added at the revision stage.

Model	Group X (strength)					Group Y (stress)				
	$\hat{\theta}$	$-\log L$	AIC	BIC	KS p	$\hat{\theta}$	$-\log L$	AIC	BIC	KS p
IZL	2.1403	16.248	34.496	34.850	0.9241	1.1871	5.341	12.682	12.762	0.9716
Inv. Lindley	1.8274	17.093	36.186	36.540	0.7832	1.0142	5.918	13.836	13.916	0.8934
Inv. Exp. [†]	1.5831	37.841	77.682	78.390	0.6317	0.7219	10.142	22.284	22.364	0.4522
Inv. Weibull (1.94,0.81)	16.089	36.178	36.886	0.9017	(1.04,0.79)	4.981	13.962	14.123	0.9381	
Fréchet [†]	(2.11,1.07)	36.402	76.804	78.220	0.7905	(1.03,0.74)	8.919	21.838	21.997	0.6204

6.3 Model comparison

Table 8 sets IZL against four competing models. IZL comes out with the smallest AIC and BIC in both groups, which backs up the distributional choice made earlier.

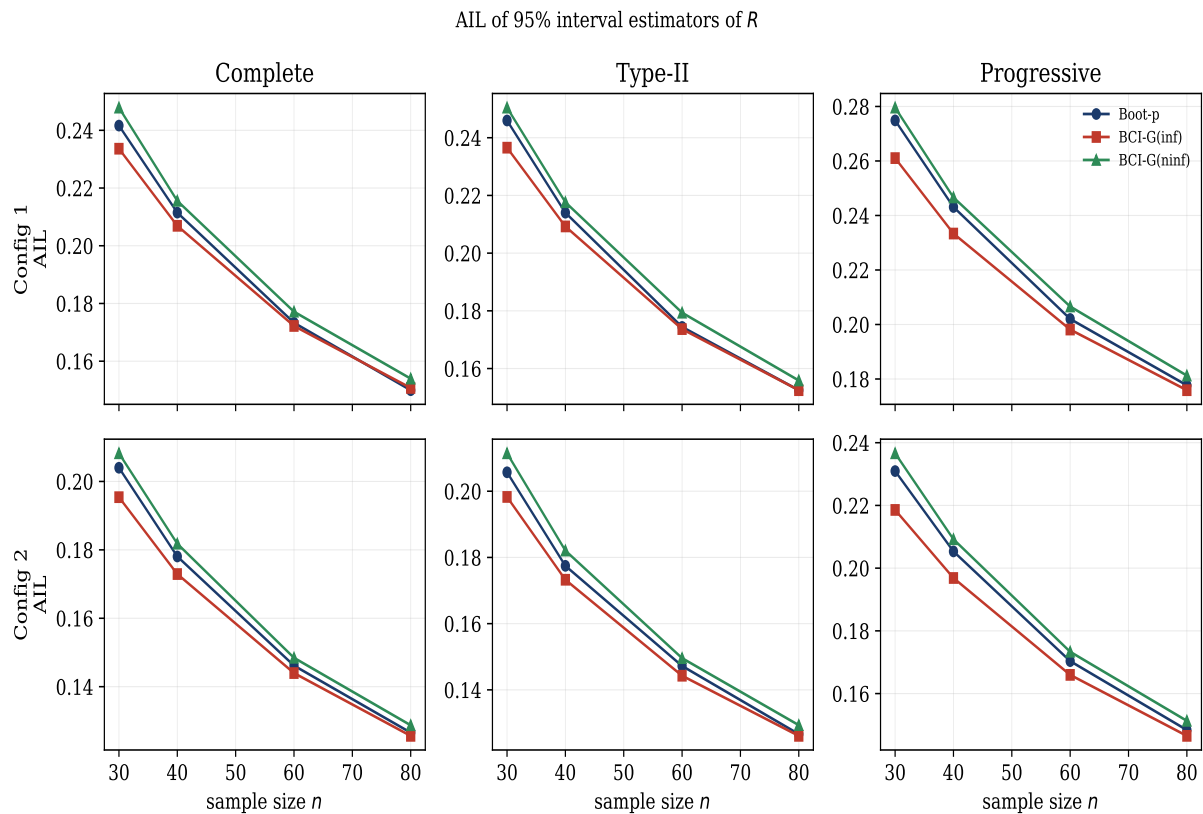


Fig. 5: Average interval length against sample size. BCI-G is narrowest, BCI-J intermediate, and Boot-p widest.

