

Unit ZLindley Distribution with Statistical Properties, Estimation, and Application to Engineering Data

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Abstract: This paper introduces a new one-parameter continuous probability model, called the Unit ZLindley (UZL) distribution, for modeling observations restricted to the unit interval (0,1). The proposed model is constructed by applying the transformation $Z = e^{-Y}$, where Y follows the ZLindley distribution. The resulting distribution possesses a simple analytical form and provides a flexible model for bounded data. Several statistical properties are investigated, including ordinary moments, central moments, conditional moments, moment generating function, characteristic function, mode, quantile function, Bowley skewness, Moore's kurtosis, mean residual life, mean past life, Bonferroni curve, Lorenz curve, and entropy measures. Five estimation procedures, namely maximum likelihood, Anderson–Darling, Cramér–von Mises, ordinary least-squares, and weighted least-squares, are considered. A Monte Carlo simulation framework is developed to assess the finite-sample behavior of the proposed estimators under different parameter values and sample sizes. Finally, the usefulness of the UZL distribution is illustrated using two real datasets associated with the P3 and SC16 algorithms. The proposed distribution is compared with several existing unit-interval distributions using likelihood-based criteria and goodness-of-fit statistics. It is evident from the results that the UZL distribution offers a convenient choice to fit positively skewed data on the unit interval.

Keywords: ZLindley distribution, bounded data, statistical inference, entropy, reliability

1 Introduction

Probability distributions are key components in statistical modeling because they provide a mathematical structure for describing uncertainty and representing the behavior of random variables. The choice of an appropriate probability distribution is particularly important when the response variable has a restricted support. An inappropriate distribution may assign probability to impossible values and may fail to adequately represent important characteristics such as skewness, concentration, tail behavior, and boundary effects [1].

A broad range of applications involves continuous observations constrained to the unit interval (0,1). Such results are normally presented in the form of proportions, rates, probabilities, indices, efficiencies, and standardizations. Some examples include cure rates in medical studies, environmental indices, reliability indices, economic proportions, and efficiency of algorithms. Conventional distributions defined on the positive real line or the entire real line are not always appropriate for such data because they do not respect the natural bounds of the response variable [2].

Consequently, considerable attention has been devoted to the advancement of probability distributions defined on the unit interval. Several unit models have been proposed, including the unit Birnbaum–Saunders [3], unit Weibull [4], unit log-logistic [5], unit Gompertz [6], unit Gamma [7], unit-half-normal [8], unit exponentiated Fréchet [9], unit Teissier [10], unit power Burr X [11], modified unit-half-normal [12], unit two-parameter Bilal [13], unit compound Rayleigh [14], unit Muth [15], sine power unit inverse Lindley [16], unit Haq [17], unit

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Komal [18], unit Mgamma [19], unit Shiha [20], unit Jamal [21], among others. Such models provide useful alternatives for bounded continuous data.

Despite the substantial development of unit-interval distributions, there remains a need for parsimonious models that combine mathematical tractability with adequate flexibility. Existing unit-interval models differ in their parameterization, complexity, density shapes, and ease of inference. A model with extra parameters allows for more flexibility; however, it can also mean the likelihood function becomes more complex and computationally difficult. This is especially important in the case of a very small or medium-sized sample. Moreover, data on the unit interval may have quite different skewness and concentration patterns, so there still exists a need for a simple model able to generate a variety of density shapes.

In recent research, Saaïdia et al. [22] presented the ZLindley (ZL) distribution, which is based on the polynomial exponential class of distributions. The ZL distribution is an analytically simple distribution with several important statistical properties. Its simple parameterization and tractable mathematical form make it a suitable parent distribution for constructing a bounded counterpart. In particular, the one-parameter structure is attractive because it allows the construction of a unit-interval model without introducing an additional shape parameter solely to obtain bounded support.

To obtain a probability model with unit interval support, we consider the transformation, $Z = e^{-Y}$, where Y follows the ZL distribution. This transformation is selected because it keeps a simple monotone mapping from the positive real line to the bounded support. Since $Y > 0$, $0 < e^{-Y} < 1$, and hence $0 < Z < 1$.

One of the most significant features of such a transformation is the change in the support of the parent distribution without altering the one-parameter structure of the distribution. As a result, unlike transformations which need extra parameters for the control of the boundaries of the transformation, the log transformation supplies the required unit support automatically. Also, the monotonicity of the transformation makes it easy to derive distributional functions from it.

The resultant distribution is called the Unit ZLindley (UZL) distribution. The suggested UZL distribution model possesses three properties, which are: bounded support, parameterization with only one parameter, and analytical simplicity. One parameter of the distribution affects the shape and concentration of the density function within the unit interval, as shown in density graphs and described in Table 1. It should be noted that the skewness property depends strongly on the parameter value, which means that asymmetry can be represented in many ways with the help of this model. Therefore, the UZL distribution provides a simple alternative to more highly parameterized unit-interval models when a parsimonious model is preferred.

The proposed distribution may be useful in several fields where continuous measurements naturally lie between zero and one. In medical and biomedical studies, potential applications include response proportions, recovery rates, and other normalized clinical measures. In environmental sciences, the model may be considered for environmental indices, normalized measurements, and proportions associated with environmental processes. In reliability and engineering, possible applications include reliability indices, normalized system performance measures, and efficiency measures. In economics and finance, bounded proportions, rates, and normalized indicators provide additional potential applications. The model may also be useful in computer science and algorithm evaluation, where accuracy, efficiency, and normalized performance measures can take values in $(0, 1)$. The two algorithm datasets considered later in this study therefore represent one practical setting in which the proposed distribution can be applied.

The principal contributions of this paper are:

- construction of a new one-parameter unit-interval distribution;
- derivation of its fundamental distributional functions;
- investigation of its moments and other statistical properties;
- derivation of reliability-related characteristics;
- development of five parameter-estimation procedures;
- assessment of estimator performance through Monte Carlo simulation; and
- demonstration of the practical usefulness of the proposed model using real datasets.

The remainder of the paper is organized as follows. The UZL distribution is introduced, and its main functions are obtained in Section 2. Statistical characteristics are studied in Section 3. Different estimation methods are developed in Section 4. The Monte Carlo simulation study is explained in Section 5. The applications are presented in Section 6. Section 7 concludes the paper.

2. Derivation of UZL Distribution

Let Y be a random variable following the ZL distribution with parameter $\zeta > 0$. Define a transformation $Z = e^{-Y}$, where $X \in (0, 1)$. For $0 < z < 1$, $Z \leq z \iff e^{-Y} \leq z \iff Y \geq -\ln(z)$.

Consequently,

$$F_Z(z; \zeta) = P(Y \geq -\ln(z)) = 1 - F_Y(-\ln(z)).$$

For the UZL distribution, the resulting cumulative distribution function (CDF) is

$$F(z; \zeta) = z^\zeta \left[1 - \frac{\zeta \ln(z)}{2(1+\zeta)} \right], \quad 0 < z < 1, \quad \zeta > 0. \quad (1)$$

The probability density function (PDF), which corresponds to the above equation, can be found using:

$$f(z; \zeta) = \frac{\zeta z^{\zeta-1}}{2(1+\zeta)} [1 + 2\zeta - \zeta \ln(z)]. \quad (2)$$

It is easy to confirm that $f(z; \zeta) > 0$ for $0 < z < 1$, $\zeta > 0$, and $\int_0^1 f(z; \zeta) dz = 1$. Hence, equation (2) defines a proper PDF.

Figure 1 provides the graphical representation of the UZL distribution.

