

Statistical Inference for the Induced Moment Exponential Distribution: Bayesian and Non-Bayesian Approaches with Applications to Physics and Reliability Data

Mohammed Elgarhy^{1,2}, Doaa Basalamah^{3,*}, Ahmed Mohamed El Gazar⁴, Dina A. Ramadan⁵, Ghareeb A. Marei¹ and Said G. Nassr⁶

¹Faculty of Computers and Information Systems, Egyptian Chinese University, Nasr City, Egypt

²Department of Computer Engineering, Biruni University, 34010, Istanbul, Turkey

³Mathematics Department, Faculty of Sciences, Umm Al-Qura University, Makkah, Saudi Arabia

⁴Department of Basic Sciences, Higher Institute for Commercial Sciences, Almahlla Alkubra, Egypt

⁵Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 33516, Egypt

⁶Department of Statistics and Insurance, Faculty of Commerce, Arish University, El-Arish 45511, Egypt

Received: 2 Apr. 2026, Revised: 18 May 2026, Accepted: 21 Jun. 2026

Published online: 1 Jul. 2026

Abstract: The induced moment exponential distribution is introduced with a flexible probability density function that can decrease, while its hazard rate exhibits increasing patterns. These properties make it suitable for survival analysis, reliability, and uncertainty modeling. Several distributional characteristics are derived, including moments, moment generating function, probability-weighted moments, and measures of skewness and kurtosis. Parameter estimation is addressed using maximum likelihood, maximum product of spacings, and Bayesian methods. To evaluate the efficacy of these estimating approaches across varying sample sizes, Monte Carlo simulations are performed. The efficacy of each estimator is evaluated through comparisons of average relative bias and mean squared error, highlighting their suitability for the samples used. Three applications utilizing real-world datasets related to physics and reliability data demonstrate the significance of the new model in relation to several established competitive models. This empirical investigation further supports the utility and adaptability of the induced moment exponential model in capturing the intricacies of distinct datasets, hence delivering significant insights for practitioners in numerous domains.

Keywords: Moment exponential distribution; moments; quantile function; classical and Bayesian estimation; real datasets

1 Introduction

One of the primary challenges in contemporary research is modeling real-world data. This requires the application of appropriate statistical distributions that accurately represent the data's underlying structure. These distributions allow researchers to estimate key demographic features and extract important functions from the sampled data. In addition to its distribution, the data's failure rate or hazard rate function (HRF) behavior is sometimes referred to as its shape. Numerous statistical distributions, each with its own characteristics and applications, have been developed. Among these, the exponential family of distributions is the most widely used one-parameter family for modeling lifetime data. Furthermore, the first lifespan model for which inference techniques were widely used was the one-parameter exponential distribution (ED), established in the reliability studies and life-testing literature; refer to Barlow and Proschan [1]. According to Marshall and Olkin [2], extended exponential distributions that are obtained by adding one or more parameters are the most widely utilized families of life distributions. The ED's memory-less nature and steady failure rate are two factors contributing to its significance. Since it is rare to see any system or object with a time-independent hazard rate, the applicability of ED is restricted to a constant hazard rate. As a result, numerous ED generalizations that are inappropriate for the real-world issue have been put forth in previous studies.

* Corresponding author e-mail: dabasalamah@uqu.edu.sa

By giving well-known exponential distributions linear weights, Dara and Ahmad [3] created the moment exponential distribution. The length-biased distribution is another name for the moment exponential distribution. Thus, the length biased distribution is a particular instance of the weighted distribution, which is a more generic form. There are several uses for the length biased distribution notion in the biomedical field, including disease survival and family history. In addition to certain fundamental theorems for weighted distributions and length bias as particular cases, Patil and Ord [4] provided a list of the most popular weight function types that are helpful in scientific and statistical literature. The length-biased form of a mixture of discrete distributions is a mixture of the length-biased versions of these distributions, according to the researchers' findings. Patil and Rao [5] provide a table for a few fundamental distributions, including the lognormal, gamma, Pareto, and beta distributions, along with their length-biased versions. A novel class of weighted exponential distribution was examined by Gupta and Kundu [6]. This distribution has uses in a variety of domains, including species abundance research, social and behavioral sciences, and ecology. By establishing a connection between the initial random variable X and its length-biased variant Y , Khatree [7] provided a helpful conclusion. Additionally, Elgazar et al. [8] introduced the truncated version of the moment exponential distribution.

A random variable X follows a moment exponential distribution, if its probability density function (PDF), cumulative distribution function (CDF), and expectation are expressed as follows:

$$f(x; \theta) = \frac{x}{\theta^2} e^{-\frac{x}{\theta}}, x > 0, \theta > 0, \quad (1)$$

$$F(x; \theta) = 1 - \left(1 + \frac{x}{\theta}\right) e^{-\frac{x}{\theta}}, x > 0, \theta > 0, \quad (2)$$

and

$$E(X) = 2\theta. \quad (3)$$

This distribution combines mathematical simplicity with the flexibility to model lifetime data characterized by an increasing hazard rate. Consequently, it serves as a practical alternative to classical models such as the exponential, gamma, and Weibull distributions in applications involving reliability analysis and biological or clinical studies. Moreover, several of its statistical properties, including the moments, survival function, and moment generating function can be obtained in closed forms, which facilitates efficient parameter estimation and statistical inference, see [9, 10, 11].

The idea of weighted distributions is one way to create new distributions. The analysis of weighted distributions is useful in distribution theory because it helps to better understand normal probability distributions and provides methods for extending them to represent lifetime data by adding a parameter, which adds flexibility to their properties. When clustered sampling, heterogeneity, and extraneous variation within the dataset are modeled, weighted distributions are produced. In order to combat determination biases, the concept of weighted distributions was first put up by [12] and then codified by [4] in a thorough theory that addresses problems in which observations are made in a non-experimental, non-replicated, and non-random manner. If a researcher uses a certain stochastic model to record observations in nature, the distribution of those observations will differ from the original distribution unless every observation has an equal chance of being recorded.

The Induced Moment Exponential Distribution (IMED) is constructed based on the weighted distribution framework. Generally, if X is a random variable with probability density function (PDF), the PDF of the weighted distribution, denoted by $f_w(x)$, is defined as:

$$f_w(x) = \frac{w(x)f(x)}{\int_{-\infty}^{\infty} w(x)f(x)dx}.$$

where $w(x) \geq 0$ is a non-negative weight function. For the IMED, we employ the moment inducing mechanism by setting the weight function as $w(x) = x^r$. In the specific case where $r = 1$, this simplifies to the length-biased (or size-biased) exponential distribution. By incorporating the r th moment, the resulting IMED provides additional flexibility in the shape of the density and hazard functions compared to the baseline exponential distribution.

Weight functions are situation-specific and have been proposed by a number of scholars. For example, a few properties of bivariate weighted distributions were examined in [13]. [14] introduced the length-biased weighted new Quasi Lindley distribution. The weighted Nwike distribution was suggested by [15]. [16] proposed the weighted Zeghdoudi distribution. The Suja distribution ought to be weighted, per [17]. A review of weighted distributions was given by [18]. [19] introduced the weighted generalized Quasi Lindley distribution. The induced distribution is a variation of the weighted distribution that Singh and Das [20] study. It is distinguished by the following PDF:

$$f_I(x) = \frac{1 - F(x)}{E(X)}, \quad x > 0, \quad (4)$$

where $E(X)$ is the expectation of the underlying random variable X and $F(x)$ is the cumulative distribution function (or CDF). Using the idea of an induced distribution, Singh and Das [20] suggested the induced Garima distribution as a

variation of the Garima distribution. A variation of the induced model of the Ailamujia distribution was recently introduced by Abdelwahab et al. [21].

To demonstrate the practical relevance of the proposed model, two real datasets are analyzed. Analyzing these datasets not only highlights the flexibility of the proposed distribution in modeling skewed and heavy-tailed data but also underscores its potential applicability in material science, reliability engineering, and survival analysis.

The limitations of the classical exponential distribution, particularly its constant hazard rate, have driven the development of more flexible models such as weighted and length-biased distributions. Among these, the induced moment exponential distribution (IMED) provides greater adaptability, with the ability to capture decreasing, skewed, and heavy-tailed behaviors, as well as diverse hazard rate shapes. These features make the IMED especially relevant for survival analysis, biomedical studies, reliability, and uncertainty modeling. However, its statistical properties and estimation procedures remain underexplored, motivating a comprehensive study to establish its theoretical foundations, assess estimation techniques, and demonstrate its applicability to real-world data.

This paper is organized into seven interconnected sections. Section 2 introduces the formulation of the proposed distribution. Section 3 explores its fundamental statistical properties, supported where appropriate by graphical illustrations and numerical tables. Section 4 presents three estimation approaches, namely maximum likelihood (ML), maximum product of spacings (MPS), and Bayesian estimation (BE). Section 5 reports the findings of a comprehensive simulation study. Section 6 demonstrates the practical utility and flexibility of the distribution through the analysis of two real datasets. Finally, Section 7 concludes the paper with a summary of the main results and possible directions for future research.

2 Induced Moment Exponential Distribution

In this part, we propose a novel enhancement of the moment exponential distribution by taking into account the idea of induced distribution, which is defined in Eq. (4). The PDF and CDF of the IMED may be found by replacing Eqs. (2) and (3) in (4) as follows:

$$f(x; \theta) = \frac{1}{2\theta^2}(\theta + x)e^{-\frac{x}{\theta}}, \quad x > 0, \theta > 0, \quad (5)$$

and

$$F(x; \theta) = 1 - \left(1 + \frac{x}{2\theta}\right)e^{-\frac{x}{\theta}}, \quad x > 0, \theta > 0. \quad (6)$$

Figure 1 shows the forms of the PDF and CDF for the IMED. The possible formats of the IMED's PDF for specific parameter values of θ are shown in this Figure. The PDF's plot makes it evident that it displays a declining, right-skewed, and reversed J-shaped pattern. Additionally, the IMED's CDF plot shows a rising function for every parameter value that was selected. Additionally, Figure 2 provides the 3D plots of the PDF and CDF of the IMED.