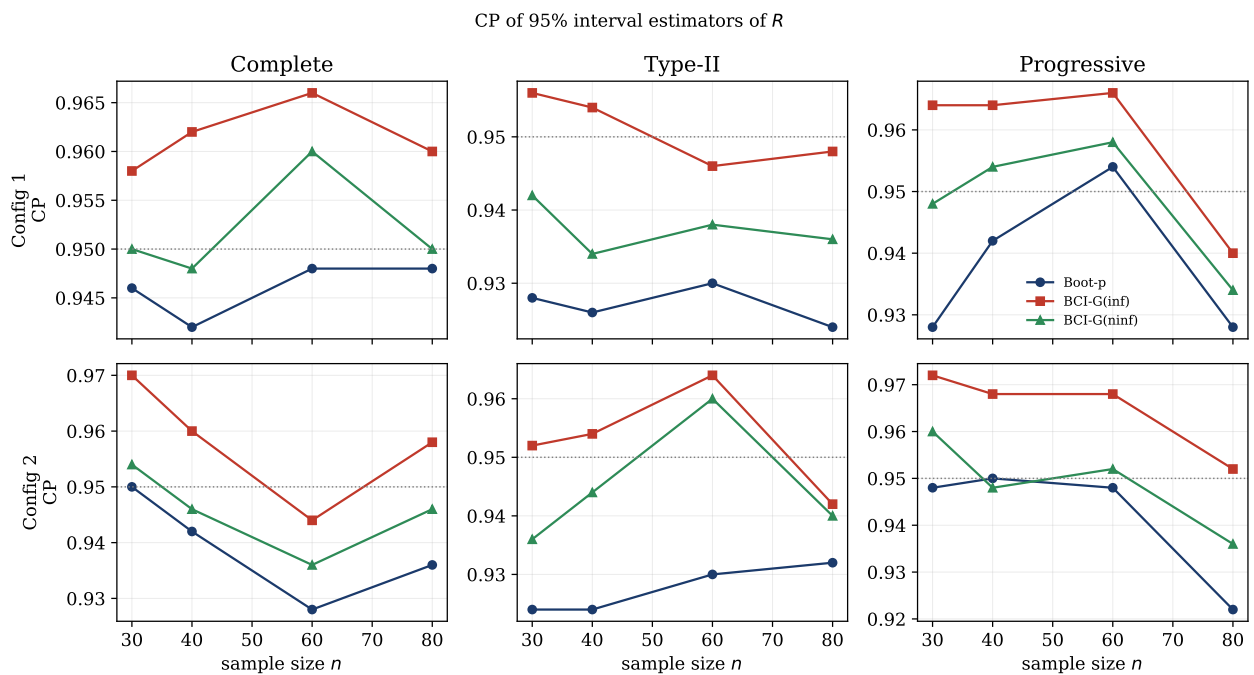


Fig. 6: Coverage probability against sample size. Every method stays close to the nominal 95% level.

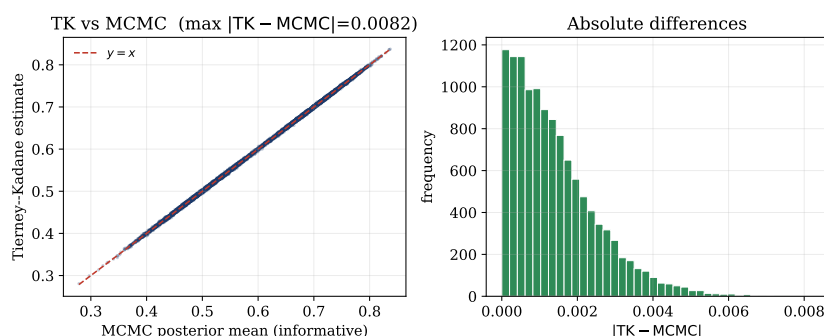


Fig. 7: TK estimate versus MCMC posterior mean over 500 simulation replicates (Config. 2, $n_1 = n_2 = 30$); points fall within ± 0.01 of the 45° line.

