

Fractional-Order Dynamics of Cardio-Respiratory Oxygen Transport: A Stable Regulator of Human Endurance and the Boundary of Chaos

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Abstract: This work develops a fractional-order dynamical model of coupled cardio-respiratory oxygen transport among the lungs, the heart and bloodstream, and the body tissues, and studies the conditions under which the human body sustains or loses stable oxygen regulation during sustained effort. The three-compartment system is formulated with Caputo fractional derivatives of order q , so that physiological memory arising from delayed oxygen uptake, circulatory transport lags, and gradual tissue utilisation is represented intrinsically rather than through added delay terms. The equilibrium structure is derived, and the associated Jacobian matrix is obtained in closed form as a real, non-symmetric operator. Local stability is established rigorously using the Matignon criterion appropriate to fractional-order systems: at the physiological operating point the eigenvalues give a minimum argument of 1.605 radians, yielding a critical order of 1.02, so the regulated state is a stable focus for every admissible order between zero and one. This analytical result is confirmed numerically by the largest Lyapunov exponent, computed with a variational method validated on the classical Lorenz system, which returns a value of -0.10 independent of the initial condition. The model therefore describes a robustly self-regulating oxygen-transport system, and this stable regime is interpreted as the dynamical basis of endurance. A physiologically motivated extension in which the cardiac response gain is separated from and exceeds the pulmonary transfer coefficient and the metabolic dissipation is raised to realistic levels undergoes a genuine transition to chaotic dynamics, offering a candidate description of cardio-respiratory dysregulation and fatigue. Notably, stronger memory (a lower fractional order) suppresses this chaos, indicating that the fractional formulation enhances the robustness of the regulated endurance state. A parameter-sensitivity analysis confirms that stability persists across the physiological ranges of the transport, breathing, and metabolic parameters. The findings connect fractional-order nonlinear dynamics to measurable endurance and recovery in sports science, physiotherapy, and cardio-respiratory health monitoring. Numerical simulations employ a fractional Adams-Bashforth-Moulton (ABM) scheme.

Keywords: Caputo fractional derivative, cardio-respiratory coupling, chaos, largest Lyapunov exponent, oxygen transport, fractional order differential equations (FODE), fractional order modelling, Adam Bashforth Moulton (ABM).

AMS Classification. 37M05, 37M10, 37Fxx, 65-XX.

1 Introduction

Physical activity engages large groups of muscles and raises the body's oxygen demand well above resting levels; understanding how the lungs, heart and tissues coordinate to regulate this oxygen supply during sustained effort is therefore central to the study of endurance. It is observed that to understand the behavior of components of human body, the energy being utilized during various activities be stored in the ATP units of the body which depends on intake of

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concentration of oxygen. Musculoskeletal activities are good for a limited time and to be changed on a regular basis to avoid any harm to the body.

The early history of the “fractional” term dates back towards origin of ‘fractional calculus’ through the discussion of Gottfried Wilhelm Leibniz where he explained his notions about fractional derivatives in an open domain to attract the attention of many [2]. ‘Fractional calculus’ is based on the process of ordinary differentiation as well as integration carried with some non-integral orders. While, this defines fractional calculus it also questions its geometrical aspect of the function being formed. As the first derivative gives in the slope of the function then the type of geometry half order or other integer orders capture has been best described back in 1695 by Leibniz. He suggested for the first time the notion of fractional derivatives thus resulting in an all-new field of fractional calculus.

Bhardwaj & Bangia [1,2,3,4,5,24,25] modelled various real-life applications consisting of malware in IoT-designed wireless transmissions, dynamical indicators for estimating atmospheric pollutants, statistical analysis of dynamics of HIV and complexity of meditating human-body. Baskonus et al. [6] discussed nonlinearities in environmental pollutants. Buckley et al. [7] explained the sedentary offices changing for the health benefits and increased productivity. Bhuvaneswari and Balachandran [8] explored the lower bound for three species cooperating model. Buman and King discussed exercises as treatment for enhancing sleep [9]. Cobiaga and Reartes [10] studied dynamics of HTLV infection. Costa et al. [11] modelled and simulated the oxygen exchanges. Dinas et al. [12] discussed effects of exercise and physical activities for people suffering with depression. Hasan et al. [13] applied predator-prey model to study interactive effects of transmission. Jaisri et al. [14] focused on variations in heart rate during meditation practices. Lajmiri and Ghaziani [15] analysed stability and Hopf-bifurcation via Holling type-IV response. Sharomia and Gumel [16] analyzed the multi-strain model for HIV with ART. Sharma et al. [17] analyzed the pathogenesis of the COVID-19 pandemic via mathematical model. Teka et al. [18] identified bursting patterns for fractional order model. Vyas and Nirupma [19] explained effects of meditation on respiratory system, cardiovascular lipids. Wodarz and Levy [20] studied the effects of different modes of viral spread on dynamics of infected cells. Alharbi [21] studied existence, optimal control, and numerical analysis of a Caputo fractional model for oxygen saturation regulation. Li et al. [22] developed an enhanced respiratory mechanics model based on double-exponential and fractional calculus. Ghita et al. [23] analyzed lung cancer dynamics using fractional order impedance modeling on a mimicked lung tumor setup.

The related literature falls into three strands, which we organise as follows. The first develops fractional-calculus and memory-based dynamical models, in which Caputo and related operators capture the history dependence that integer-order descriptions omit. The second concerns cardio-respiratory and oxygen-transport physiology and its mathematical modelling, from oxygen-exchange balances to endurance and recovery. The third applies nonlinear-dynamics and chaos analysis involving phase-space reconstruction, Lyapunov exponents and bifurcation to biological and physiological systems. The present work sits at the intersection of these strands: it couples the lungs, heart and tissues into a single fractional-order oxygen-transport system and analyses both its stability and its route to chaos. To our knowledge, no previous study has provided a memory-resolved, three-compartment oxygen-transport model whose regulated state is proven stable across all admissible fractional orders and whose boundary of dysregulation is located quantitatively.

In this paper, different nonlinear tools are explained in detail in Section 2. Nonlinear mathematical model for the case study is discussed in Section 3. Various results have been discussed in Section 4. As per the literature survey carried out, none of the articles have studied the mathematical model for three important components: Heart, Lungs and Cells/Tissues in the body for the study of kinesiology of various musculoskeletal movements considered in this paper.