The survival function (SF) and hazard rate function (HRF), as determined by PDF (5) and CDF (6), are provided by:

$$S(x; \theta) = 1 - F(x; \theta) = \left(1 + \frac{x}{2\theta}\right)e^{-\frac{x}{\theta}},$$

and

$$h(x; \theta) = \frac{f(x; \theta)}{s(x; \theta)} = \frac{\theta + x}{\theta(2\theta + x)}.$$

The behavior of the SF and HRF is illustrated in Figure 3 using a variety of values for θ . The IMED's HRF exhibit slightly increasing form, while the shape of the SF function exhibit decreasing pattern. To support these plots, the 3D plots of the SF and HRF are shown in Figure 4.

In this subsection, we derive the Cumulative Hazard Rate Function (CHRF) and the Reverse Hazard Rate Function (RHRF) for the IMED

$$H(x; \theta) = -\ln[S(x; \theta)] = -\ln\left(\left(1 + \frac{x}{2\theta}\right)e^{-\frac{x}{\theta}}\right)$$

and

$$\tau(x; \theta) = \frac{f(x; \theta)}{F(x; \theta)} = \frac{(\theta + x)e^{-\frac{x}{\theta}}}{2\theta^2 \left[1 - \left(1 + \frac{x}{2\theta}\right)e^{-\frac{x}{\theta}}\right]}.$$

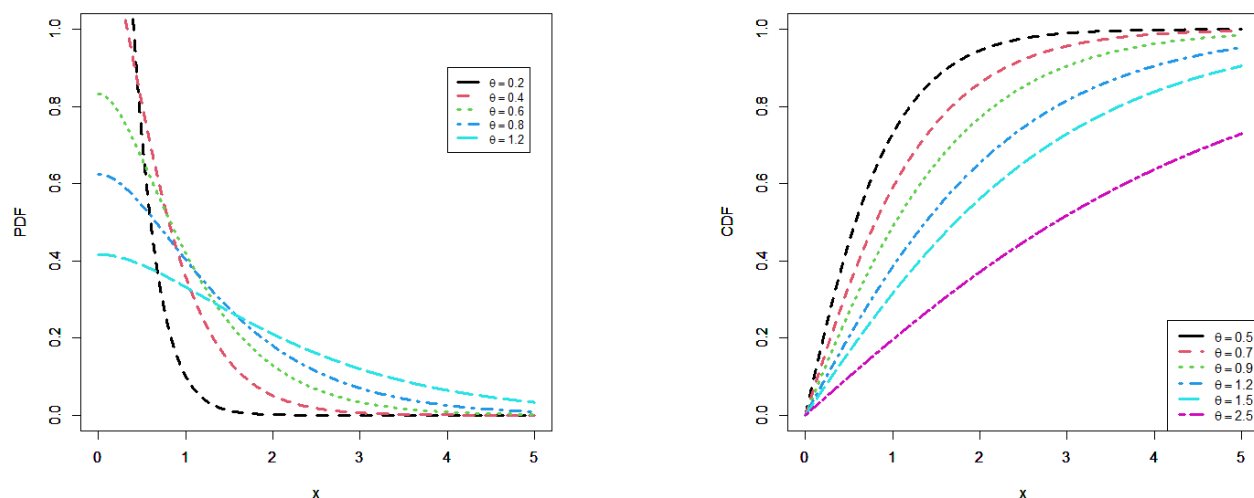


Fig. 1: The plots of PDF and CDF for the IMED at different values of the parameter θ .

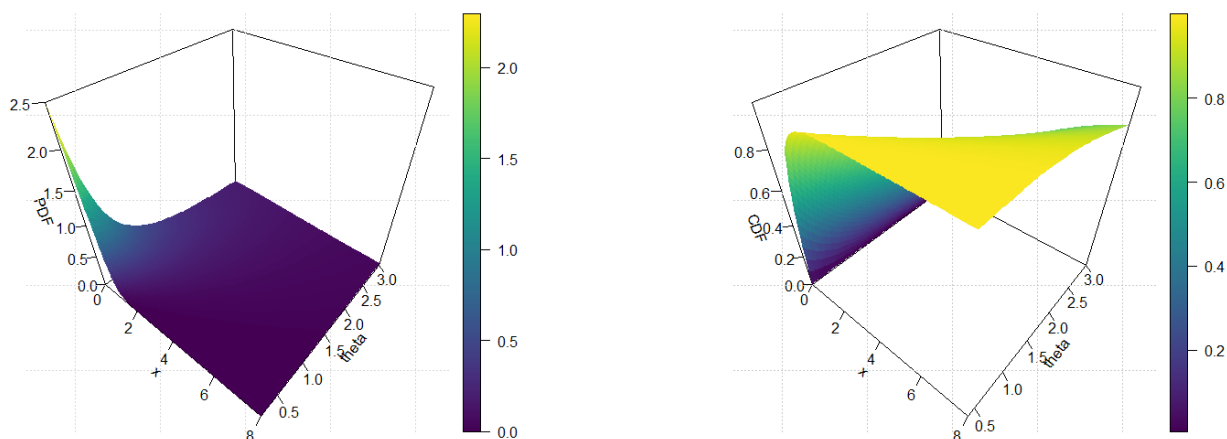


Fig. 2: 3D plots of the PDF and CDF for the IMED at different values of the parameter θ .

3 Basic Properties

Some significant statistical characteristics of the IMED are derived in this section.

3.1 Moments and Related Measures

Distributional moments of the IMED, a crucial part of any statistical investigation, are obtained in this subsection. By using moments to determine skewness and kurtosis coefficients, one can examine some of the most significant features and characteristics of a distribution. The function that generates moments is derived. The following is the result of $\hat{\mu}_r$ if X possesses the PDF in Eq. (5):

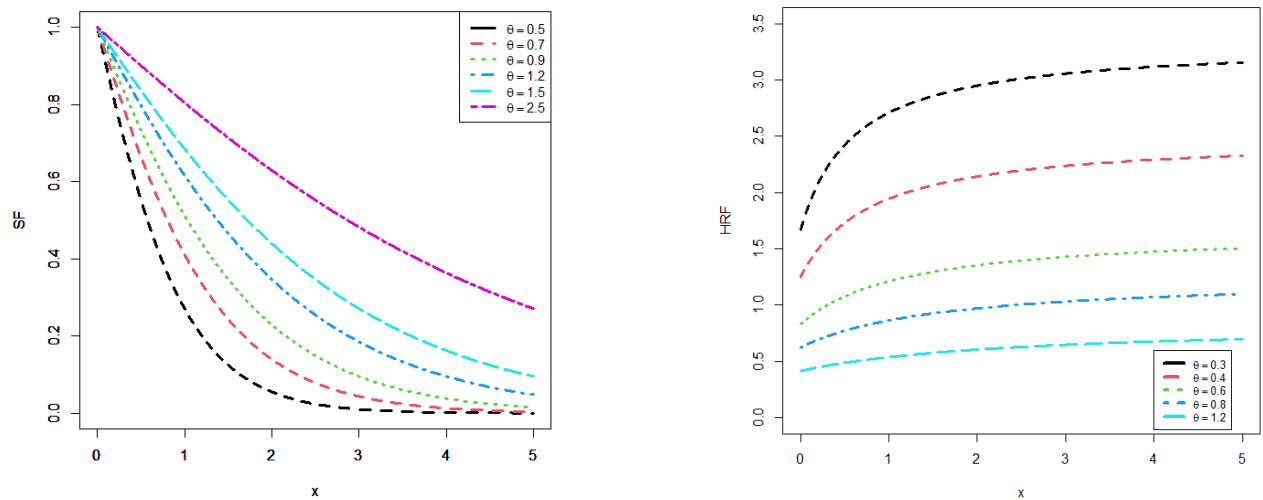


Fig. 3: The plots of SF and HRF for the IMED at different values of the parameter θ .

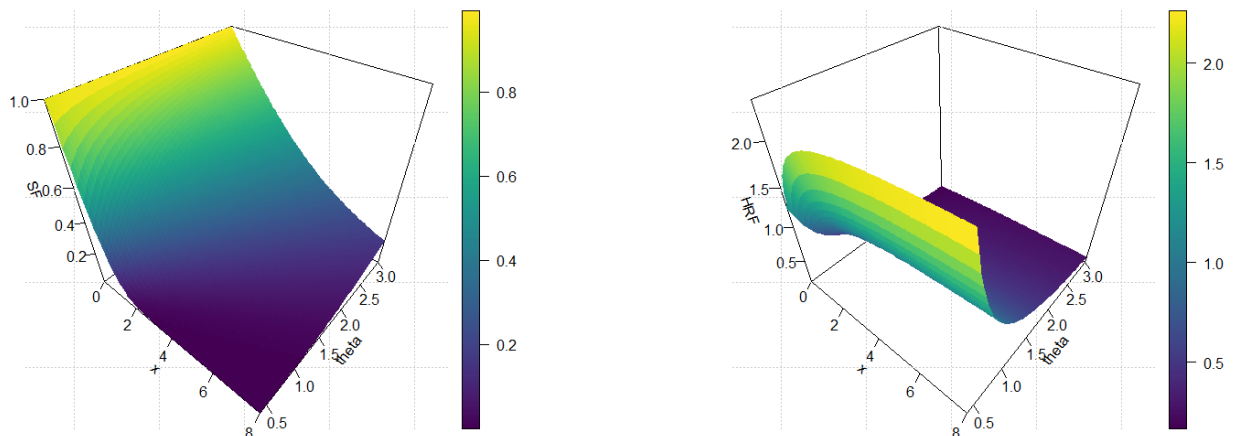


Fig. 4: 3D plots of the SF and HRF for the IMED.

