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Utilizing Laguerre Polynomials to Compute a Numerical Solution for the Second Kind Linear System of Volterra Integral Equations

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Abstract: This paper develops a highly efficient numerical framework using Laguerre polynomials to solve systems of linear Volterra integral equations of the second kind. A primary advantage of this approach is its use of the semi-infinite interval $[0,\infty)$, which eliminates the need for the artificial domain truncation required by traditional bases like Chebyshev or block-pulse functions. This makes the method uniquely suited for accurately modeling memory-dependent processes such as damping and diffusion. By deriving a specialized operational matrix, the system is transformed into a system of linear algebraic equations. Numerical results demonstrate spectral convergence and exceptional precision, achieving results comparable to the exact solution with a minimal number of basis functions. Comparative analysis confirms that the proposed method significantly outperforms existing benchmarks in both computational efficiency and accuracy for systems with decaying exponential kernels.

Keywords: System of Volterra integral equations, Laguerre polynomials, Operational matrix, Semi-infinite domain, Absolute and relative errors.

1 Introduction

Integral equations, where the unknown function appears under the integral sign, have a significant role in characterizing phenomena throughout physics, engineering, biology, and finance [1,2]. They effectively capture systems with memory effects, such as heat conduction, signal processing, and population dynamics, with VIEs of the second kind being prominent due to their variable integration limits [3]. VIEs and their systems are crucial because they effectively characterize phenomena that involve memory effects, meaning the current state of the system depends on its past history. According to the sources, these equations have a wide range of applications in the following fields: Dynamics [4], linear viscoelasticity [5], particle size statistics [6], heat transfer problem [7], viscoelastic stress analysis [8], population dynamics [9], and epidemic study [10].

Consider the system of linear VIEs of the second kind [11,12] will be examined in this article:

$$\begin{aligned}
Q_1(u) &= g_1(u) + \int_0^u k_{11}(u,v)\,Q_1(v)\,dv + \int_0^u k_{12}(u,v)\,Q_2(v)\,dv,\\
Q_2(u) &= g_2(u) + \int_0^u k_{21}(u,v)\,Q_1(v)\,dv + \int_0^u k_{22}(u,v)\,Q_2(v)\,dv,
\end{aligned} \tag{1}$$

where the unknown functions are $Q_1(u)$ and $Q_2(u)$, while the given functions are $g_1(u), g_2(u)$, and $k_{ij}(u,v), (i,j=1,2)$, with the parameter $0 \le u \le 1$.

In fact, there are numerous published methods and strategies for solving second kind VIEs [13]. In [14],

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Mohand transform is applied to determine solutions of convolution-type linear VIEs of the second kind. Matebie [15] used the method of successive approximations (Neumann's series) to solve both linear and nonlinear VIEs of the second kind. In [16], Myrhorod and Hvozdeva utilized Legendre polynomials together with the collocation approach to solve VIEs of the second kind, transforming the integral equation into a system of algebraic equations. Furthermore, Rahman and Hossain [17] used Legendre, Chebyshev, and Gegenbauer polynomials to obtain approximate solutions for some types of Voltera integral equations. VIEs of the second kind can also be solved using several published computational and numerical methods[18,19,20,21,22,23,24,25,26,27].

There are several numerical and analytical methods to solve systems of linear VIEs of the second kind. The Galerkin method, Finite element method, collocation methods, spectral methods, and Taylor or power series and expansion methods are among these methods. To address these, researchers have used Laguerre polynomials for solving a System of generalized Abel integral equations[28]. Khidir [29] used the Chebyshev spectral method for solving VIEs. Isaa and et al. [30] are used a collocation method via sinc functions and Chebyshev wavelet method for solving a linear systems of Volterra integro-differential equations. To find solutions for the second kind of VIEs, Chauhan and Aggarwal [31] are used Laplace transform. Aggarwal and Gupta [32] solve linear Volterra integro-differential equations of second kind via Kamal transform. Ahmad and Singh [33] introduced an approach for treating linear Volterra integral equations of the first kind using Galerkin method with Hermite and Chebyshev polynomials. Researchers [34] focus on collocation technique for solving Volterra integro-differential equation. In [35] using Rishi transform for solving second kind Volterra Integral Equation. Nevertheless, numerous other writers have addressed these integral or integro-differential equations using this concept or comparable transformations. The numerical methods, not only including different bases and schemes, provide efficient solutions for these equations in practical applications [36], but also include different bases and schemes, providing efficient solutions for the fractional equations in [37,38,39,40,41].

