Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/120301

49 Amplified Generalized Fractional Neural Network Approximations at a Reduced Finite Domain

George A. Anastassiou

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A.

Received: 7 Mar. 2026, Revised: 12 May 2026, Accepted: 27 May 2026
Published online: 1 Jul. 2026

Abstract: This research deals with the determination of the rate of generalized fractional pointwise, uniform and L_p, $p \geq 1$, convergences to the unit operator of 49 specific "normalized cusp neural network operators". The cusp is a compact support activation function, which derives from the composition of two specific activation functions having as domain the whole real line. We employ 7 different known activation functions. These convergences are given via the moduli of continuity of the right and left generalized fractional derivatives of Caputo type of the engaged function in the form of Jackson type inequalities. The composition of activation functions aims to more flexible and powerful neural networks utilizing the reduction of infinite domains to the one domain of compact support.

Keywords: Amplified Neural network approximation, cusp activation function, modulus of continuity, reduction of domain, generalized fractional derivatives.

1 Introduction

From AI and computer science we have the following: In essence, composing activation functions in neural networks offers the advantage of potentially tailoring the network's ability to learn and model complex, non-linear relationships in data. Here's a breakdown of the potential benefits:

1. Enhanced Capacity for Complex Modeling:

–Diversification of Non-linearity: Different activation functions have different characteristics. For example, ReLU introduces sparsity, while Sigmoid squashes values into a range. By composing them, the network potentially can learn a wider variety of non-linear transformations and capture more intricate patterns in the data.

2. Improved Training Dynamics:

–Mitigating Gradient Problems: Activation functions influence gradient flow during training. Using different activation functions can potentially help address issues like vanishing or exploding gradients, which hinder learning in deep networks.
–Faster Convergence: Certain activation functions, like ReLU, can accelerate the convergence of the training process compared to others like Sigmoid or Tanh. Combining different functions can potentially lead to faster training and competitive performance.

3. Enhanced Generalization and Robustness:

–Better Generalization: By learning richer representations of the data through diverse activation functions, the network's ability to generalize well to unseen data improves, reducing the risk of overfitting.
–Increased Robustness: Networks with carefully chosen activation functions can handle variations in input data more effectively, adapting to noise, missing data, or unexpected perturbations.

4. Adaptation to Input Characteristics:

* Corresponding author e-mail: ganastss@memphis.edu

–Handling Diverse Data: Different activation functions can be suited to different data characteristics. For instance, tanh can be useful when dealing with data containing both positive and negative values.

5. Potential for Architectural Interpretability:

–Insight into Learning: By using distinct activation functions, different parts of the network might become responsible for capturing specific features, which can potentially offer insights into how the model learns.

In summary, composing activation functions potentially allows for a more flexible and powerful neural network capable of:

–Learning more complex patterns.
–Faster and more stable training.
–Better generalization to new data.
–Greater adaptability to diverse data.

Attention: While composing activation functions can offer benefits, it's important to choose them judiciously and with consideration for the specific problem at hand, as some combinations might not be beneficial or could even lead to unwanted behaviors like exploding gradients. Empirical testing and validation are crucial when exploring different activation function compositions.

The author was greatly inspired and motivated by [8] and he was the pioneer of quantitative neural network approximation, see [2], and since then he has published numerous of papers and books, e.g. see [3]- [7].

In this article we continue this trend at the generalized fractional level.

In mathematical neural network approximation AMS Mathscinet lists no articles related to composition of activation functions. So this is a seminal article.

By using composition of known and specific activation functions he achieves the first long part of the introduction, and most notably these compositions lead to activation functions of compact support, though the initial activation functions had an infinite domain, the whole real line.

Now each resulting bi-composed activation function is an open cusp of compact support $[-1,1]$. Our involved 7 activation functions generate 49 amplified neural network operators, which resemble to the squashing operators in [2], [4], and so do the produced quantitative results.

Of great inspiration are the articles [12]-[14]. References [15], [16], [18] are foundational. Finally references [19]-[28] represent recent important works.

2 Foundations

We need

Definition 1._([7], p. 530) The sigmoid activation functions we deal with here are described as follows_ $g:\mathbb{R}\to[-1,1]$, _strictly increasing_ $g(0)=0$, $g(-x)=-g(x)$, $x\in\mathbb{R}$, $g(+\infty)=1$, $g(-\infty)=-1$. _Also_ g _is strictly convex over_ $(-\infty,0]$ _and strictly concave over_ $[0,+\infty)$, _with_ $g^{(2)}\in C(\mathbb{R})$.

More precisely we will deal with the following sigmoid activation functions which possess all of the above properties: the algebraic activation function,

$$g_1(x)=\frac{x}{\sqrt[2m]{1+x^{2m}}},\quad m\in\mathbb{N},\, x\in\mathbb{R}, \tag{1}$$

(see [6], pp. 2-3);

the normalized and parametrized arctangent

$$g_2(x)=\frac{2}{\pi}\arctan\left(\frac{x}{2}\lambda x\right)=\frac{2}{\pi}\int_0^{\frac{\pi\lambda x}{2}}\frac{dz}{1+z^2},\quad x\in\mathbb{R},\, \lambda>0, \tag{2}$$

(see [7], p.p. 156-157);

the normalized and parametrized Gudermannian function

$$g_3(x)=\frac{2}{\pi}gd(\lambda x)=\frac{4}{\pi}\arctan\left(\tanh\left(\frac{\lambda x}{2}\right)\right)$$

$$= \frac{2}{\pi}\int_0^{\lambda x}\frac{dt}{\cosh t} = \frac{4}{\pi}\int_0^{\lambda x}\frac{dt}{e^t+e^{-t}}, \ \ x\in\mathbb{R}, \lambda>0, \tag{3}$$

where

$$gd(x) := \int_0^x \frac{dt}{\cosh t} = 2\arctan\left(\tanh\left(\frac{x}{2}\right)\right), \ \ \forall\, x\in\mathbb{R},$$

is the Gudermannian function (see [7], p. 198-199);

the parametrized error function

$$g_4(x) = erf\lambda x = \frac{2}{\sqrt{\pi}}\int_0^{\lambda x} e^{-t^2}dt, \ \ x\in\mathbb{R}, \lambda>0, \tag{4}$$

(see [7], pp. 226-227);

the parametrized hyperbolic tangent function

$$g_5(x) = \tanh\lambda x = \frac{e^{\lambda x}-e^{-\lambda x}}{e^{\lambda x}+e^{-\lambda x}} = \frac{e^{2\lambda x}-1}{e^{2\lambda x}+1}$$

$$= \frac{1-e^{-2\lambda x}}{1+e^{-2\lambda x}}, \ \ x>0, \lambda>0, \tag{5}$$

(see [7], p. 72);

for small $0<\lambda<1$, $\tanh\lambda x$ is expected to behave better than *ReLu* and *Leaky ReLu* activation functions;

the parametrized hyperbolic tangent like activation function,

$$g_6(x) = \frac{A^{\lambda x}-A^{-\lambda x}}{A^{\lambda x}+A^{-\lambda x}}, \ \ A>1, \lambda>0, x\in\mathbb{R} \tag{6}$$

(see [7], pp. 262-263);

finally, the β-parametrized half hyperbolic tangent function

$$g_7(x) = \frac{1-e^{-\beta x}}{1+e^{-\beta x}}, \ \ \beta>0, \forall\, x\in\mathbb{R}, \tag{7}$$

(see [7], p. 509-510).

