

Analysis of Chaos and Solitons Stability in the (2+1)-Dimensional Sakovich Equation for Nonlinear Wave Motion

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Received: 7 Jun. 2026, Revised: 20 Jul. 2026, Accepted: 28 Aug. 2026

Published online: 1 Sep. 2026

Abstract: In this work, the nonlinear dynamics and soliton structures of the integrable (2+1)-dimensional Sakovich equation (2-dSE), which arises in nonlinear wave propagation and fluid dynamics, are investigated. This equation effectively models describes nonlinear wave phenomena in shallow water and stratified fluids, including wave breaking, soliton formation, and periodic wave patterns. By employing the extended simplest equation and the Φ^6 -model expansion methods, several new families of exact analytical solutions are derived, including solitary waves, periodic structures, kink and anti-kink solitons, breather-type waves, and multi-peak solitons. These methods effectively handle the nonlinearities in the model and provide a systematic framework for deriving closed-form solutions. The dynamical behavior of the model is further explored through chaotic analysis under external perturbations, revealing the system's sensitivity, long-term evolution, and complex dynamics. In addition, conservation laws are derived to identify key invariants governing the equation, and modulational instability analysis confirms the stability and physical relevance of the obtained solutions. Graphical simulations are presented to illustrate the dynamical features, stability regimes, and diverse soliton structures. In particular, Figures support the main objectives of this work by clearly demonstrating the analytically constructed soliton families, their dynamical behavior, and their physical relevance in nonlinear wave propagation. The analytical and numerical results demonstrate the effectiveness of the proposed methods in generating rich classes of exact solutions for complex multidimensional nonlinear partial differential equations.

Keywords: 2-dSE, Analytical methods, Fluid Dynamics, Energy Conservation, Chaos and Stability, Solitons, Periodic solutions.

1 Introduction

The investigation of nonlinear partial differential equations (NPDEs) is becoming more and more important in modern mathematical physics because of its critical role in simulating complicated natural phenomena such as fluid dynamics, optical processes, and nonlinear wave propagation. Beyond of all of these, the 2-dSE is a key model for explaining complex wave interactions in

dispersive media [1, 2]. Its applications in nonlinear optics, plasma physics, and shallow water wave theory make it essential for comprehending multi-dimensional wave structures [3]. Numerous facets of the equation's intricate solution space remain unexplored despite its significance, including its rich soliton dynamics, modulational instability features, and bifurcation tendencies, especially in higher-dimensional regimes.

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Significant advancements in the solution of NPDEs with higher-order nonlinearities and dispersive effects have been made possible by recent developments in analytical and semi-analytical approaches [4, 5]. The simplest equation and expansion methods are one of these that has become a potent and methodical foundation for creating precise solutions. Novel soliton solutions, breather-type excitations, and complex periodic wave patterns can be found using these methods, which extends traditional solution techniques by incorporating auxiliary equations, symmetry reductions, and transformation strategies. Numerous higher-order models, including as the Zakharov-Kuznetsov (ZK) and nonlinear Schrödinger-type equations, have been analyzed with its help, yielding a variety of localized solution families, including multi-peak solitons and breather waves [6, 7]. These advancements highlight how the technique might help us better comprehend intricate wave dynamics and find novel physical phenomena in nonlinear dispersive systems [8].

The study of nonlinear dispersive waves relies heavily on the KdV equation, which was developed by Dutch mathematicians Diederik Korteweg and Gustav de Vries. Originally created to model long-wavelength, small-amplitude shallow-water waves, this theory offered the first rigorous explanation of the solitary wave famously observed by John Scott Russell in 1834. Certain localized disturbances could develop into stable solitary waves, later referred to as solitons, as the KdV equation demonstrated by describing the complex interaction between nonlinear steepening and dispersive spreading. This finding established the KdV equation as a basic nonlinear evolution model by bridging the gap between theoretical analysis and practical findings [9–11].

Beyond shallow-water dynamics, the KdV equation has shown remarkable versatility, providing a unifying framework for comprehending finite-amplitude dispersive waves in a variety of physical contexts, such as optical pulse propagation in nonlinear media, ion-acoustic waves in plasmas, and acoustic radiation in crystalline solids [11–13]. Its mathematical structure, which includes infinitely many conservation rules and total integrability, has sparked in-depth research on integrable systems and soliton theory. Additional extensions, such as Sakovich integrable equations for KdV-type solitons [14], keep making it more relevant. With its profound insights into nonlinear wave interactions, soliton dynamics, and the universal laws controlling dispersive wave propagation [15, 16], the KdV equation is still a key model in mathematical physics today.

The development of techniques to get solutions of NPDEs in the form of solitary waves has been the subject of extensive research in recent years [17, 18]. The (G'/G) -expansion scheme, the auxiliary methods, the generalized $\exp(-\phi(\xi))$ approach, the inverse scattering scheme, direct algebraic technique, Bäcklund transform and Hirota techniques, the extended tanh scheme, and rational expansion methods, and elliptic function

techniques are just a few of the many analytical approaches that have been proposed for building solitary wave and soliton solutions [19–27]. Simultaneously, a variety of effective numerical techniques have been created to solve NPDEs, each with special benefits in terms of precision, computational effectiveness, and usefulness. Many homotopy-based techniques, the Adomian technique, and many variations of the differential transform scheme are some of the most successful approximate schemes, all of which have demonstrated significant success in handling the complexities of nonlinear wave dynamics [28–33].

Comprehensive studies have concentrated on analysis of bifurcation and the formulation of analytical wave results for time dependent NPDEs. Chaos, a defining characteristic of nonlinear systems, is widely observed across diverse scientific disciplines [34]. Conservation laws, which serve as cornerstones in describing the dynamics of mechanical, electrical, optical, chemical, biological, and atmospheric systems, remain central to their analysis. Despite variations in physical origin, these systems often demonstrate comparable qualitative and quantitative behaviors, frequently triggered by the breakdown of periodicity. Moreover, sensitivity analysis has emerged as a vital methodology for evaluating the influence of parameter variations on model outcomes, thereby enhancing predictive accuracy, ensuring model validation, and reducing uncertainty [35].

The analysis of NPDEs heavily relies on conservation laws [36], which reveal fundamental invariants such as energy, momentum, and mass. These invariants serve as essential tools for validating analytical derivations and numerical computations, ensuring both correctness and physical relevance. Within multidimensional nonlinear wave models, conservation laws are instrumental in characterizing stability, integrability, and the long-term evolution of solutions. They further facilitate the discovery of physically meaningful wave structures and underpin the design of computational schemes that rigorously conserve energy.

Recent studies have extensively explored integrable and non-integrable multidimensional nonlinear wave equations, revealing a wide range of soliton interactions, breather-type excitations, and chaotic dynamics. In particular, Sakovich-type equations and their higher-dimensional extensions have attracted considerable attention due to their rich mathematical structures and applications in shallow water theory, plasma physics, and nonlinear optics. Various analytical approaches, including Painlevé analysis, Hirota bilinear methods, and expansion techniques, have been successfully applied to construct exact wave solutions and investigate their stability properties. Despite these developments, a comprehensive analysis combining exact soliton construction, conservation laws, chaotic behavior under perturbations, and stability analysis for the 2-dSE remains limited.

In this work, the ϕ^6 -model expansion method and the extended simplest equation method (eSEM) are employed to construct exact wave solutions of the 2-dSE. These approaches effectively handle nonlinearities involving both polynomial and non-polynomial terms, demonstrating their capability to generate a broad spectrum of solution families. The outcomes of this work not only enrich the solution space of the 2-dSE but also demonstrate the effectiveness of the proposed analytical techniques for studying complex multidimensional nonlinear partial differential equations. To provide a deeper understanding of the obtained solutions, graphical representations including 3D surface plots along with contour plots (CP) and 2-dimensional are presented. Furthermore, the conservation laws of the proposed dynamical model are systematically analyzed to uncover its intrinsic physical properties. The study also examines the chaotic behavior of the system to gain insight into its nonlinear dynamics. Finally, a stability analysis confirms that all derived solutions are both exact and stable.

The structure of this study is organized as follows: Section 2 outlines the necessary mathematical background and details the analytical methods employed. Section 3 introduces the governing equation, while Section 4 focuses on deriving its exact wave solutions. In Section 5, the conservation laws associated with the proposed model are analyzed, followed by an examination of its chaotic behavior in Section 6. Section 7 addresses the stability characteristics of the obtained solutions, and Section 8 provides a comprehensive discussion of the findings, supported by various graphical illustrations. Finally, Section 9 summarizes the key conclusion.

2 Methodology

In this part, we describe the extended SEM to find exact solutions nonlinear evolution equations. Considering the higher order NPDE below:

$$G(\mathcal{U}, \mathcal{U}_t, \mathcal{U}_x, \mathcal{U}_y, \mathcal{U}_{tt}, \mathcal{U}_{tx}, \mathcal{U}_{xx}, \mathcal{U}_{yy}, \dots) = 0, \quad (1)$$

where G stands for a polynomial in $\mathcal{U}(x, y, t)$ and all of its partial derivatives, including nonlinear terms and highest order derivatives.