$$\dot{\mu}_r = \int_0^{\infty} x^r f(x; \theta) dx = \frac{1}{2\theta^2} \int_0^{\infty} x^r (\theta + x) e^{-\frac{x}{\theta}} dx = \frac{1}{2\theta^2} \left[\theta \int_0^{\infty} x^r e^{-\frac{x}{\theta}} dx + \int_0^{\infty} x^{r+1} e^{-\frac{x}{\theta}} dx \right].$$

To evaluate the integral, let $u = \frac{x}{\theta}$, which implies $x = \theta u$ and $dx = \theta du$. As $x \rightarrow 0, u \rightarrow 0$ and as $x \rightarrow \infty, u \rightarrow \infty$. The integral becomes:

$$\mu'_r = \frac{\theta^r}{2} \left[\int_0^{\infty} u^r e^{-u} du + \int_0^{\infty} u^{r+1} e^{-u} du \right].$$

Using the definition of the gamma function $\gamma(r) = \int_0^{\infty} u^{r-1} e^{-u} du$. Thus, the $\dot{\mu}_r$ is given by:

$$\mu'_r = \frac{\theta^r}{2} [\gamma(r+1) + \gamma(r+2)]. \quad (7)$$

Also, by entering $r = 1, 2, 3, 4$, into Eq. (7), the first four moments of the IMED may be obtained. The following can be used to determine the variance and the first four moments' respective relations:

$$\dot{\mu}_1 = \frac{\theta}{2} [\gamma(2) + \gamma(3)] = \frac{3\theta}{2},$$

$$\dot{\mu}_2 = \frac{\theta^2}{2} [\gamma(3) + \gamma(4)] = 4\theta^2,$$

$$\dot{\mu}_3 = \frac{\theta^3}{2} [\gamma(4) + \gamma(5)] = 15\theta^3,$$

$$\dot{\mu}_4 = \frac{\theta^4}{2} [\gamma(5) + \gamma(6)] = 72\theta^4,$$

and

$$var = 4\theta^2 - \frac{9\theta^2}{4} = \frac{7}{4}\theta^2.$$

The initial four moments, variance (σ^2), standard deviation (SD), and index of dispersion (ID) are numerically represented for various values of the parameter θ in Table 1. All moment measures, variance, standard deviation, and index of dispersion rise, as the parameter increases. Additionally, we can observe that the IMED is right skewed, and that the distribution is greatly leptokurtic.

Table 1: Moments and some measures of the IMED.

θ	μ_1	μ_2	μ_3	μ_4	σ^2	SD	ID
0.5	0.75	1	1.875	4.5	0.4375	0.6614	0.5833
1.5	2.25	9	50.625	364.5	3.9375	1.9843	1.75
2.5	3.75	25	234.375	2812.5	10.9375	3.3072	2.9167
3.5	5.25	49	643.125	10804.5	21.4375	4.6301	4.0833
4.5	6.75	81	1366.875	29524.5	35.4375	5.9529	5.25
5.5	8.25	121	2495.625	65884.5	52.9375	7.2758	6.4167
6.5	9.75	169	4119.375	128524.5	73.9375	8.5987	7.5833
7.5	11.25	225	6328.125	227812.5	98.4375	9.9216	8.75
8.5	12.75	289	9211.875	375844.5	126.4375	11.2444	9.9167
9.5	14.25	361	12860.625	586444.5	157.9375	12.5673	11.083

3.2 Moment generating function

The moment-generating function (MGF) is a key tool in probability theory, as it provides moments, facilitates the study of sums of random variables, and uniquely determines the distribution. The MGF of the IMED, based on the PDF (5) in Eq. (8), is given by:

$$M_x(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x; \theta) dx = \frac{1}{2\theta^2} \int_0^\infty (\theta + x) e^{-(\frac{1}{\theta} - t)x} dx. \quad (8)$$

Then, the MGF of the IMED is given as:

$$\begin{aligned}
 M_X(t) &= \frac{1}{2\theta^2} \left[\theta \int_0^\infty e^{-(\frac{1}{\theta}-t)x} dx + \int_0^\infty x e^{-(\frac{1}{\theta}-t)x} dx \right] = \frac{1}{2\theta^2} \left[\frac{\theta}{\frac{1}{\theta}-t} + \frac{1}{(\frac{1}{\theta}-t)^2} \right] \\
 &= \frac{1}{2} \left[\frac{1}{1-\theta t} + \frac{1}{(1-\theta t)^2} \right] = \frac{2-\theta t}{2(1-\theta t)^2}, \quad t < \frac{1}{\theta}.
 \end{aligned} \tag{9}$$

3.3 Quantile function

A frequently used statistical method for identifying key percentiles and assessing a distribution's statistical characteristics is the quantile function (QF). The inverse of the distribution's CDF yields the quantile function. Similar to the CDF, the QF offers a thorough analysis of the distribution's statistical properties. Therefore, understanding every facet of it is essential. The quantile function of X is defined as $Q_X(p) = F_X^{-1}(p)$, where $p \in [0, 1]$, as it is important to remember that the CDF described in Eq. (6) is continuous and strictly rising. The QF is determined as follows:

$$\left(1 + \frac{x}{2\theta}\right) e^{-\frac{x}{\theta}} = 1 - p,$$

By substituting $z = \frac{x}{\theta}$, then $x = z\theta$, hence we can get:

$$(-2 - z)e^{-2-z} = -2(1 - p)e^{-2},$$

Hence, the solution of z is given by:

$$z = -[W_{-1}[-2(1 - p)e^{-2}] + 2],$$

where, $W(\cdot)$ is the Lambert function. Then the quantile function of the IMED is given as follows:

$$x = -\theta [W_{-1}[-2(1 - p)e^{-2}] + 2].$$

Specifically, by substituting $p = 0.25, 0.5$, and 0.75 , we obtain the first, second (median), and third quantiles. Furthermore, predicated on the quantiles, Bowley's skewness (ω_5), and Moor's kurtosis (ω_6), are provided respectively by the following relations:

$$\omega_5 = \frac{Q(0.75) - 2Q(0.5) + Q(0.25)}{Q(0.75) - Q(0.25)},$$

and

$$\omega_6 = \frac{Q(0.875) - Q(0.625) - Q(0.375) + Q(0.125)}{Q(0.75) - Q(0.25)}.$$

3.4 Order statistics

There are several uses for order statistics in reliability analysis and statistical estimation. In the context of the IMED, we present the fundamentals of order statistics. Using the PDF and CDF given in Eqs. (5) and (6), respectively. Let $X_1, X_2, \dots, X_n$ be independent and identically distributed random variables taken from the IME distribution. Next, the following formula is used to calculate the density of the j^{th} order statistic:

$$f_{X_{(j)}} = \frac{n!}{(j-1)!(n-j)!} f_X(x) (F_X(x))^{j-1} (1 - F_X(x))^{n-j},$$

for $j=1, \dots, n$. The PDF of the j^{th} order statistic for the IMED is determined as follows:

$$f_{X_{(j)}} = \frac{n!}{(j-1)!(n-j)!} \times \left[\frac{1}{2\theta^2} (\theta + x) e^{-\frac{x}{\theta}} \right] \left[1 - \left(1 + \frac{x}{2\theta}\right) e^{-\frac{x}{\theta}} \right]^{j-1} \times \left[\left(1 + \frac{x}{2\theta}\right) e^{-\frac{x}{\theta}} \right]^{n-j},$$

thus, the PDF of the largest order statistic $X_{(n)}$ is expressed as follows:

$$f_{X_{(n)}} = n \left[\frac{1}{2\theta^2} (\theta + x) e^{-\frac{x}{\theta}} \right] \left[1 - \left(1 + \frac{x}{2\theta}\right) e^{-\frac{x}{\theta}} \right]^{n-1},$$

and, the PDF of the smallest order statistic $X_{(1)}$ is determined as follows:

$$f_{X_{(1)}} = \left[\frac{1}{2\theta^2} (\theta + x) e^{-\frac{x}{\theta}} \right] \left[\left(1 + \frac{x}{2\theta}\right) e^{-\frac{x}{\theta}} \right]^{n-1}.$$

4 Classical Estimation

Currently under investigation are various non-Bayesian estimate techniques for the parameter θ .

4.1 Notation and Preliminaries

In this section, we present various estimation techniques for the unknown parameter θ of the IMED. For the sake of clarity and consistency throughout the manuscript, the following notations are adopted: Let $X_1, X_2, \dots, X_n$ be a random sample of size n drawn from the IMED. We denote the Maximum Likelihood Estimator (MLE) as $\hat{\theta}_{ML}$, the Maximum Product of Spacings Estimator (MPS) as $\hat{\theta}_{MPS}$, and the Bayesian Estimator (BE) as $\hat{\theta}_B$. Additionally, the log-likelihood function is represented by $\ell(\theta)$, and the observed Fisher information is denoted by $I(\theta)$. All statistical properties and simulation results in subsequent tables and figures will adhere to these specified notations (see [22, 23, 24, 25, 26, 27, 28]).