So, we can denote then by $g_i(x)$, $x\in\mathbb{R}$, $i=1,2,...,7$.

Here we are concerned with the compositions of these activation functions

$$g_i\circ g_j = g_i|_{(-1,1)}\circ g_j, \ \text{ and } g_i\circ g_i = g_i|_{(-1,1)}\circ g_i, \tag{8}$$

where $i,j\in\{1,2,...,7\}$.

There are $P(7,2)$ permutations, i.e.

$$P(7,2) = \frac{7!}{5!} = 6\cdot 7 = 42, \text{ of ordered compositions;}$$

plus 7 self-compositions; so we have a total of 49 total such compositions.

So here a $g_i\circ g_j$ means any of the compositions in (8), where $i,j\in\{1,...,7\}$, $i\neq j$ or $i=j$.

Clearly, $g_i\circ g_j = g_i|_{(-1,1)}\circ g_j$ is strictly increasing and $(g_i\circ g_j)(0)=0$, and

$$g_i\circ g_j(-x) = g_i(g_j(-x)) = g_i(-g_j(x)) = -g_i(g_j(x)) = -(g_i\circ g_j)(x),$$

that is

$$(g_i\circ g_j)(-x) = -(g_i\circ g_j)(x), \ \ \forall\, x\in\mathbb{R}.$$

Furthermore

$$(g_i\circ g_j)(+\infty) = g_i(g_k(+\infty)) = g_i(1),$$
$$(g_i\circ g_j)(-\infty) = g_i(g_j(-\infty)) = g_i(-1).$$

Next acting over $(-\infty,0]$: let $\lambda,\mu\geq 0:\lambda+\mu=1$. Then by convexity of g_j there we have

$$g_j(\lambda x+\mu y) \leq \lambda g_j(x)+\mu g_j(y), \ \ x,y\in\mathbb{R}_-;$$

and

$$g_i(g_j(\lambda x+\mu y)) \leq g_i(\lambda g_j(x)+\mu g_j(y)) \leq \lambda g_i(g_j(x))+\mu g_i(g_j(y)),$$

i.e.

$$(g_i \circ g_j)(\lambda x+\mu y) \leq \lambda (g_i \circ g_j)(x)+\mu (g_i \circ g_j)(y),$$

$x, y \in \mathbb{R}_-$.

So that $g_i \circ g_j$ is convex over $(-\infty, 0]$.

Similarly, over $[0,+\infty)$ we get: let $\lambda, \mu \geq 0 : \lambda+\mu = 1$. Then by concavity of g_j there we have

$$g_j(\lambda x+\mu y) \geq \lambda g_j(x)+\mu g_j(y), \quad x, y \in \mathbb{R}_+;$$

and

$$g_i(g_j(\lambda x+\mu y)) \geq g_i(\lambda g_j(x)+\mu g_j(y)) \geq \lambda g_i(g_j(x))+\mu g_i(g_j(y)).$$

Therefore $g_i \circ g_j$ is concave over $[0,+\infty)$.

Also, it is

$$(g_i(g_j(x)))'' = g_i''(g_j(x))\left(g_j'(x)\right)^2+g_i'(g_j(x))g_j''(x) \in C(\mathbb{R}), \quad x \in \mathbb{R}.$$

So $g_i \circ g_j$ is a sigmoid activation function with asymptotes $g_i(\pm 1)$.

Next we consider the function

$$\Phi_{i,j}(x) := \frac{1}{4}(g_i \circ g_j(x+1)-g_i \circ g_j(x-1)) > 0, \quad \forall\, x \in \mathbb{R}.$$

We observe that

$$\Phi_{i,j}(-x) = \frac{1}{4}(g_i(g_j(-x+1))-g_i(g_j(-x-1))) =$$

$$\frac{1}{4}(g_i(g_j(-(x-1)))-g_i(g_j(-(x+1)))) =$$

$$\frac{1}{4}(g_i(-g_j(x-1))-g_i(-g_j(x+1))) =$$

$$\frac{1}{4}(-g_i(g_j(x-1))+g_i(g_j(x+1))) =$$

$$\frac{1}{4}(g_i g_j(x+1)-g_i g_j(x-1)) = \Phi_{i,j}(x),$$

that is

$$\Phi_{i,j}(-x) = \Phi_{i,j}(x), \quad \forall\, x \in \mathbb{R}.$$

So $\Phi_{i,j}$ can serve as a density function in general.

So we have $g_j : \mathbb{R} \to (-1,1)$, $g_i|_{(-1,1)} : (-1,1) \to (-1,1)$, and the strictly increasing function $H_{i,j} := g_i|_{(-1,1)} \circ g_j : \mathbb{R} \to (-1,1)$, with the graph of $H_{i,j}$ containing an arc of finite length, such that $H_{i,j}(0) = 0$, starting at $(-1, g_i(g_j(-1)))$ and terminating at $(1, g_i(g_j(1)))$ .We call this arc also $H_{i,j}$. In particular $H_{i,j}$ is negative and convex over $(-1,0]$, and it is positive and concave over $[0,1)$.

So it has compact support $[-1,1]$ and it is like a squashing function, see [4], Ch. 1, p. 8.

We will work from now on with $\left|H_{i,j}\right|$, which has as a graph a cusp joining the points $\left(-1, \left|g_i(g_j(-1))\right|\right)$, $(0,0), (1, g_i(g_j(1)))$ and with compact support, again, $[-1,1]$. The points $\left(-1, \left|g_i(g_j(-1))\right|\right)$, $(1, g_i(g_j(1)))$ belong to the graph of $\left|H_{i,j}\right|$ and $(0,0)$ too.

Typically $H_{i,j}$ has a steeper slope than of g_j, but it is flatter and closer to the x-axis than g_j is, e.g. $\tanh(\tanh x)$ has asymptotes ± 0.76, while $\tanh x$ has asymptotes ± 1 , notice that $\tanh(1) = 0.76$. Clearly $H_{i,j}$ has applications in spiking neural networks.

Here we consider functions $f : \mathbb{R} \to \mathbb{R}$ to be either continuous and bounded, or uniformly continuous.

The first modulus of continuity is given by

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in \mathbb{R}: \\ |x-y| \leq \delta}} |f(x)-f(y)|, \quad \delta > 0.$$

Here we have that $\omega_1(f,\delta) < +\infty$, $\delta > 0$.

In this article we study the generalized fractional pointwise and uniform convergences with rates over the real line, to the unit operator, of the 49 different "normalized cusp neural network operators",

$$\left(A_n^{(i,j)}(f)\right)(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right)\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{\sum_{k=-n^2}^{n^2}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}, \tag{9}$$

where $0<\alpha<1$ and $x\in\mathbb{R}$, $n\in\mathbb{N}$; $i,j\in\{1,...,7\}$, $i\neq j$ or $i=j$.

Notice $A_n^{(i,j)}$ is a positive linear operator with $A_n^{(i,j)}(1)=1$.

