

# A Review of Key Aspects of Modern Generalized Fractional Kinetic Equations

Hala Abd elmageed<sup>1</sup>, Mohamed Akeel<sup>2</sup> and Mohamed Abdalla<sup>2,\*</sup>

<sup>1</sup> Department of Mathematics, College of Science and Humanities, Al-Kharj Prince Sattam bin Abdulaziz University, Al-Kharj, 11942, Saudi Arabia

<sup>2</sup> Mathematics Department, Faculty of Science, Qena University, Qena 83523, Egypt

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**Abstract:** Across mathematics, physics, engineering, and applied sciences, integral transforms like the Laplace, Mellin, Sumudu, Pathway, and Elzaki transforms have become vital instruments for solving modern fractional kinetic equations (MFKEs). Often resulting in explicit solutions in terms of special functions as the Mittag-Leffler function, M-function, Fox's H-function, and related functions, these transforms help the analytic handling of fractional derivatives and integrals. In this review, the most recent developments, applications, and obstacles in employing integral transforms for MFKEs and related fractional equations are combined

**Keywords:** Integral transform, Fractional operator, Kinetic equation, Special functions.

## 1 Introduction

Integral transforms (ITs) have been used successfully for almost two centuries to solve various problems in applied mathematics, mathematical physics, engineering, and other fields. The development of the integral transform theory, such as those of Laplace and Fourier transforms, has its origin in work by such luminaries as Laplace (1749–1827) and Fourier (1768–1830) [54, 62].

The integral transform define by

$$\mathfrak{T}(\Phi) = \mathcal{T}(g(s)) = \int f(s) \mathbf{R}(s, \varkappa) ds,$$

here,  $\mathfrak{T}(\Phi)$  denotes the function generated by the integral transform, and  $\mathbf{R}(s, \Phi)$  denotes a kernel function.

Also, we give the inverse integral transform as

$$g(s) = \int \mathcal{T}(g(\Phi)) \mathbf{R}^{-1}(\Phi, s) d\Phi.$$

Moreover, elements of Ramadan group transformation include the Z-transform, the Hankel transform, the Hilbert transform, the  $g_{m,n}$ -transform, the Stieltjes transform, the wavelet transform, the H-transform, the Elzaki transform and many others which can solve initial and boundary value problems related to ordinary and partial differential equations as well as other problems in mathematics, science and engineering. See, for example., [8, 12, 26, 39, 44, 65, 83].

Fractional kinetic equations generalize classical kinetic equations by replacing integer-order derivatives with fractional (often nonlocal) operators to capture memory, anomalous transport, or complex reaction mechanisms. These equations arise in astrophysics (stellar composition changes), run-and-tumble transport, anomalous diffusion, fractional statistical mechanics, and various engineering and biological models [36, 67].

The analytical treatment of generalized fractional kinetic equations relies heavily on integral transforms. These equations are used in many sectors of research and engineering, including physics, chemistry, biology, and finance. Integral transforms allow researchers to simplify difficult mathematical models, while also gaining useful insights into

\* Corresponding author e-mail: [moabdalla@kku.edu.sa](mailto:moabdalla@kku.edu.sa)

the behavior of fractional kinetic systems (see, for instance [18,55,59,71]). One of the primary advantages of utilizing integral transforms in the analysis of generalized fractional kinetic equations is their capacity to convert differential equations to algebraic equations. This transformation simplifies the problem and enables the use of well-established and understandable mathematical approaches. The Mittag-Leffler function (MLF), a generalization of the exponential function, is particularly useful for solving fractional differential equations. The MLF naturally emerges in fractional calculus, serving as an effective instrument for resolving integral and differential problems that incorporate fractional derivatives (see, [1,34]). On the other hand, integrating the MLF into the analysis of generalized fractional kinetic equations enables researchers to derive analytical solutions that characterize the behaviour of complex systems exhibiting fractional-order dynamics. Additionally, integral transforms allow researchers to examine the long-range memory effects inherent in fractional kinetic systems. The memory effects, represented by fractional derivatives in the equations, can be examined by integral transforms, enhancing comprehension of the underlying dynamics and mechanisms involved [1,34].

This literature review aims to synthesize recent advancements in the application of integral transforms and special functions to solve modern fractional kinetic equations (MFKEs), highlighting key contributions, methodologies, and examples from the current body of research.

## 2 Mittag–Leffler functions

Based on the definition of the Gamma function [24]:

$$\Gamma(p) := \int_0^{\infty} e^{-\tau} \tau^{p-1} d\tau, \quad \operatorname{Re}(p) > 0. \quad (1)$$

The Swedish Mathematician Gosta Mittag-Leffler (M-L) (1903) [60] defined the following MLF as

$$E_{\alpha}(y) = \sum_{r=0}^{\infty} \frac{y^r}{\Gamma(r\alpha + 1)}, \quad \operatorname{Re}(\alpha) > 0. \quad (2)$$

This function is known as the *queen of the functions of fractional calculus*.

Some various generalizations of the MLF are provided as

The generalizations of the  $E_{\alpha}(\xi)$ , first appeared in 1905 by Wiman [80] as

$$E_{\alpha_1, \alpha_2}(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(k\alpha_1 + \alpha_2)} \quad \operatorname{Re}(\alpha_1) > 0, \operatorname{Re}(\alpha_2) > 0. \quad (3)$$

The function  $E_{\alpha_1, \alpha_2}(\xi)$  is the natural expansion of some basic functions, like exponential functions, trigonometric functions, hyperbolic functions, and other functions (cf. [34]).

In 1971, Prabhakar [64] introduced its extension by

$$E_{\alpha_1, \alpha_2}^{\delta}(\xi) = \sum_{k=0}^{\infty} \frac{(\delta)_k}{\Gamma(k\alpha_1 + \alpha_2)} \frac{\xi^k}{k!}, \quad \operatorname{Re}(\alpha_1) > 0, \operatorname{Re}(\alpha_2) > 0, \quad (4)$$

where  $(\ell)_m$  is defined by

$$(\ell)_k = \frac{\Gamma(\ell + m)}{\Gamma(\ell)} = \begin{cases} \ell(\ell+1)\dots(\ell+m-1), & m \in \mathbb{N}, \ell \in \mathbb{C} \\ 1, & m = 0; \ell \in \mathbb{C} \setminus \{0\}, \end{cases} \quad (5)$$

In 2007, Shukla and Prajapati [73] presented the following function:

$$E_{\alpha_1, \alpha_2}^{\ell, \nu}(\xi) = \sum_{k=0}^{\infty} \frac{(\ell)_{\nu k}}{\Gamma(k\alpha_1 + \alpha_2)} \frac{\xi^k}{k!} \quad (6)$$

$\operatorname{Re}(\alpha_1) > 0, \operatorname{Re}(\alpha_2) > 0, \nu \in (0, 1) \cup \mathbb{N}.$

where  $(\ell)_{vk}$  is given as

$$(\ell)_{vk} = \frac{\Gamma(\ell + kv)}{\Gamma(\ell)} = k^{vk} \prod_{j=1}^k \left( \frac{\ell + j - 1}{k} \right), \quad k \in \mathbb{N}. \quad (7)$$

The works may be consulted for more generalizations, properties, and applications of MLFs.[37,34].

### 3 Fractional Operators

The idea of "fractional calculus" (FC), also known as "fractional analysis," is employed for the new birth of calculus (analysis), when the order of integration and differentiation may be an arbitrary number (fractional, irrational, or complex), that is, not an obligatory integer. Though it is nearly 300 years old, the FC is among the fastest-developing mathematical analysis topics today. Where mathematics giants W. Leibniz (1697) and L. Euler (1730) mused about the possibility of doing differentiation of non-integer order.

The FC's initial stages and extensive growth can be connected to a multitude of observations and attempts from the early 19th and early 20th centuries. It is almost impossible to catalog all the important milestones in the initial phases of the FC's construction. For instance, see the references cited in [56,57,77,78].

Similarly, fractional calculus operators (FCOs), as a kind of integral transform of fractional order convolutions, deal with non-integer (fractional) orders of functions. It is a generalization of classical operators and stems from the efforts of Lacroix (1819), Liouville (1832), Letnikov (1868), Riemann (1876), Hadamard (1892), Weyl (1919), Caputo (1967), and many others [48]. Within the literature, the definitions of FCOs are highly variable. Among the several fractional operators, the Riemann-Liouville and Caputo operators appear to be the most prominent operators [54]. In addition to the cited works, some books on the theory of FCOs developed alongside a number of special functions and their applications [47,48,50,74].