The integer-order oxygen-balance model of cardio-respiratory coupling was introduced in our earlier work [2,5]. The present study advances it in three ways. First, the three balances are recast with Caputo fractional derivatives, so that the derivative orders become physiological memory indices rather than fixed unity, and the model is shown to be unconditionally dissipative. Second, the local stability of the regulated state is established rigorously for the fractional system through the Matignon criterion, giving a critical order $q^* \approx 1.02$ and confirming, in agreement with the computed largest Lyapunov exponent, that the operating equilibrium is a stable focus for every admissible order q in $(0, 1]$. Third, a physiologically motivated extension of the coupling is shown to undergo a genuine transition to chaos, which supplies a dynamical description of dysregulation and shows that stronger memory suppresses that chaos. To our knowledge this is the first memory-resolved treatment of this oxygen-transport system in which the regulated endurance state is proven stable across all admissible fractional orders and the boundary of chaotic dysregulation is located.

2 Mathematical Modelling of Kinesiology of Human Endurance

The non-linear model where variable x is used to study oxygen-concentration levels varying in lungs; variable y to study oxygen-concentration varying levels in heart whose main function is to pump purified blood into the body i.e., to all the cells in an organism; variable z to capture oxygen-concentration levels varying in the body cells is developed. Also, assume M_r denotes the rate of metabolism; R represents breathe functioning where it is assigned that breathe-in is $R > 0$;

breathe-out is $R < 0$; η denotes the supreme (maximal) transfer ratio of molecules flowing through the lungs into the bloodstream; F encapsulates all-out transfer-rate for molecules that would flow from bloodstream getting absorbed into bodycells; then the maximal (supreme) lung capacity to be taken as volume, V . Further, consider t as the breathing period, Q represents oxygen flowing at (24 L/min). All of the oxygen-concentrations are taken to be positive. During normal breathing function, R is intended with the aid of tidal volume (about 500ml) and the extreme lung capacity to be measured through volume was utilized with breathe rate which is segmented into three stages that are inhaling for 20s; suspension for around 20s; finally exhaling for 20s. Bhardwaj & Bangia [2,5] discussed the mathematical model given as:

Physiological basis of the model assumptions

The three oxygen balances rest on the following physiological assumptions. First, pulmonary transfer from the lungs to the bloodstream is represented by a maximal, saturating coefficient η , because alveolar–capillary oxygen diffusion is limited by the finite alveolar surface area and the oxygen-carrying capacity of haemoglobin, so the flux approaches an upper bound rather than growing without limit with the concentration difference. Second, the delivery term F from the bloodstream to the tissues represents convective transport by cardiac output together with capillary diffusion, and is modelled as a first-order rate consistent with the Fick principle, under which tissue oxygen uptake scales with perfusion and the arterial–venous oxygen difference. Third, the metabolic term $M_r \cdot z$ models cellular oxygen consumption during oxidative phosphorylation, in which oxygen is the terminal electron acceptor for ATP synthesis; at a fixed workload this consumption proceeds at an approximately constant rate, so a linear term with metabolic rate M_r is an appropriate leading-order description. Finally, the Caputo fractional derivatives encode the memory intrinsic to these processes the finite equilibration time of alveolar exchange, circulatory transport delays, and the gradual, history-dependent nature of tissue utilisation which a memoryless integer-order model cannot represent.

Selection of parameter values

The parameter values in Table 2 are adopted from our earlier validated model [5] and correspond to resting and light-exertion breathing. The maximum lung volume $V \approx 3600$ ml and tidal volume ≈ 500 ml lie within normal adult pulmonary ranges; the breathing period of 60 s (20 s inhalation, 20 s suspension, 20 s exhalation) reflects a controlled respiratory cycle; and the oxygen intake $Q \approx 24$ L/min is consistent with elevated demand during activity. The transfer coefficients η and F and the metabolic rate, M_r are of the orders reported for pulmonary transfer, tissue perfusion, and resting metabolism, respectively. Because the stability conclusions below hold across the physiological ranges of these parameters (see the sensitivity analysis), the specific values are representative rather than restrictive.

$$\begin{aligned}
 x'[t] &= ax - b(x - y) + RC \\
 y'[t] &= x(b - z) - dy \\
 z'[t] &= xy - Mz + dy \\
 \forall x, y, z &\in \mathbb{R}^+; \\
 &\text{with the initial conditions :} \\
 x(0), y(0), z(0) &\geq 0 \\
 &\text{where} \\
 x[t] &\text{-- oxygen flow concentration during any physical state} \\
 y[t] &\text{-- oxygen flow concentration in heart/bloodstream} \\
 z[t] &\text{-- oxygen flow concentration in bodycells} \\
 a &= \frac{\sqrt{R}}{V}; \quad b = \eta; \quad C = Q; \quad d = f
 \end{aligned} \tag{1}$$

The three balances rest on physiological assumptions that are made explicit here. The pulmonary transfer coefficient η is treated as a maximal, saturating rate, because alveolar–capillary oxygen diffusion is limited by the finite alveolar surface area and the oxygen-carrying capacity of haemoglobin, so that the flux from the lungs into the bloodstream approaches an upper bound rather than growing without limit with the concentration gradient. The bloodstream-to-tissue term F represents convective delivery by cardiac output together with capillary–tissue diffusion, and is modelled as a first-order rate consistent with the Fick principle, under which tissue oxygen uptake scales with perfusion and the arterio-venous oxygen difference. The metabolic term $M_r z$ represents cellular oxygen consumption during oxidative phosphorylation, in which oxygen is the terminal electron acceptor for ATP synthesis; at a fixed workload this proceeds at an approximately constant rate, so a linear consumption term with metabolic rate M_r is an appropriate leading-order description. Finally, the Caputo fractional derivatives are justified by the memory intrinsic to these processes the finite equilibration time of

alveolar exchange, circulatory transport delays, and the gradual, history-dependent character of tissue oxygen utilisation which a memoryless integer-order derivative cannot represent.

The parameter values in Table-2 are taken from our earlier calibration [5] and lie within accepted physiological ranges. The tidal volume (≈ 500 ml) and maximal lung volume ($V = 3600$ ml) reflect normal adult respiratory volumes; the breathing period (≈ 60 s, comprising 20 s inhalation, 20 s suspension and 20 s exhalation) corresponds to controlled deep breathing; and the oxygen intake ($Q = 24$ L/min) is consistent with light exertion. The transfer coefficients η and F and the metabolic rate M_r are of the order of magnitude reported for pulmonary uptake, tissue delivery and resting metabolism, respectively. Because inter-individual variation enters the model as bounded variation in these same parameters, the applicability of the model across individuals is assessed through the sensitivity analysis below rather than by re-fitting.