The terms in the ratio of sums (9) can be nonnegative and make sense, iff $0<\left|n^{1-\alpha}\left(x-\frac{k}{n}\right)\right|\leq 1$, i.e. $0<\left|x-\frac{k}{n}\right|\leq \frac{1}{n^{1-\alpha}}$, iff

$$nx-n^{\alpha}\leq x\leq nx+n^{\alpha},\ \ x\neq\frac{k}{n}. \tag{10}$$

In order to have the desired order of numbers

$$-n^2\leq nx-n^{\alpha}\leq nx+n^{\alpha}\leq n^2,\ \ x\neq\frac{k}{n}, \tag{11}$$

it is sufficient to assume that

$$n\geq 1+|x|,\ \ x\neq\frac{k}{n}. \tag{12}$$

When $x\in[-1,1]$, $x\neq\frac{k}{n}$, it is enough to assume $n\geq 2$, which implies (11), and $x\neq\frac{k}{n}$.

But the unique case $x=\frac{k}{n}$ contributes nothing and can be ignored.

Thus, without loss of generality we can take always that $x\neq\frac{k}{n}$.

Proposition 1.([2]) *Let $a,b\in\mathbb{R}$, $a\leq b$. Let $card(k)$ (≥ 0) be the maximum number of integers contained in $[a,b]$. Then*

$$\max(0,(b-a)-1)\leq card(k)\leq (b-a)+1.$$

*Note 1.*We would like to establish a lower bound on $card(k)$ over the interval $[nx-n^{\alpha},nx+n^{\alpha}]$. By Proposition 1 we get that

$$card(k)\geq\max(2n^{\alpha}-1,0).$$

We obtain $card(k)\geq 1$, if $2n^{\alpha}-1\geq 1$ iff $n\geq 1$, which is always true.

So to have the desired order (11) and $card(k)\geq 1$ over $[nx-n^{\alpha},nx+n^{\alpha}]$, it is enough to consider

$$n\geq\max(1+|x|,1)=1+|x|. \tag{13}$$

Also notice that $card(k)\to+\infty$, as $n\to+\infty$.

Denote by $[\cdot]$ the integral part of a number and $\lceil\cdot\rceil$ its ceiling.

Thus, it is clear that

$$\left(A_n^{(i,j)}(f)\right)(x) := \frac{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]} f\left(\frac{k}{n}\right)\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}, \tag{14}$$

$0<\alpha<1$, and $n\in\mathbb{N}: n\geq 1+|x|$, $x\in\mathbb{R}$; all $i,j\in\{1,...,7\}$, $i\neq j$ or $i=j$.

3 About Generalized Fractional Calculus

We will use

Definition 2.([5]) *Let $\beta > 0$, $\lceil \beta \rceil = N$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^N([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^N([g(a), g(b)])$.*

We define the left generalized g-fractional derivative X-valued of f of order β as follows:

$$\left(D^{\beta}_{*a;g} f\right)(x) := \frac{1}{\Gamma(N-\beta)} \int_a^x (g(x) - g(t))^{N-\beta-1} g'(t) \left(f \circ g^{-1}\right)^{(N)}(g(t))\, dt, \tag{15}$$

$\forall\, x \in [a,b]$. The last integral is of Bochner type.

If $\beta \notin \mathbb{N}$, by [5], we have that $\left(D^{\beta}_{*a;g} f\right) \in C([a,b], X)$.

We set

$$D^{N}_{*a;g} f(x) := \left(\left(f \circ g^{-1}\right)^{(N)} \circ g\right)(x) \in C([a,b], X), \ N \in \mathbb{N}, \tag{16}$$

$$D^{0}_{*a;g} f(x) = f(x), \ \forall\, x \in [a,b].$$

When $g = id$, then

$$D^{\beta}_{*a;g} f = D^{\beta}_{*a;id} f = D^{\beta}_{*a} f, \tag{17}$$

the usual left X-valued Caputo fractional derivative, see [9,11].

We will use

Definition 3.([5])*Let $\beta > 0$, $\lceil \beta \rceil = N$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^N([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^N([g(a), g(b)])$.*

We define the right generalized g-fractional derivative X-valued of f of order β as follows:

$$\left(D^{\beta}_{b-;g} f\right)(x) := \frac{(-1)^N}{\Gamma(N-\beta)} \int_x^b (g(t) - g(x))^{N-\beta-1} g'(t) \left(f \circ g^{-1}\right)^{(N)}(g(t))\, dt, \tag{18}$$

$\forall\, x \in [a,b]$. The last integral is of Bochner type.

If $\beta \notin \mathbb{N}$, by [5], we have that $\left(D^{\beta}_{b-;g} f\right) \in C([a,b], X)$.

We set

$$D^{N}_{b-;g} f(x) := (-1)^N \left(\left(f \circ g^{-1}\right)^{(N)} \circ g\right)(x) \in C([a,b], X), \ N \in \mathbb{N}, \tag{19}$$

$$D^{0}_{b-;g} f(x) = f(x), \ \forall\, x \in [a,b].$$

When $g = id$, then

$$D^{\beta}_{b-;g} f = D^{\beta}_{b-;id} f = D^{\beta}_{b-} f, \tag{20}$$

the usual right X-valued Caputo fractional derivative, see [3,10,11].

We mention the following g-left generalized X-valued Taylor's formula:

Theorem 1.([5]) *Let $\beta > 0$, $N = \lceil \beta \rceil$ and $f \in C^N([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^N([g(a), g(b)])$. Then*

$$\begin{gathered} f(x) = f(a) + \sum_{j_*=1}^{N-1} \frac{(g(x) - g(a))^{j_*}}{j_*!} \left(f \circ g^{-1}\right)^{(j_*)}(g(a)) + \\ \frac{1}{\Gamma(\beta)} \int_a^x (g(x) - g(t))^{\beta-1} g'(t) \left(D^{\beta}_{*a;g} f\right)(t)\, dt = \\ f(a) + \sum_{j_*=1}^{N-1} \frac{(g(x) - g(a))^{j_*}}{j_*!} \left(f \circ g^{-1}\right)^{(j_*)}(g(a)) + \\ \frac{1}{\Gamma(\beta)} \int_{g(a)}^{g(x)} (g(x) - z)^{\beta-1} \left(\left(D^{\beta}_{*a;g} f\right) \circ g^{-1}\right)(z)\, dz, \ \forall\, x \in [a,b]. \end{gathered} \tag{21}$$

We also mention the following g-right generalized X-valued Taylor's formula:

Theorem 2.([5]) *Let $\beta > 0$, $N = \lceil \beta \rceil$ and $f \in C^N([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^N([g(a), g(b)])$. Then*

$$f(x) = f(b) + \sum_{j_*=1}^{N-1} \frac{(g(x) - g(b))^{j_*}}{j_*!} \left(f \circ g^{-1}\right)^{(j_*)}(g(b)) +$$

$$\frac{1}{\Gamma(\beta)} \int_x^b (g(t) - g(x))^{\beta-1} g'(t) \left(D_{b-;g}^{\beta} f\right)(t)\, dt =$$

$$f(b) + \sum_{j_*=1}^{N-1} \frac{(g(x) - g(b))^{j_*}}{j_*!} \left(f \circ g^{-1}\right)^{(j_*)}(g(b)) + \tag{22}$$

$$\frac{1}{\Gamma(\beta)} \int_{g(x)}^{g(b)} (z - g(x))^{\beta-1} \left(\left(D_{b-;g}^{\beta} f\right) \circ g^{-1}\right)(z)\, dz, \quad \forall\, x \in [a,b].$$

Integrals in (15)-(22) are of Bochner type. For the Bochner integral excellent resources are [17] and [1, pp. 422-428].