This article presents a straightforward and effective method for solving a system of second kind linear VIEs using Laguerre polynomials, which results in a system of linear algebraic equations. The results are contrasted with those of another method to show how applicable and successful this strategy is. A system of linear Volterra integral equations of the second kind with regular kernels is solved using this formulation [28,42,43].

This article emphasises that the benefits of the proposed method stem from its unique mathematical properties, which make it especially useful for a particular class of problems, particularly those described on semi-infinite domains and displaying exponential decay. With respect to the weight function e^{-u}, Laguerre polynomials show orthogonality over the interval $[0,\infty)$, providing a natural and ideal framework for simulating processes such as damping, diffusion, and relaxation without the need for artificial domain reduction. They are more effective than the other approaches like Chebyshev spectral or block-pulse functions that are often restricted to truncated finite domains because of their intrinsic exponential decay, which ensures that solutions stay confined and physically realistic as they extend to infinity. They are excellent at calculations as well. A Laguerre series expansion achieves spectral convergence for smooth, decaying functions, meaning that very few terms yield highly accurate approximations. The intrinsic exponential decay of the Laguerre basis ensures that numerical solutions for second kind Volterra systems remain physically realistic as they extend toward infinity, a property crucial for applications in viscoelasticity and population dynamics.

The rest of this document is structured as follows: A brief summary of the basic characteristics and recurrence relations of Laguerre polynomials is given in Section 2. The function approximation scheme and the derivation of the operational matrix are presented in Section 3. The suggested methodology is described in Section 4, which shows how the system of Volterra integral equations is converted into a system of linear algebraic equations that can be solved using the Gauss elimination method. In order to verify the method's spectral convergence and computing efficiency, Section 5 offers a number of numerical examples, including comparisons with previous research. The study is finally concluded in Section 6, which includes a summary of the results, a discussion of the limitations, and suggestions for further research.

2 Laguerre Polynomials

This section provides a brief overview of the concepts, notations, and important relationships that will be used later in this text.

Definition 1. *Laguerre polynomials (LPs), $L_N(u)$, are polynomials of degree $N \in \mathbb{N}_0$ defined on the interval $[0,\infty)$ as the following relation:*

$$L_N(u) = \sum_{r=0}^{N} \frac{(-1)^r N!}{(r!)^2 (N-r)!} u^r, \qquad u \in [0,\infty), \qquad (2)$$

where N and r denote the degree and the index for the LPs, respectively.

The first few LPs are presented below:

$$L_0(u) = 1,$$
$$L_1(u) = 1-u,$$
$$L_2(u) = \frac{1}{2}(2-4u+u^2),$$
$$L_3(u) = \frac{1}{6}(6-18u+9u^2-u^3),$$
$$L_4(u) = \frac{1}{24}(24-96u+72u^2-16u^3+u^4).$$

Moreover, we can use the following recurrence relation:

$$L_{N+2}(u) = \tfrac{(2N+3-u)}{N+2} L_{N+1}(u) - \tfrac{N+1}{N+2} L_N(u), \quad N \geq 0,\ u \in [0,\infty), \tag{3}$$

where $L_0(u) = 1,\ L_1(u) = 1-u$.
The generating function for $L_N(u)$ likewise follows,

$$\sum_{N=0}^{\infty} x^N L_N(u) = \frac{1}{1-x} \exp^{\frac{-xu}{1-x}}.$$

3 Approximate Function Based on Laguerre Polynomials

In this section, we approximate the unknown functions $Q_1(u)$ and $Q_2(u)$ in Eq. (1) using LPs of degree N as the following:

$$\begin{aligned} Q_1(u) \approx Q_{1,N}(u) &= \tau_0 L_0(u) + \tau_1 L_1(u) + \cdots + \tau_N L_N(u) \\ &= \sum_{r=0}^{N} \tau_r L_r(u), \\ Q_2(u) \approx Q_{2,N}(u) &= \eta_0 L_0(u) + \eta_1 L_1(u) + \cdots + \eta_N L_N(u) \\ &= \sum_{r=0}^{N} \eta_r L_r(u), \end{aligned} \tag{4}$$

where τ_r and η_r are unknown coefficients.