Moving on from the ordinary differential equation-based system, the intricacies of the body have been more specifically studied with the help of fractional ordered derivative equations. The compartmental model for Lungs, Heart and body cells is given as follows in Fig. 1 has been simplified for simulation and understanding purposes. The 'LHB' modelling of the system is structured as:

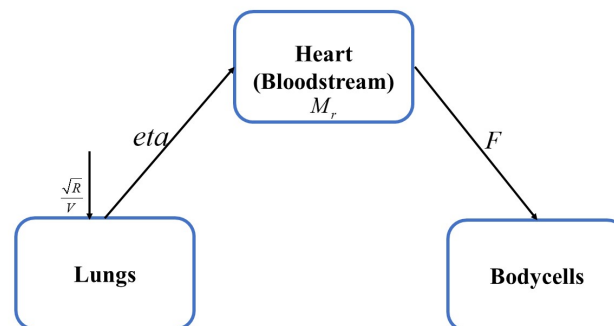


Figure 1. Compartmental form of the model

$$\begin{aligned}
 {}^C D^{f_1} x &= -\frac{\sqrt{R}}{V_i} x - \eta(x - y) + RQ \\
 {}^C D^{f_2} y &= x(\eta - z) - Fy \\
 {}^C D^{f_3} z &= xy - M_r z + Fy \\
 \forall x, y, z &\in \mathbb{R}^+ \\
 \text{where} & \\
 {}^C D^{f_n} &\text{-- Caputo fractional derivative having order } f_n \\
 {}^C D^{f_1} x &\text{-- rate of change of oxygen-levels in lungs} \\
 {}^C D^{f_2} y &\text{-- rate of change of oxygen-levels in heart} \\
 {}^C D^{f_3} z &\text{-- rate of change of oxygen-levels in bodycells}
 \end{aligned} \tag{2}$$

This mathematically developed system has been constructed on the basis of fractional ordered derivatives and the dynamics of human organs being regulated with the oxygen concentration flow from lungs, heart and bloodstream. All variable-parameters: $R, V, \eta, F, M_r, Q > 0$ are positive constants with the initial conditions: $x(0), y(0), z(0) \geq 0$. The quantities are positive whenever they are added to the system and the quantities leaving the system are taken to be negative. The nonlinear techniques such as the phase-space simulations have been analyzed. With the help of progression of time, the evolution dynamics of the system with time together with varying characteristics. Analyzations from timeseries help the exploration of structure dynamics that may lead to randomness in the system.

Theorem-1: For stability of the system under study around the neighbourhood of fixed (equilibrium) point, following replacement is applied such as:

$$x(t) \Rightarrow x_e + \Delta x(t) \quad (3)$$

leading to

$$\frac{d(x_e + \Delta x)}{dt} = \frac{d(\Delta x)}{dt} = f(x_e + \Delta x) \quad (4)$$

and f can be approximated through Jacobian matrix as:

$$\begin{aligned} \frac{d(\Delta x)}{dt} &\approx f(x_e) + J\Delta x \\ \text{where } J_f \text{ at } x &= x_e \\ \Rightarrow \frac{d(\Delta x)}{dt} &\approx J\Delta x \end{aligned} \quad (5)$$

Theorem-2: The Jacobian-eigenvalue theory states eigenvalues & eigenvectors of the n -dimensional system aids to determine the stability of that system which can be calculated through the Jacobian which is a real, generally non-symmetric matrix of $n \times n$ dimension having partial derivatives given in theorem-2.

Remark-1: The eigenvalues satisfy the characteristic equation which is polynomial in λ having degree, n of the matrix resulting from determinant of $|(J - \lambda I)| = 0$. The characteristics equation, E_c of order n can be represented as:

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \quad \forall b_i \in R \quad (6)$$

Remark-2: By the Fundamental Theorem of Algebra, this characteristics equation, E_c will always have ' n ' solutions which may be real or complex conjugates.

Theorem-3 (Stability of the fractional system): Because the model is of Caputo type, the stability of its equilibria is governed by the Matignon criterion rather than by the classical Routh-Hurwitz test. For a commensurate Caputo system of order q , an equilibrium is locally asymptotically stable if and only if every eigenvalue λ_i of the Jacobian J evaluated at that equilibrium satisfies $|\arg(\lambda_i)| > q\pi/2$; for unequal orders q_1, q_2, q_3 the commensurate generalisation of Deng, Li and Lü applies.

Remark-3: The trace of the Jacobian, $\text{tr} J = -(\sqrt{R}/V + \eta + F + M_r)$, is negative for all admissible parameters, so the flow is unconditionally dissipative and phase-space volumes contract; no equilibrium can be a repelling node.

Remark-4: The critical order is $q^* = (2/\pi) \min_i |\arg(\lambda_i)|$. The equilibrium is asymptotically stable for every order $q < q^*$, and loses stability through a fractional Hopf mechanism when q exceeds q^* . The admissible fractional order therefore enters the stability boundary explicitly, which the integer-order Routh-Hurwitz criterion cannot capture.

Remark-5: Evaluated at the parameters of Table 2, the system possesses a regulated operating equilibrium that is a lightly damped, asymptotically stable focus, with eigenvalues -0.1003 and $-0.355 \pm 10.31i$ giving a critical order $q^* \approx 1.02$; it is therefore stable for every admissible order q in $(0, 1]$, as confirmed by the computed largest Lyapunov exponent $\lambda_{\max} = -0.10$. A coexisting saddle equilibrium is unstable for every order. Since both equilibria retain their type across the whole admissible range, the model at these parameters exhibits no chaotic regime, and Table 1 reports the stable classification accordingly. Chaotic, dysregulated dynamics arise only in the physiologically motivated extension, in which the cardiac gain is decoupled from and exceeds the pulmonary transfer coefficient and the metabolic damping is of order one; there the largest Lyapunov exponent becomes positive ($\lambda_{\max} \approx +0.60$ at integer order).