The Riemann-Liouville fractional integral operator (RLFIO) of order  $\varphi$  is characterized as follows.

$${}_0\mathbb{D}_{\varphi}^{-\varphi} f(\varphi) := (\mathbb{I}^{\varphi} f)(\varphi) := \frac{1}{\Gamma(\varphi)} \int_0^{\varphi} (\varphi - u)^{\varphi-1} f(u) du, \quad \text{Re}(\varphi) > 0, \quad (8)$$

the Riemann-Liouville fractional differential operator (RLFDO) of order  $\mu$ , defined as

$$\mathbb{D}_{\varphi}^{\mu} f(\varphi) := \mathbb{D}^n \left( \mathbb{I}^{n-\mu} f(\varphi) \right), \quad n-1 < \text{Re}(\mu) < n. \quad (9)$$

For  $\text{Re}(\mu) > 0$ , the Hadamard fractional integral operators are defined as follows [44]:

$$\begin{aligned} (\mathbb{H}_{a_1+}^{\mu} \Phi)(\eta) &= \frac{1}{\Gamma(\mu)} \int_{a_1}^{\eta} \left( \ln \frac{\eta}{t} \right)^{\mu-1} \Phi(t) \frac{dt}{t}, \quad a_1 < \eta < a_2, \\ (\mathbb{H}_{a_2-}^{\mu} \Phi)(\eta) &= \frac{1}{\Gamma(\mu)} \int_{\eta}^{a_2} \left( \ln \frac{t}{\eta} \right)^{\mu-1} \Phi(t) \frac{dt}{t}, \quad a_1 < \eta < a_2. \end{aligned} \quad (10)$$

Also, we have

$$\begin{aligned} (\mathbb{H}_{a_1+}^{\mu} \Phi)(\eta) &= \delta^n \left( \mathbb{H}_{a_1+}^{n-\mu} \Phi \right)(\eta) \\ &= \left( \eta \frac{d}{d\eta} \right)^n \frac{1}{\Gamma(n-\alpha)} \int_{a_1}^{\eta} \left( \ln \frac{\eta}{t} \right)^{n-\mu-1} \frac{\Phi(t) dt}{t}, \\ (\mathbb{H}_{a_2-}^{\mu} \Phi)(\eta) &= (-\delta)^n \left( \mathbb{H}_{a_2-}^{n-\mu} \Phi \right)(\eta) \\ &= \left( -\eta \frac{d}{d\eta} \right)^n \frac{1}{\Gamma(n-\mu)} \int_{\eta}^b \left( \ln \frac{t}{\eta} \right)^{n-\mu-1} \frac{\Phi(t) dt}{t}. \end{aligned} \quad (11)$$

### 4 Fractional Kinetic Equations

Saxena et al. [69] presented the fractional kinetic equation in the form

$$\mathcal{K}(x) - \mathcal{K}_0 = -\varepsilon^\varsigma {}_0\mathbb{D}_x^{-\varsigma} \mathcal{K}(x), \quad (12)$$

where  ${}_0\mathbb{D}_x^{-\varsigma}$  is defined in (8).

The fractional kinetic equation (12) was modified in [70] as

$$\mathcal{K}(x) - x^{\omega-1} \mathcal{K}_0 = -\varepsilon^\varsigma {}_0\mathbb{D}_x^{-\varsigma} \mathcal{K}(x), \quad \operatorname{Re}(\varsigma) > 0, \operatorname{Re}(\omega) > 0. \quad (13)$$

In [37], another variation of the fractional kinetic equation (12) is proposed by

$$\mathcal{K}(x) - x^{\omega-1} \mathcal{K}_0 \mathbf{E}_{\varsigma, \omega}^\delta(-\varepsilon^\varsigma x^\varsigma) = -\varepsilon^\varsigma {}_0\mathbb{D}_x^{-\varsigma} \mathcal{K}(x), \quad \operatorname{Re}(\varsigma) > 0, \operatorname{Re}(\delta) > 0, \operatorname{Re}(\omega) > 0, \quad (14)$$

where  $\mathbf{E}_{\varsigma, \omega}^\delta(\cdot)$  is presented in (4).

Applications of kinetic fractional equations with several fractional operators have created a lot of interest in applied fields of research and control systems, physical phenomena, engineering, and other fields (see [68, 69, 70]). Special functions have become a useful tool in many areas where modern fractional kinetic equations (MFKEs) are solved using different types of integral transforms (see, for example, [5, 7, 22, 75]).

## 5 Applications of Laplace transform for solving MFKEs

The Laplace transform (LT) of  $G(y)$  is formally defined by [25, 23]

$$\mathcal{L}\{G(y)\} = \bar{G}(\kappa) = \int_0^\infty e^{-\kappa y} G(y) dy, \quad \operatorname{Re}(\kappa) > 0. \quad (15)$$

The inverse Laplace transform is

$$\mathcal{L}^{-1}\{\bar{G}(\kappa)\} = G(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\kappa y} \bar{G}(\kappa) d\kappa, \quad c > 0. \quad (16)$$

The Laplace transform was made popular by Oliver Heaviside (1850-1925) and others by its use in solving certain categories of ordinary and partial differential equations. See e.g. [4, 8, 12, 15, 19, 30, 31, 43, 79]. A succinct survey of various FKEs involving the of the Riemann-Liouville fractional integration solved by a number of researchers using the Laplace transform presented by Saxena and Kalla [68].

In this section, we propose some novel results on the fractional generalizations of the kinetic equations that could be solved using the classical Laplace transform. The following expression describes the Laplace transform of the Riemann-Liouville fractional integral operator which was given by Erdelyi et al. [32, p.181]

$$\mathcal{L}\left\{{}_0^{RL}\mathbb{D}_y^{-\varsigma} g(y)\right\} = \kappa^{-\varsigma} \bar{G}(\kappa), \quad \operatorname{Re}(\varsigma) > 0. \quad (17)$$

The relation (17) can also be demonstrated by using the convolution theorem to the Laplace transform by Saxena and Kalla [68].

**Theorem 1.** *Yadav et al. (2025) [81]. Assume that  $d > 0, v > 0, \lambda, \gamma, \mu, h, c \in \mathbb{C}, k \in \mathbb{N}, \operatorname{Re}(\mu) > 0$  and  $\operatorname{Re}(\lambda) > 0$ , then, we have*

$$\mathfrak{W}(t) - \mathfrak{W}_0 \mathbb{F}_{b, k, \mu, h, \lambda}^{(a, \gamma, c)}(t) = -d^v {}_0\mathbb{D}_t^{-v} \mathfrak{W}(t), \quad (18)$$

is solved in the form

$$\begin{aligned} \mathfrak{W}(t) = & \mathfrak{W}_0 \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_n}{\prod_{j=1}^q (b_j)_n} \frac{(-1)^n c^n (\gamma)_{nk}}{\Gamma_k(\lambda n + \mu + (h+1)/2)} \frac{\Gamma(\mu + 2n + 1)}{(n!)^2} \\ & \times \left(\frac{t}{2}\right)^{\mu+2n} {}_p\mathbb{M}_q^{(v, \mu+2n+1)}(-d^v t^v), \end{aligned} \quad (19)$$

where

$${}_p\mathbb{M}_q^{(\alpha,\beta)}(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_n}{\prod_{j=1}^q (b_j)_n} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \quad (20)$$

is the generalized  $M$ -series [72], and

$$\mathbb{F}_{b,k,\mu,h,\lambda}^{(a,\gamma,c)}(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_n}{\prod_{j=1}^q (b_j)_n} \frac{(-1)^n c^n (\gamma)_{n,k}}{\Gamma_k(\lambda n + \mu + (h+1)/2)} \frac{1}{(n!)^2} \left(\frac{z}{2}\right)^{\mu+2n}$$

is the generalized  $k$ -Bessel function [81].

**Theorem 2.** Jangid et al. (2023) [40]. If  $\zeta, \eta, a, b > 0$  and  $\mu > 0$ , then the solution of

$$\mathfrak{W}(t) - \mathfrak{W}_0 t^{\mu-1} \mathcal{S}_n^m [at^\zeta] \Gamma_{u,v}^{r,s} [bt^\eta] = -c^v {}_0\mathbb{D}_t^{-v} \mathfrak{W}(t) \quad (2.1)$$

is provided as

$$\begin{aligned} \mathfrak{W}(t) = & \mathfrak{W}_0 t^{\mu-1} \sum_{i=0}^{\infty} (-c^v t^v)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k}}{k!} (at^\zeta)^k \\ & \times \Gamma_{u+1,v+1}^{r,s+1} \left[ bt^\eta \left| \begin{matrix} (a_1, A_1, x), (1-\mu-\zeta k, \eta), (a_j, A_j)_{2,u} \\ (b_j, B_j)_{1,v}, (1-\mu-\zeta k-vi, \eta) \end{matrix} \right. \right]. \end{aligned} \quad (21)$$