Step-I: The travelling wave transformation

$$\mathcal{U}(x, y, t) = \mathcal{P}(\xi), \quad \xi = \mu_1 x + \mu_2 y - \tau t, \quad (2)$$

where μ_1, μ_2 and τ are wave numbers and frequency of waves. This transformation converts equation (1) into a nonlinear ODE represented as follows:

$$H(\mathcal{P}, -\tau \mathcal{P}', \mu_1 \mathcal{P}', \mu_2 \mathcal{P}', \tau^2 \mathcal{P}'', -\mu_1 \tau \mathcal{P}'', \mu_1^2 \mathcal{P}'', \mu_2^2 \mathcal{P}'', \dots) = 0. \quad (3)$$

2.1 The Φ^6 -model expansion Method

This method essential steps are as

Step-II: Considering that Eq.(3) has a solution of the form

as:

$$\mathcal{P}(\xi) = \sum_{i=-N}^N A_i f(\xi)^i, \quad (4)$$

where A_i are constants, and $f(\xi)$ is derived from the Bäcklund transformation, $f(\xi) = \frac{p_1 Z(\xi) - p_2 \lambda}{p_1 + p_2 Z(\xi)}$. In this context, λ, p_1 , and p_2 are constants with $p_2 \neq 0$, and $Z(\xi)$ serves as the solution to the ensuing ODE.

$$\frac{dZ}{d\xi} = \lambda + Z(\xi)^2. \quad (5)$$

Step-III: Applying the homogeneous balance principle on equation (3) yields the positive integer N , as shown in equation (4). First, it must be recognized that the balance number of a processor can be determined as follows

$$D \left[\frac{d^k g}{d\Psi^k} \right] = N + k, D \left[g^J \left(\frac{d^k g}{d\Psi^k} \right)^s \right] = NJ + s(k + N). \quad (6)$$

Step-IV: We substitute equation (4) into equation (3), and then gather all terms containing $g(\Phi)$. The polynomial coefficients are then set to zero, yielding a system of algebraic equations involving parameters.

Step-V: These equations are resolved with the computational tools, and wave results for Eq.(1) are attained.

2.2 Extended simplest equation approach

This method essential steps are as

Step-II: Presuming that the solution to Eq.(3) takes the following form:

$$\mathcal{P}(\xi) = \sum_{i=-N}^N A_i \Psi^i(\xi), \quad (7)$$

here A_i are arbitrary constants and N is a +ve integer attained via employing the balancing principle to Eq.(3), and $\Psi(\xi)$ satisfies the below new ansatz equation

$$\Psi'(\xi) = \alpha_0 + \alpha_1 \Psi + \alpha_2 \Psi^2, \quad (8)$$

where the arbitrary constants are α_0, α_1 and α_2 .

Step-III: Merging Equations (7) and (8) into equation (3) and setting the coefficients of powers of Ψ^i to zero, yields an algebraic system of equations with parameters $A_i, \alpha_0, \alpha_1, \alpha_2, \mu_1, \mu_2$ and τ . Mathematica solves the algebraic system and the values of the parameters to be determined.

Step-IV: By replacing the values of the parameters obtained in Stage 3rd and $\Psi(\xi)$ into Eq.(7), one can derive the solution to Eq.(1).

3 Governing Model

The new extended (2 + 1)-dimensional Sakovich equation [37] has the form as

$$\mathcal{U}_t + \mathcal{U}_{xx} + \mathcal{U}_{yy} + \mathcal{U}_{xy} + 2\mathcal{U}\mathcal{U}_{xy} + 6\mathcal{U}^2\mathcal{U}_{xx} + 2(\mathcal{U}_{xx})^2 = 0, \quad (9)$$

where the unknown scalar function is $\mathcal{U} = \mathcal{U}(x, y, t)$, and the spatial and temporal variables are x, y , and t . In terms of nonlinear interactions between wave amplitude and its spatial curvature, the words $2\mathcal{U}\mathcal{U}_{xy}$ and $6\mathcal{U}^2\mathcal{U}_{xx}$ are used. A coupling between dimensions is suggested by the mixed derivatives $\mathcal{U}_{xy}$ and $\mathcal{U}_{xt}$, which indicate that the behavior in one spatial direction affects the behavior in the other. One quadratic higher-order dispersive term that leads to complex wavefronts and rich dynamics such as shock development, wave breaking, or pattern generation is $(\mathcal{U}_{xx})^2$. Nonlinear shallow water waves, plasma wave packets, nonlinear optics, some quantum field models, and nonlinear acoustic waves are all described by this equation, which describes nonlinear and dispersive wave evolution in multi-dimensional media. Additionally, this equation is widely acknowledged as an important component of fluid dynamics [41, 42]. We get the following by using the wave transformation in Eq.(2) on Eq.(9) as

$$2\mu_1^4 \mathcal{P}'' + 6\mu_1^2 \mathcal{P}^2 + 2\mu_1 \mu_2 \mathcal{P} + \mu_1 (\mu_2 - \tau) + \mu_1^2 + \mu_2^2 = 0. \quad (10)$$

4 Mathematical Formulation of Solitons

In this section, we construct the soliton solutions of equation (10) using two mathematical techniques.

4.1 By using ϕ^6 -model expansion method

Using Bäcklund transformation and the Riccati-Bernoulli sub-ODE process, we can obtain wave solutions for the considered model. This approach has advantage of intrinsic properties in the system balancing equations, to minimize and solve for wave structures. Integrating specific phrases enables the extraction of individual features of wave events. Substituting equations (4) and (5) into equation (10) and collecting the coefficients of $Z(\xi)$ provides the system of equations. Resolving the attained system of equation via software and solutions sets are obtained which shown below:

$$\begin{aligned} A_{-2} &= -2\lambda^2 \mu_1^2, \quad A_{-1} = 0, \\ A_0 &= \frac{-8\lambda \mu_1^3 - \mu_2}{6\mu_1}, \quad A_1 = 0, \\ A_2 &= 0, \quad \tau = \frac{16\lambda^2 \mu_1^6 + 6\mu_1^2 + 6\mu_2 \mu_1 + 5\mu_2^2}{6\mu_1}. \end{aligned} \quad (11)$$

$$\begin{aligned} A_{-2} &= A_{-1} = 0, \\ A_0 &= \frac{-8\lambda \mu_1^3 - \mu_2}{6\mu_1}, \quad A_1 = 0, \\ A_2 &= -2\mu_1^2, \quad \tau = \frac{16\lambda^2 \mu_1^6 + 6\mu_1^2 + 6\mu_2 \mu_1 + 5\mu_2^2}{6\mu_1}. \end{aligned} \quad (12)$$

$$\begin{aligned} A_{-2} &= -2\lambda^2 \mu_1^2, \quad A_{-1} = 0, \\ A_0 &= \frac{-8\lambda \mu_1^3 - \mu_2}{6\mu_1}, \quad A_1 = 0, \\ A_2 &= -2\mu_1^2, \quad \tau = \frac{256\lambda^2 \mu_1^6 + 6\mu_1^2 + 6\mu_2 \mu_1 + 5\mu_2^2}{6\mu_1}. \end{aligned} \quad (13)$$

From solution sets (11), (12) and (13), we obtain the below families of results for equation (9) as

Family-I:

From solution set (11), got

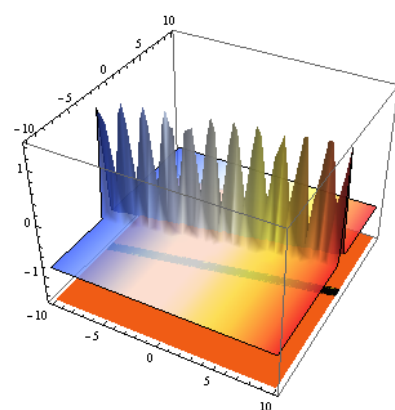
$$\begin{aligned} \mathcal{U}_{1,1}(x, y, t) &= -\frac{8\lambda \mu_1^3 + \mu_2}{6\mu_1} \\ &\quad - \frac{2\lambda^2 \mu_1^2 (p_1 - \sqrt{-\lambda} p_2 \tanh(\sqrt{-\lambda} \xi))^2}{(\sqrt{-\lambda} p_1 \tanh(\sqrt{-\lambda} \xi) + \lambda p_2)^2}, \quad \lambda < 0. \end{aligned} \quad (14)$$

$$\begin{aligned} \mathcal{U}_{1,2}(x, y, t) &= -\frac{8\lambda \mu_1^3 + \mu_2}{6\mu_1} \\ &\quad - \frac{2\lambda^2 \mu_1^2 (p_1 - \sqrt{-\lambda} p_2 \coth(\sqrt{-\lambda} \xi))^2}{(\sqrt{-\lambda} p_1 \coth(\sqrt{-\lambda} \xi) + \lambda p_2)^2}, \quad \lambda < 0. \end{aligned} \quad (15)$$