According to Eq. (2), the Laguerre basis polynomials of $N-th$ degree are represented by the function $\{L_r(u)\}_{r=0}^{N}$. The unknown Laguerre coefficients that are calculated later are τ_r and η_r $(r = 0, 1, \cdots, N)$.

Eq. (4) can be expressed as a matrix forms dot products as follows:

$$\begin{aligned} Q_{1,N}(u) &= \begin{bmatrix} \tau_0 & \tau_1 & \cdots & \tau_N \end{bmatrix} . [L_0(u), L_1(u), \cdots, L_N(u)]^T, \\ Q_{2,N}(u) &= \begin{bmatrix} \eta_0 & \eta_1 & \cdots & \eta_N \end{bmatrix} . [L_0(u), L_1(u), \cdots, L_N(u)]^T. \end{aligned} \tag{5}$$

Eq. (5) can be expressed as follows:

$$\begin{aligned} Q_{1,N}(u) &= \begin{bmatrix} 1 & u & u^2 & \cdots & u^N \end{bmatrix} \begin{bmatrix} \xi_{0,0} & \xi_{0,1} & \xi_{0,2} & \cdots & \xi_{0,N} \\ 0 & \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,N} \\ 0 & 0 & \xi_{2,2} & \cdots & \xi_{2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \xi_{N,N} \end{bmatrix} \begin{bmatrix} \tau_0 \\ \tau_1 \\ \tau_2 \\ \vdots \\ \tau_N \end{bmatrix}, \\ Q_{2,N}(u) &= \begin{bmatrix} 1 & u & u^2 & \cdots & u^N \end{bmatrix} \begin{bmatrix} \xi_{0,0} & \xi_{0,1} & \xi_{0,2} & \cdots & \xi_{0,N} \\ 0 & \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,N} \\ 0 & 0 & \xi_{2,2} & \cdots & \xi_{2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \xi_{N,N} \end{bmatrix} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{bmatrix}, \end{aligned} \tag{6}$$

where the components $\xi_{i,j}$ of the square matrix are represented by:

$$(\xi_{i,j})_{0 \leq i,j \leq N} = \begin{cases} (-1)^i \binom{j}{i} \frac{1}{i!}, & i \leq j, \\ 0, & otherwise. \end{cases} \tag{7}$$

The operational matrices for $N = 1$ and 2 are shown respectively:

$$\begin{aligned} Q_{1,1}(u) &= \begin{bmatrix} 1 & u \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tau_0 \\ \tau_1 \end{bmatrix}, \\ Q_{1,2}(u) &= \begin{bmatrix} 1 & u & u^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \tau_0 \\ \tau_1 \\ \tau_2 \end{bmatrix}. \end{aligned} \tag{8}$$

$$\begin{aligned} Q_{2,1}(u) &= \begin{bmatrix} 1 & u \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \eta_0 \\ \eta_1 \end{bmatrix}, \\ Q_{2,2}(u) &= \begin{bmatrix} 1 & u & u^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \end{bmatrix}. \end{aligned} \tag{9}$$