When $X = \mathbb{R}$, then the Bochner integral collapses to the Lebesgue integral. So all the above apply also to real valued functions which we treat here.

*Example 1.*The following function

$$f(x) = x + \sqrt{x^2+1}, \ \ x \in \mathbb{R},$$

has derivative $f'(x) = 1 + \frac{x}{\sqrt{x^2+1}}$ with range $(0,2)$.

Indeed we have: $1 + \frac{x}{\sqrt{x^2+1}} > 0$, iff $\frac{x}{\sqrt{x^2+1}} > -1$, iff $x > -\sqrt{x^2+1}$, iff $-x < \sqrt{x^2+1}$. The last is true if $x \geq 0$, and if $x < 0$, then $|x| < \sqrt{x^2+1}$, iff $x^2 < x^2+1$, iff $0 < 1$ true.

And we have $1 + \frac{x}{\sqrt{x^2+1}} < 2$, iff $\frac{x}{\sqrt{x^2+1}} < 1$, iff $x < \sqrt{x^2+1}$, if $x < 0$ the last is true, if $x > 0$ then $x^2 < x^2+1$, iff $0 < 1$ true. Thus, f is strictly increasing, differentiable and uniformly continuous. It has C^1 inverse, i.e. f^{-1} is continuous and differentiable, with $\left(f^{-1}\right)'(y) = \frac{1}{f'\left(f^{-1}(y)\right)}$, by continuous f' and $f' \neq 0$.

*Example 2.*Let $f(x) = x + \arctan(x)$. Then $f'(x) = 1 + \frac{1}{1+x^2}$, with $1 < f'(x) \leq 2$, $\forall\, x \in \mathbb{R}$. Thus f is C^1, strictly increasing and uniformly continuous.

By the inverse function theorem f^{-1} is differentiable and $\left(f^{-1}\right)'$ is continuous.

*Example 3.*Let $f(x) = 2x + \sin(x)$, it is $1 \leq f'(x) \leq 3$, thus f is uniformly continuous, strictly increasing, continuously differentiable with a C^1 inverse.

So our assumption on g in Definitions 2, 3 and Theorems 1, 2 is possible.

4 Main Results

We present the following generalized fractional approximation result.

Theorem 3.*Let $\beta > 0$, $\beta \notin \mathbb{N}$, $N = \lceil \beta \rceil$. Let also a strictly increasing $g \in C^1(\mathbb{R})$, such that $g^{-1} \in C^N(\mathbb{R})$; g is also assumed to be uniformly continuous. Let also $f \in C^N(\mathbb{R})$ and assume that $\left(f \circ g^{-1}\right)^{(N)} \circ g \in L_\infty(\mathbb{R})$. Let also $x \in \mathbb{R}$, $n \in \mathbb{N} : n \geq 1 + |x|$. We further assume that $D_{*x;g}^{\beta} f$, $D_{x-;g}^{\beta} f$ are uniformly continuous or continuous and bounded on $[x, +\infty)$, $(-\infty, x]$, respectively. Then*

1)

$$\max_{\substack{i,j \in \{1,\ldots,7\}, \\ i \neq j \text{ or } i = j}} \left| A_n^{(i,j)}(f)(x) - f(x) - \sum_{j_*=1}^{N-1} \frac{\left(f \circ g^{-1}\right)^{(j_*)}(g(x))}{j_*!} \right. \tag{23}$$

$$\left. \left(\frac{\sum_{k=\lceil nx-n^{\alpha} \rceil}^{[nx+n^{\alpha}]} \left(g\left(\frac{k}{n}\right) - g(x)\right)^{j*} \left|H_{i,j}\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)\right|}{\sum_{k=\lceil nx-n^{\alpha} \rceil}^{[nx+n^{\alpha}]} \left|H_{i,j}\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)\right|} \right) \right| \leq$$

$$\frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)}\left[\omega_1\left(D_{x-;g}^{\beta}f;\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}+\omega_1\left(D_{*x;g}^{\beta}f;\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}\right],$$

above it is $\sum_{j=1}^{0}\cdot=0$.

2) *If* $N=1$ *or* $\left(f\circ g^{-1}\right)^{(j)}(g(x))=0,\ j=1,...,N-1,$ *we have*

$$\max_{\substack{i,j\in\{1,...,7\},\\ i\neq j\ or\ i=j}}\left|A_n^{(i,j)}(f)(x)-f(x)\right|\leq \tag{24}$$

$$\frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)}\left[\omega_1\left(D_{x-;g}^{\beta}f;\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}+\omega_1\left(D_{*x;g}^{\beta}f;\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}\right],$$

3)

$$\max_{\substack{i,j\in\{1,...,7\},\\ i\neq j\ or\ i=j}}\left|A_n^{(i,j)}(f)(x)-f(x)\right|\leq\sum_{j_*=1}^{N-1}\frac{\left|\left(f\circ g^{-1}\right)^{(j_*)}(g(x))\right|}{j_*!}\omega_1^{j_*}\left(g,\frac{1}{n^{1-\alpha}}\right) \tag{25}$$
$$+\frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)}\left[\omega_1\left(D_{x-;g}^{\beta}f;\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}+\omega_1\left(D_{*x;g}^{\beta}f;\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}\right].$$

Here we get generalized fractionally with rates the pointwise convergence of $A_n^{(i,j)}(f)(x)\to f(x)$*, as* $n\to+\infty$*,* $x\in\mathbb{R}$*;* $i,j\in\{1,...,7\}$*.*

*Proof.*Let $x\in\mathbb{R}$. We have that $D_{x-;g}^{\beta}f(x)=D_{*x;g}^{\beta}f(x)=0$, explanation at the end of this proof.

From Theorem 6, by using the left generalized Caputo fractional Taylor formula we get that

$$f\left(\frac{k}{n}\right)=\sum_{j_*=0}^{N-1}\frac{\left(f\circ g^{-1}\right)^{(j_*)}(g(x))}{j_*!}\left(g\left(\frac{k}{n}\right)-g(x)\right)^{j_*}+$$

$$\frac{1}{\Gamma(\beta)}\int_x^{\frac{k}{n}}\left(g\left(\frac{k}{n}\right)-g(J)\right)^{\beta-1}g'(J)\left[\left(D_{*x;g}^{\beta}f\right)(J)-\left(D_{*x;g}^{\beta}f\right)(x)\right]dJ, \tag{26}$$

for all $x\leq\frac{k}{n}\leq x+n^{\alpha-1}$, iff $\lceil nx\rceil\leq k\leq[nx+n^{\alpha}]$, where $k\in\mathbb{Z}$.