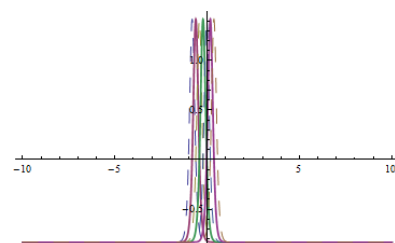
From solution set (12), got

$$\begin{aligned} \mathcal{U}_{1,3}(x, y, t) &= -\frac{8\lambda \mu_1^3 + \mu_2}{6\mu_1} \\ &\quad - \frac{2\mu_1^2 (\sqrt{-\lambda} p_1 \tanh(\sqrt{-\lambda} \xi) + \lambda p_2)^2}{(p_1 - \sqrt{-\lambda} p_2 \tanh(\sqrt{-\lambda} \xi))^2}, \quad \lambda < 0. \end{aligned} \quad (16)$$

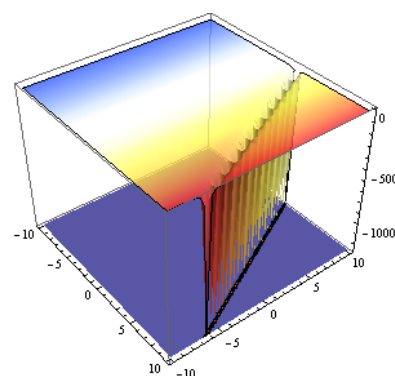
$$\begin{aligned} \mathcal{U}_{1,4}(x, y, t) &= -\frac{8\lambda \mu_1^3 + \mu_2}{6\mu_1} \\ &\quad - \frac{2\mu_1^2 (\sqrt{-\lambda} p_1 \coth(\sqrt{-\lambda} \xi) + \lambda p_2)^2}{(p_1 - \sqrt{-\lambda} p_2 \coth(\sqrt{-\lambda} \xi))^2}, \quad \lambda < 0. \end{aligned} \quad (17)$$



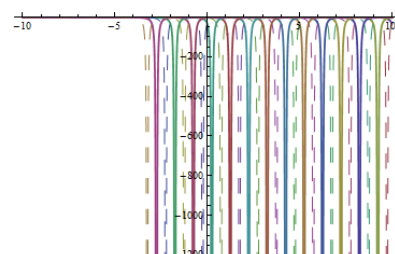
(a)



(b)

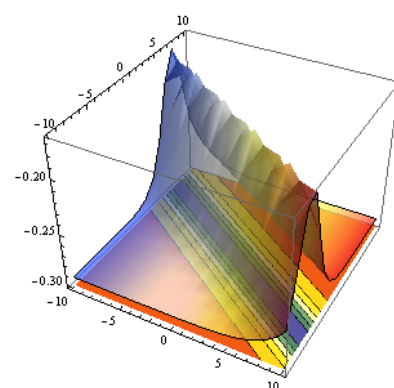


(c)

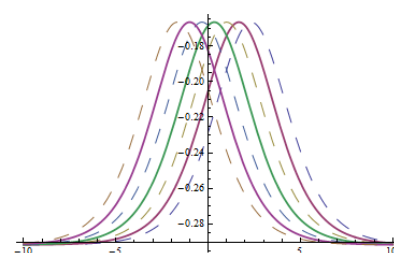


(d)

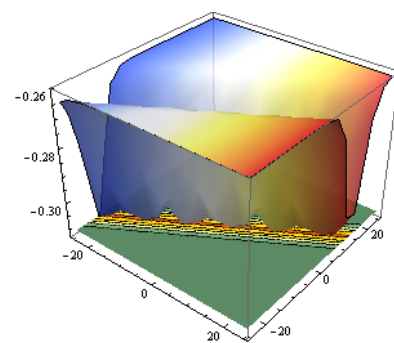
Fig. 1: The solutions (14) and (15) are depicted appropriate parameter values: (a) shows bright multi-peak soliton with contourplot (CP) and its 2-dimensional (2D) in (b), (c) depicts dark multi-peak solitons with CP and its 2D in Fig(1.D), respectively.



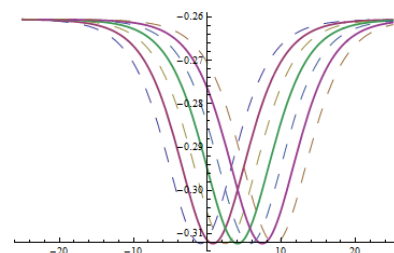
(a)



(b)



(c)



(d)

Fig. 2: The solutions (16) and (17) are showed by giving appropriate parameter values: (a) shows bright soliton with CP and its 2D in (b), (c) depicts dark soliton with CP and its 2D in (d), respectively.

From solution set (13), got

$$\mathcal{U}_{1,5}(x, y, t) = -\frac{8\lambda\mu_1^3 + \mu_2}{6\mu_1} - \frac{2\lambda^2\mu_1^2(p_1 - \sqrt{-\lambda}p_2 \tanh(\sqrt{-\lambda}\xi))^2}{(\sqrt{-\lambda}p_1 \tanh(\sqrt{-\lambda}\xi) + \lambda p_2)^2} - \frac{2\mu_1^2(\sqrt{-\lambda}p_1 \tanh(\sqrt{-\lambda}\xi) + \lambda p_2)^2}{(p_1 - \sqrt{-\lambda}p_2 \tanh(\sqrt{-\lambda}\xi))^2}, \lambda < 0. \quad (18)$$

$$\mathcal{U}_{1,6}(x, y, t) = -\frac{8\lambda\mu_1^3 + \mu_2}{6\mu_1} - \frac{2\lambda^2\mu_1^2(p_1 - \sqrt{-\lambda}p_2 \coth(\sqrt{-\lambda}\xi))^2}{(\sqrt{-\lambda}p_1 \coth(\sqrt{-\lambda}\xi) + \lambda p_2)^2} - \frac{2\mu_1^2(\sqrt{-\lambda}p_1 \coth(\sqrt{-\lambda}\xi) + \lambda p_2)^2}{(p_1 - \sqrt{-\lambda}p_2 \coth(\sqrt{-\lambda}\xi))^2}, \lambda < 0. \quad (19)$$

Family-II:

From solution set (11), got

$$\mathcal{U}_{2,1}(x, y, t) = -\frac{8\lambda\mu_1^3 + \mu_2}{6\mu_1} - \frac{2\lambda\mu_1^2(\sqrt{\lambda}p_2 \tan(\sqrt{\lambda}\xi) + p_1)^2}{(\sqrt{\lambda}p_2 - p_1 \tan(\sqrt{\lambda}\xi))^2}, \lambda > 0. \quad (20)$$

$$\mathcal{U}_{2,2}(x, y, t) = -\frac{8\lambda\mu_1^3 + \mu_2}{6\mu_1} - \frac{2\lambda\mu_1^2(p_1 - \sqrt{\lambda}p_2 \cot(\sqrt{\lambda}\xi))^2}{(p_1 \cot(\sqrt{\lambda}\xi) + \sqrt{\lambda}p_2)^2}, \lambda > 0. \quad (21)$$

From solution set (12), got

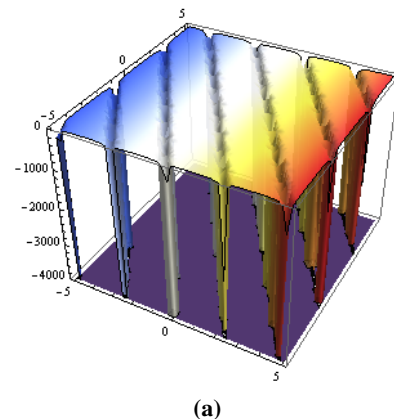
$$\mathcal{U}_{2,3}(x, y, t) = -\frac{8\lambda\mu_1^3 + \mu_2}{6\mu_1} - \frac{2\mu_1^2(\lambda p_2 - \sqrt{\lambda}p_1 \tan(\sqrt{\lambda}\xi))^2}{(\sqrt{\lambda}p_2 \tan(\sqrt{\lambda}\xi) + p_1)^2}, \lambda > 0. \quad (22)$$

$$\mathcal{U}_{2,4}(x, y, t) = -\frac{8\lambda\mu_1^3 + \mu_2}{6\mu_1} - \frac{2\mu_1^2(\sqrt{\lambda}p_1 \cot(\sqrt{\lambda}\xi) + \lambda p_2)^2}{(p_1 - \sqrt{\lambda}p_2 \cot(\sqrt{\lambda}\xi))^2}, \lambda > 0. \quad (23)$$