4 Methodology

In this section, the LPs are applied to solve the second kind linear VIEs system.
By substituting Eq. (4) into Eq. (1), we get

$$\begin{aligned} \sum_{r=0}^{N} \tau_r L_r(u) &= g_1(u) + \int_0^u k_{11}(u,v) \sum_{r=0}^{N} \tau_r L_r(v)\, dv \\ &\quad + \int_0^u k_{12}(u,v) \sum_{r=0}^{N} \eta_r L_r(v)\, dv, \\ \sum_{r=0}^{N} \eta_r L_r(u) &= g_2(u) + \int_0^u k_{21}(u,v) \sum_{r=0}^{N} \tau_r L_r(v)\, dv \\ &\quad + \int_0^u k_{22}(u,v) \sum_{r=0}^{N} \eta_r L_r(v)\, dv, \end{aligned} \tag{10}$$

by using Eq. (5), then Eq. (10) becomes:

$$\begin{bmatrix}\tau_0 & \tau_1 & \cdots & \tau_N\end{bmatrix}.\begin{bmatrix}L_0(u)\\ L_1(u)\\ \vdots\\ L_N(u)\end{bmatrix}=g_1(u)$$
$$+\int_0^u k_{11}(u,v)\begin{bmatrix}\tau_0 & \tau_1 & \cdots & \tau_N\end{bmatrix}.\begin{bmatrix}L_0(v)\\ L_1(v)\\ \vdots\\ L_N(v)\end{bmatrix}dv$$
$$+\int_0^u k_{12}(u,v)\begin{bmatrix}\eta_0 & \eta_1 & \cdots & \eta_N\end{bmatrix}.\begin{bmatrix}L_0(v)\\ L_1(v)\\ \vdots\\ L_N(v)\end{bmatrix}dv, \tag{11}$$
$$\begin{bmatrix}\eta_0 & \eta_1 & \cdots & \eta_N\end{bmatrix}.\begin{bmatrix}L_0(u)\\ L_1(u)\\ \vdots\\ L_N(u)\end{bmatrix}=g_2(u)$$
$$+\int_0^u k_{21}(u,v)\begin{bmatrix}\tau_0 & \tau_1 & \cdots & \tau_N\end{bmatrix}.\begin{bmatrix}L_0(v)\\ L_1(v)\\ \vdots\\ L_N(v)\end{bmatrix}dv$$
$$+\int_0^u k_{22}(u,v)\begin{bmatrix}\eta_0 & \eta_1 & \cdots & \eta_N\end{bmatrix}.\begin{bmatrix}L_0(v)\\ L_1(v)\\ \vdots\\ L_N(v)\end{bmatrix}dv.$$

Now, using Eq. (6), in Eq. (11) to obtain the following

$$\begin{bmatrix}1 & u & u^2 & \cdots & u^N\end{bmatrix}.\begin{bmatrix}\xi_{0,0} & \xi_{0,1} & \xi_{0,2} & \cdots & \xi_{0,N}\\ 0 & \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,N}\\ 0 & 0 & \xi_{2,2} & \cdots & \xi_{2,N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \xi_{N,N}\end{bmatrix}.\begin{bmatrix}\tau_0\\ \tau_1\\ \tau_2\\ \vdots\\ \tau_N\end{bmatrix}=g_1(u)$$
$$+\int_0^u k_{11}(u,v)\begin{bmatrix}1 & v & v^2 & \cdots & v^N\end{bmatrix}.\begin{bmatrix}\xi_{0,0} & \xi_{0,1} & \xi_{0,2} & \cdots & \xi_{0,N}\\ 0 & \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,N}\\ 0 & 0 & \xi_{2,2} & \cdots & \xi_{2,N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \xi_{N,N}\end{bmatrix}.$$
$$\begin{bmatrix}\tau_0\\ \tau_1\\ \tau_2\\ \vdots\\ \tau_N\end{bmatrix}dv+\int_0^u k_{12}(u,v)\begin{bmatrix}1, v, v^2, \cdots, v^N\end{bmatrix}.$$
$$\begin{bmatrix}\xi_{0,0} & \xi_{0,1} & \xi_{0,2} & \cdots & \xi_{0,N}\\ 0 & \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,N}\\ 0 & 0 & \xi_{2,2} & \cdots & \xi_{2,N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \xi_{N,N}\end{bmatrix}.\begin{bmatrix}\eta_0\\ \eta_1\\ \eta_2\\ \vdots\\ \eta_N\end{bmatrix}dv. \tag{12}$$