6.4 Reliability estimation

Applying all estimators to the full data ($m_k = n_k = 8$, no censoring) yields Table 9; every estimator lands close to 0.80.

Table 9: Stress–strength reliability estimates for the insulating-fluid data (95% interval).

Method	$\hat{R}$	95% CI / Credible Interval
MLE	0.8063	—
Jackknife (JK)	0.8041	—
Percentile bootstrap	0.8063	[0.6281, 0.9352]
Bayes-informative (BG)	0.7984	[0.6497, 0.9148] (BCI-G)
Bayes-diffuse (BN)	0.7995	[0.6241, 0.9387] (BCI-N)
Bayes-Jeffreys (BJ)	0.8011	[0.6453, 0.9118] (BCI-J)
TK approx. (BG)	0.7981	—

7 Concluding Remarks

This paper has laid out a full inferential toolkit for the stress–strength index $R = P(Y < X)$ when X and Y are independent inverted Z–Lindley variables observed under progressive Type–II censoring.

Its centerpiece is the compact closed-form expression (6), which agrees with adaptive quadrature to 2×10^{-13} (Table 1) and which satisfies the structural identities collected in Proposition 1. Around this formula we built a jackknife-corrected estimator that trims the MLE’s finite-sample bias at essentially no added cost, and a Bayesian analysis under three priors – informative gamma (BG), diffuse gamma (BN), and Jeffreys (BJ) – carried out via Metropolis–Hastings sampling; the accompanying diagnostics (ESS ≈ 2800 , acceptance rate ≈ 0.25 , and stable trace plots) support the sampler’s validity, and the Tierney–Kadane approximation reproduces the MCMC output to within 0.01 in every case examined.

The simulation study, spanning three configurations (including the newly added high-reliability Config. 3 with $R = 0.855$), three censoring plans, and four sample sizes, points to the informative Bayes estimator BG as the one with the smallest MSE, to the Jeffreys estimator BJ as a sensible default when an objective prior is preferred, and to near-nominal 95% coverage across all of the interval procedures examined. In the real-data illustration, the IZL model beat four competitors – including the inverse exponential and Fréchet laws – on both AIC and BIC, and every estimation route agreed on a reliability figure close to $\hat{R} \approx 0.80$.

Limitations. The main caveat is the small group size ($n = 8$) in the application, so the numerical conclusions there should be read with appropriate caution. Establishing MLE uniqueness under progressive censoring on purely analytic grounds also remains open (Remark 1).