4.2 Maximum likelihood estimation

Let $x_1, x_2, \dots, x_n$ represent a random sample from a random variable w that adheres to the IMED. The parameter θ is estimated using the maximum likelihood (MLE) approach by maximizing the next equation:

$$L(\theta|x) = \prod_{i=1}^n \left(\frac{1}{2\theta^2} (\theta + x_i) e^{-\frac{x_i}{\theta}} \right). \quad (10)$$

The log-likelihood function is given by

$$\ell = \log(L) = -n \log(2\theta^2) + \sum_{i=1}^n \log(\theta + x_i) - \sum_{i=1}^n \frac{x_i}{\theta}.$$

The MLEs of the parameter θ , are derived by maximizing the function ℓ . We can verify this by calculating the first derivatives of the function ℓ and equating the outcomes to zero, which will provide the estimates of the IMED as below:

$$\frac{\partial \ell}{\partial \theta} = \frac{-n}{\theta} + \sum_{i=1}^n \frac{1}{(\theta + x_i)} + \sum_{i=1}^n \frac{x_i}{\theta^2}. \quad (11)$$

By setting the Eq. (11) to zero, we can obtain the estimates of the parameter θ . To determine the ML estimates of the parameter, we utilize computational tools by solving Eq. (11) where its derivative is set to zero. Since it is not feasible to derive exact confidence intervals for this parameter, we rely on approximate methods. As a necessary step, we compute the second-order partial derivatives, which are essential for constructing the Fisher information matrix. This matrix is then used to approximate the confidence intervals.

$$I(\theta) = -E \left[\frac{\partial^2 \ell}{\partial \theta^2} \right],$$

where,

$$\frac{\partial^2 \ell}{\partial \theta^2} = \frac{n}{\theta^2} - \sum_{i=1}^n \frac{1}{(\theta + x_i)^2} - 2 \sum_{i=1}^n \frac{x_i}{\theta^3}.$$

The $(1-\gamma)$ 100% confidence interval for the parameter θ and can be shown as below:

$$(\hat{\theta}_L, \hat{\theta}_U) = \hat{\theta} \pm z_{1-\frac{\gamma}{2}} \sqrt{\text{var}(\hat{\theta})},$$

where, $\hat{\theta}$ is ML estimators of θ , $z_{1-\frac{\gamma}{2}}$ is the percent of the standard normal distribution, and $\text{var}(\hat{\theta})$ is the asymptotic variance of ML computed utilizing the inverse for the information matrix.

4.3 Maximum product of spacings method

Cheng and Amin [29] and Ranneby [30] separately created the MPS approach, which provides an alternative to ML estimate. The researchers looked at a number of MPS estimator properties, showing that MPS offers reliable and asymptotically efficient estimates in scenarios where ML estimators might struggle. This is especially noticeable when the probability function is unbounded, as in the case of mixed distributions and heavy-tailed distributions with unknown size and location parameters as described by [31]. Therefore, the MPS strategy preserves nearly all of the ML method's features in large datasets while mitigating some of its intrinsic constraints [32]. For more details see [33,34,35]. According to the IMED, a random sample's uniform intervals are defined as follows:

$$D_i(\theta) = F(x_{i:n}|\theta) - F(x_{i-1:n}|\theta), \quad i = 1, 2, \dots, n,$$

where, $F(x_{0:n}|\theta) = 0$ and $F(x_{n+1:n}|\theta) = 1$. Obviously $\sum_{i=1}^{n+1} D_i(\theta) = 1$. Consequently, the following are the MPS estimators for the unknown parameter:

$$G(\theta) = \left[\prod_{i=1}^{n+1} D_i(\theta) \right]^{\frac{1}{n+1}}.$$

In contrast, by improving the function

$$H(\theta) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log D_i(\theta).$$

The MPS estimators for the parameters θ are obtained by employing numerical methods to solve the non-linear equation $\frac{\partial H(\theta)}{\partial \theta} = 0$, necessitating a numerical solution. The established asymptotic equivalence between the ML and the MPS approach, as indicated by [29], is the basis for the asymptotic confidence intervals for the parameters and that are obtained from the MPS estimation. Thus, the parameter using MPS have a $(1-\gamma)$ 100% confidence interval developed as follows:

$$\hat{\theta}_{MPS} \pm z_{1-\frac{\gamma}{2}} \sqrt{\text{var}(\hat{\theta}_{MPS})},$$

where, $\hat{\theta}_{MPS}$ is the MPS estimators of θ , $z_{1-\frac{\gamma}{2}}$ is the percent of the standard normal distribution, and $\text{var}(\hat{\theta}_{MPS})$ is the asymptotic variance, computed utilizing the inverse for the information matrix.

4.4 Bayesian estimation

This sub section covers the difficulty of developing Bayes estimators (BE) for the parameter θ of the IMED. In Bayesian inference, it is assumed that the unknown parameters are random variables with a joint prior PDF. The prior PDF may be computed utilizing antecedent information and experience. In the absence of prior knowledge, non-informative priors may be employed for Bayesian inference. We assume that θ is independent gamma random variable with prior PDF as follows:

$$\pi(\theta) \propto \theta^{a-1} \exp(-b\theta), \quad \theta > 0, a > 0, b > 0,$$

where, a and b are designated hyper parameters intended to represent the prior distribution of the unknown parameter. Therefore, we suggest that you use independent gamma priors for the parameter θ of the IMED, as the gamma distribution is extremely flexible. The posterior density function of the parameter θ is derived by multiplying the likelihood function by the prior according to Bayes' theorem, and it is described as follows:

$$\pi^*(\theta|x) \propto \frac{\pi(\theta)L(\theta)}{\int_0^\infty \pi(\theta)L(\theta)d\theta}. \quad (12)$$

Hence, it can be expressed as follows:

$$\pi^*(\theta|x) = \theta^{a-1} \exp(-b\theta) \prod_{i=1}^m \frac{1}{2\theta^2} (\theta + x_i) e^{-\frac{x_i}{\theta}}.$$

To enhance the practicality and significance of Bayesian statistical inference, it is essential to account for both symmetric and asymmetric loss functions. According to [36], a loss function is a real-valued function that encompasses all feasible parameters and estimates. Employing an asymmetric loss function augments the flexibility and applicability

of Bayesian inference in scenarios where the consequences of overestimation and underestimating diverge. These functions guarantee that the inference process stays resilient and pertinent across diverse parameter estimators. Examine the squared error loss (SEL) function, defined as:

$$L(\theta - \hat{\theta}) = (\theta - \hat{\theta})^2,$$

This loss function is symmetric, meaning that it equally penalizes both overestimation and underestimation. For any function of θ , the Bayesian estimation $g(\theta)$ under the SEL function is expressed as:

$$\hat{g}_{BS}(\theta|x) = E_{\theta|x}(g(\theta))$$

and

$$E_{\theta}(g(\theta)) = \frac{\int_0^{\infty} g(\theta)\pi(\theta)L(\theta|x)d\theta}{\int_0^{\infty} \pi(\theta)L(\theta|x)d\theta}. \quad (13)$$

The ratio of various integrals in Eq. (13) remains ambiguous. The MCMC approach is utilized to generate samples from the joint posterior density function as specified in Eq. (12). The MCMC methodology is specifically executed by the Gibbs sampling method within the Metropolis-Hastings (M-H) framework. The Bayesian estimation of the parameter θ for the IMED is analytically unmanageable due to the model's complexity. Obtaining closed-form solutions is challenging under the parameters of the IMED. To address this difficulty, we recommend use the MCMC method, a robust computational technique for estimating posterior distributions in the absence of analytical solutions; refer to Eq. (13). The MCMC approach produces a series of samples that estimate the posterior distribution of model parameters by simulating a Markov chain that converges to the desired distribution. These samples can be utilized to estimate posterior statistics, such as the mean, which functions as Bayesian estimates for θ . The histograms of the posterior samples were analyzed to evaluate convergence and the characteristics of the posterior distributions, as illustrated in Figure 5.

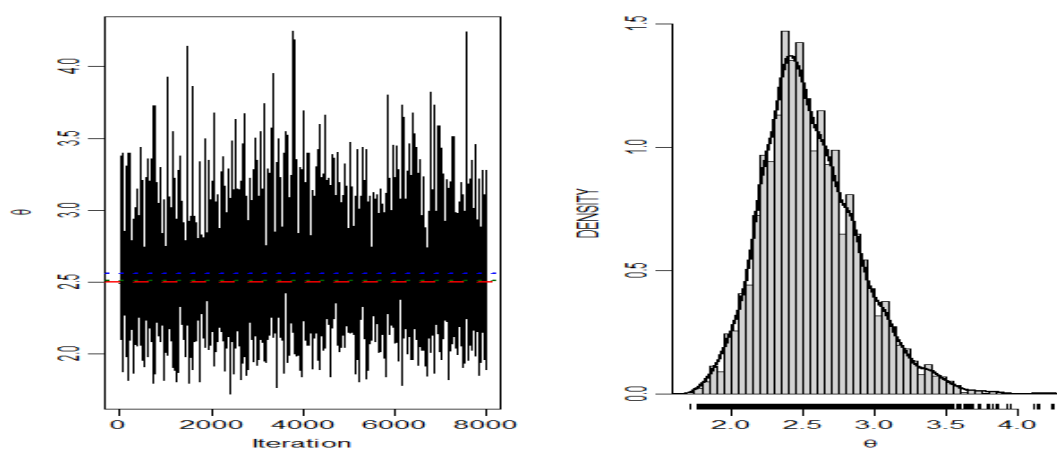


Fig. 5: Plots of trace and histogram for $\theta = 2.5$ and $n = 50$.