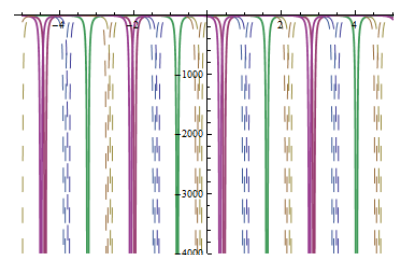
From solution set (13), got

$$\left\{ \mathcal{U}_{2,5}(x, y, t) = \frac{2\lambda\mu_1^2}{3} \times \left(4 - \frac{12(\lambda p_2^2 + p_1^2)}{((p_1^2 - \lambda p_2^2) \sin(2\sqrt{\lambda}\xi) - 2\sqrt{\lambda}p_1 p_2 \cos(2\sqrt{\lambda}\xi))^2} \right) - \frac{\mu_2}{6\mu_1}, \lambda > 0. \right. \quad (24)$$

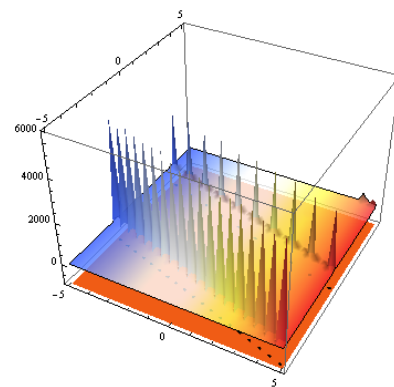
$$\left\{ \mathcal{U}_{2,6}(x, y, t) = \frac{2\lambda\mu_1^2}{3} \times \left(4 - \frac{12(\lambda p_2^2 + p_1^2)}{((p_1^2 - \lambda p_2^2) \sin(2\sqrt{\lambda}\xi) - 2\sqrt{\lambda}p_1 p_2 \cos(2\sqrt{\lambda}\xi))^2} \right) - \frac{\mu_2}{6\mu_1}, \lambda > 0. \right. \quad (25)$$



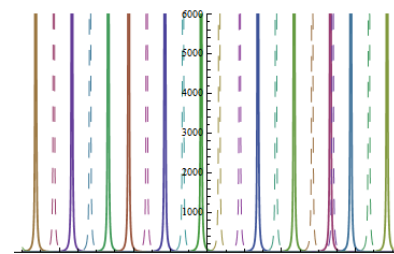
(a)



(b)

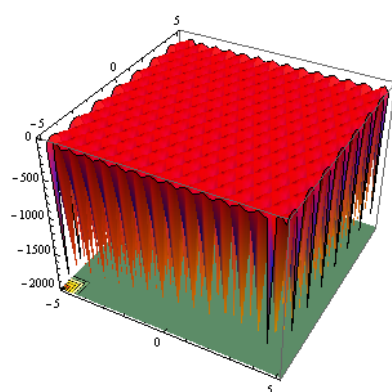


(c)

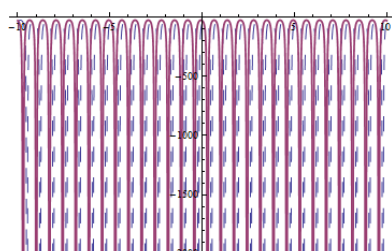


(d)

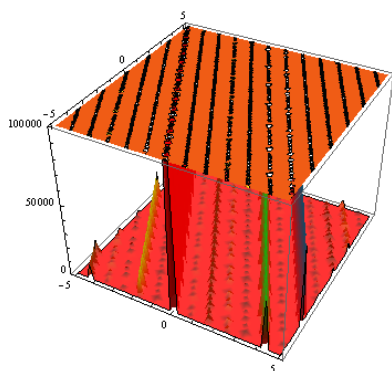
Fig. 3: The solutions (20) and (21) are depicted by giving appropriate parameter values: (a) shows depicts periodic solitary waves with CP and its 2D in (b), (c) depicts periodic multi-peak solitary wave having diverse amplitude with CP and its 2D in (d), respectively.



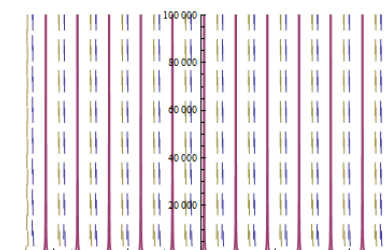
(a)



(b)



(c)



(d)

Fig. 4: The solutions (24) and (25) are depicted by assigning suitable parameter values as follows: (a) shows depicts periodic multi-peak waves with CP and its 2D in (b), (c) depicts periodic solitons having different amplitudes with CP and its 2D in (d), respectively.

Family-III:

From solution set (12), got

$$\mathcal{U}_{3,1}(x, y, t) = -\frac{\mu_2}{6\mu_1} - \frac{2\mu_1^2 p_1^2}{(p_2 - \xi p_1)^2}, \quad \lambda = 0. \quad (26)$$

From solution set (13), got

$$\mathcal{U}_{3,1}(x, y, t) = -\frac{2\mu_1^2 p_1^2}{(p_2 - \xi p_1)^2} - \frac{\mu_2}{6\mu_1}, \quad \lambda = 0. \quad (27)$$

4.2 By using extended simple equation approach

By employing the balance principle on Eq.(10), obtain $N = 2$. Thus, the results appropriate parameter values is as

$$\mathcal{P}(\xi) = \frac{A_{-2}}{\Psi^2} + \frac{A_{-1}}{\Psi} + A_0 + A_1 \Psi + A_2 \Psi^2. \quad (28)$$

By substituting Eq.(28) and Eq.(8) into Eq.(10) and setting the coefficients of Ψ^i to zero, obtain a system of algebraic equations involving $A_i, \alpha_0, \alpha_1, \alpha_2, \mu_1, \mu_2$ and τ . Solving these equations yields the following families of solutions:

Family-I: If $\alpha_1 = 0$

$$\begin{aligned} A_{-2} &= -2\alpha_0^2 \mu_1^2, \quad A_{-1} = 0, \\ A_0 &= \frac{-8\alpha_0 \alpha_2 \mu_1^3 - \mu_2}{6\mu_1}, \quad A_1 = A_2 = 0, \\ \tau &= \frac{16\alpha_0^2 \alpha_2^2 \mu_1^6 + 6\mu_1^2 + 6\mu_2 \mu_1 + 5\mu_2^2}{6\mu_1}. \end{aligned} \quad (29)$$

$$\begin{aligned} A_{-2} &= A_{-1} = A_1 = 0, \\ A_0 &= \frac{-8\alpha_0 \alpha_2 \mu_1^3 - \mu_2}{6\mu_1}, \\ A_2 &= -2\alpha_2^2 \mu_1^2, \quad \tau = \frac{16\alpha_0^2 \alpha_2^2 \mu_1^6 + 6\mu_1^2 + 6\mu_2 \mu_1 + 5\mu_2^2}{6\mu_1}. \end{aligned} \quad (30)$$

$$\begin{aligned} A_{-2} &= -2\alpha_0^2 \mu_1^2, \quad A_{-1} = 0, \\ A_0 &= -\frac{8\alpha_0 \alpha_2 \mu_1^3 + \mu_2}{6\mu_1}, \quad A_1 = 0, \end{aligned} \quad (31)$$

$$A_2 = -2\alpha_2^2 \mu_1^2, \quad \tau = \frac{128}{3} \alpha_0^2 \alpha_2^2 \mu_1^5 + \mu_1 + \frac{5\mu_2^2}{6\mu_1} + \mu_2.$$

The wave results of Eq.(9) can be achieved from solutions sets (29), (30) and (31) in the form as

$$\mathcal{U}_{1,1}(x, y, t) = -\frac{4\alpha_0 \alpha_2 \mu_1^3 (3 \cot^2(\sqrt{\alpha_0 \alpha_2}(\xi + \xi_0)) + 2) + \mu_2}{6\mu_1}, \quad \alpha_0 \alpha_2 > 0. \quad (32)$$

$$\begin{aligned} \mathcal{U}_{1,2}(x, y, t) &= -\frac{4\alpha_0 \alpha_2 \mu_1^3 \left(2 - 3 \coth^2 \left(\sqrt{-\alpha_0 \alpha_2} \xi + \frac{\rho \log(\xi_0)}{2} \right) \right)}{6\mu_1} \\ &+ \frac{\mu_2}{6\mu_1}, \quad \xi_0 > 0, \quad \rho = \pm 1, \quad \alpha_0 \alpha_2 < 0. \end{aligned} \quad (33)$$

$$\begin{aligned} \mathcal{U}_{1,3}(x, y, t) &= -\frac{4\alpha_0 \alpha_2 \mu_1^3 (3 \tan^2(\sqrt{\alpha_0 \alpha_2}(\xi + \xi_0)) + 2) + \mu_2}{6\mu_1}, \\ &\alpha_0 \alpha_2 > 0. \end{aligned} \quad (34)$$