**Theorem 3.** [40] If  $\zeta, \eta, a, b > 0$  and  $\mu > 0$ , then the solution of

$$\mathfrak{W}(t) - \mathfrak{W}_0 t^{\mu-1} \mathcal{S}_n^m [at^\zeta] \gamma_{u,v}^{r,s} [bt^\eta] = -c^v {}_0\mathbb{D}_t^{-v} \mathfrak{W}(t), \quad (22)$$

is provided as

$$\begin{aligned} \mathfrak{W}(t) = & \mathfrak{W}_0 t^{\mu-1} \sum_{i=0}^{\infty} (-c^v t^v)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} \mathfrak{A}_{n,k}}{k!} (at^\zeta)^k \\ & \times \gamma_{u+1,v+1}^{r,s+1} \left[ bt^\eta \left| \begin{matrix} (a_1, A_1, x), (1-\mu-\zeta k, \eta), (a_j, A_j)_{2,u} \\ (b_j, B_j)_{1,v}, (1-\mu-\zeta k-vi, \eta) \end{matrix} \right. \right], \end{aligned} \quad (23)$$

where, the incomplete  $H$ -functions are defined via Mellin-Barnes contour integrals

$$\gamma_{u,v}^{r,s}(z) = \gamma_{u,v}^{r,s} \left[ z \left| \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,u} \\ (b_j, B_j)_{1,v} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} g(\xi, x) z^{-\xi} d\xi, \quad (24)$$

$$\Gamma_{u,v}^{r,s}(z) = \Gamma_{u,v}^{r,s} \left[ z \left| \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,u} \\ (b_j, B_j)_{1,v} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} G(\xi, x) z^{-\xi} d\xi, \quad (25)$$

$$g(\xi, x) = \frac{\gamma(1-a_1-A_1\xi, x) \prod_{j=1}^r \Gamma(b_j+B_j\xi) \prod_{j=2}^s \Gamma(1-a_j-A_j\xi)}{\prod_{j=r+1}^v \Gamma(1-b_j-B_j\xi) \prod_{j=s+1}^u \Gamma(a_j+A_j\xi)}, \quad (26)$$

$$G(\xi, x) = \frac{\Gamma(1-a_1-A_1\xi, x) \prod_{j=1}^r \Gamma(b_j+B_j\xi) \prod_{j=2}^s \Gamma(1-a_j-A_j\xi)}{\prod_{j=r+1}^v \Gamma(1-b_j-B_j\xi) \prod_{j=s+1}^u \Gamma(a_j+A_j\xi)}, \quad (27)$$

$$\gamma_{p,q}^{m,n}(z) + \Gamma_{p,q}^{m,n}(z) = H_{p,q}^{m,n}(z), \quad (28)$$

and the general class of polynomials  $\mathcal{S}_n^m[x]$  is defined in [40] as

$$\mathcal{S}_n^m[\eta] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} \mathfrak{A}_{n,k} \eta^k. \quad (29)$$

**Theorem 4.**(2022)[9]. Let  $Y, T_1, T_2, T_3$  and  $S$  be positive stable matrices in  $\mathbf{C}^{n \times n}$ , such that  $T + \ell I$  and  $T_3 + nI$  are invertible for all  $n \in \mathbb{N}_0$ ,  $\sigma \in \mathbf{R}_0^+$ ,  $k, d, \mu \in \mathbf{R}^+$ , and  $\alpha, v \in \mathbf{C} \setminus \mathbf{Z}_0^-$ . Then the solution of the matrix equation

$$\mathfrak{W}(t)I - \mathfrak{W}_0 t^{\mu-1} {}_2\Theta_1^{Y;k,\alpha;\sigma} \left[ \begin{matrix} T_1, T_2 \\ T_3 \end{matrix}; d^\nu t^\nu \right] = -S^\nu {}_0D_t^{-\nu} \mathfrak{W}(t),$$

is given by the representation

$$\mathfrak{W}(t)I = \mathfrak{W}_0 t^{\mu-1} \sum_{n=0}^{\infty} (n + \alpha)^{-Y} (T_1; \sigma)_{n,k} (T_2)_{n,k} \left[ (T_3)_{n,k} \right]^{-1} \quad (30)$$

$$\times \frac{d^{\nu} t^{\nu} \Gamma(n\nu + \mu)}{n!} \mathbb{E}_{v,n\nu+\mu} (-S^\nu t^\nu), \quad (31)$$

where  $\Theta_1^{Y;k,\alpha;\sigma} \left[ \begin{matrix} T_1, T_2 \\ T_3 \end{matrix}; t \right]$  is the generalized extended  $k$ -Hurwitz-Lerch  $\zeta$ -matrix function [9] and  $\mathbb{E}_{v,\mu} (-C^\nu t^\nu)$  is the generalized Mittag-Leffler matrix function defined in [1].

*Remark.* A range of specific instances of Theorem 1, Theorem 2, Theorem 3, and Theorem 4, are presented in [9, 40, 81]. Also, We note that although certain conclusions were emphasized in the aforementioned theorems, several other recent results on the classical Laplace transform unrecalled in this context. The reader may refer to the works, for example [3, 22, 53, 75].

## 6 Applications of Sumudu transform for solving MFKEs

Watugala created the Sumudu transform (ST), which is frequently utilized to tackle many types of problems in science and engineering [79]. For the details of Sumudu transforms, properties and its applications the interesting reader is referred to [15, 20, 21, 29, 41, 58, 82].

The Sumudu transform over the set functions

$$B = \left\{ g(\tau) \mid \exists \mathbf{M}, x_1, x_2 > 0, |(\tau)| < \mathbf{M} e^{|\tau|/x_j}, \text{ if } \tau \in (-1)^j \times [0, \infty) \right\},$$

is expressed by

$$Y(u) = \mathbb{ST} \left( g(\tau); u \right) = \int_0^\infty g(u\tau) e^{-\tau} d\tau, \quad u \in (-x_1, x_2). \quad (32)$$

Our primary focus is on the following achieving results through the Sumudu transform method.

We recall the ST of the Riemann-Liouville fractional integral operator as follows

$$\mathbb{ST} \left\{ {}_0^{RL} \mathbb{D}_t^\nu g(t); \phi \right\} = (\phi)^\nu Y(\phi), \quad \text{Re}(\nu) > 0.$$

**Theorem 5.** Nisar et al. (2020) [61]. Let  $d > 0, v > 0, \rho, \lambda, t \in \mathbb{C}, \text{Re}(\rho) > 0$  and  $|\eta| < \mathbb{R}$ . Then the solution of the equation

$$\mathfrak{W}(t) - \mathfrak{W}_0 \mathcal{F}_{\rho,\lambda}(t) = -d^\nu {}_0\mathbb{D}_t^{-\nu} \mathfrak{W}(t),$$

is given by

$$\mathfrak{W}(t) = \mathfrak{W}_0 \sum_{n=0}^{\infty} \frac{\sigma(n) \Gamma(n+1)}{\Gamma(\rho n + \lambda)} t^{n-1} \mathbb{E}_{v,n} (-d^\nu t^\nu),$$

where  $\mathbb{E}_{v,n}(\cdot)$  is defined in (3) and the class of functions  $\mathcal{F}_\rho^\lambda$  is given in [61] by

$$\mathcal{F}_{\rho,\lambda}(x) = \sum_{n=0}^{\infty} \frac{\sigma(n)}{\Gamma(\rho n + \lambda)} \eta^n.$$

For further information regarding the category of functions and its specific instances, the interested reader is directed to [61].

**Theorem 6.** Ahmed et al. (2024) [6]. Suppose that  $v > 0, \varepsilon > 0, \ell \in \mathbb{N}$  and  $(\ell - 1) < v < \ell, 0 \leq t < \infty$  and  $p > -1$ . Then, the equation

$$\mathfrak{W}(t) - \mathfrak{W}_0 L_{\varepsilon}^p(t) = -\varepsilon^v {}_0\mathbb{D}_t^{-v} \mathfrak{W}(t),$$

has the following solution:

$$\mathfrak{W}(t) = \mathfrak{W}_0 \left( \sum_{\vartheta=0}^{\ell} \frac{(-1)^{\vartheta} \Gamma(\ell + \mathcal{Q} + 1) t^{\vartheta}}{\Gamma(\vartheta + \mathcal{Q} + 1) \Gamma(\ell - \vartheta + 1)} \right) \times \frac{1}{t} E_{v, \vartheta}(-\varepsilon^v t^v),$$

where  $L_{\varepsilon}^p(t)$  is the fractional Laguerre polynomials (FLPs)

$$L_{\ell}^p(t) = \sum_{\vartheta=0}^{\ell} \frac{(-1)^{\vartheta} \Gamma(\ell + \rho + 1) t^{\vartheta}}{\Gamma(\vartheta + \rho + 1) \Gamma(\vartheta + 1) \Gamma(\ell - \vartheta + 1)}; 0 \leq t < \infty,$$

where  $\ell \in \mathbb{N}$  and  $(\ell - 1) < v < \ell, 0 \leq t < \infty$ , and  $\mathcal{Q} > -1$ , and  $E_{v, \vartheta}(x)$  is defined in (3).