5 Simulation Study

The effectiveness of the estimation techniques described in Section 4 is evaluated in this section. The IME model was used to create simulated datasets, and the estimating techniques discussed were used to determine the unknown values. Over the point estimation of the parameter θ , two different metrics were used to evaluate the correlation performance: relative bias (RBIAS), and mean square errors (MSEs). Also, average widths (AWs) and coverage probabilities (CPs) are used to gauge how well 95% confidence interval's function The following describes how to use the inversion approach to gather random samples from the IMED:

The IMED yields random samples $X_1, X_2, \dots, X_n$ with sizes $n = 50, 100, 150$, and 200 . We look at the values of the parameter $\theta = 1.5, 2, 2.5, 3, 3.5$ and 4 . These values and sample sizes variables are used to evaluate the IME model's

estimates. The parameter values were chosen to cover a range of behaviors observed in real-world applications, such as reliability and survival studies, based on values commonly used in similar research. Sample sizes ranging from 50 to 200 were selected to evaluate the impact of sample size on estimation accuracy, with larger samples providing more reliable results and smaller samples reflecting typical practical scenarios. In each simulation iteration, we produce 500 bootstrap samples and gather 8000 observations for the Markov chain Monte Carlo (MCMC) process. The initial 1000 observations were utilized as burn-in to mitigate the effects of the initial distribution. Furthermore, throughout the thinning procedure, every fifth observation is chosen to diminish reliance and improve the robustness of the study. For different parameter values, the estimations' relative biases and MSEs are computed. Tables 2 to 7 display the outcomes of the simulation analysis. All simulation results were obtained using R language scripts. The following is a summary of the accuracy and basic results from these tables:

- The minimal values of RBIAS and MSEs which are tracked for its parameter, show that the IMED is stable.
- As the sample size increases, the MSEs for all estimation methods decreases.
- With higher sample sizes, the RBIAS typically decreases, indicating that estimators become less biased.
- MPS estimates demonstrate comparatively low MSEs and stable CPs, establishing it as a dependable option across various sample sizes.
- MLE exhibits competitive MSEs and RBIAS for the MPSE method.
- Bayesian estimation yields favorable MSEs and elevated CPs, frequently surpassing alternative methods in sustaining high coverage.
- The AWs of the confidence intervals diminishes as the sample size escalates across all estimation methods, signifying enhanced precision in estimations with larger datasets.

The CPs approximate the nominal level, indicating that the confidence intervals generated by these approaches effectively encompass the genuine parameter values.

Table 2: Point and interval estimation results for $\theta = 1.5$.

n	Methods	Point Estimation			Interval Estimation			
		Mean	MSEs	RBIAS	Lower	Upper	AWs	CPs
50	MLE	1.5065	0.0318	0.0065	1.1814	1.9211	0.7397	0.956
	BE	1.5363	0.0340	0.0363	1.1788	1.9226	0.7438	0.956
	MPS	1.5604	0.0378	0.0604	1.2462	2.0272	0.7810	0.934
100	MLE	1.4919	0.0172	0.0081	1.2561	1.7719	0.5158	0.960
	BE	1.5067	0.0174	0.0067	1.2564	1.7711	0.5147	0.960
	MPS	1.5234	0.0184	0.0234	1.3001	1.8304	0.5303	0.938
150	MLE	1.4995	0.0109	0.0005	1.3031	1.7256	0.4225	0.960
	BE	1.5091	0.0111	0.0091	1.3039	1.7228	0.4189	0.964
	MPS	1.5223	0.0117	0.0223	1.3359	1.7670	0.4311	0.938
200	MLE	1.5052	0.0081	0.0052	1.3329	1.6999	0.3671	0.966
	BE	1.5128	0.0083	0.0128	1.3347	1.6984	0.3638	0.964
	MPS	1.5232	0.0088	0.0232	1.3600	1.7314	0.3714	0.942

To provide a concise overview of the comprehensive simulation results presented in the preceding tables, Table 8 summarizes the performance trends of the Maximum Likelihood (ML), Maximum Product of Spacings (MPS), and Bayesian (BE) estimation methods. The comparison is based on the Mean Squared Error (MSE) and Relative Bias (RBIAS) across various sample sizes (n).

Overall, the simulation study reveals that the MSE for all estimators decreases as the sample size n increases, confirming the consistency property. Specifically, the Bayesian estimator (BE) exhibits the smallest MSE in most cases, particularly for small samples, whereas the MPS estimator shows a slight advantage over the MLE in terms of relative bias.

6 Applications on Real Datasets

This section illustrates the significance of the IMED through two applications utilizing real datasets. The model's effectiveness, demonstrated through data plots, is likely more compelling, as these graphical representations highlight the

Table 3: Point and interval estimation results for $\theta = 2$.

n	Methods	Point Estimation			Interval Estimation			
		Mean	MSEs	RBIAS	Lower	Upper	AWs	CPs
50	MLE	2.0087	0.0566	0.0087	1.5752	2.5614	0.9863	0.956
	BE	2.0451	0.0598	0.0451	1.5705	2.5574	0.9870	0.958
	MPS	2.0805	0.0672	0.0805	1.6616	2.7029	1.0413	0.934
100	MLE	1.9892	0.0305	0.0108	1.6748	2.3626	0.6878	0.960
	BE	2.0074	0.0308	0.0074	1.6735	2.3586	0.6851	0.956
	MPS	2.0313	0.0327	0.0313	1.7335	2.4406	0.7071	0.938
150	MLE	1.9994	0.0194	0.0006	1.7374	2.3008	0.5634	0.960
	BE	2.0112	0.0197	0.0112	1.7379	2.2965	0.5586	0.966
	MPS	2.0297	0.0209	0.0297	1.7812	2.3560	0.5748	0.938
200	MLE	2.0070	0.0143	0.0070	1.7771	2.2666	0.4894	0.966
	BE	2.0163	0.0147	0.0163	1.7786	2.2632	0.4846	0.966
	MPS	2.0309	0.0156	0.0309	1.8134	2.3086	0.4952	0.942

Table 4: Point and interval estimation results for $\theta = 2.5$.

n	Methods	Point Estimation			Interval Estimation			
		Mean	MSEs	RBIAS	Lower	Upper	AWs	CPs
50	MLE	2.5108	0.0884	0.0108	1.9690	3.2018	1.2328	0.956
	BE	2.5522	0.0924	0.0522	1.9609	3.1914	1.2305	0.956
	MPS	2.6007	0.1050	0.1007	2.0770	3.3786	1.3016	0.934
100	MLE	2.4865	0.0477	0.0135	2.0935	2.9532	0.8597	0.960
	BE	2.5073	0.0481	0.0073	2.0901	2.9454	0.8553	0.956
	MPS	2.5391	0.0512	0.0391	2.1668	3.0507	0.8839	0.938
150	MLE	2.4992	0.0303	0.0008	2.1718	2.8760	0.7042	0.960
	BE	2.5127	0.0306	0.0127	2.1714	2.8680	0.6966	0.962
	MPS	2.5371	0.0326	0.0371	2.2265	2.9450	0.7185	0.938
200	MLE	2.5087	0.0224	0.0087	2.2214	2.8332	0.6118	0.966
	BE	2.5195	0.0229	0.0195	2.2229	2.8282	0.6053	0.964
	MPS	2.5385	0.0243	0.0385	2.2667	2.8856	0.6189	0.942

Table 5: Point and interval estimation results for $\theta = 3$.

n	Methods	Point Estimation			Interval Estimation			
		Mean	MSEs	RBIAS	Lower	Upper	AWs	CPs
50	MLE	3.0130	0.1273	0.0130	2.3628	3.8422	1.4794	0.956
	BE	3.0575	0.1316	0.0575	2.3492	3.8202	1.4710	0.958
	MPS	3.1208	0.1512	0.1208	2.4923	4.0543	1.5620	0.934
100	MLE	2.9838	0.0686	0.0162	2.5122	3.5438	1.0317	0.960
	BE	3.0065	0.0689	0.0065	2.5067	3.5319	1.0252	0.962
	MPS	3.0469	0.0737	0.0469	2.6002	3.6608	1.0606	0.938
150	MLE	2.9991	0.0436	0.0009	2.6061	3.4512	0.8451	0.960
	BE	3.0137	0.0440	0.0137	2.6039	3.4391	0.8351	0.964
	MPS	3.0446	0.0470	0.0446	2.6717	3.5340	0.8622	0.938
200	MLE	3.0105	0.0322	0.0105	2.6657	3.3998	0.7341	0.966
	BE	3.0220	0.0328	0.0220	2.6650	3.3913	0.7263	0.968
	MPS	3.0463	0.0350	0.0463	2.7200	3.4627	0.7427	0.942

Table 6: Point and interval estimation results for $\theta = 3.5$.

n	Methods	Point Estimation			Interval Estimation			
		Mean	MSEs	RBIAS	Lower	Upper	AWs	CPs
50	MLE	3.5152	0.1732	0.0152	2.7565	4.4825	1.7260	0.956
	BE	3.5610	0.1772	0.0610	2.7407	4.4499	1.7091	0.960
	MPS	3.6410	0.2058	0.1410	2.9077	4.7300	1.8223	0.934
100	MLE	3.4810	0.0934	0.0190	2.9309	4.1345	1.2036	0.960
	BE	3.5047	0.0934	0.0047	2.9234	4.1173	1.1939	0.960
	MPS	3.5547	0.1003	0.0547	3.0336	4.2710	1.2374	0.938
150	MLE	3.4989	0.0593	0.0011	3.0405	4.0264	0.9859	0.960
	BE	3.5141	0.0597	0.0141	3.0378	4.0113	0.9735	0.964
	MPS	3.5520	0.0640	0.0520	3.1170	4.1229	1.0059	0.938
200	MLE	3.5122	0.0439	0.0122	3.1100	3.9665	0.8565	0.966
	BE	3.5242	0.0445	0.0242	3.1086	3.9552	0.8465	0.964
	MPS	3.5540	0.0477	0.0540	3.1733	4.0399	0.8665	0.942

Table 7: Point and interval estimation results for $\theta = 4$.

n	Methods	Point Estimation			Interval Estimation			
		Mean	MSEs	RBIAS	Lower	Upper	AWs	CPs
50	MLE	4.0173	0.2263	0.0173	3.1503	5.1229	1.9726	0.956
	BE	4.0635	0.2292	0.0635	3.1305	5.0792	1.9487	0.958
	MPS	4.1611	0.2688	0.1611	3.3231	5.4057	2.0826	0.934
100	MLE	3.9783	0.1220	0.0217	3.3496	4.7251	1.3756	0.960
	BE	4.0022	0.1217	0.0022	3.3389	4.7005	1.3616	0.960
	MPS	4.0625	0.1310	0.0625	3.4670	4.8811	1.4142	0.938
150	MLE	3.9987	0.0775	0.0013	3.4748	4.6016	1.1268	0.960
	BE	4.0140	0.0776	0.0140	3.4710	4.5822	1.1112	0.964
	MPS	4.0594	0.0836	0.0594	3.5623	4.7119	1.1496	0.938
200	MLE	4.0140	0.0573	0.0140	3.5543	4.5331	0.9788	0.966
	BE	4.0264	0.0580	0.0264	3.5530	4.5194	0.9664	0.966
	MPS	4.0617	0.0623	0.0617	3.6267	4.6170	0.9903	0.942

Table 8: Summary of estimators performance trends.