$$\mathcal{U}_{1,4}(x, y, t) = -\frac{4\alpha_0\alpha_2\mu_1^3 \left(2 - 3 \tanh^2 \left(\sqrt{-\alpha_0\alpha_2}\xi + \frac{\rho \log(\xi_0)}{2}\right)\right) + \mu_2}{6\mu_1} + \frac{\mu_2}{6\mu_1}, \quad \xi_0 > 0, \quad \rho = \pm 1, \alpha_0\alpha_2 < 0. \quad (35)$$

$$\mathcal{U}_{1,5}(x, y, t) = -\frac{8\alpha_0\alpha_2\mu_1^3 (\cos(4\sqrt{\alpha_0\alpha_2}(\xi + \xi_0)) + 5)}{6\mu_1} \times \frac{\csc^2(2\sqrt{\alpha_0\alpha_2}(\xi + \xi_0))}{6\mu_1} + \frac{\mu_2}{6\mu_1}, \quad \alpha_0\alpha_2 > 0. \quad (36)$$

$$\mathcal{U}_{1,6}(x, y, t) = \frac{8\alpha_0\alpha_2\mu_1^3 (\cosh(4\sqrt{-\alpha_0\alpha_2}\xi + 2\rho \log(\xi_0)) + 5)}{6\mu_1} \times \frac{\operatorname{csch}^2(2\sqrt{-\alpha_0\alpha_2}\xi + \rho \log(\xi_0))}{6\mu_1} - \frac{\mu_2}{6\mu_1}, \quad \xi_0 > 0, \quad \rho = \pm 1, \quad \alpha_0\alpha_2 < 0. \quad (37)$$

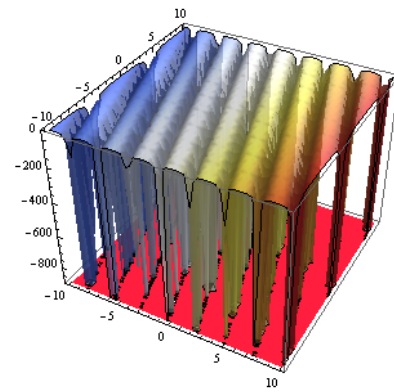
Family-II: if we take $\alpha_0 = 0$

$$\begin{aligned} A_{-2} = A_{-1} = 0, \quad A_0 &= -\frac{\alpha_1^2\mu_1^3 + \mu_2}{6\mu_1}, \\ A_1 &= -2\alpha_1\alpha_2\mu_1^2, \quad A_2 = -2\alpha_2^2\mu_1^2, \\ \tau &= \frac{\alpha_1^4\mu_1^5}{6} + \mu_1 + \frac{5\mu_2^2}{6\mu_1} + \mu_2. \end{aligned} \quad (38)$$

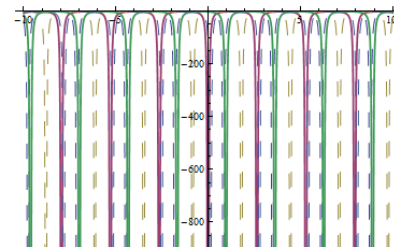
The solitary wave results of Eq.(9) can be attained from solution set (38) in the form as

$$\mathcal{U}_{2,1}(x, y, t) = -\frac{\alpha_1^2\mu_1^3 \left(\alpha_2^2 e^{2\alpha_1(\xi + \xi_0)} + 10\alpha_2 e^{\alpha_1(\xi + \xi_0)} + 1\right)}{6\mu_1 (\alpha_2 e^{\alpha_1(\xi + \xi_0)} - 1)^2} + \frac{\mu_2 (\alpha_2 e^{\alpha_1(\xi + \xi_0)} - 1)^2}{6\mu_1 (\alpha_2 e^{\alpha_1(\xi + \xi_0)} - 1)^2}, \quad \alpha_1 > 0. \quad (39)$$

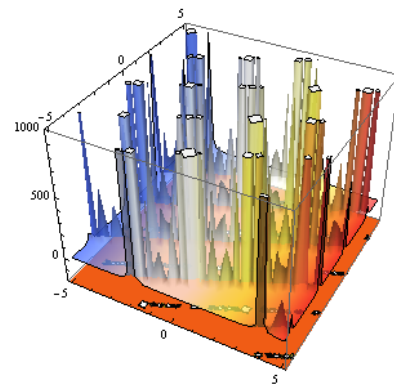
$$\mathcal{U}_{2,2}(x, y, t) = -\frac{\alpha_1^2\mu_1^3 \left(\alpha_2^2 e^{2\alpha_1(\xi + \xi_0)} - 10\alpha_2 e^{\alpha_1(\xi + \xi_0)} + 1\right)}{6\mu_1 (\alpha_2 e^{\alpha_1(\xi + \xi_0)} + 1)^2} + \frac{\mu_2 (\alpha_2 e^{\alpha_1(\xi + \xi_0)} + 1)^2}{6\mu_1 (\alpha_2 e^{\alpha_1(\xi + \xi_0)} + 1)^2}, \quad \alpha_1 < 0. \quad (40)$$



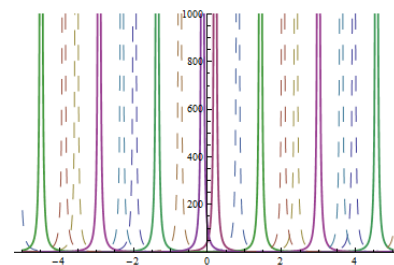
(a)



(b)

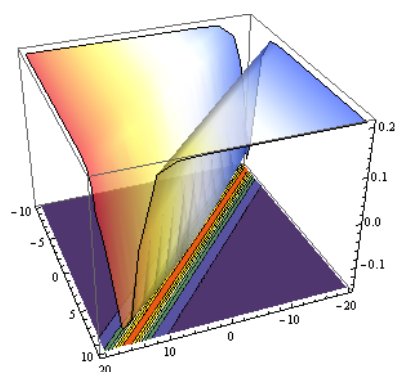


(c)

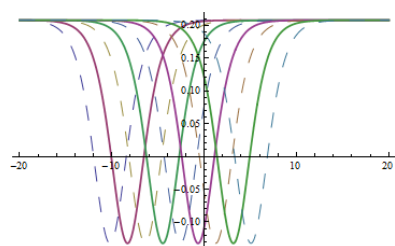


(d)

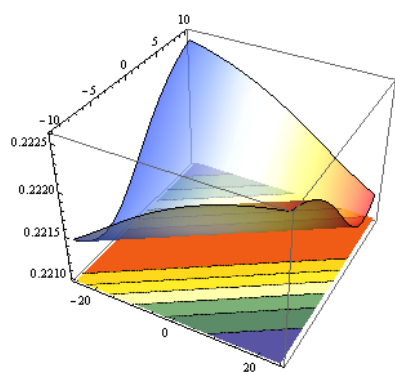
Fig. 5: The results (32) and (34) are depicted by assigning suitable parameter values as follows: (a) shows a singular periodic solitary wave with CP and its 2D representation in (b), (c) depicts a mix singular and non-singular periodic solitary waves with CP, and its 2D representation is provided in (d), respectively.



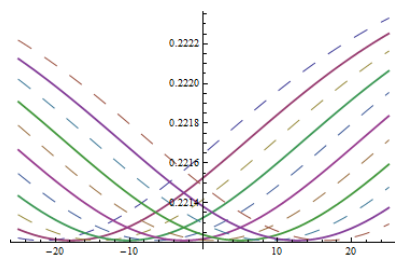
(a)



(b)



(c)



(d)

Fig. 6: The solutions (39) and (43) are defined by giving appropriate parameter values: (a) shows a dark soliton with CP and its 2D representation in (b), and (c) shows a solitary wave with CP and its 2D representation in (d) respectively.