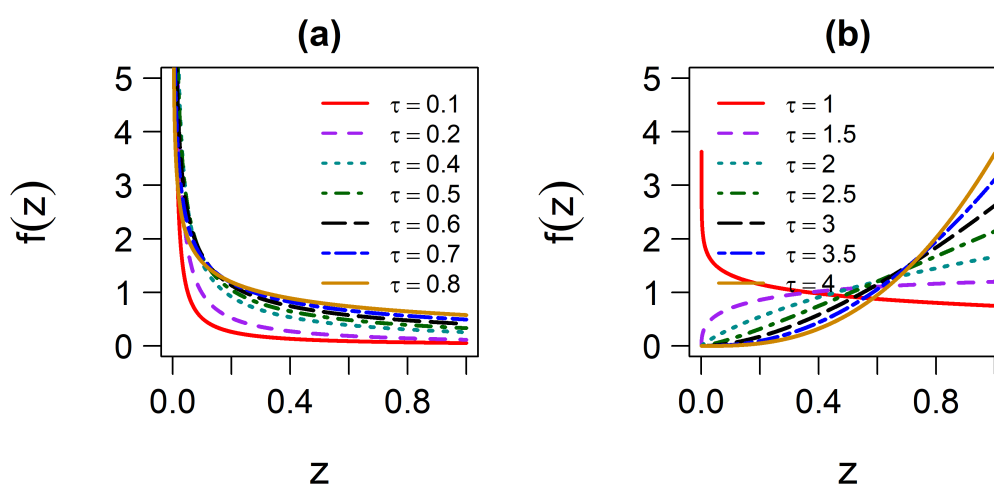


Figure 1: Graphs of PDFs at various parameter values of ζ .

2.1. Survival Function (SF)

The SF is $S(z; \zeta) = 1 - F(z; \zeta)$, and hence

$$S(z; \zeta) = 1 - z^\zeta \left[1 - \frac{\zeta \ln(z)}{2(1+\zeta)} \right]. \quad (3)$$

2.2. Hazard Rate Function (HRF)

The HRF is

$$h(z; \zeta) = \frac{\frac{\zeta z^{\zeta-1}}{2(1+\zeta)} [1 + 2\zeta - \zeta \ln(z)]}{1 - z^\zeta \left[1 - \frac{\zeta \ln(z)}{2(1+\zeta)} \right]}. \quad (4)$$

The hazard rate tends to infinity as $z \rightarrow 1^-$. Its behavior near zero depends on ζ , and numerical plots can be used to investigate the different shapes produced by the model. Figure 2 provides the failure rate behavior of the UZL distribution.

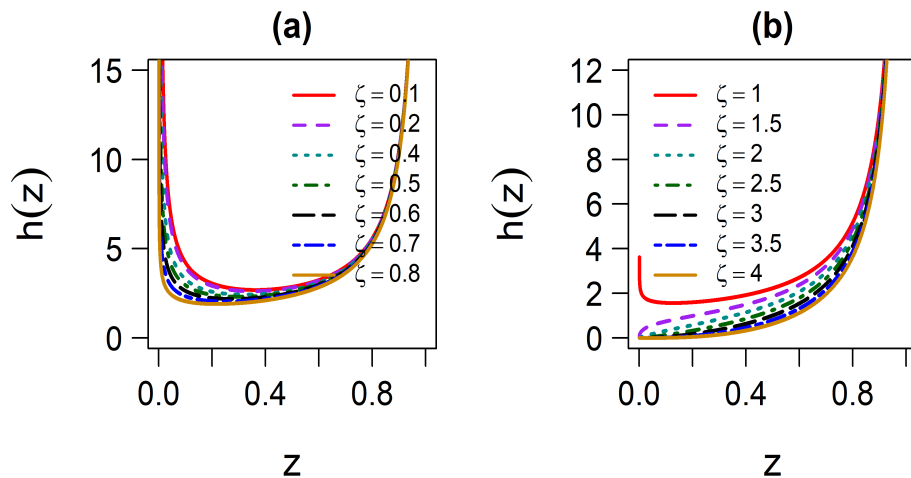


Figure 2: Hazard Rate plot for different values of parameter ζ .

2.3. Cumulative Hazard Function (CHF)

The CHF is

$$H(z; \zeta) = \int_0^z h(t) dt,$$

so that

$$H(z; \zeta) = -\ln \left\{ 1 - z^\zeta \left[1 - \frac{\zeta \ln(z)}{2(1+\zeta)} \right] \right\}. \quad (5)$$

2.4. Reverse Hazard Function (RHF)

The RHF is

$$r(z; \zeta) = \frac{f(z; \zeta)}{F(z; \zeta)}.$$

Hence,

$$r(z; \zeta) = \frac{\zeta (1 + 2\zeta - \zeta \ln(z))}{z(2(1+\zeta) - \zeta \ln(z))}. \quad (6)$$

2.5. Boundary Behavior

From equation (2),

$$f(z; \zeta) = \frac{\zeta}{2(1+\zeta)} z^{\zeta-1} (1 + 2\zeta - \zeta \ln(z)).$$

As $z \rightarrow 0^+$,

$$\lim_{z \rightarrow 0} f(z; \zeta) = \begin{cases} \infty & 0 < \theta \leq 1 \\ 0 & \theta > 1 \end{cases} \quad (8)$$

On the other hand, as $x \rightarrow 1$, the probability density function converges to a finite value.

$$\lim_{z \rightarrow 1^-} f(z; \zeta) = \frac{\zeta(1+2\zeta)}{2(1+\zeta)}. \quad (9)$$

Thus, unlike the earlier incorrectly transcribed CDF, the PDF does not converge to zero at $z = 1$. This is consistent with the PDF used to produce the original numerical moment table.

3. Statistical Properties

This section investigates several important statistical properties of the UZL distribution. Closed-form expressions are obtained for the raw moments, central moments, conditional moments, generating functions, characteristic function, and mode. The quantile function is also derived in terms of the Lambert W function. In addition, quantile-based measures of skewness and kurtosis, mean residual life, mean past life, Bonferroni and Lorenz curves, and several entropy measures are presented. These properties provide useful information concerning the location, dispersion, shape, reliability, and uncertainty characteristics of the UZL distribution.

3.1. Raw Moments

Let $Z \sim UZL(\zeta)$. The r -th raw moment of the Z is defined by

$$\mu'_r = E(Z^r) = \int_0^1 z^r f(z; \zeta) dz.$$

Using the PDF in Equation (2),

$$\mu'_r = \int_0^1 z^r \frac{\zeta}{2(1+\zeta)} z^{\zeta-1} (1 + 2\zeta - \zeta \ln(z)) dz.$$

Since $\int_0^1 z^{\alpha-1} dz = \frac{1}{\alpha}$ and $\int_0^1 z^{\alpha-1} \ln(z) dz = -\frac{1}{\alpha^2}$, we obtain

$$\mu'_r = \frac{\zeta}{2(1+\zeta)} \left[\frac{1+2\zeta}{r+\zeta} + \frac{\zeta}{(r+\zeta)^2} \right]. \quad (10)$$

Equivalently,

$$\mu'_r = \frac{\zeta \{ (1+2\zeta)(r+\zeta) + \zeta \}}{2(1+\zeta)(r+\zeta)^2}. \quad (11)$$

The first four moments are

$$\mu'_1 = \frac{\zeta (2\zeta^2 + 4\zeta + 1)}{2(1+\zeta)^3}. \quad (12)$$

$$\mu'_2 = \frac{\zeta (\zeta^2 + 3\zeta + 1)}{(1+\zeta)(2+\zeta)^2}. \quad (13)$$

$$\mu'_3 = \frac{\zeta (2\zeta^2 + 8\zeta + 3)}{2(1+\zeta)(3+\zeta)^2}. \quad (14)$$

and

$$\mu'_4 = \frac{\zeta (\zeta^2 + 5\zeta + 2)}{(1+\zeta)(4+\zeta)^2}. \quad (15)$$

3.2. Central Moments

Moments (or central moments) are significant descriptive statistics that describe the dispersion and shape of a probability distribution. In the case of the UZL distribution, the central moments are determined using the raw moments as follows.

$$\mu = \mu'_1.$$

$$\mu_2 = \mu'_2 - \mu^2,$$

$$\mu_3 = \mu'_3 - 3\mu\mu'_2 + 2\mu^3,$$

and

$$\mu_4 = \mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4.$$

In particular,

$$\text{Var}(Z) = \frac{\zeta (4 + 28\zeta + 68\zeta^2 + 67\zeta^3 + 28\zeta^4 + 4\zeta^5)}{4(1 + \zeta)^6(2 + \zeta)^2}. \quad (16)$$

The coefficient of variation (CV) is

$$\text{CV}(Z) = \frac{\sqrt{\mu_2}}{\mu}.$$

The coefficient of skewness is

$$\gamma_1 = \frac{\mu_2}{(\mu_3)^{\frac{3}{2}}},$$

and the coefficient of kurtosis is

$$\gamma_2 = \frac{\mu_4}{(\mu_2)^2}.$$

The numerical characteristics reported in the original Table 1 show that the mean increases with ζ , while the variance and coefficient of variation generally decrease. The skewness also changes from positive to negative as ζ increases.