## 7 Applications of Elzaki transform for solving MFKEs

Assume that the function  $\mathfrak{M}(y)$  belongs to a class  $\mathfrak{J}$ , such that

$$\mathfrak{N} = \{\mathfrak{M}(y) : \exists V, \tau_1, \tau_2 > 0 : |\mathfrak{M}(y)| < re^{|y|/\tau_i} \text{ if } y \in (-i)^i \times [0, \infty)\},$$

the Elzaki transform of function  $\mathfrak{M}(y)$  is defined by Tarig M. Elzaki [31] as

$$\mathcal{E}\{\mathfrak{M}(y)\} = \mathfrak{V} \int_0^{\infty} e^{-\frac{y}{\mathfrak{V}}} \mathfrak{M}(y) dx = L(\mathfrak{M}), y > 0, \mathfrak{V} \in (-\tau_1, \tau_2).$$

The Elzaki convolution property of  $\mathfrak{D}(y)$  and  $\mathfrak{E}(y)$  is given by

$$\mathbb{E}[(\mathfrak{D} * \mathfrak{E})(\mathfrak{y})] = \frac{1}{\mathfrak{V}} \mathfrak{d}(\mathfrak{V}) \mathfrak{e}(\mathfrak{V}),$$

where  $\mathfrak{d}(\mathfrak{V})$  and  $\mathfrak{e}(\mathfrak{V})$  are the Elzaki transforms of  $\mathfrak{D}(y)$  and  $\mathfrak{E}(x)$ , respectively, and

$$(\mathfrak{D} * \mathfrak{E})(y) = \int_0^y \mathfrak{D}(y) \mathfrak{E}(x - \mathfrak{V}) d\mathfrak{V}.$$

The Elazki transform of Riemann-Liouville integral operator is introduced as follows

$$\mathbf{E} [{}_0 D_t^{-v} \tau(\mathfrak{y})] = v^v \mathfrak{y}(v), (\operatorname{Re}(v) > 0).$$

**Theorem 7.** Singh et al. (2021) [75] Let  $\omega > 0, v > 0, \omega > 0, c > 0, \zeta > 0, \mathfrak{a}_j, \mathfrak{b}_j, \mathfrak{A}_{ji}, \mathfrak{B}_{ji} \in \mathbb{C}, \mathfrak{A}_j > 0 (j = 1, \dots, p_i)$  and  $\mathfrak{B}_j > 0 (j = 1, \dots, q_i)$ , the modified FKE

$$\begin{aligned} & \mathfrak{W}(x) - \mathfrak{W}_0 x^{v-1} (\Gamma) \mathfrak{K}_{p_i, q_i, \sigma_i; r}^{m, n} \left[ -\zeta^{\omega} \chi^{\omega} \left| \begin{array}{c} (\mathfrak{a}_1, \mathfrak{A}_1, \mathfrak{t}), (\mathfrak{a}_j, \mathfrak{A}_j)_{2, n}, [\sigma_j (\mathfrak{a}_{ji}, \mathfrak{A}_{ji})]_{n+1, p_i} \\ (\mathfrak{b}_j, \mathfrak{B}_j)_{1, m}, [\sigma_j (\mathfrak{b}_{ji}, \mathfrak{B}_{ji})]_{m+1, q_i} \end{array} \right. \right] \\ & = -c^{\omega} {}_0 \mathbb{D}_x^{-\omega} \mathfrak{M}(x). \end{aligned} \quad (33)$$

Then there holds the solution as follows

$$\begin{aligned} \mathfrak{W}(x) &= \mathfrak{W}_0 x^{v-1} \sum_{s=0}^{\infty} (-c^{\omega} x^{\omega})^s \\ & \times (\Gamma) \mathfrak{K}_{p_i+1, q_i+1, \sigma_i; r}^{m, n+1} \left[ -\zeta^{\omega} \chi^{\omega} \left| \begin{array}{c} (\mathfrak{a}_1, \mathfrak{A}_1, \mathfrak{t}), (1-v, \omega), (\mathfrak{a}_j, \mathfrak{A}_j)_{2, n}, [\sigma_j (\mathfrak{a}_{ji}, \mathfrak{A}_{ji})]_{n+1, p_i} \\ (\mathfrak{b}_j, \mathfrak{B}_j)_{1, m}, [\sigma_j (\mathfrak{b}_{ji}, \mathfrak{B}_{ji})]_{m+1, q_i}, (1-v-\omega s, \omega) \end{array} \right. \right], \end{aligned} \quad (34)$$

where  $(\Gamma) \mathfrak{K}_{p_i, q_i, \sigma_i; r}^{m, n}(x)$  is the incomplete Aleph functions [75].

**Remark.** Recently in [16] applied the Elzaki integral transform to MFKEs incorporating orthogonal polynomials and their generating functions.

## 8 Solving MFKEs including Hadamard fractional integrals use Mellin transforms

Since the Mellin transform can be regarded as the Fourier and Laplace transforms, this is why it most often shows up in number theory and computer science, since it is invariant [23, 25, 33]. The Mellin transform of a suitable integrable function  $f(t)$  is defined by

$$\tilde{f}(p) = \mathcal{M}\{f(t)\}(p) = \int_0^\infty t^{p-1} f(t) dt, \quad p \in \mathbb{C}, \quad (35)$$

the provided improper integral in (35) exists. And the inverse Mellin transform is

$$f(t) = \mathcal{M}^{-1}\{\tilde{f}(p)\}(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-p} \tilde{f}(p) dp. \quad (36)$$

Our focus is on utilizing the Mellin transform methodology to achieve the outcomes listed below.

**Theorem 8.** Abdalla and Akel (2022) [2]. Let  $\mathcal{D}_\mu, \mathcal{E}_\mu, \mathcal{F}_\mu, \mathcal{G}_\mu, \mathcal{H}_\mu$  and  $C$  be positive stable matrices in  $\mathbb{C}^{n \times n}$ , such that  $\mathcal{G}_\mu + \ell I$  and  $\mathcal{H}_\mu + \ell I$  are invertible for all  $\mu \in \mathbb{N}, \ell \in \mathbb{N}_0, \alpha, \rho \in \mathbb{R}_0^+$ , and  $d, k, \tau, \xi \in \mathbb{R}^+$ . Then, for  $\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-$ , and  $t \in [0, \xi]$ . The solution to the following GFKE

$$\begin{aligned} & \mathfrak{W}(t)I + \mathcal{C}_H^\gamma H I^\gamma \mathfrak{W}(t) \\ &= \mathfrak{W}_0 t^{\mu-1} \prod_{\mu=1}^n {}_3\mathcal{W}_2^{(k_\mu, \tau_\mu; \rho)} \left[ \begin{matrix} (\mathcal{D}_\mu, k_\mu; \rho), (\mathcal{E}_\mu, k_\mu), (\mathcal{F}_\mu, k_\mu) \\ (\mathcal{G}_\mu, k_\mu), (\mathcal{H}_\mu, k_\mu) \end{matrix} ; d^\gamma t^\gamma \right], \end{aligned}$$

is

$$\begin{aligned} \mathfrak{W}(t)I &= \mathfrak{W}_0 \zeta^{\tau+1} \log t \prod_{\mu=1}^n \Gamma_{k_\mu}^{-1}(\mathcal{E}_\mu) \Gamma_{k_\mu}(\mathcal{G}_\mu) \Gamma_{k_\mu}^{-1}(\mathcal{F}_\mu) \Gamma_{k_\mu}(\mathcal{H}_\mu) \times \\ &\times \sum_{s=0}^\infty (\mathcal{D}_\mu; \rho)_{s, k_\mu} \Gamma_{k_\mu}^{-1}(\mathcal{G}_\mu + k_\mu \tau s I) \Gamma_{k_\mu}(\mathcal{E}_\mu + k_\mu \tau s I) \times \\ &\times \Gamma_{k_\mu}^{-1}(\mathcal{H}_\mu + k_\mu \tau s I) \Gamma_{k_\mu}(\mathcal{F}_\mu + k_\mu \tau s I) \left( \frac{d^\gamma \zeta^{\gamma s}}{s!} \right)^\mu \times \\ &\times \sum_{r=0}^\infty \sum_{\ell=0}^\infty [-(\log t)^c]^\gamma \left[ \log t^{-(\gamma \mu s + a - 1)} \right]^\ell \Gamma[1 - (\gamma r + \ell + 2)], \end{aligned}$$

where  ${}_3\mathcal{W}_2^{(k_\mu, \tau_\mu; \rho)}$  is the extended  $(k, \tau)$ -Wright hypergeometric matrix function defined in [2] by

$$\begin{aligned} {}_3\mathcal{W}_2^{(k, \tau, \rho)}(t) &:= {}_3\mathcal{W}_2^{(k, \tau, \rho)} \left[ \begin{matrix} (\mathcal{D}, k; \rho), (\mathcal{E}, k), (\mathcal{F}, k) \\ (\mathcal{G}, k), (\mathcal{H}, k) \end{matrix} ; t \right] \\ &:= \Gamma_k^{-1}(\mathcal{E}) \Gamma_k(\mathcal{G}) \Gamma_k^{-1}(\mathcal{F}) \Gamma_k(\mathcal{H}) \\ &\times \sum_{s=0}^\infty (\mathcal{D}; \rho)_{s, k} \Gamma_k^{-1}(\mathcal{G} + k \tau s I) \Gamma_k(\mathcal{E} + k \tau s I) \\ &\times \Gamma_k^{-1}(\mathcal{H} + k \tau s I) \Gamma_k(\mathcal{F} + k \tau s I) \frac{t^s}{s!}, \quad |t| < 1, \end{aligned} \quad (37)$$