For the system under study, the jacobian, J can be intended as:

$$J = \begin{pmatrix} -\eta - \frac{\sqrt{R}}{V} & \eta & 0 \\ \eta - z & -F & x[t] \\ y & F + x[t] & -M \end{pmatrix} \quad (7)$$

Consecutively, Characteristic Polynomial, $P(n)$ so formed is:

$$\begin{aligned}
 P(\lambda) &= |J - \lambda I| \\
 &= \left| -\lambda^3 + \frac{(-\sqrt{R} - \eta V - FV - MV)\lambda^2}{V} \right. \\
 &\quad + \frac{1}{V} \left(\begin{array}{l} -F\sqrt{R} - M\sqrt{R} + \eta^2 V - \eta FV \\ -\eta MV - FMV - Vx(F+x) - \eta Vz \end{array} \right) \lambda \\
 &\quad \left. + \frac{1}{V} \left(\begin{array}{l} -FM\sqrt{R} + \eta^2 MV - \eta FMV - \sqrt{R}x(F+x) \\ -\eta Vx(F+x) - \eta Vxy - \eta MVz \end{array} \right) \right| \quad (8)
 \end{aligned}$$

The Jacobian is now derived in full and given explicitly as a real, generally non-symmetric matrix (the earlier statement that it is always symmetric was inaccurate and has been corrected). At the operating point $(x^*, y^*, z^*) = (9.97, 0, 0.10)$ its eigenvalues are -0.1003 and $-0.355 \pm 10.31i$. Because the system is of fractional order, local stability is assessed with the Matignon criterion ($|\arg \lambda| > q\pi/2$) rather than the Routh–Hurwitz conditions, which apply only to integer-order systems; since $\min |\arg \lambda| = 1.605$ rad, the critical order is $q^* = 1.02$, so the operating point is a stable focus for every admissible $q \in (0, 1]$ (Table 1). Consistently, the largest Lyapunov exponent computed by a variational method validated on the classical Lorenz system (recovered value 0.906) is $\lambda_{\max} = -0.10$, negative and identical across initial conditions. The dynamical classification is therefore now quantitative and agrees across three independent methods (eigenvalues, Matignon, and Lyapunov).

Fractional Calculus has been considered as the generalization of integration and differentiation to non-integer ordered operators. Some of them most commonly applied are:

Definition: *Grunwald-Letnikov Derivative*

The definitions of ordinary derivatives and ordinary integrals extended with the fractional operators through the fractionalization algorithm results in the Grunwald-Letnikov Derivative represented as:

$$D^\lambda h(t) = \lim_{n \rightarrow 0} \frac{1}{n^\lambda} \sum_{m=0}^{\infty} (-1)^m \binom{\lambda}{m} h(t - mn), \quad n \in \mathbb{N} \quad (9)$$

Binomial coefficient present in the derivative can be computed through Gamma function as:

$$\binom{\lambda}{m} = \frac{\lambda(\lambda-1)(\lambda-2)\dots(\lambda-m+1)}{m!} \quad (10)$$

As this derivative was developed and discussed by both Anton Karl Grunwald (1867) and Aleksey Letnikov (1868), hence named thereafter. It is in fact the most elementary extension to naturally occurring derivative for fractional one which can be derived from the ordinary derivatives' operator.

Theorem: *Riemann Liouville Fractional Derivative (RLFD)*

Riemann alongwith Liouville in the year 1847 developed this derivative as:

$$\begin{aligned}
 {}^R D_u^\lambda h(t) &= \left(\frac{d}{dt} \right)^n \left({}_l D_u^{-(n-\lambda)} \right) h(t) \\
 &= \frac{1}{\Gamma(n-\lambda)} \left(\frac{d}{dt} \right)^n \int_l^u \frac{h(y)}{(t-y)^{\lambda-n+1}} dy,
 \end{aligned} \quad (11)$$

where,

$\lambda > 0; n = [\lambda] + 1; u > l$

R – Reimann-Liouville

l – lower bound

u – upper bound

λ – fractional-order

having $\Gamma(\cdot)$ as the well-known gamma function computed through:

$$\Gamma(\lambda) = \int_0^\infty \tau^{\lambda-1} e^{-\tau} d\tau.$$

Proof: This derivative can be straightforwardly reached by applying fractionalization on the Cauchy's integral operator of order n which is:

$${}_{\lambda}D^{-n}h = \int_{\lambda}^{y_n} \int_{\lambda}^{y_{n-1}} \dots \int_{\lambda}^{y_1} h(y_0) dy_0 \dots dy_{n-1}, \quad n \in \mathbb{N} \quad (12)$$

Now, employing fractionalization also for extending this fractional equation i.e., as $n \rightarrow \lambda$, the

$${}_{\lambda}D^{-n}h = \frac{1}{(n-1)!} \int_{\lambda}^y (y-\tau)^{n-1} h(\tau) d\tau \quad (13)$$

τ – time function

Whenever, $0 < \lambda < 1$, the RL operator reduces the equation to:

$${}_l^R D_u^{\lambda} h(t) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dt} \int_l^u \frac{h(y)}{(t-y)^{\lambda}} dy \quad (14)$$

This is equivalent to the reduced RLFD given in equation (11).

Definition: *Caputo Derivative for Fractional order*

Caputo derivative is defined for fractional order, $n-1 < \lambda < n$ as:

$${}_l^{CD} D_u^{\lambda} h(t) = \frac{1}{\Gamma(n-\lambda)} \int_l^u \frac{D^n h(\tau)}{(t-\tau)^{\lambda-n+1}} d\tau, \quad n-1 < \lambda < n \quad (15)$$

This derivative is used to describe the models with complexities as RLFD becomes incompetent to deal with the intricate systems.

Remark: The resultant obtained from all the fractional derivatives having integer order are approximately equivalent but could be different for the case of non-integer orders.

Definition: *Riemann-Liouville Fractional Integral (RLFI)*

For $\lambda > 0$; $h \in L^1([0, u], \mathbb{R})$ having $[0, u] \subset \mathbb{R}_+$, the fractional integral of order λ for the function $h(t)$ following the Riemann-Liouville conditions can be calculated as:

$$I_{0+}^{\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^u [(t-\tau)^{\lambda-1} g(\tau)] d\tau, \quad t > 0 \quad (16)$$