Also from Theorem 7, by using the right generalized Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right)=\sum_{j_*=0}^{N-1}\frac{\left(f\circ g^{-1}\right)^{(j_*)}(g(x))}{j_*!}\left(g\left(\frac{k}{n}\right)-g(x)\right)^{j_*}+$$

$$\frac{1}{\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(g(J)-g\left(\frac{k}{n}\right)\right)^{\beta-1}g'(J)\left[\left(D_{x-;g}^{\beta}f\right)(J)-\left(D_{x-;g}^{\beta}f\right)(x)\right]dJ, \tag{27}$$

for all $x-n^{\alpha-1}\leq\frac{k}{n}\leq x$, iff $\lceil nx-n^{\alpha}\rceil\leq k\leq[nx]$, where $k\in\mathbb{Z}$.

Notice that $\lceil nx\rceil\leq[nx]+1$.

Call

$$W_{i,j}(x):=\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|,\ \ i,j\in\{1,...,7\}.$$

Hence we have

$$\frac{f\left(\frac{k}{n}\right)\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}=$$

$$\sum_{j_*=0}^{N-1}\frac{\left(f\circ g^{-1}\right)^{(j_*)}(g(x))}{j_*!}\left(g\left(\frac{k}{n}\right)-g(x)\right)^{j*}\frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}+ \tag{28}$$

$$\frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(g\left(\frac{k}{n}\right)-g(J)\right)^{\beta-1}g'(J)\left[D_{*x;g}^{\beta}f(J)-D_{*x;g}^{\beta}f(x)\right]dJ,$$

and

$$\frac{f\left(\frac{k}{n}\right)\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}=$$

$$\sum_{j_*=0}^{N-1}\frac{\left(f\circ g^{-1}\right)^{(j_*)}(g(x))}{j_*!}\left(g\left(\frac{k}{n}\right)-g(x)\right)^{j*}\frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}+ \tag{29}$$

$$\frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(g(J)-g\left(\frac{k}{n}\right)\right)^{\beta-1}g'(J)\left[D_{x-;g}^{\beta}f(J)-D_{x-;g}^{\beta}f(x)\right]dJ.$$

Therefore we obtain

$$\frac{\sum_{k=[nx]+1}^{[nx+n^{\alpha}]}f\left(\frac{k}{n}\right)\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}=$$

$$\sum_{j_*=0}^{N-1}\frac{\left(f\circ g^{-1}\right)^{(j_*)}(g(x))}{j_*!}\left(\frac{\sum_{k=[nx]+1}^{[nx+n^{\alpha}]}\left(g\left(\frac{k}{n}\right)-g(x)\right)^{j*}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}\right)$$

$$+\sum_{k=[nx]+1}^{[nx+n^{\alpha}]}\frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)}$$

$$\int_{x}^{\frac{k}{n}}\left(g\left(\frac{k}{n}\right)-g(J)\right)^{\beta-1}g'(J)\left[D_{*x;g}^{\beta}f(J)-D_{*x;g}^{\beta}f(x)\right]dJ, \tag{30}$$

and

$$\frac{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx]}f\left(\frac{k}{n}\right)\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}=$$

$$\sum_{j_*=0}^{N-1}\frac{\left(f\circ g^{-1}\right)^{(j_*)}(g(x))}{j_*!}\left(\frac{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx]}\left(g\left(\frac{k}{n}\right)-g(x)\right)^{j*}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}\right)$$

$$+\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx]}\frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)}$$

$$\int_{\frac{k}{n}}^{x}\left(g(J)-g\left(\frac{k}{n}\right)\right)^{\beta-1}g'(J)\left[D_{x-;g}^{\beta}f(J)-D_{x-;g}^{\beta}f(x)\right]dJ. \tag{31}$$

Adding the two equalities (30) and (31) we obtain

$$\left(A_n^{(i,j)}(f)\right)(x)=\sum_{j_*=0}^{N-1}\frac{\left(f\circ g^{-1}\right)^{(j_*)}(g(x))}{j_*!} \tag{32}$$

$$\left(\frac{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}\left(g\left(\frac{k}{n}\right)-g(x)\right)^{j*}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}\right)+\widehat{M}_n^{(i,j)}(x),$$

where $i,j\in\{1,\ldots,7\}$, and

$$\widehat{M}_n^{(i,j)}(x):=\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx]}\frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)} \tag{33}$$

$$\int_{\frac{k}{n}}^{x}\left(g(J)-g\left(\frac{k}{n}\right)\right)^{\beta-1} g'(J)\left[D_{x-;g}^{\beta}f(J)-D_{x-;g}^{\beta}f(x)\right]dJ+$$

$$\sum_{k=[nx]+1}^{[nx+n^{\alpha}]}\frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)}$$

$$\int_{x}^{\frac{k}{n}}\left(g\left(\frac{k}{n}\right)-g(J)\right)^{\beta-1} g'(J)\left[D_{*x;g}^{\beta}f(J)-D_{*x;g}^{\beta}f(x)\right]dJ.$$

We call

$$\widehat{M}_{1n}^{(i,j)}(x):=\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx]}\frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)} \tag{34}$$

$$\int_{\frac{k}{n}}^{x}\left(g(J)-g\left(\frac{k}{n}\right)\right)^{\beta-1} g'(J)\left[D_{x-;g}^{\beta}f(J)-D_{x-;g}^{\beta}f(x)\right]dJ,$$

and

$$\widehat{M}_{2n}^{(i,j)}(x):=\sum_{k=[nx]+1}^{[nx+n^{\alpha}]}\frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)} \tag{35}$$

$$\int_{x}^{\frac{k}{n}}\left(g\left(\frac{k}{n}\right)-g(J)\right)^{\beta-1} g'(J)\left[D_{*x;g}^{\beta}f(J)-D_{*x;g}^{\beta}f(x)\right]dJ.$$

That is

$$\widehat{M}_{n}^{(i,j)}(x)=\widehat{M}_{1n}^{(i,j)}(x)+\widehat{M}_{2n}^{(i,j)}(x), \tag{36}$$

where $i,j\in\{1,...,7\}$.

Furthermore we get

$$E_n^{(i,j)}(f)(x):=\left(A_n^{(i,j)}(f)\right)(x)-f(x)-\sum_{j_*=1}^{N-1}\frac{\left(f\circ g^{-1}\right)^{(j_*)}(g(x))}{j_*!}$$

$$\left(\frac{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}\left(g\left(\frac{k}{n}\right)-g(x)\right)^{j*}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}\right)=\widehat{M}_n^{(i,j)}(x), \tag{37}$$

that is

$$\left|E_n^{(i,j)}(f)(x)\right|\leq\left|\widehat{M}_n^{(i,j)}(x)\right|\leq\left|\widehat{M}_{1n}^{(i,j)}(x)\right|+\left|\widehat{M}_{2n}^{(i,j)}(x)\right|. \tag{38}$$

If $N=1$, then

$$E_n^{(i,j)}(f)(x)=A_n^{(i,j)}(f)(x)-f(x)=\widehat{M}_n^{(i,j)}(x). \tag{39}$$

If $\left(f\circ g^{-1}\right)^{j_*}(g(x))=0$, for $j_*=1,...,N-1$, then again

$$E_n^{(i,j)}(f)(x)=A_n^{(i,j)}(f)(x)-f(x)=\widehat{M}_n^{(i,j)}(x), \tag{40}$$

where $i,j\in\{1,...,7\}$.