Table 1: Descriptive characteristics of the UZL distribution

ζ	Mean	$E(Z^2)$	$E(Z^3)$	$E(Z^4)$	Var(Z)	SD(Z)	CV(Z)	CS(Z)	CK(Z)	DI(Z)
0.1	0.0533	0.0270	0.0181	0.0136	0.0242	0.1554	2.9138	3.7416	17.4000	0.4529
0.3	0.1625	0.0868	0.0591	0.0448	0.0604	0.2458	1.5125	1.7098	4.9436	0.3717
0.5	0.2592	0.1467	0.1020	0.0782	0.0794	0.2819	1.0872	1.0189	2.8459	0.3064
0.7	0.3405	0.2080	0.1441	0.1117	0.0868	0.2946	0.8653	0.6213	2.1434	0.2549
0.9	0.4081	0.2540	0.1840	0.1442	0.0875	0.2958	0.7248	0.3472	1.8806	0.2144
1.1	0.4644	0.3003	0.2215	0.1754	0.0846	0.2909	0.6264	0.1401	1.8059	0.1823
1.3	0.5118	0.3420	0.2565	0.2050	0.0801	0.2830	0.5530	-0.0250	1.8255	0.1565
1.5	0.5520	0.3796	0.2889	0.2331	0.0749	0.2737	0.4957	-0.1620	1.8966	0.1357
1.7	0.5864	0.4135	0.3189	0.2595	0.0695	0.2637	0.4497	-0.2780	1.9972	0.1186
1.9	0.6162	0.4441	0.3468	0.2844	0.0644	0.2537	0.4117	-0.3780	2.1147	0.1044
2.1	0.6422	0.4719	0.3727	0.3078	0.0595	0.2439	0.3799	-0.4660	2.2419	0.0927
2.3	0.6650	0.4972	0.3967	0.3299	0.0550	0.2345	0.3527	-0.5440	2.3740	0.0827
2.5	0.6851	0.5203	0.4191	0.3508	0.0509	0.2256	0.3292	-0.6140	2.5080	0.0743
2.7	0.7031	0.5414	0.4400	0.3705	0.0471	0.2171	0.3087	-0.6770	2.6420	0.0670
2.9	0.7191	0.5609	0.4595	0.3890	0.0437	0.2090	0.2907	-0.7350	2.7746	0.0608
3.1	0.7336	0.5788	0.4777	0.4066	0.0406	0.2015	0.2746	-0.7870	2.9050	0.0553
3.3	0.7464	0.5953	0.4948	0.4232	0.0378	0.1944	0.2603	-0.8350	3.0325	0.0506
3.5	0.7586	0.6106	0.5108	0.4390	0.0352	0.1877	0.2474	-0.8790	3.1569	0.0464
3.7	0.7694	0.6249	0.5259	0.4540	0.0329	0.1814	0.2357	-0.9190	3.2779	0.0427
3.9	0.7793	0.6381	0.5401	0.4682	0.0308	0.1754	0.2251	-0.9580	3.3954	0.0395

3.3. Conditional Moments

The conditional moment of order r , given by $Z > z$, is

$$E(Z^r | Z > z) = \frac{1}{S(z; \zeta)} \int_z^1 t^r f(t; \zeta) dt.$$

Substituting the density function from equation (2) yields

$$E(Z^r | Z > z) = \frac{1}{S(z; \zeta)} \int_z^1 t^r \frac{\zeta t^{\zeta-1}}{2(1+\zeta)} (1 + 2\zeta - \zeta \ln(t)) dt,$$

Separating the integral gives

$$E(Z^r | Z > z) = \frac{\zeta}{2(1+\zeta)S(z; \zeta)} \left[(1+2\zeta) \int_z^1 t^{r+\zeta-1} dt - \zeta \int_z^1 t^{r+\zeta-1} \ln(t) dt \right].$$

Considering the following standard integrals

$$\int_z^1 t^{\alpha-1} dt = \frac{1-z^\alpha}{\alpha},$$

and

$$\int_z^1 t^{\alpha-1} \ln(t) dt = \frac{z^\alpha \ln(z)}{\alpha} + \frac{z^\alpha}{\alpha^2},$$

We obtain

$$E(Z^r | Z > z) = \frac{\zeta}{2(1+\zeta)S(z; \zeta)} \left[\frac{(1-z^{r+\zeta})(1+2\zeta)}{r+\zeta} - \frac{\zeta z^{r+\zeta} \ln(z)}{r+\zeta} - \frac{\zeta(z^{r+\zeta}-1)}{(r+\zeta)^2} \right]. \quad (17)$$

3.4. Moment Generating Function (MGF)

The MGF is calculated from $M_Z(t) = E(e^{tZ})$. Using $e^{tz} = \sum_{r=0}^{\infty} \frac{t^r z^r}{r!}$, we obtain

$$M_Z(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r.$$

Consequently,

$$M_Z(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\zeta \{(1+2\zeta)(r+\zeta) + \zeta\}}{2(1+\zeta)(r+\zeta)^2}. \quad (18)$$

This series representation provides a straightforward method to compute moments of any order and analyze the distribution's behavior under linear transformations of the random variable.

3.5. Characteristic Function (CF)

The CF is computed from next formula $\phi_Z(t) = E(e^{itZ})$, and therefore

$$\phi_Z(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mu'_r.$$

Thus,

$$\phi_Z(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{\zeta \{(1+2\zeta)(r+\zeta) + \zeta\}}{2(1+\zeta)(r+\zeta)^2}. \quad (19)$$

3.6 Mode

Differentiating the density gives

$$\frac{\partial f(z; \zeta)}{\partial z} = \frac{\zeta z^{\zeta-1}}{2(1+\zeta)} [(\zeta-1)(1+2\zeta - \zeta \ln(z)) - \zeta].$$

Setting the derivative equal to zero yields

$$\ln(z) = \frac{2\zeta^2 - 2\zeta - 1}{\zeta(\zeta-1)}.$$

Therefore, the interior stationary point is

$$z_m = \exp\left(\frac{2\zeta^2 - 2\zeta - 1}{\zeta(\zeta-1)}\right).$$

An interior mode exists for

$$1 < \zeta < \frac{1 + \sqrt{3}}{2}.$$

For $0 < \zeta \leq 1$, the density is unbounded at zero, while for sufficiently large ζ , the density increases toward the upper boundary.