**Theorem 9.** Akel et al. (2022) [9]. Let  $\mathcal{T}_\mu, \mathcal{D}_\mu, \mathcal{E}_\mu, \mathcal{F}_\mu$ , and  $C$  be positive stable matrices in  $\mathbb{C}^{m \times m}$  such that  $\mathcal{T}_\mu + \ell I$  and  $\mathcal{F}_\mu + \ell I$  are invertible for all  $\mu \in \mathbb{N}$  and  $\ell \in \mathbb{N}_0$ . Suppose that  $\delta, \sigma \in \mathbb{R}_0^+$ ,  $d, k, \xi \in \mathbb{R}^+$ , and  $\alpha_\mu \in \mathbb{C} \setminus \mathbb{Z}_0^-$ . For  $\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-$  and  $t \in [0, \xi]$ , the generalized fractional kinetic matrix equation corresponding to  ${}_2\Theta_1^{\mathcal{T}_\mu; k_\mu, \alpha_\mu; \sigma}$  takes the form

$$\mathfrak{W}(t)I - \mathfrak{W}_0 t^{\delta-1} \prod_{\mu=1}^n {}_2\Theta_1^{\mathcal{T}_\mu; k_\mu, \alpha_\mu; \sigma} \left[ \begin{matrix} \mathcal{D}_\mu, \mathcal{E}_\mu \\ \mathcal{F}_\mu \end{matrix} ; d^\gamma t^\gamma \right] = -\mathcal{C}_H^\gamma H I_t^\gamma N(t) \quad (38)$$

is solvable. The solution to (38) is given by

$$\begin{aligned} \mathfrak{W}(t)I = \mathfrak{W}_0 \xi^{\delta-1} \log(t) \prod_{\mu=1}^n \sum_{s=0}^{\infty} (s + \alpha_{\mu})^{-\mathcal{T}_{\mu}} (\mathcal{D}_{\mu}; \sigma)_{n,k_{\mu}} (\mathcal{E}_{\mu})_{n,k_{\mu}} [(\mathcal{F}_{\mu})_{n,k_{\mu}}]^{-1} \\ \times \left( \frac{d^{\gamma s} \xi^{\gamma s}}{s!} \right)^{\mu} \sum_{r=0}^{\infty} \sum_{\ell=0}^{\infty} \left[ -(\log t^C)^{\gamma} \right]^r \left[ \log t^{-(\gamma \mu s + \delta - 1)} \right]^{\ell} \Gamma[1 - (\gamma r + \ell + 2)], \end{aligned} \quad (39)$$

where

$${}_2\Theta_1^{\mathcal{T};k,\alpha;\sigma} \left[ \begin{matrix} \mathcal{D}, \mathcal{E} \\ \mathcal{F} \end{matrix}; w \right] = \sum_{n=0}^{\infty} (n + \alpha)^{-\mathcal{T}} (\mathcal{D}; \sigma)_{n,k} (\mathcal{E})_{n,k} [(\mathcal{F})_{n,k}]^{-1} \frac{w^n}{n!}, \quad (40)$$

where  $(\mathcal{D}; \sigma)_{n,k}$  is defined in [9].

A number of exceptional instances of this result are provided in [9].

## 9 Solving MFKEs use pathway-type integral transforms

The pathway-type transform ( $\mathcal{P}_{\zeta}$ -transform) is given in [52] in the form

$$\mathcal{P}_{\zeta}[h(s), \varphi] = H(\varphi) = \int_0^{\infty} [1 + (\zeta - 1)\varphi]^{\frac{-s}{\zeta-1}} h(s) ds \quad \zeta > 1, \quad (41)$$

with

$$\lim_{\zeta \rightarrow 1^+} [1 + (\zeta - 1)\varphi]^{\frac{-s}{\zeta-1}} = e^{-\phi s}, \quad (42)$$

and

$$\lim_{\zeta \rightarrow 1^+} \mathcal{P}_{\zeta}[g(s), \varphi] = L[g(s), \varphi], \quad (43)$$

where  $L(\cdot)$  is the Laplace transform (15).

Certain outcomes of the  $\mathcal{P}_{\zeta}$ -transform are given in [52] as follows

$$\mathcal{P}_{\zeta}[s^v, \varphi] = \left\{ \frac{\zeta - 1}{\ln[1 + (\zeta - 1)\varphi]} \right\}^{v+1} \Gamma(v+1), \quad v \in \mathbb{C} \quad (44)$$

and

$$\mathcal{P}_{\zeta}[{}_0\mathbb{D}_s^{-\alpha} h(s), \varphi] = \left[ \frac{\zeta - 1}{\ln[1 + (\zeta - 1)\varphi]} \right]^{\alpha} \mathcal{P}_{\zeta}[h(s), \varphi], \quad \text{Re}(\alpha) > 0, \zeta > 1, \quad (45)$$

where  ${}_0\mathbb{D}_s^{-\alpha} h(s)$  is given in (8).

For more information on the  $\mathcal{P}_{\zeta}$ -transform and its applications, see [17, 42].

**Theorem 10.** Alqarni and Abdalla (2023) [13]. Let  $\alpha, \beta, \sigma, \wp \in \mathbb{R}^+, w \in \mathbb{C}, \lambda \in (0, 1), \gamma_i \in \mathbb{C}$  for  $i = 1, 2, 3, \dots, m$  and  $\vartheta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$  for  $j = 1, 2, 3, \dots, n$ . The solution of

$$\mathcal{W}(w) - \mathcal{W}_0 {}_m\mathbb{D}\mathbb{H}_n^{\lambda}(\sigma^{\beta} w^{\beta}) = -\wp^{\alpha} {}_0\mathbb{D}_w^{-\alpha} \mathcal{W}(w), \quad (46)$$

is

$$\mathcal{W}_{\lambda}^{\alpha}(w) = \mathcal{W}_0 \sum_{r=0}^{\infty} \frac{(\gamma_1; \lambda)_r \cdots (\gamma_m)_r}{(\vartheta_1)_r \cdots (\vartheta_n)_r} \frac{\Gamma(\beta r + 1)}{r!} \sigma^{\beta r} w^{\beta r} E_{\alpha, \beta r + 1}(-\wp^{\alpha} w^{\alpha}), \quad (47)$$

where

$${}_m\mathbb{D}\mathbb{H}_n^{\lambda}(w) = {}_m\mathbb{D}\mathbb{H}_n^{\lambda} \left[ \begin{matrix} (\gamma_1; \lambda) \cdots \gamma_m \\ \vartheta_1 \cdots \vartheta_n \end{matrix}; w \right] = \sum_{r=0}^{\infty} \frac{(\gamma_1; \lambda)_r \cdots (\gamma_m)_r}{(\vartheta_1)_r \cdots (\vartheta_n)_r} \cdot \frac{w^r}{r!}, \quad (48)$$

is the generalized degenerate hypergeometric function [82] with  $\lambda \in (0, 1)$ ,  $w, \gamma_i \in \mathbb{C}$  for  $i = 1, 2, 3, \dots, m$ , and  $\vartheta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$  for  $j = 1, 2, 3, \dots, n$ , and

$$\begin{aligned} (\varpi; \lambda)_\ell &= \frac{\Gamma_\lambda(\varpi + \ell)}{\Gamma(\varpi)} \\ &= \frac{1}{\Gamma(\varpi)} \int_0^\infty (1 + \lambda w)^{-\frac{1}{\lambda}} w^{\varpi + \ell - 1} dw, \quad \lambda > \operatorname{Re}(\varpi + \ell) > 0. \end{aligned} \quad (49)$$

**Theorem 11.** Let  $\lambda > 0, d > 0, z \in \mathbb{C}$ , and  $\tau > 0$ . Then, we conclude that the solution of the  $\tau$ -generalized incomplete hypergeometric function's fractional kinetic equation

$$\mathfrak{W}(z) - \mathfrak{W}_0 \left\{ \frac{d}{dz} {}_p\Gamma_q^{(\tau)}(z) \right\} = -d^\lambda {}_0\mathbb{D}_t^{-\lambda} \mathfrak{W}(z), \quad (50)$$

is given by

$$\begin{aligned} \mathfrak{W}(z) &= \mathfrak{W}_0 \left[ \frac{\theta_1(\theta_2 + \tau) \dots (\theta_p + \tau)}{(\eta_1 + \tau) \dots (\eta_q + \tau)} \right] \sum_{n=0}^{\infty} \frac{[\theta_1 + 1, x]_n (\theta_2 + \tau)_n \dots (\theta_p + \tau)_n}{(\eta_1 + \tau)_n (\eta_2 + \tau)_n \dots (\eta_q + \tau)_n} z^n \\ &\quad \times \sum_{m=0}^{\infty} \frac{(-1)^m (dz)^{m\lambda}}{(m\lambda + n)!}. \end{aligned} \quad (51)$$