Family-III:

$$\begin{aligned} A_{-2} = A_{-1} = 0, \quad A_0 &= -\frac{\alpha_1^2 \mu_1^3 + 8\alpha_0 \alpha_2 \mu_1^3 + \mu_2}{6\mu_1}, \\ A_1 &= -2\alpha_1 \alpha_2 \mu_1^2, \quad A_2 = -2\alpha_2^2 \mu_1^2, \\ \tau &= \frac{(\alpha_1^2 - 4\alpha_0 \alpha_2)^2 \mu_1^6 + 6\mu_1^2 + 6\mu_2 \mu_1 + 5\mu_2^2}{6\mu_1}. \end{aligned} \quad (41)$$

$$\begin{aligned} A_{-2} &= -2\alpha_0^2 \mu_1^2, \quad A_{-1} = -2\alpha_0 \alpha_1 \mu_1^2, \\ A_0 &= -\frac{\alpha_1^2 \mu_1^3 + 8\alpha_0 \alpha_2 \mu_1^3 + \mu_2}{6\mu_1}, \quad A_1 = A_2 = 0, \\ \tau &= \frac{(\alpha_1^2 - 4\alpha_0 \alpha_2)^2 \mu_1^6 + 6\mu_1^2 + 6\mu_2 \mu_1 + 5\mu_2^2}{6\mu_1}. \end{aligned} \quad (42)$$

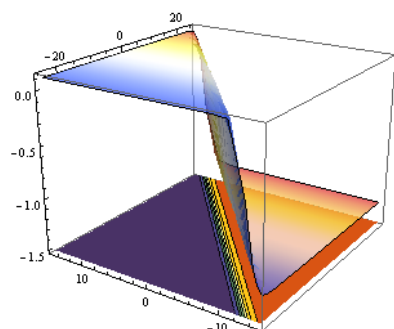
The solitary wave solutions of Eq.(9) are derived from solution sets (41) and (42) as:

$$\begin{aligned} \mathcal{U}_{3,1}(x, y, t) &= \frac{(\alpha_1^2 - 4\alpha_0 \alpha_2) \mu_1^3 \left(3 \tan^2 \left(\frac{\sqrt{4\alpha_0 \alpha_2 - \alpha_1^2}}{2} (\xi + \xi_0) \right) + 2 \right)}{6\mu_1} \\ &\quad - \frac{\mu_2}{6\mu_1}, \quad 4\alpha_0 \alpha_2 > \alpha_1^2, \quad \alpha_2 > 0. \end{aligned} \quad (43)$$

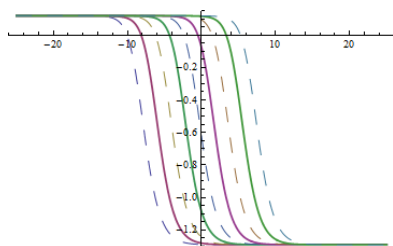
$$\begin{cases} \mathcal{U}_{3,2}(x, y, t) = -\frac{\alpha_1^2 \mu_1^3 + 8\alpha_0 \alpha_2 \mu_1^3 + \mu_2}{6\mu_1} \\ \quad - \alpha_1 \mu_1^2 \left(\sqrt{4\alpha_0 \alpha_2 - \alpha_1^2} \tan \left(\frac{\sqrt{4\alpha_0 \alpha_2 - \alpha_1^2}}{2} (\xi + \xi_0) \right) + \alpha_1 \right) \\ \quad - \frac{\mu_1^2}{2} \left(\sqrt{4\alpha_0 \alpha_2 - \alpha_1^2} \tan \left(\frac{\sqrt{4\alpha_0 \alpha_2 - \alpha_1^2}}{2} (\xi + \xi_0) \right) + \alpha_1 \right)^2, \\ \quad 4\alpha_0 \alpha_2 > \alpha_1^2, \quad \alpha_2 < 0. \end{cases} \quad (44)$$

$$\begin{cases} \mathcal{U}_{3,3}(x, y, t) = \frac{4\alpha_0 \alpha_1 \alpha_2 \mu_1^2}{\alpha_1 - \sqrt{4\alpha_0 \alpha_2 - \alpha_1^2} \tan \left(\frac{\sqrt{4\alpha_0 \alpha_2 - \alpha_1^2}}{2} (\xi + \xi_0) \right)} \\ \quad - \frac{8\alpha_0^2 \alpha_2^2 \mu_1^2}{\left(\alpha_1 - \sqrt{4\alpha_0 \alpha_2 - \alpha_1^2} \tan \left(\frac{\sqrt{4\alpha_0 \alpha_2 - \alpha_1^2}}{2} (\xi + \xi_0) \right) \right)^2} \\ \quad - \frac{\alpha_1^2 \mu_1^3 + 8\alpha_0 \alpha_2 \mu_1^3 + \mu_2}{6\mu_1}, \quad 4\alpha_0 \alpha_2 > \alpha_1^2, \quad \alpha_2 > 0. \end{cases} \quad (45)$$

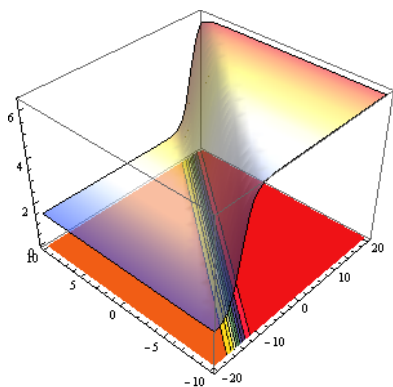
$$\begin{cases} \mathcal{U}_{3,4}(x, y, t) = -\frac{\alpha_1^2 \mu_1^3 + 8\alpha_0 \alpha_2 \mu_1^3 + \mu_2}{6\mu_1} \\ \quad - \frac{8\alpha_0^2 \alpha_2^2 \mu_1^2}{\left(\sqrt{4\alpha_0 \alpha_2 - \alpha_1^2} \tan \left(\frac{\sqrt{4\alpha_0 \alpha_2 - \alpha_1^2}}{2} (\xi + \xi_0) \right) + \alpha_1 \right)^2} \\ \quad - \frac{4\alpha_0 \alpha_1 \alpha_2 \mu_1^2}{\sqrt{4\alpha_0 \alpha_2 - \alpha_1^2} \tan \left(\frac{\sqrt{4\alpha_0 \alpha_2 - \alpha_1^2}}{2} (\xi + \xi_0) \right) + \alpha_1}, \\ \quad 4\alpha_0 \alpha_2 > \alpha_1^2, \quad \alpha_2 < 0. \end{cases} \quad (46)$$



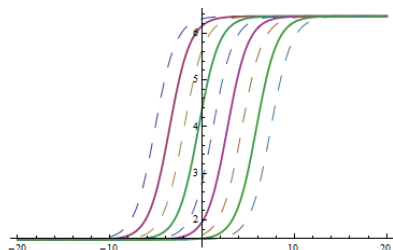
(a)



(b)



(c)



(d)

Fig. 7: The solutions (44) and (45) are defined by giving appropriate parameter values: (a) shows a anti-Kink soliton with CP and its 2D representation in (b), and (c) depicts an kink soliton with CP and its 2D representation in (d), respectively.

5 Conservation Laws

In this section, we derive the conservation laws for equation for dynamical model (9).

5.1 First Conservation Law

In this sub-part, we derive the first conservation law for the equation (9). Multiply the equation (9) by $\mathcal{U}_x$, yields as

$$\mathcal{U}_x (\mathcal{U}_{xt} + \mathcal{U}_{xx} + \mathcal{U}_{yy} + \mathcal{U}_{xy} + 2\mathcal{U} \mathcal{U}_{xy} + 6\mathcal{U}^2 \mathcal{U}_{xx} + 2(\mathcal{U}_{xx})^2) = 0. \quad (47)$$

Taking the subsequent formulas into consideration

$$\begin{aligned} \mathcal{U}_x \mathcal{U}_{xt} &= \frac{\partial}{\partial t} \left(\frac{1}{2} \mathcal{U}_x^2 \right), \\ \mathcal{U}_x \mathcal{U}_{xx} &= \frac{\partial}{\partial x} \left(\frac{1}{2} \mathcal{U}_x^2 \right), \\ \mathcal{U}_x \mathcal{U}_{yy} &= \frac{\partial}{\partial y} (\mathcal{U}_x \mathcal{U}_y) - \mathcal{U}_{xy} \mathcal{U}_y, \\ \mathcal{U}_x \mathcal{U}_{xy} &= \frac{\partial}{\partial y} \left(\frac{1}{2} \mathcal{U}_x^2 \right), \\ 2\mathcal{U} \mathcal{U}_x \mathcal{U}_{xy} &= \frac{\partial}{\partial y} (\mathcal{U} \mathcal{U}_x^2) - \mathcal{U}_y \mathcal{U}_x^2, \\ 6\mathcal{U}^2 \mathcal{U}_x \mathcal{U}_{xx} &= \frac{\partial}{\partial x} (2\mathcal{U}^3 \mathcal{U}_x) - 6\mathcal{U}^2 \mathcal{U}_x^2. \end{aligned}$$

The equation (47) can be expressed in a conservation form as:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\mathcal{U}_x^2}{2} \right) + \frac{\partial}{\partial x} \left(\frac{\mathcal{U}_x^2}{2} + 2\mathcal{U}^3 \mathcal{U}_x \right) + \frac{\partial}{\partial y} \left(\mathcal{U}_x \mathcal{U}_y + \frac{\mathcal{U}_x^2}{2} + \mathcal{U} \mathcal{U}_x^2 \right) \\ = - \left(\mathcal{U}_y (\mathcal{U}_{xy} + \mathcal{U}_x^2) - 6\mathcal{U}^2 \mathcal{U}_x^2 + 2\mathcal{U}_x (\mathcal{U}_{xx})^2 \right). \end{aligned} \quad (48)$$

The above equation can be articulated through energy conservation as:

$$\frac{\partial I}{\partial t} + \nabla \cdot \mathbf{F} = R. \quad (49)$$

From Eq.(48), we derive the conserved quantity in the following form:

$$I = \frac{1}{2} \mathcal{U}_x^2. \quad (50)$$

This represents the first conservation law (energy-like quantity) for equation (9).