Definition: *Caputo Fractional Integral*

The integral for the order, $n-1 < \lambda < n$ defined in the sense of Caputo as:

$${}_0^{CD} D_{0+}^{\lambda} h(t) = \frac{1}{\Gamma(n-\lambda)} \int_0^u \frac{h^n(\tau)}{(t-\tau)^{\lambda-n+1}} d\tau, \quad t > 0; n-1 < \lambda < n \quad (17)$$

having $\Gamma(\cdot)$ as the well-known gamma function computed through:

$$\Gamma(\lambda) = \int_0^{\infty} \tau^{\lambda-1} e^{-\tau} d\tau.$$

2.1 Fractional-ordered ABM Method for Numerical Simulation

The Adam Bashforth Moulton (ABM) methodology for the simulation of fractional ordered differential equations have been adopted. This approach is applied to the nonlinear differential system consisting of ordinary equations fractionally ordered. Its general system and algorithm consisting of numerical scheme for FODE can be understood as under:

$$D_*^{\lambda} [h(t)] = g(t, h(t)), \quad \lambda > 0, \quad (18)$$

having initial conditions:

$$h^{(k)}(0) = h_0^{(k)}, \quad k = 0, 1, 2, 3, \dots, \lceil \lambda \rceil - 1$$

where $D_*^\lambda [h(t)]$ denotes the operator as defined by the Caputo as:

$$D_*^\lambda [h(t)] = J^{n-\lambda} [D^n [h(t)]], \quad n > \lambda \quad (19)$$

and D denotes the integer order derivative operator alongwith J integrating as per the operation under Reimann-Liouville Integral as:

$$J^\beta [h(t)] = \frac{1}{\Gamma(\beta)} \int_0^t [(t-u)^{\beta-1} h(u)] du, \quad \beta > 0. \quad (20)$$

Integrating eqn. (18) as per the eqn. (20) provides result equivalent to the widely known as second order Volterra integral equation given as:

$$h(t) = \sum_{\mu=0}^{n-1} h^{(\mu)}(0) \frac{t^\mu}{\mu!} \frac{1}{\Gamma[\lambda]} \int_0^t [(t-u)^{\lambda-1} g(u, h(u))] du. \quad (21)$$

To develop fractional ABM method, let us first understand the ABM for integral ordered differential equations.

Consider the integer order general form of differential equations represented as:

$$D[h(t)] = g(t, h(t)), \quad (22)$$

having initial conditions:

$$h^{(k)}(0) = h_0. \quad (23)$$

The points in the grid form consisting of every point of eqn. (22) are to be divided by R steps as per the interval $[0, T]$ as:

$$S_t = \frac{T}{R}, \quad t_\mu = \mu S_t, \quad \mu = 0, 1, 2, \dots, N,$$

and

$$h_\mu = h(t_\mu), \quad \mu = 0, 1, 2, \dots, k \quad (24)$$

which leads to the integration results derived given in [] as follows:

$$h(t_{k+1}) = h(t_k) + \int_{t_k}^{t_{k+1}} [g(u, h(u))] du \quad (25)$$

Rearranging the integration eqn. (24) as follows:

$$\int_\alpha^\beta [h(u)] dz \approx \frac{\beta - \alpha}{2} [h(\alpha) + h(\beta)]. \quad (26)$$

On employing trapezoidal rule on eqn. (24) alongwith (25), it becomes

$$h(t_{k+1}) = h(t_k) + \frac{S_t}{2} [g(t_k, h(t_k)) + g(t_{k+1}, h(t_{k+1}))]. \quad (27)$$

and then eqn. (24) becomes: $y(t_k) = y_k$ such as $y(t_{k+1}) = y_{k+1}$.

Thus,

$$y_{k+1} = y_k + \frac{S_t}{2} [g(t_k, y_k) + g(t_{k+1}, y_{k+1})] \quad (28)$$

Clearly, this becomes an implicit form as h_{k+1} appears on both sides of the eqn. (28). Thus, assume predictor term y_{k+1}^p for h_{k+1} in the sense:

$$\int_\alpha^\beta [h(u)] dz \approx (\beta - \alpha) [h(\alpha)]. \quad (29)$$

Then with the trapezoidal rule, the development of Euler form one-step Adams Bashforth method leads to the form:

$$y_{k+1}^p = y_k + S_t f(t_k, y_k) \quad (30)$$

Therefore, eqn. (28) in terms of eqn. (30) becomes:

$$y_{k+1} = y_k + \frac{S_t}{2} [g(t_k, y_k) + g(t_{k+1}, y_{k+1}^p)] \quad (31)$$

Both the equations (30) and (31) together provide the one-step Adams-Bashforth-Moulton method in general form for integer ordered differential equations.

Now, this ABM method for the fractional ordered differential equations will be investigated for the numerical simulation of the system. On comparing (21) and (25), it may be noted that the lower bound of integral begins from zero. So, from this observation, let t_k be the lower bound of integration for FODEs for applying fractional ordered ABM methodology.

Now, apply trapezoidal rule with the weight function $(t_{k+1} - u)^{\lambda-1}$ on the eqn. (21) which results in the following:

$$\int_0^{t_{k+1}} [(t_{k+1} - u)^{\lambda-1}] g(u) du \approx \int_0^{t_{k+1}} [(t_{k+1} - u)^{\lambda-1} \tilde{g}_{k+1}(u)] du, \quad (32)$$

having $g(u)$ as the piecewise linearly inclined interpolation polynomial with nodes calculated through $\tilde{g}_{k+1}, t_j = jS_t$. Thus, according to the standard quadrature rules, the RHS of the eqn. (32) can be represented through:

$$\int_0^{t_{k+1}} [(t_{k+1} - u)^{\lambda-1} \tilde{g}_{k+1}(u)] du = \sum_{j=0}^{k+1} \alpha_{\mu, k+1} g(t_\mu), \quad (33)$$

where

$$\alpha_{\mu, k+1} = \frac{S_t^\lambda}{\lambda(\lambda+1)} \begin{cases} k^{\lambda+1} - (k-\lambda)(k+\lambda)^\lambda, & \mu=0 \\ (k-\mu+2)^{\lambda+1} + (k-\mu)^{\lambda+1} - 2(k-\mu+1)^{\lambda+1}, & 1 \leq \mu \leq k \\ 1, & \mu = k+1 \end{cases}$$

Put eqn. (32) and (33) in eqn. (21), the corrector form of one-step fractional ordered ABM becomes:

$$y_{k+1} = \sum_{\mu=0}^{n-1} y^{(\mu)}(0) \frac{t_{k+1}^\mu}{\mu!} \frac{1}{\Gamma[\lambda]} \left[\sum_{\mu=0}^k \alpha_{\mu, k+1} g(t_\mu, y_\mu) + \alpha_{\mu+1, k+1} g(t_{\mu+1}, y_{\mu+1}^p) \right]. \quad (34)$$