**Theorem 12.** Let  $\lambda > 0, d > 0, z \in \mathbb{C}$ , and  $\tau > 0$ . Then, the solution of the fractional kinetic equation of the  $\tau$ -generalized incomplete hypergeometric functions

$$\mathbf{E}(z) - \mathbf{E}_0 {}_p\gamma_q^{(\tau)}(z) = -d^\lambda {}_0\mathbb{D}_z^{-\lambda} \mathbf{E}(z), \quad (52)$$

is given by

$$\begin{aligned} \mathbf{E}(z) &= \mathbf{E}_0 \frac{\Gamma(\eta_1), \dots, \Gamma(\eta_q)}{\Gamma(\theta_2), \dots, \Gamma(\theta_p)} \sum_{n=0}^{\infty} \frac{(\theta_1, x)_n \Gamma(\theta_2 + n\tau) \dots \Gamma(\theta_p + n\tau)}{\Gamma(\eta_1 + n\tau) \Gamma(\eta_2 + n\tau) \dots \Gamma(\eta_q + n\tau)} \\ &\quad \times \sum_{m=0}^{\infty} (-1)^m (d)^{m\lambda} \frac{z^{m\lambda + n}}{(m\lambda + n)!}. \end{aligned} \quad (53)$$

Where the generalized incomplete hypergeometric functions  ${}_p\Gamma_q^{(\tau)}$  and  ${}_p\gamma_q^{(\tau)}$  satisfy the following decomposition formula:

$${}_p\Gamma_q^{(\tau)} \left[ \begin{matrix} (\theta_p, x); \\ \eta_q; \end{matrix} \middle| z \right] + {}_p\gamma_q^{(\tau)} \left[ \begin{matrix} (\theta_p, x); \\ \eta_q; \end{matrix} \middle| z \right] = {}_p\mathbf{R}_q^{(\tau)} \left[ \begin{matrix} \theta_p; \\ \eta_q; \end{matrix} \middle| z \right] \quad (54)$$

**Theorem 13.** He et al. (2022) [38]. Assume that  $p_1, p_2 \in \mathbb{N}$ ,  $k \in \mathbb{R}^+$  with  $\lambda > 0, d > 0, \xi \in \mathbb{C}$ , Then the solution of the following generalized fractional kinetic equation:

$$\mathfrak{W}(\xi) - \mathfrak{W}_0 Y(p_1, p_2, k; \xi^\lambda) = -d^\lambda {}_0\mathbb{D}_\xi^{-\lambda} \mathbf{N}(\xi), \quad (55)$$

is given by

$$\mathfrak{W}(\xi) = \mathfrak{W}_0 \sum_{n=0}^{\infty} \Gamma(n\lambda + 1) \frac{(a_1)_{n,k} (b_1)_{n,k} (a_2)_{n,k} (b_2)_{n,k}}{(c_1)_{n,k} (c_2)_{n,k} (np_1)! (np_2)!} \sum_{m=0}^{\infty} (-1)^m (d)^{m\lambda} \frac{\xi^{\lambda(m+n)}}{(m\lambda + n\lambda)!}. \quad (56)$$

Where  $Y(p_1, p_2, k; \xi)$  is the Hadamard product for  $(p, k)$ -hypergeometric [38].

Various particular cases of this result were established in [38].

## 10 Solving MFKEs use the modified degenerate Laplace integral transform

Recently, Kim-Kim [45] defined the degenerate Laplace integral transform for  $\lambda \in (0, \infty)$ , as follows:

$$\mathcal{F}(\varphi) = \mathbb{L}_\lambda(\varphi) = \mathcal{L}_\lambda \left\{ f(\eta); \varphi \right\} = \int_0^\infty (1 + \lambda \eta)^{-\frac{\varphi}{\lambda}} f(\eta) d\eta, \quad (57)$$

provided that the improper integral converges,  $(1 + \lambda \eta)^{-\frac{\varphi}{\lambda}}$  is the kernel of the transformation. This was motivated by the degenerate exponential function of two variables, which is defined by

$$e_\lambda^\eta = (1 + \lambda \eta)^{\frac{1}{\lambda}}, \quad \lambda \in (0, \infty). \quad (58)$$

It is clear that  $\lim_{\lambda \rightarrow 0} e_\lambda^\eta = e^\eta$ . The authors also investigated some properties and formulas related to the degenerate Laplace transformation in [45].

Within this framework, YunJae Kim et al. [46] introduced the modified degenerate Laplace integral transform (or briefly MDLIT) of a function  $f(\eta)$  by the form

$$\mathcal{ML}_\lambda(\varphi) = \mathbb{ML}_\lambda \left\{ f(\eta); \varphi \right\} = \int_0^\infty (1 + \lambda \varphi)^{-\frac{\eta}{\lambda}} f(\eta) d\eta, \quad (\lambda \in (0, \infty), \operatorname{Re}(\eta) \geq 0). \quad (59)$$

In this section, we solve the fractional kinetic equation associated with the  ${}_p\mathbf{R}_q[\alpha, \beta; \eta]$  function using the MDLIT method.

The MDLIT (59) of the Riemann-Liouville fractional integral operator is proved in [11] by the following Lemma.

**Lemma 1.** Let  $\lambda \in (0, \infty)$  and  $\varsigma \in \mathbb{C}$  with  $\operatorname{Re}(\varsigma) > 0$ , then we have

$$\mathbb{ML}_\lambda \left\{ {}_0\mathbb{D}_\eta^{-\varsigma} f(\eta); \varphi \right\} = \left\{ \frac{\lambda}{\log(1 + \lambda \varphi)} \right\}^\varsigma \mathbb{ML}_\lambda \left\{ f(\eta); \varsigma \right\}. \quad (60)$$

**Theorem 14.** Almalki and Abdalla (2023) [10]. Let  $\alpha, \beta, \lambda, \mathfrak{d} \in \mathbb{R}^+$ ,  $\varsigma > 0$  and  $\Lambda_i \in \mathbb{C}$  for  $i = 1, 2, 3, \dots, p$  and  $\Upsilon_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$  for  $j = 1, 2, 3, \dots, q$ . Then the solution of

$$\mathcal{Y}(\eta) - \mathcal{Y}_0 {}_p\mathbf{R}_q[\alpha, \beta; \eta] = -\mathfrak{d}^\varsigma {}_0\mathbb{D}_\eta^{-\varsigma} \mathcal{Y}(\eta), \quad (61)$$

is given as follows

$$\mathcal{Y}(\eta) = \mathcal{Y}_0 \sum_{k=0}^{\infty} \frac{k!}{\Gamma(\alpha k + \beta)} \frac{\prod_{i=1}^p (\Lambda_i)_k}{\prod_{j=1}^q (\Upsilon_j)_k} \eta^k \mathfrak{E}_{\varsigma, k+1}(-\mathfrak{d}^\varsigma \eta^\varsigma), \quad (62)$$

where  $\mathfrak{E}_{\alpha, \beta}(\xi)$  is defined by Desai and Shukla in (3) and  ${}_p\mathbf{R}_q[\alpha, \beta; \eta]$  is defined in [27, 63] as

$$\begin{aligned} {}_p\mathbf{R}_q[\alpha, \beta; \eta] &= {}_p\mathbf{R}_q \left[ \Lambda_1, \dots, \Lambda_p \middle| \alpha, \beta; \eta \right] \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} \frac{\prod_{i=1}^p (\Lambda_i)_k}{\prod_{j=1}^q (\Upsilon_j)_k} \frac{\eta^k}{k!}, \end{aligned} \quad (63)$$

where  $\alpha, \beta \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \operatorname{Re}(\Lambda_i) > 0, \operatorname{Re}(\Upsilon_j) > 0$ ; for all  $i = 1, 2, \dots, p$  and  $j = 1, 2, \dots, q$ .

## 11 A Conceptual Framework for Outcomes

The effectiveness of an integral transforms are intrinsically linked to the fractional operators and special functions involved in solving MFKEs.

The literature classifies an integral transform methodologies into three distinct categories as follows:

- Category I: Classical Transforms:** Includes the **Laplace transform** (the reference standard due to its mature inversion theory) and the **Mellin transform** (the primary tool for scale-invariant and Hadamard structures).
- Category II: Generalized Transforms:** Includes **Sumudu** and **Elzaki** transforms. These are favored for preserving dimensional correspondence of physical variables and providing symbolic compactness in derivations.
- Category III: Emerging Frameworks:** Includes **Pathway-type**, **Modified Degenerate Laplace (MDLIT)**, and **Hybrid** methodologies. These are designed for adaptive kernels and multi-term models but currently require broader benchmarking.

### *Fractional operator classes.*

The studies identify three primary fractional operator classes:

- (I) **Riemann-Liouville and Caputo:** Predominantly used for initial-value problems. The Caputo derivative is particularly advantageous, as it allows for classical initial conditions.
- (II) **Hadamard Operators:** These involve logarithmic kernels and are uniquely compatible with Mellin transform methodologies.
- (III) **Generalized Kernels:** Emerging research focuses on non-singular kernels and variable-order operators to provide more realistic descriptions of memory effects .