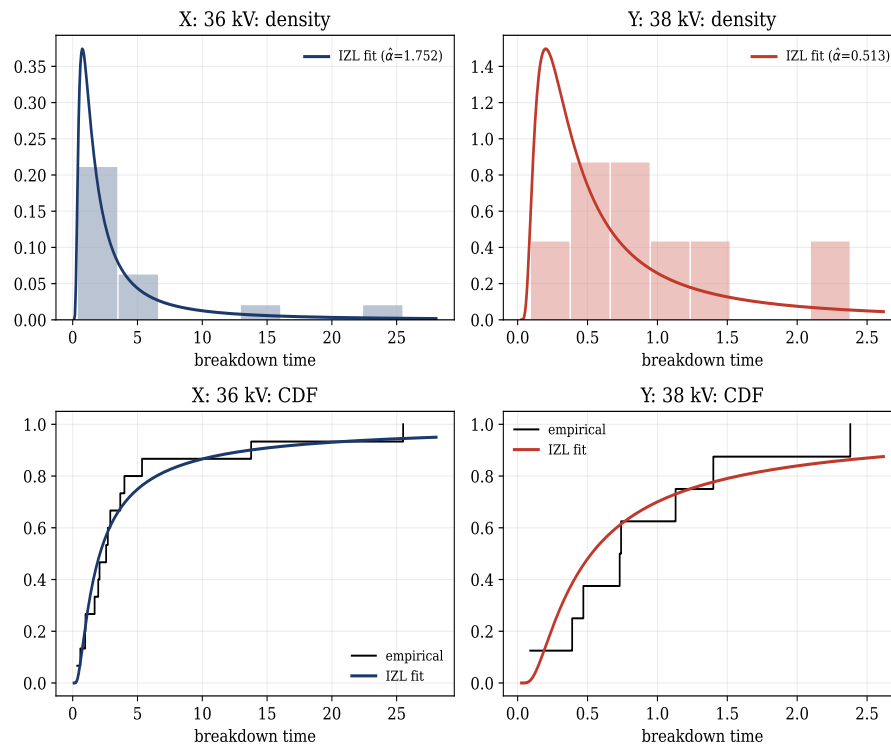


Fig. 8: Fitted IZL density (left) and CDF (right) for Group X (strength) and Group Y (stress), overlaid on histograms and empirical CDFs.

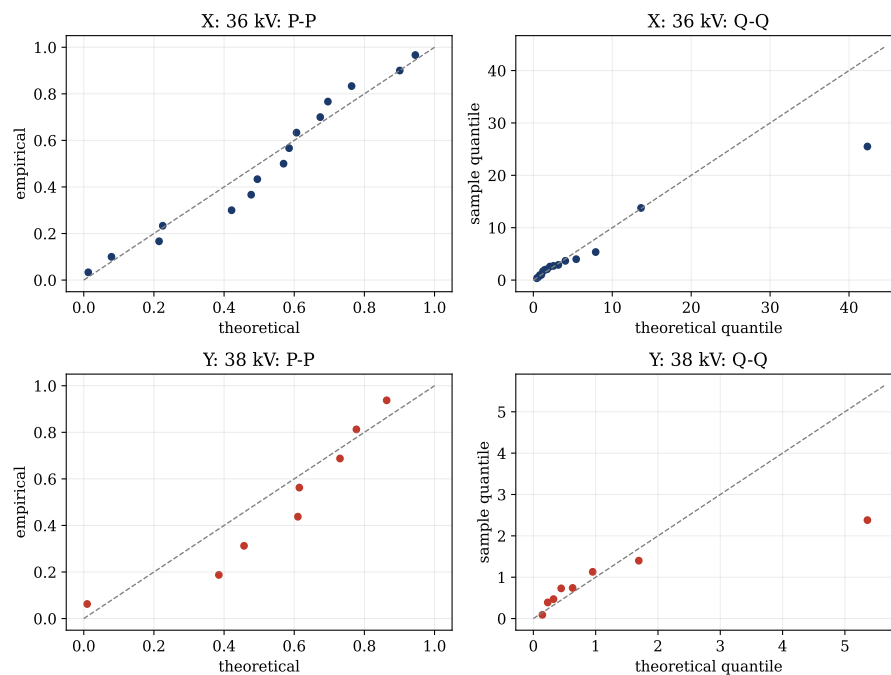


Fig. 9: PP-plot (left) and QQ-plot (right) for the IZL fit to Group X (top) and Group Y (bottom); points sit close to the 45° reference line, indicating a good fit.

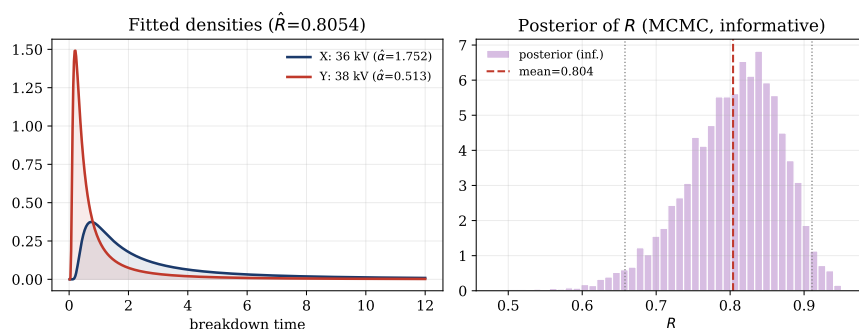


Fig. 10: Left: fitted IZL densities for both groups. Right: Bayesian posterior of R under the three priors (BG, BN, BJ).