3.7. Quantile Function

Let $Q(u)$ be the quantile function. It satisfies

$$u = Q(u)^\zeta \left[1 - \frac{\zeta \ln Q(u)}{2(1 + \zeta)} \right].$$

After transformation and application of the Lambert W function,

$$Q(u) = \exp \left[\frac{2(1 + \zeta)}{\zeta} + \frac{1}{\zeta} W_{-1} \left\{ -2u(1 + \zeta)e^{-2(1 + \zeta)} \right\} \right]. \quad (20)$$

3.8. Mean Residual Life (MRL)

The MRL function is a useful reliability and survival analysis tool. It is the mathematical expectation of the remaining life of a random variable Z , when it has survived to time t . Formally, it is defined as

$$m(t) = \frac{1}{S(t; \zeta)} \int_t^1 S(z; \zeta) dz.$$

Since

$$S(z; \zeta) = 1 - z^\zeta + \frac{\zeta \ln(z)}{2(1 + \zeta)},$$

we obtain

$$\int_t^1 S(z; \zeta) dz = 1 - t - \frac{1 - t^{\zeta+1}}{1 + \zeta} + \frac{\zeta}{2(1 + \zeta)} \left[\frac{t^{\zeta+1}}{(\zeta + 1)^2} - \frac{t^{\zeta+1} \ln(t)}{\zeta + 1} \right].$$

Hence,

$$m(t) = \frac{1 - t - \frac{1 - t^{\zeta+1}}{1 + \zeta} + \frac{\zeta}{2(1 + \zeta)} \left[\frac{t^{\zeta+1}}{(\zeta + 1)^2} - \frac{t^{\zeta+1} \ln(t)}{\zeta + 1} \right]}{S(t; \zeta)}. \quad (21)$$

3.9. Mean Past Life (MPL)

The MPL is

$$m_p(t) = \frac{1}{F(z; \zeta)} \int_0^t z f(z; \zeta) dz.$$

The numerator is

$$\int_0^t z f(z; \zeta) dz = \frac{\zeta t^{\zeta+1}}{2(1 + \zeta)} \left[\frac{1 + 2\zeta - \zeta \ln(t)}{\zeta + 1} + \frac{\zeta}{(1 + \zeta)^2} \right].$$

Therefore,

$$m_p(t) = \frac{\frac{\zeta t^{\zeta+1}}{2(1 + \zeta)} \left[\frac{1 + 2\zeta - \zeta \ln(t)}{\zeta + 1} + \frac{\zeta}{(1 + \zeta)^2} \right]}{F(z; \zeta)}. \quad (22)$$

The corresponding expected elapsed time is $t - m_p(t)$.

3.10. Bonferroni and Lorenz Curves

Let $\mu = E(Z)$. The Lorenz curve is

$$L(p) = \frac{1}{\mu} \int_0^{Q(p)} z f(z; \zeta) dz.$$

and the Bonferroni curve is

$$B(p) = \frac{1}{p\mu} \int_0^{Q(p)} z f(z; \zeta) dz.$$

Let $q = Q(p)$. Then

$$\int_0^q z f(z; \zeta) dz = \frac{\zeta q^{\zeta+1}}{2(1+\zeta)} \left[\frac{1+2\zeta - \zeta \ln(q)}{\zeta+1} + \frac{\zeta}{(1+\zeta)^2} \right].$$

Consequently,

$$L(p) = \frac{\zeta q^{\zeta+1}}{2(1+\zeta)\mu} \left[\frac{1+2\zeta - \zeta \ln(q)}{\zeta+1} + \frac{\zeta}{(1+\zeta)^2} \right],$$

and

$$B(p) = \frac{L(p)}{p}.$$

3.11. Rényi Entropy

For $v > 0, v \neq 1$, the Rényi entropy is

$$R_v = \frac{1}{1-v} \ln \left[\int_0^1 f(z; \zeta)^v dz \right].$$

For integer v ,

$$[1+2\zeta - \zeta \ln(z)]^v = \sum_{k=0}^v \binom{v}{k} (1+2\zeta)^{v-k} (-\zeta \ln(z))^k.$$

Let

$$\alpha_v = 1 + v(\zeta - 1).$$

Using

$$\int_0^1 z^{\alpha_v-1} (-\ln(z))^k dz = \frac{k!}{\alpha_v^{k+1}},$$

we obtain

$$R_v = \frac{1}{1-v} \ln \left[\left\{ \frac{\zeta}{2(1+\zeta)} \right\}^v \sum_{k=0}^v \binom{v}{k} (1+2\zeta)^{v-k} \zeta^k \frac{k!}{[1+v(\zeta-1)]^{k+1}} \right]. \quad (23)$$

The convergence condition is

$$1 + v(\zeta - 1) > 0.$$

3.12. Havrda–Charvát Entropy

The Havrda–Charvát entropy is

$$HC_v = \frac{\int_0^1 (f(z; \zeta))^v dz - 1}{2^{1-v} - 1}.$$

Hence,

$$HC_v = \frac{I_v - 1}{2^{1-v} - 1},$$

where

$$I_v = \left\{ \frac{\zeta}{2(1+\zeta)} \right\}^v \sum_{k=0}^v \binom{v}{k} (1+2\zeta)^{v-k} \zeta^k \frac{k!}{[1+v(\zeta-1)]^{k+1}}.$$

3.13. Tsallis Entropy

The Tsallis entropy is

$$T_v = \frac{1 - I_v}{v - 1},$$

Hence,

$$T_v = \frac{1 - \left\{ \frac{\zeta}{2(1+\zeta)} \right\}^v \sum_{k=0}^v \binom{v}{k} (1+2\zeta)^{v-k} \zeta^k \frac{k!}{[1+v(\zeta-1)]^{k+1}}}{v - 1}.$$

4. Estimation of the Parameter

Let $Z_1, Z_2, \dots, Z_n$ be a random sample from the UZL distribution. The unknown parameter ζ is estimated to be using five procedures.

4.1. Maximum Likelihood Estimation (MLE)

The likelihood function is

$$L(\zeta) = \left[\frac{\zeta}{2(1+\zeta)} \right]^n \prod_{i=1}^n z_i^{\zeta-1} \prod_{i=1}^n (1+2\zeta - \zeta \ln(z_i)).$$

The log-likelihood is

$$l(\zeta) = n \ln(\zeta) - n \ln[2(1+\zeta)] + (\zeta - 1) \sum_{i=1}^n \ln(z_i) + \sum_{i=1}^n \ln(1+2\zeta - \zeta \ln(z_i)). \quad (24)$$

Differentiating,

$$\frac{\partial l(\zeta)}{\partial \zeta} = \frac{n}{\zeta} - \frac{n}{1+\zeta} + \sum_{i=1}^n \ln(z_i) + \sum_{i=1}^n \frac{2 - \zeta \ln(z_i)}{1+2\zeta - \zeta \ln(z_i)}. \quad (25)$$

The MLE $\hat{\zeta}_{MLE}$ is obtained by solving $\frac{\partial l(\zeta)}{\partial \zeta} = 0$.

Because Equation $\frac{\partial l(\zeta)}{\partial \zeta}$ does not have a convenient closed-form solution, numerical optimization such as BFGS or one-dimensional bounded optimization can be employed.

The observed information is

$$J(\zeta) = -\frac{d^2 l(\zeta)}{d\zeta^2},$$

where

$$\frac{d^2 l(\zeta)}{d\zeta^2} = -\frac{n}{\zeta^2} + \frac{n}{(1+\zeta)^2} - \sum_{i=1}^n \frac{(2 - \ln(z_i))^2}{(1 + 2\zeta - \zeta \ln(z_i))^2}. \quad (26)$$

An approximate standard error is

$$SE(\hat{\zeta}_{MLE}) = J\sqrt{\hat{\zeta}_{MLE}}.$$

4.2. Anderson–Darling Estimation (ADE)

Let $z_{(1)} < z_{(2)} < \dots < z_{(n)}$ denote the ordered observations. The Anderson–Darling objective function is

$$A(\zeta) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) [\log F(z_{(i)}; \zeta) + \log S(z_{(n+1-i)}; \zeta)].$$

The ADE is

$$\hat{\zeta}_{ADE} = \arg \min_{\zeta > 0} A(\zeta).$$

This estimator places reasonably greater importance on discrepancies in the tails.