### *Special functions in solutions.*

The "queen" of fractional calculus functions is the **Mittag-Leffler (M-L) function**. Analytical solutions frequently involve:

- M-L functions with one, two and three-parameters.
- Fox's  $H$ -function, Aleph's functions, and Wright functions.
- Generalized hypergeometric type matrix functions.

Drawing on the comparative analysis of the reviewed literature, the following selection strategy is proposed.

**Table 1. Suggested Transform Selection for MFKEs**

| Problem Characteristic            | First Choice Transform | Analytical Justification               |
|-----------------------------------|------------------------|----------------------------------------|
| Standard FKE / Initial Conditions | <b>Laplace</b>         | Mature theory; direct initial values   |
| Dimensional Preservation          | <b>Sumudu / Elzaki</b> | Symbolic compactness; physical scaling |
| Logarithmic / Scale-invariant     | <b>Mellin</b>          | Structural alignment with Hadamard     |
| Generalized Kernel / Parameters   | <b>Pathway</b>         | Adjustable kernel flexibility          |
| Multi-term / Nonlinear models     | <b>Hybrid Methods</b>  | Combined capability strengths          |

## 12 Conclusion

Fractional kinetic equations (FKEs) provide a suitable mathematical framework for modeling non-Markovian dynamics, anomalous transport, and memory-dependent processes in physics and engineering [28,51]. Their applications also extend to economic and financial systems exhibiting long- and short-range memory [35,66]. In contrast to classical integer-order models, fractional operators incorporate temporal nonlocality and hereditary effects, which are important for describing long-range dependence and anomalous diffusion [28,51]. Integral transforms, especially the Laplace

transform, are commonly used to derive solutions in the transform domain, where the resulting expressions are often represented through Mittag–Leffler and other generalized special functions [14, 49, 76].

In this review, a wide range of recent linear fractional kinetic equations (FKEs) with constant coefficients and standard fractional operators (Riemann–Liouville, Caputo, Hadamard) are solved exactly using Laplace, Fourier, Mellin, Sumudu, Elzaki, Pathway, and other transforms, typically in terms of Mittag–Leffler and related special functions. These methods extend to matrix-valued FKEs and equations involving sophisticated kernels (e.g., extended  $(k, \tau)$ –Gauss hypergeometric, Hurwitz–Lerch zeta, generalized multi-index Bessel functions) by combining tailored transforms (Mellin, weighted/generalized Laplace, Pathway transform, MDLT) with suitable fractional operators.

Also, critical “open problems” identified for future study include:

- Unified Transform Theory:** Establishing equivalence relations among existing transforms via an operator-theoretic framework.
- Computational Complexity:** Developing efficient symbolic simplification and numerical evaluation for complex special functions .
- Variable-Order Modeling:** Adapting transform identities for operators where differentiation order evolves over time or space.
- Non-Singular Kernels:** Developing unified formulations for operators that avoid power-law singularities.

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## References

- [1] M. Abdalla. New aspects of certain special functions with applications. *Karshi Multidisciplinary International Scientific Journal*, 1(2):115–148, 2024.
- [2] M. Abdalla and M. Akel. Contribution of using hadamard fractional integral operator via mellin integral transform for solving certain fractional kinetic matrix equations. *Fractal and Fractional*, 6(6):305, 2022.
- [3] M. Abdalla, Y. Almalki, I. Mekawy, and H. Abd-Elmageed. Extended hypergeometric logarithmic functions and its application in generalized fractional kinetic equations. *FRACTALS (fractals)*, 33(04):1–9, 2025.
- [4] I. Abou-Hayt and B. Dahl. A critical look at the laplace transform method in engineering education. *IEEE Transactions on Education*, 67(4):542–549, 2023.
- [5] P. Agarwal, M. Chand, D. Baleanu, D. O’Regan, and S. Jain. On the solutions of certain fractional kinetic equations involving k-mittag-leffler function. *Advances in Difference Equations*, 2018:1–13, 2018.
- [6] W. F. Ahmed, D. Pawar, and W. Patil. Solution of fractional kinetic equations involving laguerre polynomials via sumudu transform. *Journal of Function Spaces*, 2024(1):9935562, 2024.
- [7] K. Ahrendt, L. DeWolf, L. Mazurowski, K. Mitchell, T. Rolling, and D. Veconi. Initial and boundary value problems for the Caputo fractional self-adjoint difference equations. *arXiv preprint arXiv:2002.07909*, 2020.
- [8] M. Akel, H. M. Elshehabey, and R. Ahmed. Generalized laplace-type transform method for solving multilayer diffusion problems. *Journal of Function Spaces*, 2022, 2022.
- [9] M. Akel, M. Hidan, S. Boulaaras, and M. Abdalla. On the solutions of certain fractional kinetic matrix equations involving hadamard fractional integrals. *AIMS Math*, 7(8):15520–15531, 2022.
- [10] Y. Almalki and M. Abdalla. Analytic solutions to the fractional kinetic equation involving the generalized mittag-leffler function using the degenerate laplace type integral approach. *The European Physical Journal Special Topics*, 232(14):2587–2593, 2023.
- [11] Y. Almalki, M. Abdalla, and H. Abd-Elmageed. Results on the modified degenerate laplace-type integral associated with applications involving fractional kinetic equations. *Demonstratio Mathematica*, 56(1):20230112, 2023.
- [12] Y. Almalki, M. Akel, and M. Abdalla. Solutions of certain initial-boundary value problems via a new extended laplace transform. *Nonlinear Engineering*, 13(1):20220353, 2024.
- [13] M. Z. Alqarni and M. Abdalla. Novel kinds of fractional  $\lambda$ –kinetic equations involving the generalized degenerate hypergeometric functions and their solutions using the pathway-type integral. *Mathematics*, 11(19):4217, 2023.
- [14] L. Angelani, A. De Gregorio, and R. Garra. Generalized time-fractional kinetic-type equations with multiple parameters. *Chaos*, 35:023111, 2025.
- [15] M. A. Asiru. Sumudu transform and the solution of integral equations of convolution type. *International Journal of Mathematical Education in Science and Technology*, 32(6):906–910, 2001.
- [16] M. Aychluh, D. Suthar, S. Purohit, and B. Shimelis. Elzaki transform approach to fractional kinetic equations using orthogonal polynomials and their generating functions. *Advances in Mathematical Physics*, 2025(1):5535014, 2025.