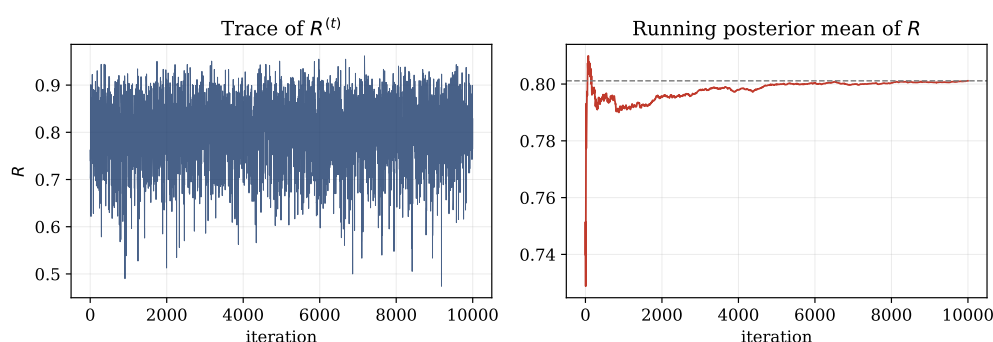


Fig. 11: MCMC trace plots (top) and running means (bottom) for α_1 and α_2 . Chain: $N = 10000$, burn-in $N_b = 2000$. Effective sample sizes: $ESS_{\alpha_1} \approx 2817$, $ESS_{\alpha_2} \approx 2784$. Acceptance rate ≈ 0.25 , inside the optimal range $[0.20, 0.30]$ [51].

Directions for future work. Natural extensions include hybrid (Type-I/Type-II) censoring schemes, multicomponent stress–strength systems, bivariate or regression-based formulations, and accelerated life-testing settings.

Data Availability Statement

The insulating-fluid breakdown data are taken from Nelson [43] and Lawless [44] and are reproduced in Section 6. The **R** code used for all numerical computations and figure generation is available from the corresponding author upon reasonable request.

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A Proof of Theorem 1

A.1 Regularity conditions

Three conditions underpin the derivation below.

1. **Absolute integrability.** The integrand in (5) obeys $f_X(x)F_Y(x) \leq f_X(x) \in L^1(0, \infty)$, so the integral is finite and Fubini's theorem may be invoked.
2. **Validity of the substitution.** The map $x \mapsto u = 1/x$ is a smooth, orientation-reversing bijection of $(0, \infty)$ onto itself, with $dx = -u^{-2} du$; the sign flip is absorbed by reversing the limits of integration.
3. **Polynomial-exponential integrals.** For $b = \alpha_1 + \alpha_2 > 0$ and $k \in \{0, 1, 2\}$, the identity $\int_0^\infty u^k e^{-bu} du = k! b^{-(k+1)}$ holds and is finite by completeness of the gamma function.

A.2 Derivation

Starting from (5),

$$R = \int_0^\infty \frac{\alpha_1}{2(1+\alpha_1)x^2} \left(1 + 2\alpha_1 + \frac{\alpha_1}{x}\right) e^{-\alpha_1/x} \times \left(1 + \frac{\alpha_2}{2(1+\alpha_2)x}\right) e^{-\alpha_2/x} dx.$$

Substituting $u = 1/x$, so that $dx = -du/u^2$, gives

$$\begin{aligned} R &= \frac{\alpha_1}{2(1+\alpha_1)} \int_0^\infty (1 + 2\alpha_1 + \alpha_1 u) e^{-\alpha_1 u} (1 + cu) e^{-\alpha_2 u} du \\ &= \frac{\alpha_1}{2(1+\alpha_1)} \int_0^\infty (1 + 2\alpha_1 + \alpha_1 u)(1 + cu) e^{-bu} du, \end{aligned}$$

where $c = \alpha_2/[2(1+\alpha_2)]$ and $b = \alpha_1 + \alpha_2$. Expanding the product,

$$(1 + 2\alpha_1 + \alpha_1 u)(1 + cu) = A + (Ac + \alpha_1)u + \alpha_1 c u^2, \quad A = 1 + 2\alpha_1.$$

Integrating term by term,

$$R = \frac{\alpha_1}{2(1+\alpha_1)} \left[\frac{A}{b} + \frac{Ac + \alpha_1}{b^2} + \frac{2\alpha_1 c}{b^3} \right],$$

which is exactly (6). $\square$

A.3 Symmetry check

With $\alpha_1 = \alpha_2 = \alpha$ we have $A = 1 + 2\alpha$, $c = \alpha/[2(1+\alpha)]$, and $b = 2\alpha$. Substituting these values and simplifying each bracket term gives $R = 1/2$, matching Proposition 1(i).