5.2 Second Conservation Law

Multiply the equation (9) by $\mathcal{U}$, yields as

$$\begin{aligned} \mathcal{U} \mathcal{U}_{xt} + \mathcal{U} \mathcal{U}_{xx} + \mathcal{U} \mathcal{U}_{yy} + \mathcal{U} \mathcal{U}_{xy} + 2\mathcal{U}^2 \mathcal{U}_{xy} \\ + 6\mathcal{U}^3 \mathcal{U}_{xx} + 2\mathcal{U} (\mathcal{U}_{xx})^2 = 0. \end{aligned} \quad (51)$$

Taking the subsequent formulas into consideration

$$\begin{cases} \mathcal{U} \mathcal{U}_{xt} = \frac{\partial}{\partial t} (\mathcal{U} \mathcal{U}_x) - \mathcal{U}_t \mathcal{U}_x, \\ \mathcal{U} \mathcal{U}_{xx} = \frac{\partial}{\partial x} (\mathcal{U} \mathcal{U}_x) - \mathcal{U}_x^2, \\ \mathcal{U} \mathcal{U}_{yy} = \frac{\partial}{\partial y} (\mathcal{U} \mathcal{U}_y) - \mathcal{U}_y^2, \\ \mathcal{U} \mathcal{U}_{xy} = \frac{\partial}{\partial y} (\mathcal{U} \mathcal{U}_x) - \mathcal{U}_y \mathcal{U}_x, \\ 2\mathcal{U}^2 \mathcal{U}_{xy} = \frac{\partial}{\partial y} (\mathcal{U}^2 \mathcal{U}_x) - 2\mathcal{U} \mathcal{U}_y \mathcal{U}_x, \\ 6\mathcal{U}^3 \mathcal{U}_{xx} = \frac{\partial}{\partial x} (2\mathcal{U}^3 \mathcal{U}_x) - 6\mathcal{U}^2 \mathcal{U}_x^2. \end{cases}$$

The equation (51) can be expressed in the form as

$$\begin{aligned} & \frac{\partial}{\partial t} (\mathcal{U} \mathcal{U}_x) + \frac{\partial}{\partial x} (\mathcal{U} \mathcal{U}_x + 2\mathcal{U}^3 \mathcal{U}_x) \\ & + \frac{\partial}{\partial y} (\mathcal{U} \mathcal{U}_y + \mathcal{U} \mathcal{U}_x + \mathcal{U}^2 \mathcal{U}_x) = P_1, \end{aligned} \quad (52)$$

where $P_1 = -(\mathcal{U}_t \mathcal{U}_x + \mathcal{U}_x^2 + \mathcal{U}_y^2 + \mathcal{U}_y \mathcal{U}_x + 2\mathcal{U} \mathcal{U}_y \mathcal{U}_x + 6\mathcal{U}^2 \mathcal{U}_x^2 - 2\mathcal{U} (\mathcal{U}_{xx})^2)$.

The equation (52) can be expressed as the conservation law as

$$\frac{\partial I_2}{\partial t} + \nabla \cdot \mathbf{F} = P_1. \quad (53)$$

From Eq.(53), we derive the conserved quantity in the following form:

$$I_2 = \mathcal{U} \mathcal{U}_x. \quad (54)$$

This represents the 2nd conservation law (momentum-like quantity) for equation (9).

5.3 Third Conservation Law

For third conservation law, multiply the above equation (9) by $\mathcal{U}^2$, yields as

$$\begin{aligned} & \mathcal{U}^2 \mathcal{U}_{xt} + \mathcal{U}^2 \mathcal{U}_{xx} + \mathcal{U}^2 \mathcal{U}_{yy} + \mathcal{U}^2 \mathcal{U}_{xy} + 2\mathcal{U}^3 \mathcal{U}_{xy} \\ & + 6\mathcal{U}^4 \mathcal{U}_{xx} + 2\mathcal{U}^2 (\mathcal{U}_{xx})^2 = 0. \end{aligned} \quad (55)$$

Taking the subsequent formulas, the equation (55) can be articulated in a conservation form as:

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{1}{2} \mathcal{U}^2 \mathcal{U}_x \right) + \frac{\partial}{\partial x} \left(\frac{1}{2} \mathcal{U}^2 \mathcal{U}_x + \frac{3}{2} \mathcal{U}^4 \mathcal{U}_x \right) \\ & + \frac{\partial}{\partial y} \left(\frac{1}{2} \mathcal{U}^2 \mathcal{U}_y + \frac{1}{2} \mathcal{U}^2 \mathcal{U}_x + \frac{2}{3} \mathcal{U}^3 \mathcal{U}_x \right) = P_2. \end{aligned} \quad (56)$$

where $P_2 = -\mathcal{U} \mathcal{U}_t \mathcal{U}_x - \mathcal{U} \mathcal{U}_x^2 - \mathcal{U} \mathcal{U}_y^2 - \mathcal{U} \mathcal{U}_y \mathcal{U}_x - 2\mathcal{U}^2 \mathcal{U}_y \mathcal{U}_x - 12\mathcal{U}^3 \mathcal{U}_x^2 + 2\mathcal{U}^2 (\mathcal{U}_{xx})^2$. The equation (56) can be expressed as the conservation law as

$$\frac{\partial I_3}{\partial t} + \nabla \cdot \mathbf{F} = P_2. \quad (57)$$

From Eq.(53), we derive the conserved quantity in the following form:

$$I_3 = \frac{\mathcal{U}^2 \mathcal{U}_x}{2}. \quad (58)$$

This represents the 3rd conservation law (higher-order conserved densities) for equation (9).

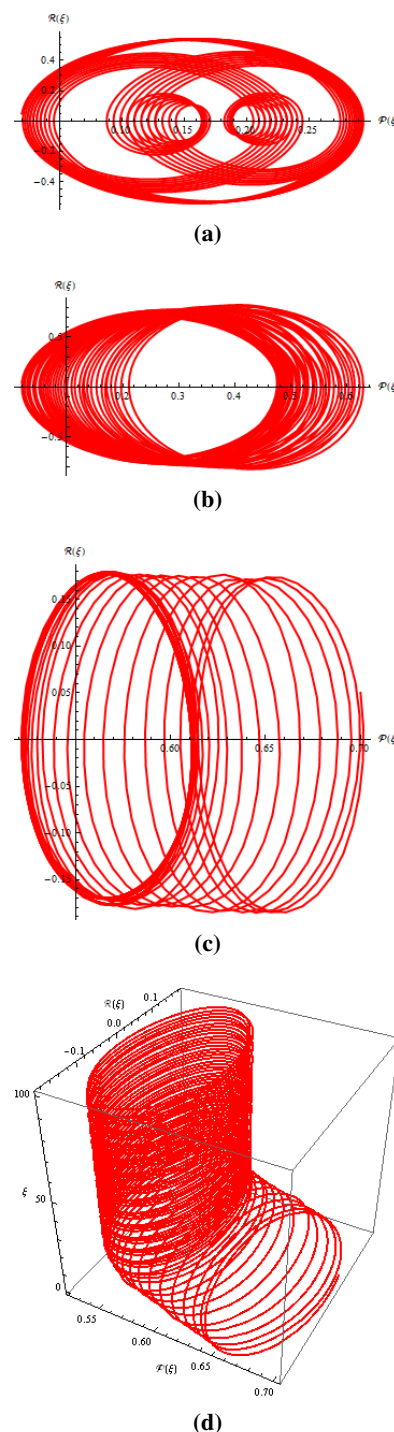


Fig. 8: The chaotic behavior of system (59) with different perturbed terms ($G(\xi)$) as: (a) Phase portrait in 2D having $G(\xi) = \cos(5\xi)$, (b) phase portrait in 2D having $G(\xi) = \cosh(0.023\xi)$, (c) and (d) are phase portraits in 2D and 3D having $G(\xi) = \frac{\sqrt{90}}{2\pi} e^{-\frac{(0.13\xi)^2}{2}}$ respectively.

6 Chaotic Behavior

Chaotic dynamics in NPDEs frequently occur through bifurcations, resulting in complicated and unpredictable behavior [38, 39]. The planar dynamical system obtained from Eq.(10) does not exhibit chaotic behavior. However, when exposed to an external disruption, the system can exhibit chaotic behavior. Substituting $\mathcal{P}'(\xi) = (\xi)$ in Eq.(10), the equation is reduces into planar system and perturbed form of this planar system is shown below:

$$\begin{aligned}\mathcal{P}'(\xi) &= \mathcal{R}(\xi), \\ \mathcal{R}'(\xi) &= h_1 - \mathcal{P}(h_2 + h_3 \mathcal{P}) + G(\xi),\end{aligned}\quad (59)$$

where $h_1 = \frac{\mu_1(\tau-\mu_2)-\mu_1^2-\mu_2^2}{2\mu_1^4}$, $h_2 = \frac{\mu_2}{\mu_1^3}$, $h_3 = \frac{3}{\mu_1^2}$ and $G(\xi)$ is perturbed term. ξ represent the magnitude and frequency of external force exerted to the system. To get the chaotic dynamics, we consider some special values for perturbation term as follows:

(i) Trigono-metric function: Letting $G(\xi) = \cos(5\xi)$, yield the phase portraits as illustrated in Figure 8(a).