Sample Size (n)	Best Performer (Lowest MSE)	Key Observation
Small ($n=30$)	Bayesian (BE)	BE shows superior performance due to informative priors.
Moderate ($n=100$)	MPS / BE	MPS starts competing closely with BE as information increases.
Large ($n=300$)	All (MLE, MPS, BE)	All estimators converge; MSE approaches zero for all methods.

IMED enhanced ability to conform to the data relative to competing models. The practical importance is demonstrated by the examination of physics data on the [tensile strength](#) of polyester fibers, and failure times (in hours) of 50 components, which highlight the IMED superiority compared to other distributions. The proposed IMED will be relevant across various fields, including medical sciences, and reliability research. The comprehensive overview of the datasets is encapsulated in the subsequent points:

- The first data set comprised 30 measurements of the tensile strength of polyester fibers, as discussed in [37]. This dataset is used to demonstrate the application of the IMED in modeling material properties. The information is presented in this data set as follows: 0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926.

–The second dataset is leptokurtic two-dimensional and moderately right-skewed, as indicated by [38], and presents the failure times (in hours) of 50 Components (FTC). This dataset is used to illustrate the IMED's application in reliability and survival analysis. The data is summarized as follows: 0.008, 0.017, 0.058, 0.061, 0.084, 0.090, 0.134, 0.238, 0.245, 0.353, 0.374, 0.480, 0.495, 0.535, 0.564, 0.681, 0.686, 0.688, 0.921, 0.959, 1.022, 1.092, 1.260, 1.284, 1.295, 1.373, 1.395, 1.414, 1.760, 1.858, 1.892, 1.921, 1.926, 1.933, 2.135, 2.169, 2.301, 2.320, 2.405, 2.506, 2.598, 2.808, 2.971, 3.087, 3.492, 3.669, 3.926, 4.446, 5.119, 8.596.

Table 9 mainly analyzes the two datasets. Figures 6 and 7 provide graphical representations of the datasets via Quantile–Quantile (QQ) plots, violin plots, box plots, histograms, kernel density estimates, and total time on test (TTT) plots. The findings indicate that the initial dataset exhibit right skewness, with the majority of values falling below 0.5. Outliers exist, and the QQ plot verifies non-normality. The TTT plot additionally indicates a declining failure rate, in accordance with reliability data. For instance, the second dataset exhibit significant right skewness, with the majority of observations concentrated at lower levels. Outliers are evident in the box and violin plots, and the QQ plot demonstrates significant divergence from normality. The TTT plot demonstrates a failure rate that is approximately constant to marginally increasing.

Table 9: Some descriptive statistics for the two datasets.

Dataset	n	Mean	Median	Variance	Skewness	Kurtosis	Range	Min	Max
I	30	0.3659	0.336	0.0721	0.5464	2.2294	0.903	0.023	0.926
II	50	1.6729	1.334	2.5495	1.9165	8.4351	8.588	0.008	8.596

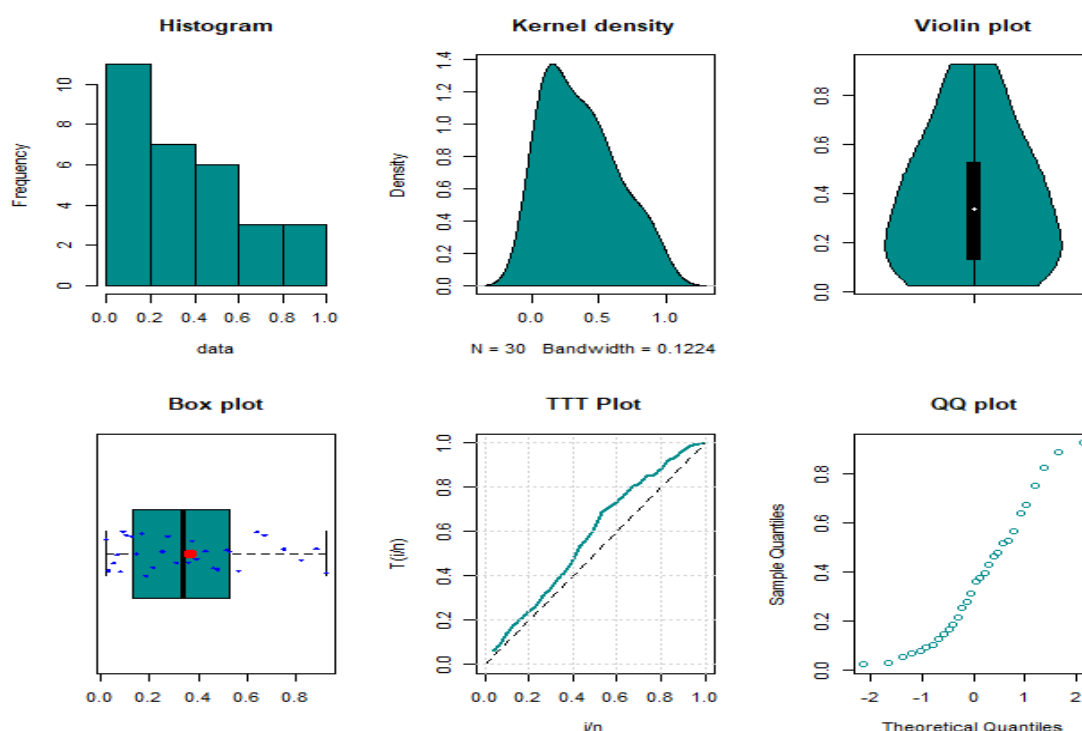
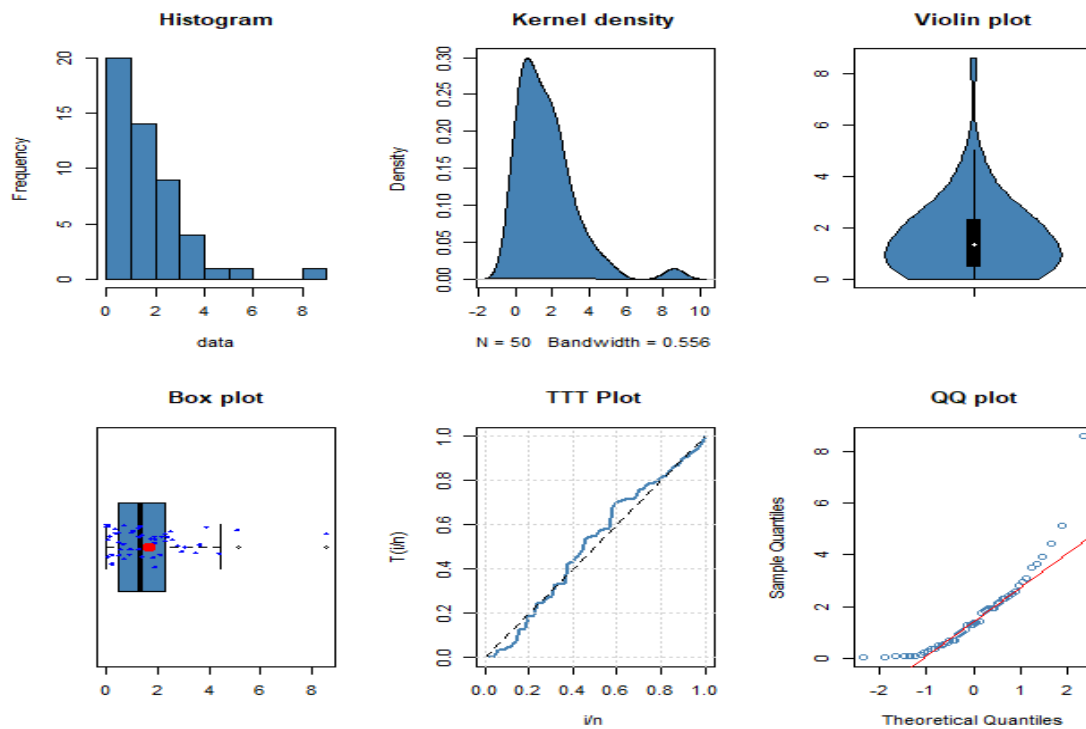


Fig. 6: Graphical representation of the first dataset.

This comparison research rigorously evaluates the IMED in relation to other competing models, including the moment exponential (ME), exponential (Ex), extended inverse exponential (EIEEx), inverse power Ailamujia (IPA), and inverse power logistic exponential (IPLE) distributions. Parameter estimation and goodness-of-fit evaluations were performed with the maximum likelihood approach in the R programming environment. Parameter estimates and their standard errors are shown in Tables 10 and 11.

Model selection was predicated on various statistical criteria, encompassing the Akaike information criterion (W_1), Bayesian information criterion (W_2), corrected Akaike information criterion (W_3), Hannan–Quinn information criterion

**Fig. 7:** Graphical representation of the second dataset.

(W_4), Kolmogorov–Smirnov distance (W_5) with corresponding P-value (W_6), and the Cramér–von Mises statistic (W_7). Tables 12 and 13 demonstrate that the IMED surpasses all competitors, attaining the lowest values for W_1 , W_2 , W_3 , W_4 , W_5 , W_7 , as well as the highest value for W_6 . Figures 8 and 9 illustrate the estimated CDF alongside the empirical CDF, the estimated PDF represented by a histogram of the data as an asymmetric ship, and probability-probability (P-P) plots for the competitive models. These figures confirm that the datasets align with the newly proposed model.

Table 10: MLEs and standard errors of the parameters for the first dataset.