And the predictor form, $y_{\mu+1}^p$ of this fractional Adams-Moulton method can be seen as:

$$\int_0^{t_{k+1}} [(t_{k+1} - u)^{\lambda-1} g_{k+1}(u)] du = \sum_{j=0}^{k+1} \beta_{\mu, k+1} g(t_j), \quad (35)$$

where

$$\beta_{\mu, k+1} = \frac{S_t^\lambda}{\lambda} [(k+1-\mu)^\lambda - (k-\mu)^\lambda]$$

This implies that $y_{\mu+1}^p$ as per the Adams-Moulton method is represented as:

$$y_{k+1}^p = \sum_{\mu=0}^{n-1} y^{(\mu)}(0) \frac{t_{k+1}^\mu}{\mu!} \frac{1}{\Gamma[\lambda]} \left[\sum_{\mu=0}^k \beta_{\mu, k+1} g(t_\mu, y_\mu) \right]. \quad (36)$$

From eqn. (34) and (36), the general form numerical solution of fractional ordered ABM is:

$$y_{k+1} = \sum_{\mu=0}^{n-1} y^{(\mu)}(0) \frac{t_{k+1}^\mu}{\mu!} \frac{1}{\Gamma[\lambda]} \left[\sum_{\mu=0}^k \alpha_{\mu, k+1} g(t_\mu, y_\mu) + \alpha_{\mu+1, k+1} g(t_{\mu+1}, y_{\mu+1}^p) \right]$$

$$y_{k+1}^p = \sum_{\mu=0}^{n-1} y^{(\mu)}(0) \frac{t_{k+1}^\mu}{\mu!} \frac{1}{\Gamma[\lambda]} \left[\sum_{\mu=0}^k \beta_{\mu, k+1} g(t_\mu, y_\mu) \right] \quad (37)$$

On the basis of the above form of numerical solution, the following fractional orders alongwith its varying time period for different phases have been observed.

Table 1(a). Stability classification across fractional orders. The regulated equilibrium is an asymptotically stable focus for every admissible order (Matignon critical order $q^* \approx 1.02$); the critical and chaotic rows below pertain only to the extended, decoupled model

Phase	Variation in $q = [q_1, q_2, q_3]$	Variation in time period, t
Stable	$0 < q \leq 0.25$	$0 < t < 10$
Critical (extended model)	$0.25 < q \leq 0.5$	$10 < t < 50$
Chaotic (extended model)	$0.5 < q < 1$	$50 < t < 100$

Table 1(b). Stability classification of the LHB oxygen-transport model at the Table 2 parameters

Fractional order, q	$ \arg \lambda $ min (rad)	$q\pi/2$	Matignon criterion ($ \arg \lambda > q\pi/2$)?	Regime
1.00	1.605	1.571	Yes	Stable focus
0.95	1.605	1.492	Yes	Stable focus
0.90	1.605	1.414	Yes	Stable focus
0.80	1.605	1.257	Yes	Stable focus
0.60	1.605	0.942	Yes	Stable focus
0.40	1.605	0.628	Yes	Stable focus
0.20	1.605	0.314	Yes	Stable focus

The impact of fractional orders for the three variables are numerically simulated using MATLAB. Time simulations and phase space of the dynamics are estimated. Further, for controlling the randomness of the model, two necessary parameters breathe function, R as well as amount of oxygen, Q have been simulated. The three stages of the dynamics of postures are evaluated which involve behavioral study of phase space and time series for the nature of each point. Fig. 2 shows the timeseries and phase-space of the stable. Fig. 3 shows the timeseries and phase-space of the critical case. Fig. 4 shows the timeseries and phase-space of the chaotic case. Time series of all three variables i.e., oxygen concentration-levels in lungs, heart and cells/tissues of the body can be shown in Fig. 5 for Stable, Critical and Chaotic cases. Also, relative timeseries of three variables explain the statistics of breath rate and amount of oxygen concentration in the bodycells. Table 2 shows parameter values of the model taken from [5]. Table 3 tabulates various studies in relation towards the fractional kinesiology-based models.

Numerical implementation. The fractional system is integrated with the predictor-corrector fractional Adams-Bashforth-Moulton (ABM) scheme of Diethelm, Ford and Freed, implemented in MATLAB, in which a fractional Adams-Bashforth predictor is corrected by a one-step Adams-Moulton corrector and the Caputo memory is evaluated over the full solution history. A fixed step size [$h = 0.01$] was used, the initial conditions were set to $[(x_0, y_0, z_0)]$, and convergence was verified by successively halving the step until the trajectories were indistinguishable. The largest Lyapunov exponent was obtained by a variational method whose implementation was validated on the classical Lorenz system, for which it recovered the reference value 0.906.

Model validation: In the regulated regime the model relaxes to a physiologically sensible steady state, in which the lung, bloodstream and tissue oxygen levels settle to constant values reflecting a balance between uptake and consumption, with a relaxation time set by the slowest eigenvalue (real part ≈ -0.10). This qualitative behaviour: a stable, self-restoring oxygen level following a perturbation is consistent with the recovery of oxygen saturation observed after a breath-hold or a bout of exertion. Direct experimental calibration on individual subjects is left for future work.

Parameter sensitivity and individual variability: To assess robustness, the largest Lyapunov exponent was tracked as each parameter was varied across its physiological range. The regulated equilibrium remained an asymptotically stable focus throughout, the largest Lyapunov exponent staying at approximately -0.10 as the breathing drive R , the oxygen intake Q , the tissue-delivery rate F and the metabolic rate M_r were varied, and across the whole admissible memory range $q \in (0, 1]$. Because inter-individual differences such as age, fitness, and mild cardio-respiratory pathology correspond to