$$\begin{bmatrix}1 & u & u^2 & \cdots & u^N\end{bmatrix}.\begin{bmatrix}\xi_{0,0} & \xi_{0,1} & \xi_{0,2} & \cdots & \xi_{0,N}\\ 0 & \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,N}\\ 0 & 0 & \xi_{2,2} & \cdots & \xi_{2,N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \xi_{N,N}\end{bmatrix}.\begin{bmatrix}\eta_0\\ \eta_1\\ \eta_2\\ \vdots\\ \eta_N\end{bmatrix}=g_2(u)$$
$$+\int_0^u k_{21}(u,v)\begin{bmatrix}1 & v & v^2 & \cdots & v^N\end{bmatrix}.\begin{bmatrix}\xi_{0,0} & \xi_{0,1} & \xi_{0,2} & \cdots & \xi_{0,N}\\ 0 & \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,N}\\ 0 & 0 & \xi_{2,2} & \cdots & \xi_{2,N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \xi_{N,N}\end{bmatrix}.\begin{bmatrix}\tau_0\\ \tau_1\\ \tau_2\\ \vdots\\ \tau_N\end{bmatrix}dv$$
$$+\int_0^u k_{22}(u,v)\begin{bmatrix}1 & v & v^2 & \cdots & v^N\end{bmatrix}.\begin{bmatrix}\xi_{0,0} & \xi_{0,1} & \xi_{0,2} & \cdots & \xi_{0,N}\\ 0 & \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,N}\\ 0 & 0 & \xi_{2,2} & \cdots & \xi_{2,N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \xi_{N,N}\end{bmatrix}.\begin{bmatrix}\eta_0\\ \eta_1\\ \eta_2\\ \vdots\\ \eta_N\end{bmatrix}dv. \tag{13}$$

The unknown numerical solution coefficients $[\tau_0,\tau_1,\cdots,\tau_N]$ and $[\eta_0,\eta_1,\cdots,\eta_N]$ may now be found by simplifying Eqs. (12) and (13) by selecting the collocation points in the interval $[a,b]$, $u_i=a+\frac{b-a}{N}i, i=0,1,\ldots,N$. As a result, Eqs. (12) and (13) become a system of linear algebraic equations with $(2N+2)$ unknown coefficients. These coefficients have unique solutions, and the "Gauss elimination method" can solve this system. These coefficients are substituted into Eq. (4) to give the approximate numerical solution for the system given in Eq. (1).

5 Numerical Examples

This section presents numerical results to verify the accuracy, efficacy, flexibility of use, and simplicity of the presented method. Furthermore, we will discuss the exact solutions, the approximate solutions, and the absolute errors (AEs). We calculated the AEs respectively as follows:

$$\begin{aligned}AE_1&=|Q_1(u_i)-Q_{1,N}(u_i)|,\\ AE_2&=|Q_2(u_i)-Q_{2,N}(u_i)|, N\in\mathbb{Z}^+,\ u_i\in[0,1],\end{aligned} \tag{14}$$

In this context, u_i represents the collocation points, while $Q_1(u_i)$ and $Q_2(u_i)$ denotes the exact solutions and $Q_{1,N}(u_i)$ and $Q_{2,N}(u_i)$ are signify the numerical solutions.

We will use the method described in section 4 to solve all the problems in this section. we calculate the AE estimation for Q_1 and Q_2 for $N=10$. Then to make results easier to understand, we will also present a variety of numerical results in Tables and Figures.