Models	Est. of θ	SE($\hat{\theta}$)	Est. of λ	SE($\hat{\lambda}$)	Est. of α	SE($\hat{\alpha}$)
IME	0.2409	0.0384	-	-	-	-
ME	0.1829	0.0236	-	-	-	-
Ex	2.7336	0.4991	-	-	-	-
EIEx	0.1801	0.1339	0.0608	0.0406	-	-
IPLE	9.852	3.5256	0.0339	0.0414	0.6332	0.0365
IPA	0.6317	0.1395	0.6614	0.0839	-	-

Table 11: MLEs and standard errors of the parameters for the second dataset.

Models	Est. of θ	SE($\hat{\theta}$)	Est. of λ	SE($\hat{\lambda}$)	Est. of α	SE($\hat{\alpha}$)
IME	1.1277	0.1403	-	-	-	-
ME	0.8365	0.0837	-	-	-	-
Ex	0.5979	0.0845	-	-	-	-
EIEx	0.0027	0.0005	0.0044	0.0008	-	-
IPLE	14.239	4.2368	0.0065	0.0031	0.6934	0.0009
IPA	1.5196	0.1709	0.4075	0.0381	-	-

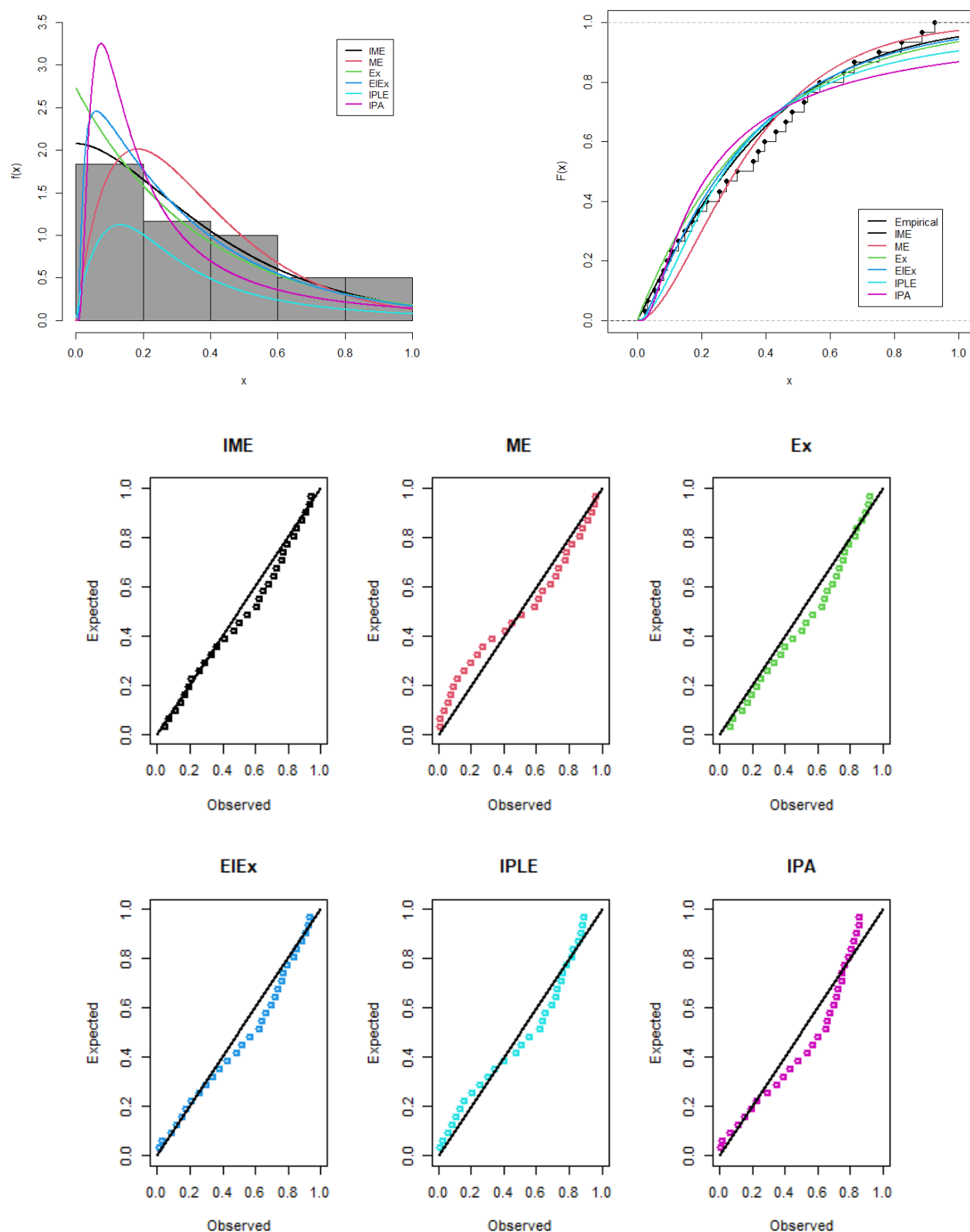


Fig. 8: Fitted plots of PDF, CDF and P-P plots for the first dataset.

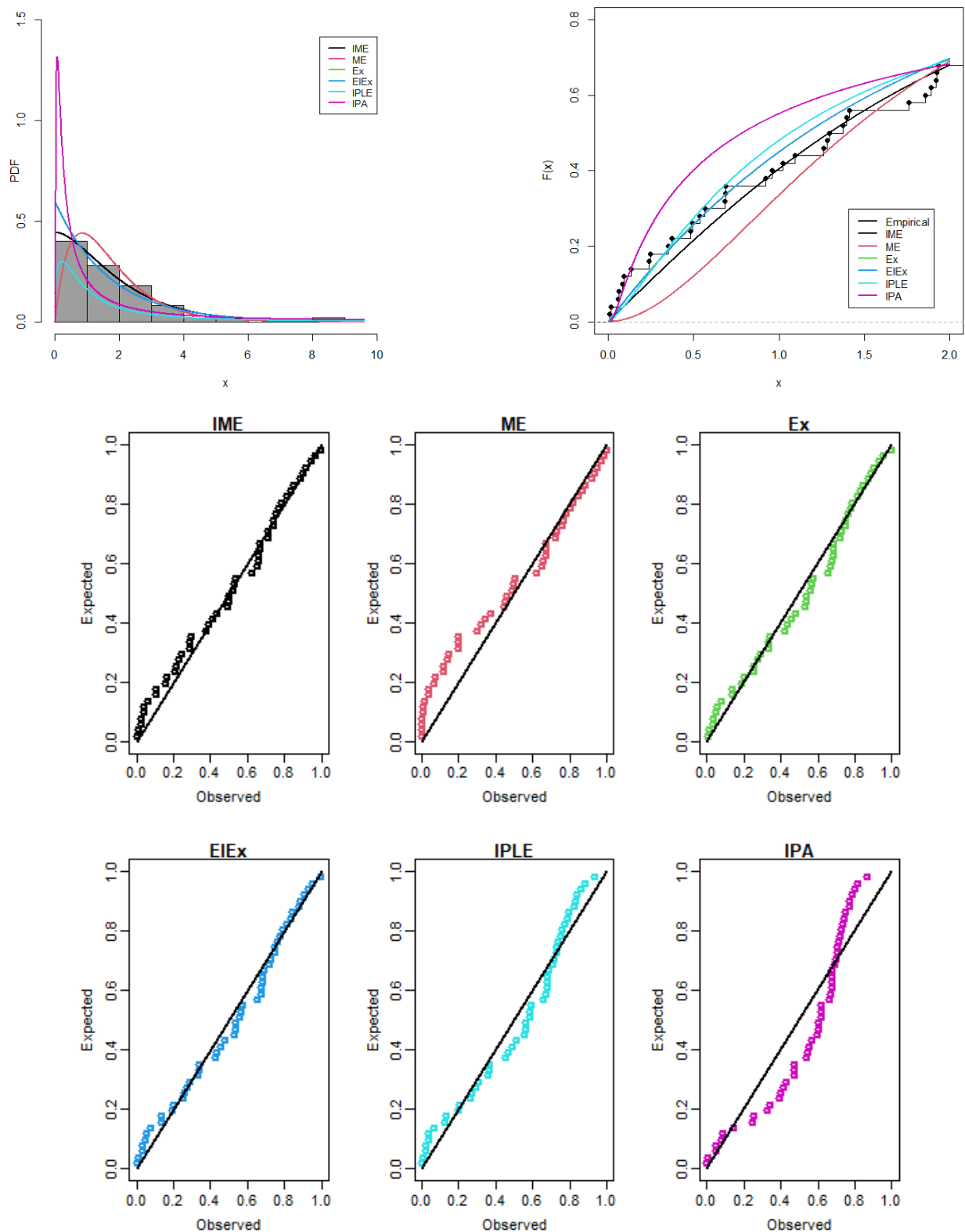


Fig. 9: Fitted plots of PDF, CDF and P-P plots for the second dataset.

Table 12: Results of goodness-of-fit of the IMED for the first dataset.

Models	W_1	W_2	W_3	W_4	W_5	W_6	W_7
IME	-0.3266	1.0746	-0.1838	0.1216	0.109	0.831	0.050
ME	0.8714	2.2726	1.0142	1.3196	0.119	0.737	0.102
Ex	1.6708	3.0721	1.8137	2.1191	0.127	0.669	0.100
EIEx	1.2579	4.0603	1.7023	2.1544	0.125	0.687	0.071
IPL	50.494	54.698	51.418	51.839	0.120	0.734	0.080
IPA	11.757	14.559	12.217	12.654	0.148	0.479	0.166

Table 13: Results of goodness-of-fit of the IMED for the second dataset.