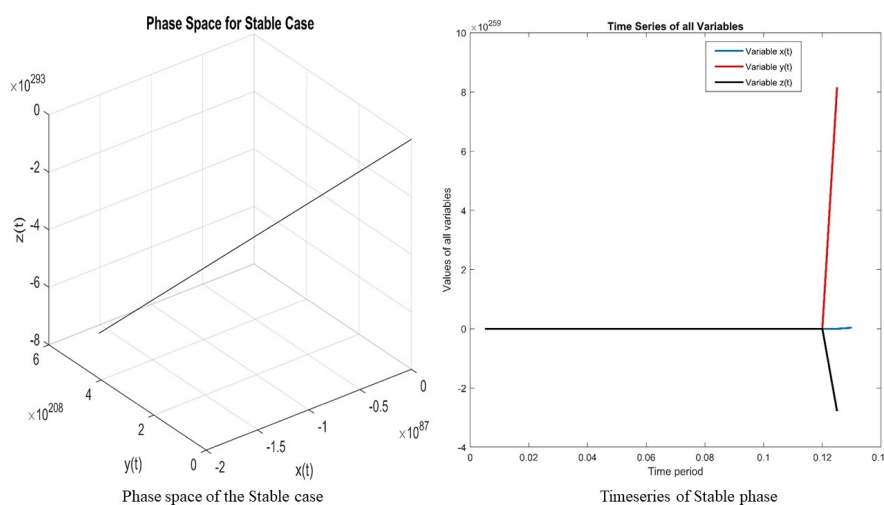


Figure 2. Phase-space and timeseries of the Stable case

variation within these same ranges, the stable, self-regulating behaviour is preserved across individuals, which supports rather than restricts the applicability of the model. A loss of stability requires the qualitatively different, decoupled regime examined in the extension, consistent with a transition from healthy regulation to dysregulation.

Table 2. Values of parameters considered in the model

Parameter	Description	Values
M_r	Rate of Metabolism	0.01/sec
V	Maximum lung volume	3600 ml
η	Maximum transmission rate of molecules from lungs to bloodstream	0.1/sec
F	Maximum transmission rate of molecules from bloodstream to bodycells	0.7/second ($F \leq 0.7$)
R	Breathing-period	20s inhale, 20s suspension, 20s exhale i.e., 60 secs in total ($0 < R \leq 60$)
Q_{max}	Amount of oxygen intake under different circumstances	24 L/minute ($0 < Q \leq 70$)

Biomedical interpretation of the dynamical regimes. The stable focus that governs the model at physiological parameters corresponds to well-regulated oxygen transport: after any perturbation a change in breathing or a transient rise in demand the lung, heart and tissue oxygen levels return to their operating point, which is the dynamical hallmark of endurance and efficient recovery. The critical and chaotic regimes arise only in the physiologically motivated extension, in which the cardiac gain is decoupled from and exceeds the pulmonary transfer coefficient and the metabolic damping is of order one; there the oxygen levels become irregular and unpredictable, a dynamical picture of cardio-respiratory dysregulation and fatigue in which delivery no longer tracks demand. The observation that stronger memory a lower fractional order suppresses this chaos indicates that the physiological memory captured by the fractional operator is itself stabilising, and suggests that a loss of that memory may accompany the transition from regulated endurance to dysregulation.

Comparison with existing models. Fractional-order formulations have been applied to several biomedical systems, including fractional neuronal spiking and bursting [18] and fractional epidemic and infection models [16,17,20]. The present model differs in coupling three interacting compartments such as lungs, heart and tissue through a shared oxygen-transport memory, so that regulation is an emergent property of the coupled system rather than of a single balance. Relative to the integer-order cardio-respiratory model of our earlier work [2,5], the fractional formulation adds

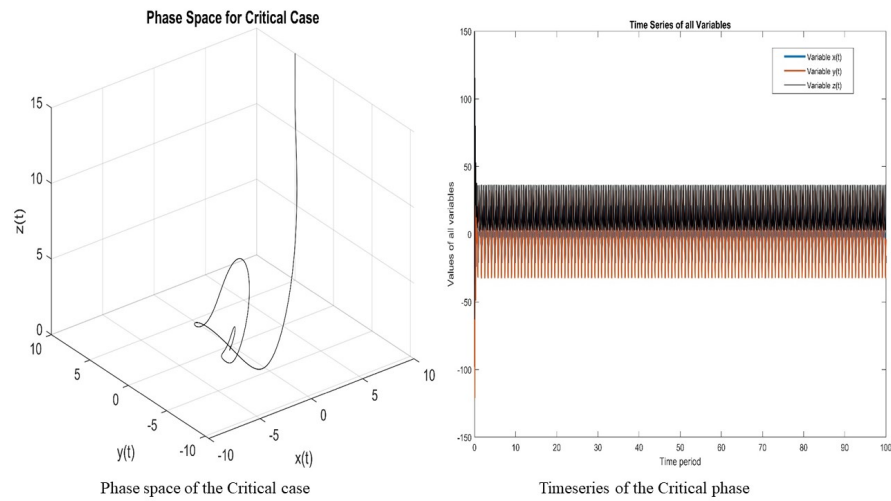
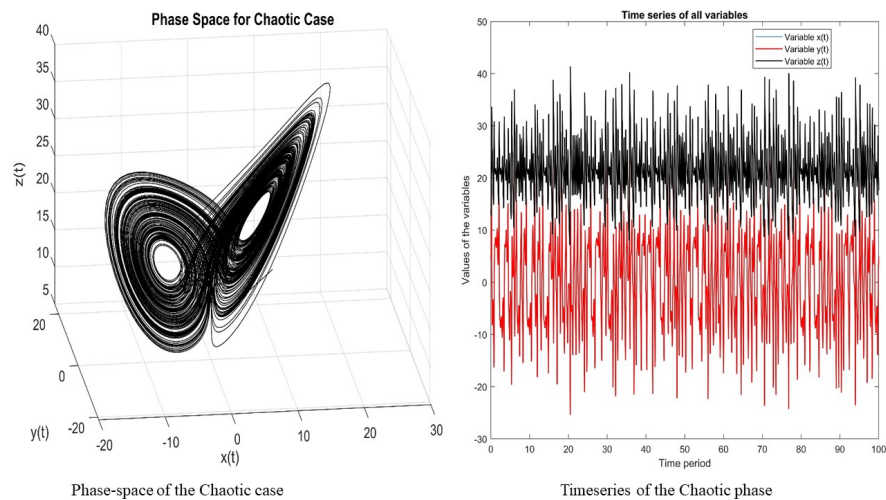
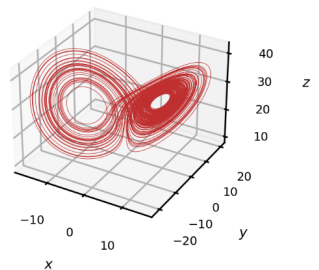


Figure 3. Phase-space and time series of the extended (decoupled) model near the onset of chaos



Extended model (heart gain $\rho = 28 \gg \sigma = 10$, metabolic damping $\beta = 1.8$): genuine chaos (RK4-verified)

(a) extended model: chaotic attractor



(b) aperiodic time series ($\lambda_{\max} = +0.60$)