4.3. Cramér–von Mises Estimation (CvME)

The CvM estimation criterion is

$$W(\zeta) = \frac{1}{12n} + \sum_{i=1}^n \left[F(z_{(i)}; \zeta) - \frac{2i-1}{2n} \right]^2.$$

The estimator is

$$\hat{\zeta}_{CvME} = \arg \min_{\zeta > 0} W(\zeta).$$

4.4. Ordinary Least-Squares Estimation (OLSE)

The OLS estimation criterion is

$$L_{OLSE}(\zeta) = \sum_{i=1}^n \left[F(z_{(i)}; \zeta) - \frac{i}{n+1} \right]^2.$$

Thus,

$$\hat{\zeta}_{OLSE} = \arg \min_{\zeta > 0} L_{OLSE}(\zeta).$$

4.5. Weighted Least-Squares Estimation (WLSE)

A WLS estimator can be obtained by assigning greater weight to observations having greater uncertainty in their empirical distribution positions.

Define

$$w_i = \frac{(n+1)^2(n+2)}{i(n+1-i)}.$$

The weighted criterion is

$$L_{WLSE}(\zeta) = \sum_{i=1}^n w_i \left[F(z_{(i)}; \zeta) - \frac{i}{n+1} \right]^2.$$

The WLS estimator is

$$\hat{\zeta}_{WLSE} = \arg \min_{\zeta > 0} L_{WLSE}(\zeta).$$

5. Monte Carlo Simulation Study

A detailed Monte Carlo simulation study is used to investigate the finite-sample behavior of the five considered estimation methods. The study considers $n = \{25, 50, 100, 200, 300\}$ and multiple parameter values, with 10,000 replications for each configuration. The parameter values considered are $\zeta \in \{0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0\}$. For each combination of ζ and n , a random sample is generated from the UZL distribution using its quantile function. For each replication, the five estimators $\hat{\zeta}_{MLE}$, $\hat{\zeta}_{ADE}$, $\hat{\zeta}_{CvME}$, $\hat{\zeta}_{LSE}$, and $\hat{\zeta}_{WLSE}$ are calculated. The following performance measures are considered.

Absolute Bias (AB)

$$AB = \left| E\left(\hat{\zeta}\right) - \zeta \right|.$$

Mean Relative Error (MRE)

$$MRE = E\left(\frac{\left|\hat{\zeta} - \zeta\right|}{\zeta}\right).$$

Mean Square Error (MSE)

$$MSE = E\left[\left(\hat{\zeta} - \zeta\right)^2\right].$$

The preferred estimator is the one that produces comparatively smaller values of these measures.

5.1. Simulation Algorithm

For each selected ζ and sample size n :

1. Generate $U_1, \dots, U_n \sim U(0, 1)$.
2. Calculate $X_i = Q(U_i; \zeta)$ using the quantile function.
3. Estimate ζ using MLE, ADE, CvME, OLSE, and WLSE.
4. Repeat the procedure 10,000 times.
5. Calculate AE, AB, MRE, and MSE.
6. Repeat for all parameter and sample-size combinations.

Table 2: AE of the estimators for different values of ζ .

ζ	n	$\hat{\zeta}_{MLE}$	$\hat{\zeta}_{ADE}$	$\hat{\zeta}_{CvME}$	$\hat{\zeta}_{LSE}$	$\hat{\zeta}_{WLSE}$
0.5	25	0.517212	0.510357	0.514558	0.511475	0.510468
	50	0.508320	0.505278	0.507367	0.505805	0.505455
	100	0.504402	0.502774	0.503818	0.503034	0.503016
	200	0.502011	0.501260	0.501768	0.501373	0.501409
	300	0.501267	0.500736	0.501040	0.500777	0.500864
1.0	25	1.036071	1.022686	1.031454	1.025283	1.023346
	50	1.015465	1.009544	1.013951	1.010821	1.010173
	100	1.005401	1.002366	1.004391	1.002819	1.002888
	200	1.003058	1.001450	1.002469	1.001682	1.001826
	300	1.003111	1.002094	1.002790	1.002263	1.002365
1.5	25	1.554642	1.534996	1.548853	1.539464	1.535741
	50	1.524248	1.514525	1.521158	1.516403	1.515560
	100	1.512713	1.507313	1.510095	1.507692	1.507805
	200	1.506947	1.504385	1.505878	1.504673	1.504853
	300	1.502951	1.501369	1.502307	1.501504	1.501797
2.0	25	2.076984	2.048093	2.067265	2.054525	2.050764
	50	2.037694	2.024493	2.034023	2.027564	2.025892
	100	2.019082	2.012475	2.017011	2.013768	2.013638
	200	2.007746	2.004352	2.006576	2.004946	2.005113
	300	2.006067	2.004427	2.006061	2.004971	2.005006
2.5	25	2.592155	2.555283	2.578705	2.562539	2.558592
	50	2.542477	2.525784	2.537200	2.528970	2.527355
	100	2.521831	2.514728	2.520631	2.516501	2.515782

	200	2.510635	2.506765	2.509520	2.507450	2.507508
	300	2.507054	2.504269	2.506322	2.504942	2.504937
	25	3.120692	3.079304	3.107441	3.087586	3.081469
	50	3.053680	3.030915	3.043909	3.033928	3.032672
3.0	100	3.021159	3.009882	3.016370	3.011373	3.011527
	200	3.012388	3.007348	3.010220	3.007702	3.008402
	300	3.009040	3.005279	3.007159	3.005480	3.006174
	25	3.620162	3.572659	3.608781	3.585786	3.576509
	50	3.563118	3.543025	3.561738	3.549983	3.545998
3.5	100	3.531582	3.517138	3.524103	3.518192	3.519125
	200	3.515929	3.509396	3.513321	3.510364	3.510541
	300	3.506022	3.502935	3.505912	3.503939	3.503770
	25	4.149202	4.089841	4.128507	4.101855	4.094407
	50	4.076582	4.047563	4.065985	4.052469	4.050320
4.0	100	4.036238	4.021147	4.029511	4.022699	4.023333
	200	4.013484	4.006431	4.011297	4.007889	4.008094
	300	4.009875	4.004081	4.006706	4.004431	4.005123

Table 3: AB of the estimators for different values of ζ .