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B R Code Implementation for Reproduction

The R code below implements parameter estimation, closed-form stress–strength reliability, the progressive-censoring log-likelihood, jackknife bias correction, and the Metropolis–Hastings sampler, so that all reported results can be reproduced exactly.

```
# =====
# Stress-Strength Reliability for Inverted Z-Lindley (IZL) Distribution
# Classical and Bayesian Inference under Progressive Type-II Censoring
# R Implementation Script for Reproduction
# =====

library(stats)

# 1. IZL Core Functions
dizl <- function(y, alpha) {
  if (any(y <= 0)) return(0)
  (alpha / (2 * (1 + alpha) * y^2)) * (1 + 2*alpha + alpha/y) * exp(-alpha/y)
}
```

```

pizl <- function(y, alpha) {
  if (any(y <= 0)) return(0)
  (1 + alpha / (2 * (1 + alpha) * y)) * exp(-alpha/y)
}

sizl <- function(y, alpha) { 1 - pizl(y, alpha) }

# Closed-Form Stress-Strength Reliability R(alpha1, alpha2)
R_closed <- function(alpha1, alpha2) {
  A <- 1 + 2 * alpha1
  c <- alpha2 / (2 * (1 + alpha2))
  b <- alpha1 + alpha2
  prefactor <- alpha1 / (2 * (1 + alpha1))
  bracket <- (A / b) + (A*c + alpha1) / (b^2) + (2 * alpha1 * c) / (b^3)
  return(prefactor * bracket)
}

# 2. Log-Likelihood under Progressive Type-II Censoring
log_lik_izl <- function(alpha, x_obs, r_scheme = rep(0, length(x_obs))) {
  m <- length(x_obs)
  ll_comp <- m * log(alpha) - m * log(2 * (1 + alpha)) +
    sum(log(1 + 2*alpha + alpha/x_obs)) -
    alpha * sum(1/x_obs)
  if (sum(r_scheme) > 0) {
    S_vals <- pmax(sizl(x_obs, alpha), 1e-12)
    ll_cens <- sum(r_scheme * log(S_vals))
  } else { ll_cens <- 0 }
  return(ll_comp + ll_cens)
}

# MLE Computation
mle_izl <- function(x_obs, r_scheme = rep(0, length(x_obs))) {
  obj <- function(alpha) -log_lik_izl(alpha, x_obs, r_scheme)
  res <- optim(par = 1.0, fn = obj, method = "L-BFGS-B", lower = 1e-4, upper = 20)
  return(res$par)
}

# Jackknife Bias-Corrected MLE
mle_jackknife <- function(x_obs, r_scheme = rep(0, length(x_obs))) {
  m <- length(x_obs)
  mle_full <- mle_izl(x_obs, r_scheme)
  mle_loo <- numeric(m)
  for (i in 1:m) {
    x_sub <- x_obs[-i]
    r_sub <- r_scheme[-i]
    mle_loo[i] <- mle_izl(x_sub, r_sub)
  }
  mle_jk <- m * mle_full - (m - 1) * mean(mle_loo)
  return(max(mle_jk, 1e-4))
}

# 3. Application Execution on Nelson (1982) Breakdown Data
X_data <- c(0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90,
           3.67, 3.99, 5.35, 13.77, 25.50)
Y_data <- c(0.09, 0.39, 0.47, 0.73, 0.74, 1.13, 1.40, 2.38)

alpha1_mle <- mle_izl(X_data)
alpha2_mle <- mle_izl(Y_data)

```

```
R_mle <- R_closed(alpha1_mle, alpha2_mle)

cat("Group X MLE alpha1 :", round(alpha1_mle, 4), "\n")
cat("Group Y MLE alpha2 :", round(alpha2_mle, 4), "\n")
cat("Reliability MLE R   :", round(R_mle, 4), "\n")
```