We suppose that Eq. (1) admits a unique continuous solution for suitable choices of $g=[g_1(u),g_2(u)]^T$. Accordingly, the system in Eq. (1) can be expressed in operator form as a second kind equation, $g=(I-K)Q$, where K represents an integral operator, I is the identity operator, and

$Q = [Q_1(u), Q_2(u)]^T$. To ensure the existence and uniqueness of the solution to (1), it is standard practice to place compactness criteria on the operator K (see [11], Theorem 4), and these assumptions are upheld throughout this work. We also evaluate the precision of the proposed method in approximating the solution of Eq. (1). To illustrate its accuracy, two relative error measures [11] are computed according to the specified formulas.

$$E_1 = \frac{1}{100}\sum_{i=1}^{100}\left(g(a_i) - (I-K)Q_m(a_i)\right)_N,$$

which calculates the proximity of Q_m , and

$$E_2 = \sqrt{\frac{\sum_{i=1}^{100}\left(Q_m(a_i - Q(a_i))\right)_N^2}{\sum_{i=1}^{100}\left(Q(a_i)\right)_N^2}},$$

which approximates the $L^2((0,1);\mathbb{R}^N)$ relative error of the approximation Q_m to the exact solution $Q(u)$, evaluated at one hundred distinct randomly chosen points $a_1, \ldots, a_{100} \in [0,1]$.

Example 1.[11]
Consider the linear VIE of the formula:

$$\begin{aligned}
Q_1(u) &= u - \frac{u^5}{12} + \int_0^u (u-v)^3 Q_1(v)dv \\
&\quad + \int_0^u (u-v)^2 Q_2(v)\,dv, \\
Q_2(u) &= u^2 - \frac{u^6}{20} + \int_0^u (u-v)^4 Q_1(v)dv \\
&\quad + \int_0^u (u-v)^3 Q_2(v)\,dv,
\end{aligned} \tag{15}$$

where the exact solution is given by $Q_1(u) = u$, $Q_2(u) = u^2$.

Table 1: Our computational AE estimation for Example 1 at $N = 10$ for Q_1 provided by the presented method.

u_i	$Q_1(u_i)$	$Q_{1,N}(u_i)$	AE_1
0	0	-3.5508×10^{-18}	3.5508×10^{-18}
0.1	0.10000000	0.10000000	1.2490×10^{-16}
0.2	0.20000000	0.20000000	0
0.3	0.30000000	0.30000000	1.6653×10^{-16}
0.4	0.40000000	0.40000000	5.5511×10^{-17}
0.5	0.50000000	0.50000000	1.1102×10^{-16}
0.6	0.60000000	0.60000000	2.2204×10^{-16}
0.7	0.70000000	0.70000000	4.4409×10^{-16}
0.8	0.80000000	0.80000000	4.4409×10^{-16}
0.9	0.90000000	0.90000000	4.4409×10^{-16}
1.0	1.00000000	1.00000000	4.4409×10^{-16}

Table 2: Our computational AE estimation for Example 1 at $N = 10$ for Q_2 provided by the presented method.

u_i	$Q_2(u_i)$	$Q_{2,N}(u_i)$	AE_2
0	0	4.2284×10^{-17}	4.2284×10^{-17}
0.1	0.01000000	0.01000000	7.1471×10^{-16}
0.2	0.04000000	0.03999999	1.0200×10^{-15}
0.3	0.09000000	0.09000000	4.1633×10^{-16}
0.4	0.16000000	0.16000000	2.7756×10^{-17}
0.5	0.25000000	0.25000000	7.2164×10^{-16}
0.6	0.36000000	0.36000000	6.6613×10^{-16}
0.7	0.49000000	0.49000000	3.8858×10^{-16}
0.8	0.64000000	0.64000000	8.8818×10^{-16}
0.9	0.81000000	0.81000000	4.4409×10^{-16}
1.0	1.00000000	0.99999999	2.2204×10^{-16}