ζ	n	$\hat{\zeta}_{MLE}$	$\hat{\zeta}_{ADE}$	$\hat{\zeta}_{CvME}$	$\hat{\zeta}_{LSE}$	$\hat{\zeta}_{WLSE}$
0.5	25	0.017212	0.010357	0.014558	0.011475	0.010468
	50	0.008320	0.005278	0.007367	0.005805	0.005455
	100	0.004402	0.002774	0.003818	0.003034	0.003016
	200	0.002011	0.001260	0.001768	0.001373	0.001409
	300	0.001267	0.000736	0.001040	0.000777	0.000864
1.0	25	0.036071	0.022686	0.031454	0.025283	0.023346
	50	0.015465	0.009544	0.013951	0.010821	0.010173
	100	0.005401	0.002366	0.004391	0.002819	0.002888
	200	0.003058	0.001450	0.002469	0.001682	0.001826
	300	0.003111	0.002094	0.002790	0.002263	0.002365
1.5	25	0.054642	0.034996	0.048853	0.039464	0.035741
	50	0.024248	0.014525	0.021158	0.016403	0.015560
	100	0.012713	0.007313	0.010095	0.007692	0.007805
	200	0.006947	0.004385	0.005878	0.004673	0.004853
	300	0.002951	0.001369	0.002307	0.001504	0.001797
2.0	25	0.076984	0.048093	0.067265	0.054525	0.050764
	50	0.037694	0.024493	0.034023	0.027564	0.025892
	100	0.019082	0.012475	0.017011	0.013768	0.013638
	200	0.007746	0.004352	0.006576	0.004946	0.005113
	300	0.006067	0.004427	0.006061	0.004971	0.005006
2.5	25	0.092155	0.055283	0.078705	0.062539	0.058592
	50	0.042477	0.025784	0.037200	0.028970	0.027355
	100	0.021831	0.014728	0.020631	0.016501	0.015782
	200	0.010635	0.006765	0.009520	0.007450	0.007508
	300	0.007054	0.004269	0.006322	0.004942	0.004937
3.0	25	0.120692	0.079304	0.107441	0.087586	0.081469
	50	0.053680	0.030915	0.043909	0.033928	0.032672
	100	0.021159	0.009882	0.016370	0.011373	0.011527
	200	0.012388	0.007348	0.010220	0.007702	0.008402
	300	0.009040	0.005279	0.007159	0.005480	0.006174
3.5	25	0.120162	0.072659	0.108781	0.085786	0.076509
	50	0.063118	0.043025	0.061738	0.049983	0.045998
	100	0.031582	0.017138	0.024103	0.018192	0.019125
	200	0.015929	0.009396	0.013321	0.010364	0.010541
	300	0.006022	0.002935	0.005912	0.003939	0.003770
4.0	25	0.149202	0.089841	0.128507	0.101855	0.094407
	50	0.076582	0.047563	0.065985	0.052469	0.050320
	100	0.036238	0.021147	0.029511	0.022699	0.023333
	200	0.013484	0.006431	0.011297	0.007889	0.008094
	300	0.009875	0.004081	0.006706	0.004431	0.005123

Table 4: MRE of the estimators for different values of ζ .

ζ	n	$\hat{\zeta}_{MLE}$	$\hat{\zeta}_{ADE}$	$\hat{\zeta}_{CvME}$	$\hat{\zeta}_{LSE}$	$\hat{\zeta}_{WLSE}$
0.5	25	0.144886	0.152779	0.163856	0.163278	0.156543

	50	0.098286	0.104426	0.111096	0.110828	0.105974
	100	0.069135	0.074379	0.078953	0.078867	0.075005
	200	0.049031	0.053112	0.056167	0.056139	0.053266
	300	0.040505	0.043993	0.046449	0.046437	0.044077
1.0	25	0.146040	0.153926	0.165167	0.164546	0.157911
	50	0.098837	0.105577	0.112380	0.112164	0.107134
	100	0.069970	0.075513	0.080010	0.079978	0.076084
	200	0.049593	0.053939	0.057020	0.057001	0.054122
	300	0.040628	0.043761	0.046219	0.046202	0.043827
1.5	25	0.148880	0.156236	0.167570	0.166790	0.160410
	50	0.102302	0.108211	0.114847	0.114593	0.109701
	100	0.071389	0.076046	0.080385	0.080304	0.076454
	200	0.050233	0.054164	0.057217	0.057183	0.054301
	300	0.040300	0.043596	0.046231	0.046223	0.043661
2.0	25	0.151393	0.159381	0.171156	0.170380	0.163792
	50	0.103739	0.111271	0.118464	0.118161	0.112607
	100	0.072406	0.077845	0.082606	0.082517	0.078422
	200	0.050726	0.054788	0.058022	0.057988	0.054975
	300	0.041601	0.044791	0.047458	0.047434	0.044931
2.5	25	0.150301	0.157821	0.169811	0.169062	0.162179
	50	0.104680	0.111411	0.118450	0.118177	0.112884
	100	0.073065	0.078734	0.083601	0.083496	0.079268
	200	0.050306	0.054677	0.057959	0.057931	0.054888
	300	0.041724	0.044737	0.047218	0.047195	0.044840
3.0	25	0.154317	0.163609	0.175696	0.174944	0.167959
	50	0.104852	0.111802	0.119036	0.118763	0.113316
	100	0.073527	0.079295	0.084107	0.084033	0.079750
	200	0.051093	0.055178	0.058403	0.058371	0.055348
	300	0.042561	0.045445	0.048148	0.048139	0.045586
3.5	25	0.153746	0.161833	0.174087	0.173353	0.166194
	50	0.105211	0.112134	0.119903	0.119599	0.113973
	100	0.074437	0.079448	0.084211	0.084114	0.080063
	200	0.051643	0.055880	0.059073	0.059033	0.056089
	300	0.042088	0.045749	0.048437	0.048427	0.045873
4.0	25	0.155176	0.165348	0.178155	0.177339	0.170080
	50	0.107045	0.113347	0.120707	0.120412	0.114892
	100	0.073924	0.079845	0.084727	0.084638	0.080334
	200	0.052401	0.056546	0.060004	0.059972	0.056708
	300	0.042852	0.046288	0.048925	0.048908	0.046343

Table 5: MSE of the estimators for different values of ζ .

ζ	n	$\hat{\zeta}_{MLE}$	$\hat{\zeta}_{ADE}$	$\hat{\zeta}_{CvME}$	$\hat{\zeta}_{LSE}$	$\hat{\zeta}_{WLSE}$
0.5	25	0.009004	0.009885	0.011666	0.011501	0.010536
	50	0.003989	0.004484	0.005121	0.005079	0.004645
	100	0.001914	0.002225	0.002530	0.002519	0.002272
	200	0.000954	0.001112	0.001249	0.001246	0.001120
	300	0.000643	0.000758	0.000848	0.000847	0.000762
1.0	25	0.036656	0.040090	0.047434	0.046743	0.043006
	50	0.015955	0.018196	0.020939	0.020770	0.018893
	100	0.007800	0.009054	0.010241	0.010205	0.009230
	200	0.003936	0.004585	0.005147	0.005138	0.004627
	300	0.002601	0.002991	0.003329	0.003324	0.003006
1.5	25	0.085667	0.093669	0.110665	0.108878	0.100198
	50	0.037967	0.042379	0.048607	0.048200	0.043980
	100	0.018413	0.020907	0.023492	0.023390	0.021212
	200	0.009002	0.010383	0.011619	0.011591	0.010452
	300	0.005884	0.006860	0.007708	0.007699	0.006895
2.0	25	0.158254	0.172551	0.206741	0.203522	0.187184
	50	0.071296	0.081203	0.093443	0.092583	0.083982
	100	0.033443	0.038887	0.044325	0.044117	0.039697
	200	0.016370	0.018969	0.021325	0.021278	0.019148
	300	0.011011	0.012790	0.014345	0.014321	0.012888
2.5	25	0.244712	0.264562	0.315514	0.310700	0.284715
	50	0.113032	0.126126	0.144266	0.142996	0.130667
	100	0.053370	0.061934	0.070361	0.070028	0.063179
	200	0.025326	0.029485	0.033149	0.033070	0.029733
	300	0.017072	0.019692	0.022094	0.022059	0.019825
3.0	25	0.373014	0.419431	0.500670	0.492155	0.451374

	50	0.162888	0.184492	0.212026	0.210187	0.190896
	100	0.077945	0.090221	0.102448	0.102043	0.091801
	200	0.037432	0.043595	0.049009	0.048896	0.043904
	300	0.025751	0.029478	0.033078	0.033027	0.029692
3.5	25	0.491665	0.537322	0.642449	0.632724	0.579195
	50	0.223742	0.254054	0.295403	0.292610	0.265249
	100	0.109697	0.125168	0.141629	0.141000	0.127825
	200	0.051879	0.060620	0.068091	0.067925	0.061188
	300	0.034630	0.040575	0.045542	0.045481	0.040846
4.0	25	0.663384	0.735771	0.882019	0.868150	0.793487
	50	0.301813	0.337913	0.388783	0.385230	0.350246
	100	0.140636	0.162657	0.184248	0.183399	0.165442
	200	0.069084	0.080979	0.091623	0.091430	0.081621
	300	0.046326	0.053818	0.060305	0.060223	0.054052