We can also have

$$\left|A_n^{(i,j)}(f)(x)-f(x)\right|\leq\sum_{j_*=1}^{N-1}\frac{\left|\left(f\circ g^{-1}\right)^{(j_*)}(g(x))\right|}{j_*!}\omega_1^{j_*}\left(g,\frac{1}{n^{1-\alpha}}\right)+\widehat{M}_n^{(i,j)}(x), \tag{41}$$

by g being also uniformly continuous we have that $\omega_1\left(g,\frac{1}{n^{1-\alpha}}\right)<+\infty$.

We observe that

$$\left|\widehat{M}_{1n}^{(i,j)}(x)\right| \leq \sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx]} \frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)}$$

$$\int_{\frac{k}{n}}^{x}\left(g(J)-g\left(\frac{k}{n}\right)\right)^{\beta-1} g'(J)\left|D_{x-;g}^{\beta}f(J)-D_{x-;g}^{\beta}f(x)\right| dJ \leq \tag{42}$$

$$\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx]} \frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)}$$

$$\int_{\frac{k}{n}}^{x}\left(g(J)-g\left(\frac{k}{n}\right)\right)^{\beta-1} g'(J)\,\omega_1\left(D_{x-;g}^{\beta}f,|J-x|\right)_{(-\infty,x]} dJ \leq$$

$$\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx]} \frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)}$$

$$\left(\int_{g\left(\frac{k}{n}\right)}^{g(x)}\left(g(J)-g\left(\frac{k}{n}\right)\right)^{\beta-1} dg(J)\right)\omega_1\left(D_{x-;g}^{\beta}f,\left|x-\frac{k}{n}\right|\right)_{(-\infty,x]} \leq$$

$$\left(\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx]} \frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)} \frac{\left(g(x)-g\left(\frac{k}{n}\right)\right)^{\beta}}{\beta}\right)\omega_1\left(D_{x-;g}^{\beta}f,\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]} \tag{43}$$

(by assuming g is uniformly continuous, then $\omega_1(g,\delta) < +\infty$, $\forall\ \delta > 0$)

$$\leq \left(\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx]} \frac{\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}\right)\frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)}\omega_1\left(D_{x-;g}^{\beta}f,\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}$$

$$\leq \left(\frac{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}\right)\frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)}\omega_1\left(D_{x-;g}^{\beta}f,\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]} \tag{44}$$

$$= \frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)}\omega_1\left(D_{x-;g}^{\beta}f,\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}.$$

Thus it holds

$$\left|\widehat{M}_{1n}^{(i,j)}(x)\right| \leq \frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)}\omega_1\left(D_{x-;g}^{\beta}f,\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}. \tag{45}$$

Next we estimate

$$\left|\widehat{M}_{2n}^{(i,j)}(x)\right| \leq \frac{\sum_{k=[nx]+1}^{[nx+n^{\alpha}]}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)}$$

$$\int_{x}^{\frac{k}{n}}\left(g\left(\frac{k}{n}\right)-g(J)\right)^{\beta-1} g'(J)\left|D_{*x;g}^{\beta}f(J)-D_{*x;g}^{\beta}f(x)\right| dJ \leq \tag{46}$$

$$\frac{\sum_{k=[nx]+1}^{[nx+n^{\alpha}]}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)}$$

$$\int_{x}^{\frac{k}{n}}\left(g\left(\frac{k}{n}\right)-g(J)\right)^{\beta-1} g'(J)\,\omega_1\left(D_{*x;g}^{\beta}f,|J-x|\right)_{[x,+\infty)} dJ \leq$$

$$\frac{\sum_{k=[nx]+1}^{[nx+n^{\alpha}]}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)} \tag{47}$$

$$\left(\int_{g(x)}^{g\left(\frac{k}{n}\right)}\left(g\left(\frac{k}{n}\right)-g(J)\right)^{\beta-1}dg(J)\right)\omega_1\left(D_{*x;g}^{\beta}f,\left|\frac{k}{n}-x\right|\right)_{[x,+\infty)}\leq$$

$$\left(\frac{\sum_{k=[nx]+1}^{[nx+n^{\alpha}]}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)\Gamma(\beta)}\frac{\left(g\left(\frac{k}{n}\right)-g(x)\right)^{\beta}}{\beta}\right)\omega_1\left(D_{*x;g}^{\beta}f,\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}\leq \tag{48}$$

$$\left(\frac{\sum_{k=[nx]+1}^{[nx+n^{\alpha}]}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}\right)\frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)}\omega_1\left(D_{*x;g}^{\beta}f,\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}\leq$$

$$\left(\frac{\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}\left|H_{i,j}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{W_{i,j}(x)}\right)\frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)}\omega_1\left(D_{*x;g}^{\beta}f,\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)} \tag{49}$$

$$=\frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)}\omega_1\left(D_{*x;g}^{\beta}f,\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}.$$

Thus

$$\left|\widehat{M}_{2n}^{(i,j)}(x)\right|\leq\frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)}\omega_1\left(D_{*x;g}^{\beta}f,\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}. \tag{50}$$

We have found that

$$\left|\widehat{M}_{n}^{(i,j)}(x)\right|\leq\frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)}$$

$$\left[\omega_1\left(D_{x-;g}^{\beta}f,\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}+\omega_1\left(D_{*x;g}^{\beta}f,\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}\right]<+\infty, \tag{51}$$

where $i,j\in\{1,...,7\}$.

Here we assumed that $\left(f\circ g^{-1}\right)^{(N)}\circ g\in L_{\infty}(\mathbb{R})$.

Next we get

$$\left|D_{*x;g}^{\beta}f(s)\right|\leq\frac{1}{\Gamma(N-\beta)}\int_x^s(g(s)-g(t))^{N-\beta-1}g'(t)\left|\left(f\circ g^{-1}\right)^{(N)}(g(t))\right|dt\leq$$

$$\frac{\left\|\left(f\circ g^{-1}\right)^{(N)}\circ g\right\|_{\infty,\mathbb{R}}}{\Gamma(N-\beta)}\int_{g(x)}^{g(s)}(g(s)-g(t))^{N-\beta-1}dg(t)= \tag{52}$$

$$\frac{\left\|\left(f\circ g^{-1}\right)^{(N)}\circ g\right\|_{\infty,\mathbb{R}}}{\Gamma(N-\beta+1)}(g(s)-g(x))^{N-\beta}.$$

That is

$$\left|D_{*x;g}^{\beta}f(s)\right|\leq\frac{\left\|\left(f\circ g^{-1}\right)^{(N)}\circ g\right\|_{\infty,\mathbb{R}}}{\Gamma(N-\beta+1)}(g(s)-g(x))^{N-\beta}, \tag{53}$$

$\forall\, s\in[x,+\infty)$.