Models	W_1	W_2	W_3	W_4	W_5	W_6	W_7
IME	151.265	152.238	151.359	152.112	0.081	0.874	0.054
ME	178.612	180.524	178.696	179.340	0.161	0.135	0.268
Ex	153.455	155.367	153.538	154.183	0.091	0.769	0.068
EIEx	155.388	159.212	155.644	156.844	0.090	0.770	0.069
IPL	238.833	244.569	239.355	241.017	0.114	0.493	0.162
IPA	187.774	191.598	188.029	189.231	0.174	0.085	0.559

7 Conclusion

In this study, we introduced and investigated the moment exponential distribution as a flexible lifetime model capable of capturing a wide variety of distributional shapes. Theoretical properties including moments, quantiles, measures of skewness and kurtosis, and several types of moments were derived, providing a solid foundation for its statistical characterization. Various estimation methods were explored, with particular focus on maximum likelihood, maximum product of spacings, and Bayesian approaches. Simulation studies highlighted the efficiency and robustness of the proposed estimators under different sample sizes and censoring schemes. Furthermore, the analysis of two real datasets demonstrated the practical applicability of the distribution in modeling skewed, heavy-tailed, and reliability-related data. Overall, the findings suggest that the moment exponential distribution is a promising alternative to traditional lifetime models, offering greater flexibility in hazard rate behavior and improved performance of estimation techniques. Future work may consider extending the distribution to regression frameworks, exploring multivariate generalizations, and applying it to broader domains such as biomedical sciences, engineering reliability, and risk management.

Acknowledgment

The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number: 26UQU4340323GSSR01.

Funding Statement

This research work was funded by Umm Al-Qura University, Saudi Arabia under grant number: 26UQU4340323GSSR01.

References

- [1] R.E. Barlow and F. Proschan, *Statistical Theory of Reliability and Life Testing: Probability Models*, To Begin With, Silver Spring, Maryland (1981).
- [2] I. Olkin, *Life Distributions: Structure of Nonparametric, Semiparametric, and Parametric Families*, Springer (2007).
- [3] S.T. Dara and M. Ahmad, *Recent Advances in Moment Distribution and Their Hazard Rates*, LAP Lambert Academic Publishing GmbH KG (2012).
- [4] G.P. Patil and J.K. Ord, On size-biased sampling and related form-invariant weighted distribution, *Sankhya: The Indian Journal of Statistics, Series B* **38**, 48–61 (1976).
- [5] G.P. Patil and C.R. Rao, Weighted distributions and size-biased sampling with applications to wildlife populations and human families, *Biometrics* **34**(2), 179–184 (1978).

- [6] R.D. Gupta and D. Kundu, A new class of weighted exponential distribution, *Statistics* **43**(6), 621–634 (2009).
- [7] R. Khatree, Characterization of inverse-Gaussian and gamma distributions through their length biased distribution, *IEEE Transactions on Reliability* **38**(5), 610–611 (1989).
- [8] A.M. El Gazar, M. Elgarhy and B.S. El-Desouky, Truncated moment exponential distribution with application, *NeuroQuantology* **20**(14), 946 (2022).
- [9] K. Adamidis and S. Loukas, A lifetime distribution with decreasing failure rate, *Statistics & Probability Letters* **39**(1), 35–42 (1998).
- [10] R.D. Gupta and D. Kundu, Exponentiated exponential family: An alternative to gamma and Weibull distributions, *Biometrical Journal* **43**(1), 117–130 (2001).
- [11] A.S. Hassan and M. Elgarhy, A new generalized exponential distribution and its applications, *Journal of Statistical Theory and Applications* **15**(1), 1–16 (2016).
- [12] R.A. Fisher, The effects of methods of ascertainment upon the estimation of frequencies, *Annals of Human Genetics* **6**(1), 13–25 (1934). <https://doi.org/10.1111/j.1469-1809.1934.tb02105.x>
- [13] B.C. Arnold and H. Nagaraja, On some properties of bivariate weighted distributions, *Communications in Statistics – Theory and Methods* **20**(5–6), 1853–1860 (1991). <https://doi.org/10.1080/03610929108830602>
- [14] R.A. Ganaie and V. Rajagopalan, Length biased weighted new quasi-Lindley distribution: Statistical properties and applications, *Pakistan Journal of Statistics and Operation Research* **17**(1), 123–136 (2021). <https://doi.org/10.18187/pjsor.v17i1.3084>
- [15] M. Mohiuddin, S.A. Dar, A.A. Khan, M. Ahajeeth and H. Al Bayatti, On weighted Nwike distribution: Properties and applications, *Information Sciences Letters* **11**(1), 85–96 (2022). <https://doi.org/10.1353/psg.2022.a904584>
- [16] B.A. Hussain and R.S. Mohammed, Weighted Zeghdoudi distribution: Properties and applications, *Research Jet Journal of Analysis and Inventions* **3**(4), 50–63 (2022).
- [17] I.K. Alsmairan and A.I. Al-Omari, Weighted Suja distribution with an application to ball bearings data, *Life Cycle Reliability and Safety Engineering* **9**(2), 195–211 (2020). <https://doi.org/10.1007/s41872-019-00106-y>
- [18] A. Saghir, G.G. Hamedani, S. Tazeem and A. Khadim, Weighted distributions: A brief review, perspective and characterizations, *International Journal of Statistics and Probability* **6**(3), 109–131 (2017). <https://doi.org/10.5539/ijsp.v6n3p109>
- [19] S.A. Benchiha, A.I. Al-Omari, N. Alotaibi and M. Shrahili, Weighted generalized quasi-Lindley distribution: Different methods of estimation, applications for COVID-19 and engineering data, *AIMS Mathematics* **6**(11), 11850–11878 (2021). <https://doi.org/10.3934/math.2021688>
- [20] B.P. Singh and U.D. Das, On an induced distribution and its statistical properties, *arXiv preprint arXiv:2010.15078* (2020).
- [21] M.M. Abdelwahab, D.A. Ramadan, S. Kumar, M.M. Hasaballah and A.M. El Gazar, Statistical analysis of the induced Ailamujia lifetime distribution with engineering and biomedical applications, *Mathematics* **13**(20), 3307 (2025).
- [22] A.S. Hassan and S.G. Nassr, Parameter estimation of an extended inverse power Lomax distribution with Type I right censored data, *Communications for Statistical Applications and Methods* **28**(2), 99–118 (2021). <https://doi.org/10.29220/CSAM.2021.28.2.099>
- [23] A.S. Hassan, S.E. Hemeda and S.G. Nassr, On the extension exponentiated Pareto distribution, *Journal of Modern Applied Statistical Methods* **19**(1) (2020). <https://doi.org/10.22237/jmasm/1619481840>
- [24] A.A.H. Ahmadini, A.S. Hassan, M. Elgarhy, M. Elshetry, S.S. Alshqaq and S.G. Nassr, Inference of truncated Lomax inverse Lomax distribution with applications, *Intelligent Automation & Soft Computing* **29**(1), 199–212 (2021). <https://doi.org/10.32604/iasc.2021.017890>
- [25] M. Kamal, M.M. Alsolmi, Nayabuddin, A. Al Mutairi, E. Hussam, M.S.A. Mustafa and S.G. Nassr, A new distributional approach: Estimation, Monte Carlo simulation and applications to the bio-medical data sets, *Networks and Heterogeneous Media* **18**(4), 1575–1599 (2023). <https://doi.org/10.3934/nhm.2023069>
- [26] A.A.H. Ahmadini, M. Elwahab, M. Elgarhy, H. Aljohani and G.A. Marei, Modulating early-life risk without breaking Weibull structure: An odds-based Weibull model for engineering failure-time data, *Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería* **42**(6) (2026).
- [27] A.H. Alenezy, L.S. Diab, K.A. Rashedi and G.A. Marei, Inference for a step-stress model under the inverted Z-Lindley distribution with progressive Type-II censoring, *AIMS Mathematics* **11**, 31919–31962 (2026).
- [28] A.H. Alenezy, B. Elkalzah, A.F.I. Alahshan and G.A. Marei, Robust inference for stress-strength reliability of the Garhy distribution under diverse record schemes with engineering and medical applications, *Mathematics* **14**, 1940 (2026).
- [29] R.C.H. Cheng and N.A.K. Amin, Estimating parameters in continuous univariate distributions with a shifted origin, *Journal of the Royal Statistical Society: Series B (Methodological)* **45**(3), 394–403 (1983). <https://www.jstor.org/stable/2345281>
- [30] B. Ranneby, The maximum spacing method: An estimation method related to the maximum likelihood method, *Scandinavian Journal of Statistics* **11**(2), 93–112 (1984).
- [31] E.J.G. Pitman, *Some Basic Theory for Statistical Inference*, Chapman and Hall (1979).
- [32] R.C.H. Cheng and L. Traylor, Non-regular maximum likelihood problems, *Journal of the Royal Statistical Society: Series B (Methodological)* **57**(1), 3–24 (1995).
- [33] O. Adubisi, Comprehensive comparative study of non-Bayesian and Bayesian estimators for the DUS Topp-Leone Burr-Hatke-Exponential model, *Innovation in Statistics and Probability* **2**(1), 86–115 (2026).
- [34] E.E. Ehizojie, Topp-Leone reduced Kies distribution for unit-interval data: Statistical properties, parameter estimation, and applications, *Modern Journal of Statistics* **2**(2), 103–128 (2026).
- [35] T.S. Helal, M.A.F. Elbarkawy, J. Zafar and A. Hydera, The Unit Komal distribution: Bathtub risk function and applications, *Computational Journal of Mathematical and Statistical Sciences* **5**(2), 731–760 (2026). <https://doi.org/10.21608/cjmss.2026.491712.1425>

- [36] S.J. Press, *Subjective and Objective Bayesian Statistics: Principles, Models, and Applications*, John Wiley & Sons (2009).
- [37] J. Mazucheli, A.F. Menezes and S. Dey, Unit-Gompertz distribution with applications, *Statistica* **79**(1), 25–43 (2019).
- [38] D.P. Murthy, M. Xie and R. Jiang, *Weibull Models*, Vol. 505, John Wiley & Sons (2004).