(ii) Hyper-bolic function: putting $G(\xi) = \cosh(0.023\xi)$, yield the phase portraits as illustrated in Figure 8(b).

(iii) Gaussian function: putting $G(\xi) = \frac{\sqrt{90}}{2\pi} e^{-\frac{(0.13\xi)^2}{2}}$, yield the phase portraits as illustrated in Figure 8(c) and 8(d) (2D and 3D), respectively.

7 Stability Analysis

A basic theoretical method for figuring out how differential equation solutions behave under minor perturbations is stability analysis. It investigates whether a particular solution such as a steady solution, equilibrium state, or wave profile maintains its shape in the face of minor disturbances or whether deviations increase infinitely over time. Finding the steady solution, adding minor perturbations, and linearizing the governing equation while ignoring higher-order terms are the usual steps [5, 40]. By examining the growth rates of perturbation modes or the eigenvalues of the associated operator, the linearized system that results is examined. If all perturbations decrease with time, the solution is stable; if any perturbation increases, instability arises. Beyond the linear approximation, stability is evaluated for highly nonlinear systems using sophisticated techniques like Lyapunov stability theory or energy methods. First, we search for constant equilibrium solutions of $\mathcal{U} = U_0$, for which the equation is identically satisfied and all derivatives vanish. To analyze stability, we introduce a small perturbation as

$$\mathcal{U}(x, y, t) = U_0 + \varepsilon \Phi(x, y, t), \quad (60)$$

where $\varepsilon \ll 1$. Substituting the above equation (60) into equation (9) and retaining only $O(\varepsilon)$ terms yields the linearized equation

$$\Phi_{xt} + (1 + 6U_0^2) \Phi_{xx} + \Phi_{yy} + (1 + 2U_0) \Phi_{xy} = 0. \quad (61)$$

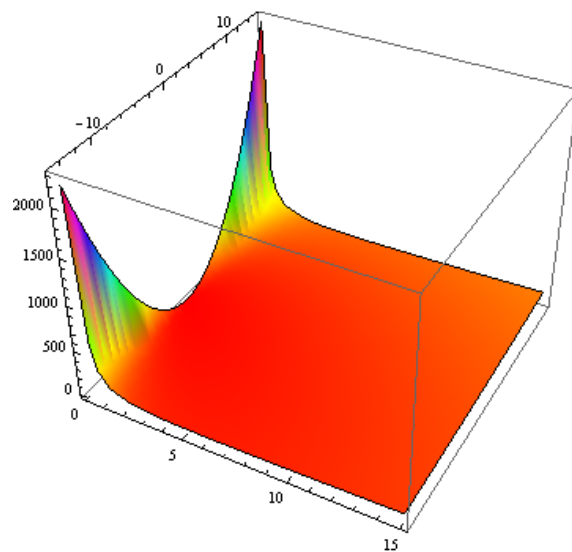
Assuming the equation (61) has solution of the form as

$$\Phi(x, y, t) = \sigma e^{i(k_1 x + k_2 y - \omega t)}, \quad (62)$$

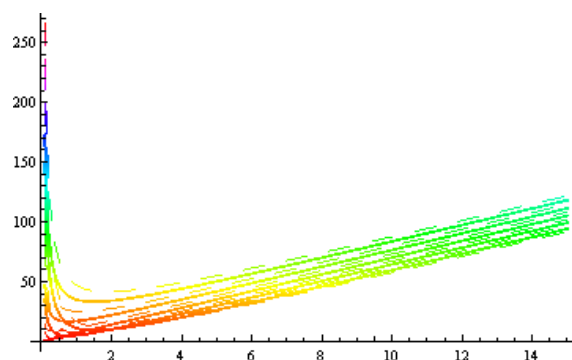
where k_1, k_2 and ω are wave numbers and frequency, respectively. The dispersion relation is obtained by substituting (62) into (61) as

$$\omega = \frac{(1 + 6U_0^2)k_1^2 + k_2^2 + (1 + 2U_0)k_1 k_2}{k_1}. \quad (63)$$

For every k_1, k_2 , and U_0 , the frequency ω is real, suggesting that perturbations do not increase or decrease exponentially with time. Consequently, $\mathcal{U} = U_0$, the constant equilibrium solution, is neutrally stable.



(a)



(b)

Fig. 9: The relation in (63) between ω and k_1, k_2 is shown in (a) and (b) (3D and 2D) respectively.

8 Physical interpretation of the results and discussion

In this work, we analyze a 2-dSE and find several families of exact solutions using the Φ^6 -model expansion and extended

simplest equation approaches. A wide range of wave phenomena, represented in hyperbolic, trigonometric, exponential, rational, and algebraic forms, are revealed by these solutions, including solitary, periodic, kink and anti-kink, bright, dark, multi-peak, peakon, and singular wave profiles. The robustness of the applied methods is shown by their ability to effectively address a high-dimensional system, suggesting their broader applicability to more intricate PDEs arising in Geophysical fluid dynamics, quantum field theory, and nonlinear acoustics [41–43]. Further understanding of wave propagation in fluids, optical media, and plasma environments is also made possible by the physical relevance of these solutions. Through careful parameter selection, we were able to produce new wave configurations, including periodic structures, breathers, multi-peak formations, and bright and dark waves. These parameters, which enable systematic investigation of how wave features change under various circumstances, were carefully selected to guarantee both mathematical rigor and physical relevance. Wave solutions for this model were obtained using Painlevé analysis in earlier works, such as Wazwaz's [1], whereas solitary wave solutions were investigated using expanded direct algebraic approaches in other studies [37]. These discoveries are unique and represent fresh contributions that haven't been documented in the literature before.

The figures display a variety of soliton structures attained via giving appropriate values to parameter to the achieved results. Figures 1 and 2 illustrate bright, multi-peak, peakon, and dark solitons as well as the CPs and 2D profiles. Figures 3 and 4 present dark multi-peakons of varying amplitudes, singular periodic, and periodic multi-peak solitons, each accompanied by their CPs and 2D visualizations. Figures 5 and 6 depict singular periodic, dark soliton, and solitary wave solutions with their associated CPs and 2D views. Figure 7 displays kink and anti-kink solitons together with their CPs and 2D representations. Figure 8 shows the phase portraits of system (59) under different perturbations. Finally, Figure 9 illustrates the relationship described by (63) between ω and k_1, k_2 , with subplots (a) and (b) showing the 3D and 2D plots, respectively.

9 Concluding Remarks

In this study, the extended simplest equation method and the Φ^6 -model expansion technique have been successfully employed to investigate the 2-dSE, an integrable model with significant uses in nonlinear wave propagation and fluid dynamics. This equation effectively describes nonlinear wave phenomena in shallow water and stratified fluids, capturing complex behaviors such as wave breaking, soliton formation, and periodic wave patterns. The applied methods yield a comprehensive set of closed-form solutions, including solitary waves, periodic structures, kink and anti-kink solitons, breather-type waves, multi-peak solitons, peakons, and other periodic patterns, each with meaningful physical interpretations. The proposed methods effectively handle the nonlinearities in the model and provide a systematic framework for deriving exact solutions. The dynamical behavior of the model is further explored through chaotic analysis under external perturbations, revealing the system's sensitivity, and complex dynamics. Conservation laws are derived to construct key invariants

governing the equation. The modulational instability analysis is used to confirm the stability and physical relevance of the obtained solutions. Graphical simulations are presented to illustrate the dynamical features, stability regimes, and diverse soliton structures. An important advantage of the proposed analytical framework, supported by graphical simulations, is its ability to generate a wide class of exact closed-form solutions without relying on mesh-based discretization or iterative numerical solvers. Compared with conventional numerical schemes, the present approach avoids numerical instability and high computational cost while providing clear insight into parameter effects and wave structures. The computational work and obtained results demonstrate the effectiveness of these methods in constructing a wide spectrum of solitary, periodic, and singular solutions for complex multidimensional NPDEs. Future work may focus on extending the current analysis to non-integrable and variable-coefficient versions of the model, investigating numerical simulations for strongly perturbed regimes, and exploring experimental or real-world applications in fluid dynamics, nonlinear optics, and plasma physics.

Availability of data and materials

The datasets analyzed during the current study are available from the corresponding author upon reasonable request.

Conflict of Interest

Authors has no conflict of interest.

Funding

No funding

Consent to Publish declaration

Not applicable

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