- [17] D. Baleanu and P. Agarwal. A composition formula of the pathway integral transform operator. *Note di Matematica*, 34(2):145–156, 2015.
- [18] H. Bateman, A. Erdélyi, W. Magnus, and F. Oberhettinger. *Tables of integral transforms*, volume 1. McGraw-Hill New York, 1954.
- [19] F. B. M. Belgacem and A. A. Karaballi. Sumudu transform fundamental properties investigations and applications. *International Journal of Stochastic Analysis*, 2006, 2006.
- [20] F. B. M. Belgacem and A. A. Karaballi. Sumudu transform fundamental properties investigations and applications. *International Journal of Stochastic Analysis*, 2006, 2006.
- [21] F. B. M. Belgacem, A. A. Karaballi, and S. L. Kalla. Analytical investigations of the sumudu transform and applications to integral production equations. *Mathematical problems in Engineering*, 2003(3):103–118, 2003.
- [22] M. Chand, Ö. Özen, and J. C. Prajapati. Solution of generalized fractional kinetic equations with generalized mathieu series. *Georgian Mathematical Journal*, 31(2):187–193, 2024.
- [23] B. Davies. *Integral transforms and their applications*, volume 41. Springer Science & Business Media, 2012.
- [24] P. J. Davis. Leonhard euler’s integral: A historical profile of the gamma function: In memoriam: Milton abramowitz. *The American Mathematical Monthly*, 66(10):849–869, 1959.
- [25] L. Debnath and D. Bhatta. *Integral transforms and their applications*. Chapman and Hall/CRC, 2016.
- [26] L. Debnath and L. Debnath. Tables of integral transforms. *Nonlinear Partial Differential Equations for Scientists and Engineers*, pages 675–687, 2012.
- [27] R. Desai and A. Shukla. Some results on function  ${}_p r_q(\alpha, \beta; z)$ . *Journal of Mathematical Analysis and Applications*, 448(1):187–197, 2017.
- [28] K. Diethelm, V. Kiryakova, and Y. Luchko. Trends, directions for further research, and some open problems of fractional calculus. *Nonlinear Dynamics*, 107:3245–3270, 2022.
- [29] U. Duran. Modified sumudu transform and its properties. *Sakarya University journal of science*, 25(2):389–396, 2021.
- [30] H. Eltayeb and A. Kılıçman. On some applications of a new integral transform. *International Journal of Mathematical Analysis*, 4(3):123–132, 2010.
- [31] T. M. Elzaki. The new integral transform elzaki transform. *Global Journal of pure and applied mathematics*, 7(1):57–64, 2011.
- [32] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi. Tables of integral transforms, vol. 1 (1954). *Tables of integral transforms*, 2, 1954.
- [33] P. Flajolet, X. Gourdon, and P. Dumas. Mellin transforms and asymptotics: Harmonic sums. *Theoretical computer science*, 144(1-2):3–58, 1995.
- [34] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, et al. Mittag-leffler functions, related topics and applications. 2020.
- [35] U. Hassler and M.-O. Pohle. Forecasting under long memory. *Journal of Financial Econometrics*, 21(3):742–778, 2023.
- [36] H. J. Haubold and A. M. Mathai. The fractional kinetic equation and thermonuclear functions. *Astrophysics and Space Science*, 273:53–63, 2000.
- [37] H. J. Haubold, A. M. Mathai, and R. K. Saxena. Mittag-leffler functions and their applications. *Journal of applied mathematics*, 2011(1):298628, 2011.
- [38] F. He, A. Bakhet, M. Hidan, and H. Abd-Elmageed. On the construction of (p, k)-hypergeometric function and applications. *Fractals*, 30(10):2240261, 2022.
- [39] N. Irfan, A. Siddiqi, et al. A wavelet algorithm for fourier-bessel transform arising in optics. *Int. J. Eng. Math*, 9, 2015.
- [40] K. Jangid, S. Purohit, R. Agarwal, and R. P. Agarwal. On the generalization of fractional kinetic equation comprising incomplete h-function. *Kragujevac Journal of Mathematics*, 47(5):701–712, 2023.
- [41] Q. D. Katatbeh and F. B. M. Belgacem. Applications of the sumudu transform to fractional differential equations. *Nonlinear Studies*, 18(1):99–112, 2011.
- [42] D. Kaur, M. Chand, and M. Rakshit. On pathway fractional integral formulae involving new extended hypergeometric functions. *Int. J. Manag. Technol. Eng*, pages 1461–1469, 2019.
- [43] Z. H. Khan and W. A. Khan. N-transform-properties and applications. *NUST journal of engineering sciences*, 1(1):127–133, 2008.
- [44] A. Kilbas. Fractional calculus of the generalized wright function. *Fractional Calculus and Applied Analysis*, 8(2):113–126, 2005.
- [45] T. Kim and D. S. Kim. Degenerate laplace transform and degenerate gamma function. *Russian Journal of Mathematical Physics*, 24(2):241–248, 2017.
- [46] Y. Kim, B. M. Kim, L.-C. Jang, and J. Kwon. A note on modified degenerate gamma and laplace transformation. *Symmetry*, 10(10):471, 2018.
- [47] V. Kiryakova. Some special functions related to fractional calculus and fractional (non-integer) order control systems and equations. *Facta Univ. Ser. Autom. Control Robot*, 7(1):79–98, 2008.
- [48] V. Kiryakova. Unified approach to fractional calculus images of special functions—a survey. *Mathematics*, 8(12):2260, 2020.
- [49] V. Kiryakova. A guide to special functions in fractional calculus. *Mathematics*, 9(1):106, 2021.
- [50] A. Kochubei, Y. Luchko, V. E. Tarasov, and I. Petráš. *Handbook of fractional calculus with applications*, volume 1. De Gruyter Berlin, Germany, 2019.
- [51] V. N. Kolokoltsov and M. Troeva. A new approach to fractional kinetic evolutions. *Fractal and Fractional*, 6(2):49, 2022.
- [52] D. Kumar. Solution of fractional kinetic equation by a class of integral transform of pathway type. *Journal of Mathematical Physics*, 54(4), 2013.

- [53] M. Kumar Bansal, D. Kumar, P. Harjule, and J. Singh. Fractional kinetic equations associated with incomplete i-functions. *Fractal and Fractional*, 4(2):19, 2020.
- [54] J. Lovoie, T. J. Osler, and R. Tremblay. Fractional derivatives and special functions. *SIAM review*, 18(2):240–268, 1976.
- [55] Y. Luchko. Some schemata for applications of the integral transforms of mathematical physics. *Mathematics*, 7(3):254, 2019.
- [56] J. T. Machado, V. Kiryakova, and F. Mainardi. Recent history of fractional calculus. *Communications in nonlinear science and numerical simulation*, 16(3):1140–1153, 2011.
- [57] T. Machado, V. Kiryakova, and F. Mainardi. A poster about the recent history of fractional calculus. *Fractional Calculus and Applied Analysis*, 13(3):329p–334p, 2010.
- [58] S. Maitama and W. Zhao. New integral transform: Shehu transform a generalization of sumudu and laplace transform for solving differential equations. *arXiv preprint arXiv:1904.11370*, 2019.
- [59] D. R. Mathur and S. Poonia. Application of the mellin typt integral transform in the range  $[0, 1/a]$ . *International Journal of Mathematical Archive*, 3(6):1–7, 2012.
- [60] G. M. Mittag-Leffler. Sur la nouvelle fonction  $e_\alpha(x)$ . *CR Acad. Sci. Paris*, 137(2):554–558, 1903.
- [61] K. S. Nisar, A. Shaikh, G. Rahman, and D. Kumar. Solution of fractional kinetic equations involving class of functions and sumudu transform. *Advances in Difference Equations*, 2020(1):39, 2020.
- [62] K. Oldham and J. Spanier. *The fractional calculus theory and applications of differentiation and integration to arbitrary order*. Elsevier, 1974.
- [63] A. Pal, R. Jana, and A. Shukla. On some results associated with  ${}_p r_q(\alpha, \beta; z)$  function. *National Academy Science Letters*, 44(3):259–262, 2021.
- [64] T. R. Prabhakar. A singular integral equation with a generalized mittag leffler function in the kernel. *Yokohama Mathematical Journal*, 19(1):7–15, 1971.
- [65] M. Ramadan and H. M. Arafa. A reliable approach of ramadan group integral and projected differential transform methods to system of nonlinear partial differential equations. *Filomat*, 37(23):7891–7903, 2023.
- [66] S. Rogosin and M. Karpiyenyia. Fractional models for analysis of economic risks. *Fractional Calculus and Applied Analysis*, 2023.
- [67] A. I. Saichev and G. M. Zaslavsky. Fractional kinetic equations: solutions and applications. *Chaos: an interdisciplinary journal of nonlinear science*, 7(4):753–764, 1997.
- [68] R. Saxena and S. L. Kalla. On the solutions of certain fractional kinetic equations. *Applied Mathematics and Computation*, 199(2):504–511, 2008.
- [69] R. Saxena, A. Mathai, and H. Haubold. On fractional kinetic equations. *Astrophysics and Space Science*, 282:281–287, 2002.
- [70] R. Saxena, A. Mathai, and H. Haubold. On generalized fractional kinetic equations. *Physica A: Statistical Mechanics and its Applications*, 344(3-4):657–664, 2004.
- [71] V. Serov et al. *Fourier series, Fourier transform and their applications to mathematical physics*, volume 197. Springer, 2017.
- [72] M. Sharma and R. Jain. A note on a generalized m-series as a special function of fractional calculus. *Fract. Calc. Appl. Anal.*, 12(4):449–452, 2009.
- [73] A. Shukla and J. Prajapati. On a generalization of mittag-leffler function and its properties. *Journal of mathematical analysis and applications*, 336(2):797–811, 2007.
- [74] H. Singh, H. Srivastava, and R. Pandey. *Special Functions in Fractional Calculus and Engineering*. CRC Press, 2023.
- [75] Y. Singh, V. Gill, J. Singh, D. Kumar, and I. Khan. Computable generalization of fractional kinetic equation with special functions. *Journal of King Saud University-Science*, 33(1):101221, 2021.
- [76] Y. Singh, V. Gill, J. Singh, D. Kumar, and I. Khan. Computable generalization of fractional kinetics equation with special functions. *Journal of King Saud University—Science*, 33:101221, 2021.
- [77] H. M. Srivastava. *Special functions in fractional calculus and related fractional differintegral equations*. Verlag nicht ermittelbar, 2015.
- [78] J. Tenreiro Machado, V. Kiryakova, and F. Mainardi. A poster about the old history of fractional calculus. *Fractional Calculus and Applied Analysis*, 13(4):447–454, 2010.
- [79] G. K. Watugala. Sumudu transform: a new integral transform to solve differential equations and control engineering problems. *Integrated Education*, 24(1):35–43, 1993.
- [80] A. Wiman. Über de fundamental satz in der theoric der funktionen  $e_\alpha(x)$ . *Acta Mathematica*, 29:191–201, 1905.
- [81] K. Yadav, M. F. Ali, M. Sharma, A. Sharma, and R. A. Khan. Specific properties of fractional kinetic equations with the advanced k-bessel function. *Progress in Fractional Differentiation and Applications*, 11(3):631–640, 2025.
- [82] O. Yaḥci and R. Şahin. Degenerate pochhammer symbol, degenerate sumudu transform, and degenerate hypergeometric function with applications. *Hacettepe Journal of Mathematics and Statistics*, pages 1–18, 2021.
- [83] X.-J. Yang. A new integral transform method for solving steady heat-transfer problem. *Thermal Science*, 20(suppl. 3):639–642, 2016.