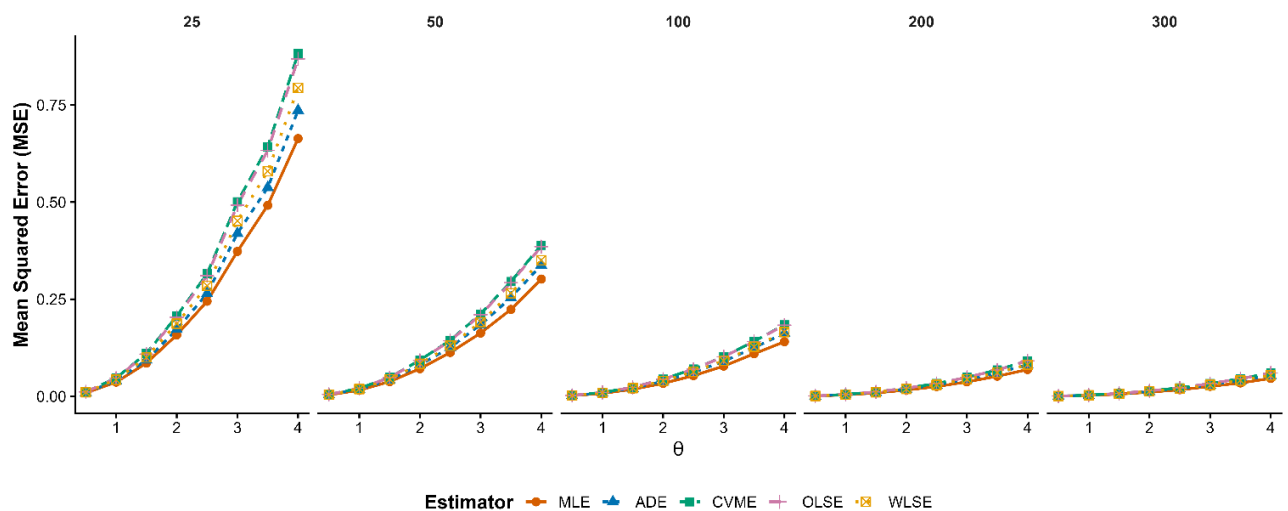


Figure 3: MSE of the MLE, ADE, CVME, OLSE, and WLSE estimators of the UZL distribution across different values of ζ and sample sizes.

It is clear from the simulation findings that all five estimators, i.e., MLE, ADE, CVME, OLSE, and WLSE, perform well in the estimation of the UZL distribution parameter. Overall, the values of AB, MRE, and MSE decrease considerably as the sample size increases from $n = 25$ to $n = 300$, thus showing better estimation performance when the sample size is increased. With a constant sample size, the value of MSE is likely to increase with an increase in ζ , while the value of MRE remains stable. On the other hand, the ADE approach usually offers the minimum value of the AB, especially when the sample size is moderate and large. WLSE shows good performance and works better than OLSE and CVME. The results between OLSE and CVME are almost identical. It can be concluded from the simulation that the new UZL model can be estimated accurately using MLE.

6. Applications

In this section, we analyze two real-world data sets to examine the practical performance of the UZL distribution. The primary goal is to study its flexibility and goodness-of-fit relative to a number of other known distributions. The applicability of this model to empirical data will be illustrated in order to prove its suitability for describing real-life situations and possible superiority over existing approaches.

For comparison, the following distributions are considered:

- Generalized exponential uniform (GEU) [1],
- Unit Burr-III (UBIII) [23],
- Unit Lindley (UL) [24],
- Unit Haq (UH) [17],
- Unit Xgamma (UXg) [25],
- Unit Zeghdoudi (UZ) [26].

The parameters of all competing models are obtained through maximum likelihood estimation (MLE). Their adequacy is subsequently assessed using several established goodness-of-fit and model-selection measures, namely

the log-likelihood value (ℓ), Akaike information criterion (AIC), Bayesian information criterion (BIC), Kolmogorov–Smirnov (K–S) statistic, and the corresponding K–S p -value.

6.1. Dataset I: P3 Algorithm

The first empirical dataset represents the unit capacity measurements obtained from the P3 algorithm. This dataset was previously investigated in [14] and comprises 22 observations. The recorded values are

0.010, 0.014, 0.019, 0.026, 0.036, 0.044, 0.056, 0.067,
0.078, 0.097, 0.118, 0.118, 0.207, 0.334, 0.399, 0.503,
0.557, 0.716, 0.759, 0.800, 0.853, 0.874.

These values fall within the unit interval range $(0, 1)$ and hence make the dataset a suitable candidate for testing the effectiveness of the bounded-data distribution model in describing real-life measurements. The dataset is then compared to other rival models on the basis of the chosen statistical measures.

For this dataset, the minimum and maximum values are 0.0100 and 0.8740, respectively. The first sample moment is approximately 0.3039, and the median is 0.1180. The variance is 0.1010, reflecting the dispersion of the observations. The skewness coefficient of 0.7114 indicates moderate positive skewness, suggesting a right-tailed distribution that is not highly skewed. The kurtosis value of 1.8837 indicates that the distribution is flatter than the normal distribution. Table 6 presents the estimated model parameters together with their subsequent goodness-of-fit statistics for this P3 algorithm dataset.

Table 6: Parameter estimates and goodness-of-fit statistics for the P3 Algorithm dataset.

Candidate model	MLE (SE)		ℓ	AIC score	BIC score	K-S distance	Significance level (p)
	ζ	β					
GEUD	0.1989 (0.1595)	0.2875 (0.2054)	2.3121	-0.6242	1.5578	0.3365	0.0137
UBIII	0.4912 (0.1444)	1.0317 (0.2533)	6.5938	-9.1877	-7.0057	0.2050	0.3134
UL	1.1929 (0.1948)	—	-6.5826	15.1652	16.2562	0.4600	0.0002
UH	3.6064 (0.6982)	—	-4.2786	-6.5572	-5.4661	0.2148	0.2623
UXg	1.3844 (0.2156)	—	-3.9081	9.8163	10.9074	0.4469	0.0003
UZ	2.0439 (0.2630)	—	-35.4901	72.9800	74.0713	0.5317	0.0000
UZL	0.6586 (0.1211)	—	6.9118	-11.8237	-10.7327	0.1969	0.3614

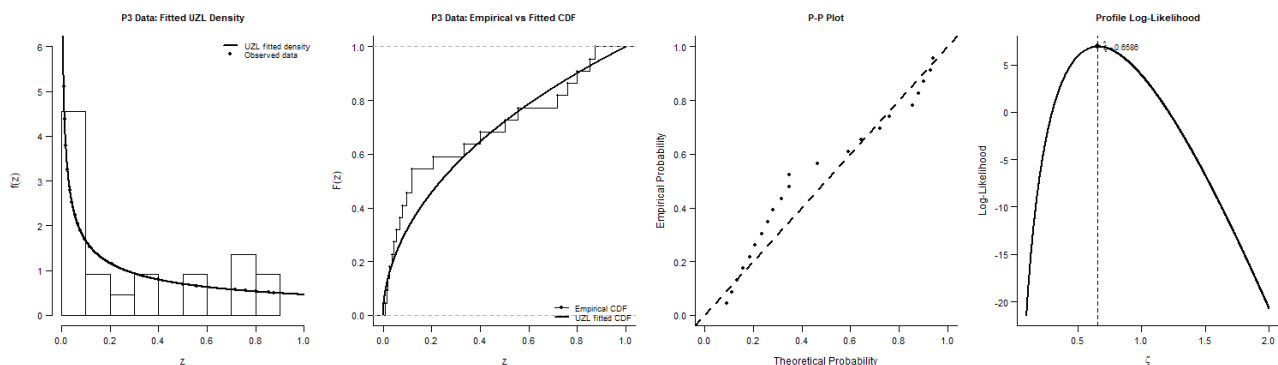


Figure 4: Fitted PDF, CDF, P-P, and profile log-likelihood plots of the UZL distribution for P3 algorithm data.