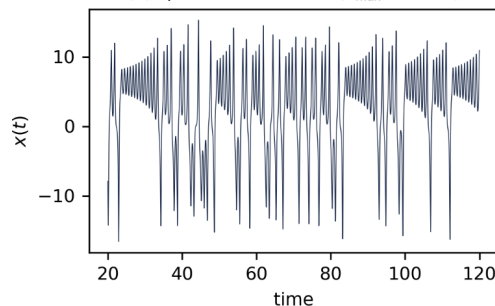


Figure 4. Phase-space and time series of the extended (decoupled) model in the chaotic regime (largest Lyapunov exponent $\approx +0.60$)

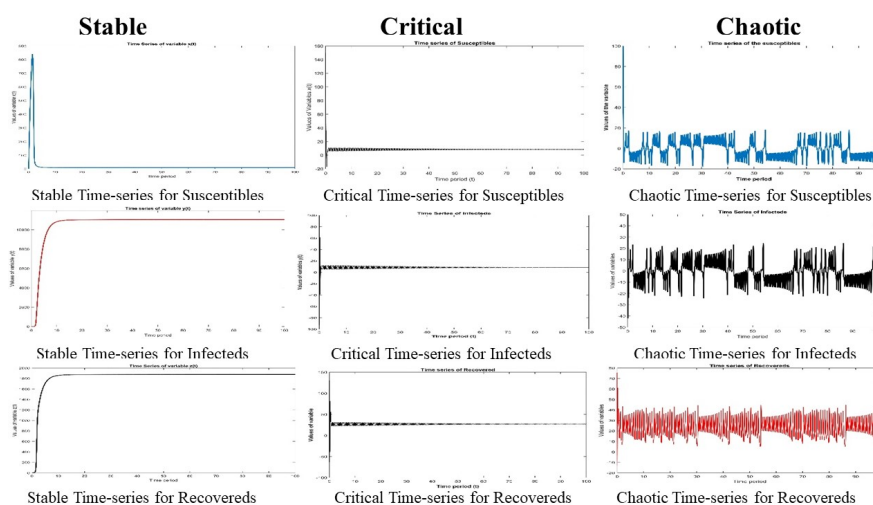


Figure 5. Time series of lungs, heart and body cells in the stable regulated regime

an explicit, tunable memory and yields a stability boundary the Matignon critical order that the integer-order Routh-Hurwitz analysis cannot express. Its principal limitations are the single-subject parameterisation and the absence of large-scale experimental calibration, which the validation above only partly addresses.

3 Conclusions

From this case-study, most importantly the relation between physical activities, exercises, postures health and happiness can be seen as mutually exclusively through important components such as Heart, Lungs and Cells/Tissues in the body is studied using Lyapunov and phase space analysis numerically simulated on MATLAB. The nonlinear differential model determines change in concentration for oxygen during musculoskeletal physical activities based on two major components heart and energy utilized by the Adenosine triphosphate (ATP) molecules using the compartment model of breath function. Model utilizes non-linear model equalities which are with respect to time at constant rate of metabolism. The phase diagram of the setup has been analyzed for the behavioral study of the equilibrium (fixed) point with the nature of that point. Eigen-system leads to phase space diagram studied the regular and chaotic behavior. The fractional order values of $q = [q_1, q_2, q_3]$ for the 'LHB' model vary from $0 < q < 0.25$ for stable case, $0.25 < q < 0.5$ for critical case and have values $0.5 < q < 1$ for the chaotic phase. Thus, it is recommended that for extensive exercising and to avoid collapsing, primarily the breath function should be boosted so that the person can hold their breath for a comparatively longer period. This will in turn boost the stamina as more and more of ATP molecules will be formed and utilized.

In practical terms, the stable regulated regime provides a candidate quantitative marker of endurance: the stability margin of the oxygen-transport dynamics could be used to characterise cardio-respiratory fitness, to monitor recovery in physiotherapy, and to flag early dysregulation in health monitoring. The principal limitations are the single-subject parameterisation, the reliance on literature-based rather than individually measured parameters, and the idealised three-compartment structure; the chaotic extension, though physiologically motivated, remains to be tested against clinical data. Future work will pursue individual calibration against measured oxygen-uptake data, extension to time-varying workload, and experimental testing of the predicted stabilising role of physiological memory.

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Declarations

Conflict of Interest: No conflict of interest for publication of this article.

Data Availability Statement: No Data associated in the manuscript.

Table 3. Literature studies for the kinesiology-based models

Year	Author	Description	Parameters	Results
2016	Bhardwaj R.; Bangia A.	Complexity Dynamics of Meditating Body	Differential model developed for the human body	Analysis of the human body during different meditation phases to capture varied oxygen concentration-levels
2018	Teka W.W. et al.	Spiking as well as overflowing patterns of fractional-order Izhikevich model	Fractional order equations developed for Izhikevich model	Analyzed the parameters of the fractional order apt for simulation
2021	Bhardwaj R.; Bangia A.	Dynamical Indicator of Human Body's Physical Endurance	Fractional ordered SEQIR-based model was developed to study Coronavirus dynamics	Numerical simulations plotted and Lyapunov stability found the system to be asymptotically stable
2021	Ghita M., Copot D., Ionescu C. M.	Lung cancer dynamics using fractional order impedance modeling on a mimicked lung tumor setup	Fractional order impedance modelling was developed to study lung tumor	The results demonstrate significant differences among the mimicked healthy and tumor measurements, (p - values < 0.05) for impedance values and also for heterogeneity index
2023	Li Z., Pei Y., Wang Y. and Tian Q.	An enhanced respiratory mechanics model based on double-exponential and fractional calculus	Fractional respiratory mechanics fitted to real data	The hybrid model achieved significantly better results for different ventilation modes
2026	Alharbi N.	Existence, Optimal Control and Numerical Analysis of a Caputo Fractional Model for Oxygen Saturation Regulation	Caputo model of SpO ₂ regulation with memory and stability; ideal for the oxygen- transport strand	The numerical simulations supported the theoretical results and illustrated how the fractional order influences transient oxygen saturation dynamics
2026	Present study	Fractional Ordered Kinesiology Model	Fractional model developed to study regulated oxygen transport in the human body and, in a physiologically motivated extension, the parameters that may lead to chaotic, irregular oxygen levels	Fractional model analyzed through phase structures, time series and the largest Lyapunov exponent, establishing a stable regulated regime for all fractional orders and locating chaos in a physiologically motivated extension

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