Clearly then

$$D_{*x;g}^{\beta}f(x)=0. \tag{54}$$

Acting similarly we have:

$$\left|D_{x-;g}^{\beta}f(s)\right|\leq\frac{1}{\Gamma(N-\beta)}\int_s^x(g(t)-g(s))^{N-\beta-1}g'(t)\left|\left(f\circ g^{-1}\right)^{(N)}(g(t))\right|dt\leq$$

$$\frac{\left\|\left(f\circ g^{-1}\right)^{(N)}\circ g\right\|_{\infty,\mathbb{R}}}{\Gamma(N-\beta)}\int_{g(s)}^{g(x)}(g(t)-g(s))^{N-\beta-1}dg(t)= \tag{55}$$

$$\frac{\left\|\left(f\circ g^{-1}\right)^{(N)}\circ g\right\|_{\infty,\mathbb{R}}}{\Gamma\left(N-\beta+1\right)}\left(g\left(x\right)-g\left(s\right)\right)^{N-\beta}.$$

That is

$$\left|D_{x-;g}^{\beta}f\left(s\right)\right|\leq\frac{\left\|\left(f\circ g^{-1}\right)^{(N)}\circ g\right\|_{\infty,\mathbb{R}}}{\Gamma\left(N-\beta+1\right)}\left(g\left(x\right)-g\left(s\right)\right)^{N-\beta},\tag{56}$$

$\forall\, s\in(-\infty,x]$.

Clearly then

$$D_{x-;g}^{\beta}f\left(x\right)=0.\tag{57}$$

As an application of Theorem 3 we get:

Theorem 4. *All as in Theorem 3, and under the assumption that $D_{*x;g}^{\beta}f\left(t\right), D_{x-;g}^{\beta}f\left(t\right)$ are both continuous and bounded in $(x,t)\in\mathbb{R}^2$; $n\in\mathbb{N}:n\geq 2$. Then*

1) if $N=1$ we have

$$\max_{\substack{i,j\in\{1,\ldots,7\},\\ i\neq j \text{ or } i=j}}\left\|A_n^{(i,j)}\left(f\right)-f\right\|_{\infty,[-1,1]}\leq\frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma\left(\beta+1\right)}$$

$$\left[\sup_{x\in[-1,1]}\omega_1\left(D_{x-;g}^{\beta}f;\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}+\sup_{x\in[-1,1]}\omega_1\left(D_{*x;g}^{\beta}f;\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}\right],\tag{58}$$

2) in general we have

$$\max_{\substack{i,j\in\{1,\ldots,7\},\\ i\neq j \text{ or } i=j}}\left\|A_n^{(i,j)}\left(f\right)-f\right\|_{\infty,[-1,1]}\leq\sum_{j_*=1}^{N-1}\frac{\left\|\left(f\circ g^{-1}\right)^{(j_*)}\circ g\right\|_{\infty,[-1,1]}}{j_*!}\omega_1^{j_*}\left(g,\frac{1}{n^{1-\alpha}}\right)\tag{59}$$

$$+\frac{\omega_1^{\beta}\left(g,\frac{1}{n^{1-\alpha}}\right)}{\Gamma\left(\beta+1\right)}\left[\sup_{x\in[-1,1]}\omega_1\left(D_{x-;g}^{\beta}f;\frac{1}{n^{1-\alpha}}\right)_{(-\infty,x]}+\sup_{x\in[-1,1]}\omega_1\left(D_{*x;g}^{\beta}f;\frac{1}{n^{1-\alpha}}\right)_{[x,+\infty)}\right].$$

An interesting case is when $\beta=\frac{1}{2}$.

Here we get generalized fractionally with rates the uniform convergence of $A_n^{(i,j)}\left(f\right)\to f$, over $[-1,1]$, as $n\to+\infty$; $i,j\in\{1,\ldots,7\}$.

Proof. By Theorem 3, and by

$$\left\|D_{*x;g}^{\beta}f\right\|_{\infty}\leq K_1,\quad\forall\, x\in\mathbb{R};$$

and

$$\left\|D_{x-;g}^{\beta}f\right\|_{\infty}\leq K_2,\quad\forall\, x\in\mathbb{R},$$

where $K_1,K_2>0$.

We get that

$$\omega_1\left(D_{x-;g}^{\beta}f;\xi\right)_{(-\infty,x]}\leq 2K_2,$$

and

$$\omega_1\left(D_{*x;g}^{\beta}f;\xi\right)_{[x,+\infty)}\leq 2K_1,\quad\forall\, x\in\mathbb{R},$$

for any $\xi\geq 0$.

It follows

Corollary 1. *Let $\beta > 0$, $\beta \notin \mathbb{N}$, $N = \lceil \beta \rceil$. Let also a strictly increasing $g \in C^1(\mathbb{R})$, such that $g^{-1} \in C^N(\mathbb{R})$; g is also assumed to be uniformly continuous. Let also $f \in C^N(\mathbb{R})$ and assume that $\left(f \circ g^{-1}\right)^{(N)} \circ g \in L_\infty(\mathbb{R})$. Let $x \in [-\phi, \phi]$, $\phi > 0$ and $n \in \mathbb{N} : n \geq 1 + \phi$, $0 < \alpha < 1$. Here $D^{\beta}_{*x;g} f(t)$, $D^{\beta}_{x-;g} f(t)$ are both continuous and bounded in $(x,t) \in \mathbb{R}^2$. Then*

1) If $N = 1$, we have

$$\max_{\substack{i,j \in \{1,\dots,7\}, \\ i \neq j \text{ or } i=j}} \left| A_n^{(i,j)}(f)(x) - f(x) \right| \leq \frac{\omega_1^{\beta}\left(g, \frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)} \tag{60}$$

$$\left[\sup_{x \in [-\phi,\phi]} \omega_1 \left(D^{\beta}_{x-;g} f; \frac{1}{n^{1-\alpha}} \right)_{(-\infty,x]} + \sup_{x \in [-\phi,\phi]} \omega_1 \left(D^{\beta}_{*x;g} f; \frac{1}{n^{1-\alpha}} \right)_{[x,+\infty)} \right],$$

2) in general we have

$$\max_{\substack{i,j \in \{1,\dots,7\}, \\ i \neq j \text{ or } i=j}} \left| A_n^{(i,j)}(f)(x) - f(x) \right| \leq \sum_{j_*=1}^{N-1} \frac{\left| \left(f \circ g^{-1}\right)^{(j_*)}(g(x)) \right|}{j_*!} \omega_1^{j_*} \left(g, \frac{1}{n^{1-\alpha}} \right)$$

$$+ \frac{\omega_1^{\beta}\left(g, \frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)} \left[\sup_{x \in [-\phi,\phi]} \omega_1 \left(D^{\beta}_{x-;g} f; \frac{1}{n^{1-\alpha}} \right)_{(-\infty,x]} + \sup_{x \in [-\phi,\phi]} \omega_1 \left(D^{\beta}_{*x;g} f; \frac{1}{n^{1-\alpha}} \right)_{[x,+\infty)} \right]. \tag{61}$$

Proof. By Theorem 3.

By Corollary 1, and monotonicity and subadditivity properties of $\|\cdot\|_p$, $p \geq 1$, we obtain the following L_p result.