The fitted results demonstrate that the proposed UZL distribution provides the most satisfactory overall fit. The UZL model gives an estimated parameter of $\zeta = 0.6586$ with SE 0.1211, log-likelihood 6.9118, AIC = -11.8237 , BIC = -10.7327 , and K–S statistic 0.1969 with $p = 0.3614$. These results are superior to those obtained from the competing models. In particular, the UZL distribution has the smallest AIC, BIC, and K–S statistic and the largest K–S p -value among the considered models. The UBIII distribution also provides a reasonable fit, whereas UL, UXg, and UZ exhibit comparatively poor goodness-of-fit results.

6.2. Dataset II: SC16 Algorithm

The second empirical dataset comprises observations corresponding to the SC16 algorithm. The data describe unit-capacity measurements and contain 23 observations within the interval $(0, 1)$. The complete set of

observations is given below:

0.006, 0.008, 0.011, 0.014, 0.021, 0.029, 0.036, 0.048,
0.059, 0.070, 0.092, 0.116, 0.116, 0.213, 0.344, 0.403,
0.492, 0.544, 0.717, 0.759, 0.809, 0.853, 0.866.

The SC16 data provide an additional bounded-data application for examining the flexibility and fitting performance of the proposed distribution. The dataset is fitted using maximum likelihood estimation, and its performance is compared with that of the selected competing distributions through the adopted model-selection and goodness-of-fit measures.

For this dataset, the minimum and maximum values are 0.0060 and 0.8660, respectively. The sample mean is 0.2881, while the median is 0.1160, indicating a clear difference between the average and the central position of the data. The variance is 0.1012, reflecting the spread of the observations. The skewness coefficient of 0.7676 indicates moderate positive skewness, suggesting a right-tailed distribution that is not highly skewed. The kurtosis value of 1.9741 indicates a distribution that is flatter than the normal distribution.

Table 7: Parameter estimates and goodness-of-fit statistics for the SC16 Algorithm dataset.

Candidate model	MLE (SE)		ℓ	AIC score	BIC score	K-S distance	Significance level (p)
	ζ	β					
GEUD	0.1757 (0.1439)	0.2435 (0.1799)	4.5600	-5.1200	-2.8490	0.3404	0.0097
UBIII	0.4536 (0.1348)	1.0018 (0.2516)	9.4845	-14.9690	-12.6980	0.1895	0.3810
UL	1.2533 (0.2009)	—	-6.8272	15.6546	16.7901	1.2527	0.0000
UH	0.7605 (0.1133)	—	9.9753	-17.9506	-16.8152	0.1841	0.4171
UXg	1.4482 (0.2217)	—	-3.8743	9.7487	10.8842	0.4624	0.0001
UZ	2.1540 (0.2718)	—	-35.4901	84.5521	85.6876	0.5466	0.0000
UZL	0.5988 (0.1076)	—	9.8174	-17.6349	-16.4994	0.1789	0.4535

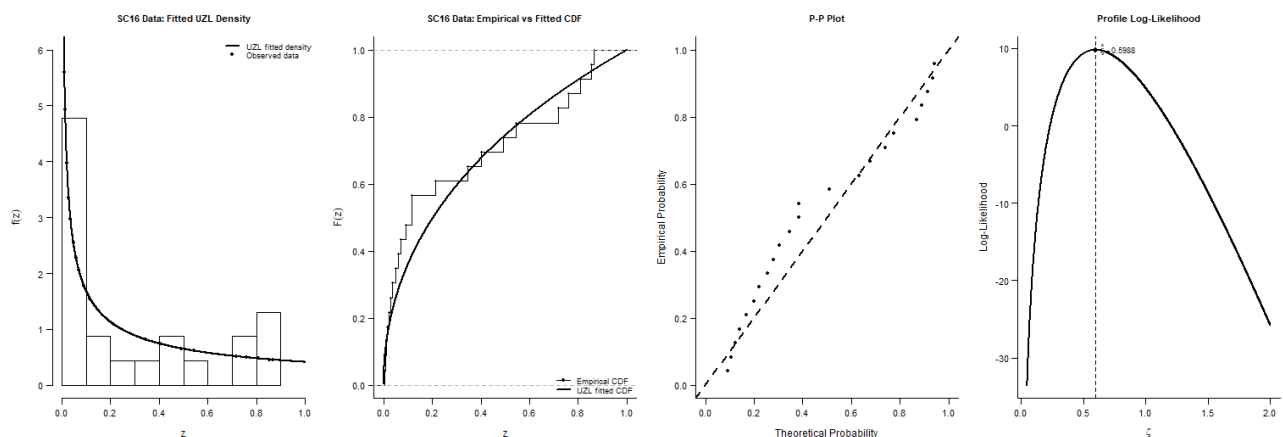


Figure 5: Fitted PDF, CDF, P-P, and profile log-likelihood plots of the UZL distribution for SC16 algorithm data.

For this dataset, the proposed UZL distribution again provides the best overall fit. Its maximum likelihood estimate is $\hat{\zeta} = 0.5988$ with SE 0.1076, and it produces a log-likelihood of 9.8174, $AIC = -17.6349$, $BIC = -16.4994$, K-S statistic 0.1789, and $p = 0.4535$. The UZL model therefore achieves the most favorable combination of information criteria and goodness-of-fit measures. Although the UH and UBIII distributions also provide acceptable fits, their K-S statistics and information criteria are slightly less favorable than those of UZL. In contrast, UL, UXg, and UZ yield considerably poorer fits, with large K-S statistics and very small p -values.

7. Conclusion

In this study, a new one-parameter UZL distribution was introduced for modeling data supported on the unit interval. The UZL model was constructed through a logarithmic transformation of the ZLindley distribution and

provides a simple and parsimonious alternative to existing unit-interval distributions. Several important statistical properties of the UZL distribution were derived, including its moments, variance, skewness, kurtosis, quantile function, mode, mean residual life, mean past lifetime, Lorenz and Bonferroni curves, and entropy measures. These analytical results demonstrate that the UZL distribution possesses a mathematically tractable structure and can be conveniently used for statistical modeling and inference.

Five estimation procedures such as MLE, ADE, CvME, OLSE, and WLSE have been examined for the estimation of the unknown parameter. The results of the Monte Carlo simulation indicate that all the estimations perform well, and the efficiency of the estimation becomes better with an increase in the sample size. In addition, the values of AB, MRE, and MSE become smaller with an increase in the sample size. Among the alternative approaches, MLE gives the best performance in terms of AB, MRE, and MSE, while ADE tends to give small bias.

The practical applicability of the UZL distribution has been verified using two real data sets corresponding to the P3 and SC16 algorithms. The comparison of the UZL distribution has been made with other existing unit-interval distributions based on the likelihood-based selection criteria and goodness-of-fit tests. The results obtained empirically from both data sets are a strong indicator supporting the UZL distribution. For example, in the case of the P3 data set, the fit of the UZL distribution is quite satisfactory, with relatively good performance of the likelihood, small values of AIC and BIC, and a nonsignificant Kolmogorov-Smirnov test. This is also true in the case of the SC16 data set. On a broader front, findings from the application show that the UZL distribution can do well for both data sets, offering adequate fitting of the observed unit interval data, with the additional benefit of being a one-parameter model.

In general, the theoretical characteristics, simulations, and data applications support the fact that the UZL distribution is an attractive, parsimonious, and practical model for modeling continuous data on the $(0, 1)$ interval. The simplicity of the mathematical form, single parameter, and its good empirical behavior make it appealing for applications that result in bounded data. Further research may investigate how the proposed model can be generalized for regression models, zero-one inflated data, Bayesian approaches, censored data, and multivariate/hierarchical unit-data settings.

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