Corollary 2. *Here all as in Corollary 1 and $p \geq 1$. Then*

1) If $N = 1$, we have

$$\max_{\substack{i,j \in \{1,\dots,7\}, \\ i \neq j \text{ or } i=j}} \left\| A_n^{(i,j)} f - f \right\|_{p,[-\phi,\phi]} \leq 2^{\frac{1}{p}} \phi^{\frac{1}{p}} \frac{\omega_1^{\beta}\left(g, \frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)} \tag{62}$$

$$\left[\sup_{x \in [-\phi,\phi]} \omega_1 \left(D^{\beta}_{x-;g} f; \frac{1}{n^{1-\alpha}} \right)_{(-\infty,x]} + \sup_{x \in [-\phi,\phi]} \omega_1 \left(D^{\beta}_{*x;g} f; \frac{1}{n^{1-\alpha}} \right)_{[x,+\infty)} \right],$$

and

2) in general we get

$$\max_{\substack{i,j \in \{1,\dots,7\}, \\ i \neq j \text{ or } i=j}} \left\| A_n^{(i,j)} f - f \right\|_{p,[-\phi,\phi]} \leq \sum_{j_*=1}^{N-1} \frac{\left\| \left(f \circ g^{-1}\right)^{(j_*)} \circ g \right\|_{p,[-\phi,\phi]}}{j_*!} \omega_1^{j_*} \left(g, \frac{1}{n^{1-\alpha}} \right) \tag{63}$$

$$+ 2^{\frac{1}{p}} \phi^{\frac{1}{p}} \frac{\omega_1^{\beta}\left(g, \frac{1}{n^{1-\alpha}}\right)}{\Gamma(\beta+1)}$$

$$\left[\sup_{x \in [-\phi,\phi]} \omega_1 \left(D^{\beta}_{x-;g} f; \frac{1}{n^{1-\alpha}} \right)_{(-\infty,x]} + \sup_{x \in [-\phi,\phi]} \omega_1 \left(D^{\beta}_{*x;g} f; \frac{1}{n^{1-\alpha}} \right)_{[x,+\infty)} \right].$$

By (62), (63) we derive the generalized fractional L_p, $p \geq 1$, convergence with rates of $A^{(i,j)} f$ to f, all $i,j \in \{1,\dots,7\}$.

References

[1] C.D. Aliprantis, K.C. Border, *Infinite Dimensional Analysis*, Springer, New York, 2006.
[2] G.A. Anastassiou, *Rate of Convergence of some neural network operators to the unit - univariate case*, Journal of Mathematical Analysis and Applications, Vol 212 (1997), 237-262.
[3] G.A. Anastassiou, *On right fractional calculus*, Chaos, Solitons and Fractals, Vol. 42 (209), 365-376.
[4] G.A. Anastassiou, *Intelligent Systems II: Complete Approximation by Neural Network Operators*, Springer, Heidelberg, New York, 2016.
[5] G.A. Anastassiou, *Principles of General Fractional Analysis for Banach space valued functions*, Bull. Allahabad Math. Soc., 32(1) (2017), 71-145.
[6] G.A. Anastassiou, *Banach Space Valued Neural Network*, Springer, Heidelberg, New York, 2023.
[7] G.A. Anastassiou, *Parametrized, Deformed and General Neural Networks*, Springer, Heidelberg, New York, 2023.
[8] P. Cardaliaguet and G. Euvrard, *Approximation of a function and its derivative with a neural network*, Neural Networks, Vol. 5 (1992), 207-220.
[9] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics 2004, Springer-Verlag, Berlin, Heidelberg, 2010.
[10] A.M.A. El-Sayed and M. Gaber, *On the finite Caputo and finite Riesz derivatives*, Electronic Journal of Theoretical Physics, Vol. 3, No. 12 (2006), 81-95.
[11] G.A. Anastassiou, *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018.
[12] Z. Chen and F. Cao, *The approximation operators with sigmoidal functions*, Computers and Mathematics with Applications, 58 (2009), 758-765.
[13] D. Costarelli, R. Spigler, *Approximation results for neural network operators activated by sigmoidal functions*, Neural Networks 44 (2013), 101-106.
[14] D. Costarelli, R. Spigler, *Multivariate neural network operators with sigmoidal activation functions*, Neural Networks 48 (2013), 72-77.
[15] S. Haykin, *Neural Networks: A Comprehensive Foundation* (2 ed.), Prentice Hall, New York, 1998.
[16] W. McCulloch and W. Pitts, *A logical calculus of the ideas immanent in nervous activity*, Bulletin of Mathematical Biophysics, 7 (1943), 115-133.
[17] J. Mikusinski, *The Bochner Integral*, Academic Press, New York, 1978.
[18] T.M. Mitchell, *Machine Learning*, WCB-McGraw-Hill, New York, 1997.
[19] D. S. Yu; F. L. Cao, *Construction and approximation rate for feed-forward neural network operators with sigmoidal functions*, J. Comput. Appl. Math. 453 (2025), paper No. 116150.
[20] Cen, Siyu; Jin, Bangti; Quan, Qimeng; Zhou, Zhi; *Hybrid neural-network FEM approximation of diffusion coefficient in elliptic and parabolic problems*. IMA J. Numer. Anal. 44 (2024), no. 5, 3059-3093.
[21] Coroianu Lucian; Costarelli, Danillo; Natale, Mariarosaria; Pantiş, Alexandra; *The approximation capabilities of Durrmeyer-type neural network operators*. J. Appl. Math. Comput. 70 (2024), no. 5, 4581-4599.
[22] Warin, Xavier; *The GroupMax neural network approximation of convex functions*. IEEE Trans. Neural Netw. Learn. Syst. 35 (2024), no. 8, 11608-11612.
[23] Fabra, Arnau; Guasch, Oriol; Baiges, Joan; Codina, Ramon; *Approximation of acoustic black holes with finite element mixed formulations and artificial neural network correction terms*. Finite Elem. Anal. Des. 241 (2024), paper No. 104236.
[24] Grohs, Philipp; Voigtlaender, Felix. *Proof of the theory-to-practice gap in deep learning via sampling complexity bounds for neural network approximation spaces*. Found. Comput. Math. 24 (2024), no. 4, 1085-1143.
[25] Basteri, Andrea; Trevisan, Dario; *Quantitative Gaussian approximation of randomly initialized deep neural networks*. Mach. Learn. 113 (2024), no. 9, 6373-6393.
[26] De Ryck, T.; Mishra, S. *Error analysis for deep neural network approximations of parametric hyperbolic conservation laws*. Math. Comp. 93 (2024), no. 350, 2643-2677.
[27] Liu, Jie; Zhang, Baoji; Lai, Yuyang; Fang, Liqiau. *Hull form optimization research based on multi-precision back-propagation neural network approximation model*. Internal. J. Numer. Methods Fluid 96 (2024), no. 8, 1445-1460.
[28] Yoo, Jihahm; Kim, Jaywon; Gim, Minjung; Lee, Haesung. *Error estimates of physics-informed neural networks for initial value problems*. J. Korean Soc. Ind. Appl. Math. 28 (2024), no. 1, 33-58.