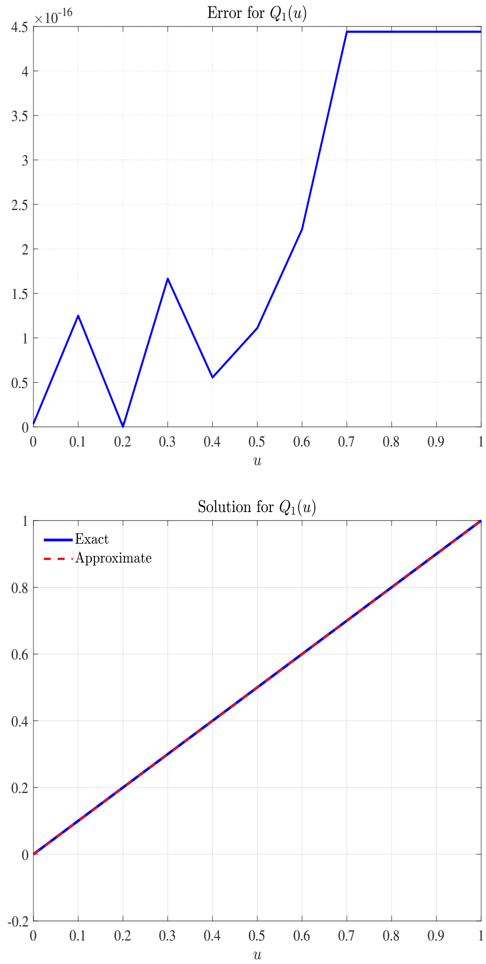

Fig. 1: For Example 1 with $N = 10$, the exact, approximate solutions, and the AE for Q_1.

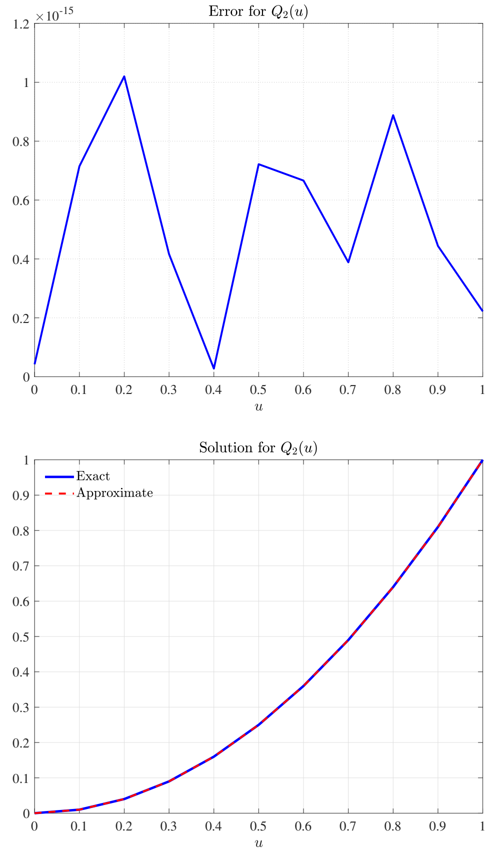

Fig. 2: For Example 1 with $N = 10$, the exact, approximate solutions, and the AE for Q_2.

The error norms produced by the proposed method for various values of $N+1$ are presented in Table 1 for Q_1 and Table 2 for Q_2. The AE outcomes for Example 1 for $N = 10$ are shown in Table 1 and Table 2. The AE_1 and the analytical and numerical solutions for Q_1 are plotted in Figure 1 at $N = 10$, where the AE_2 and the analytical and numerical solutions for Q_2 are plotted in Figure 2. It is clear that the numerical solution is almost similar to the exact solution.

Table 3: Our computational relative error estimations for Example 1 at $N = 5,10,20,30,40,50$ provided by the presented method are compared with the relative error estimations provided by the method in [11].

N	Method in [11]		Our proposed method	
	E_1	E_2	E_1	E_2
5	2.19×10^{-2}	3.11×10^{-2}	4.84×10^{-16}	6.93×10^{-16}
10	3.60×10^{-3}	4.80×10^{-3}	3.84×10^{-16}	5.47×10^{-16}
20	6.27×10^{-4}	8.30×10^{-4}	1.52×10^{-15}	2.40×10^{-15}
30	2.07×10^{-4}	3.03×10^{-4}	1.20×10^{-15}	1.87×10^{-15}
40	1.02×10^{-4}	1.25×10^{-4}	2.18×10^{-14}	3.23×10^{-14}
50	6.45×10^{-5}	9.28×10^{-5}	7.50×10^{-15}	1.08×10^{-14}

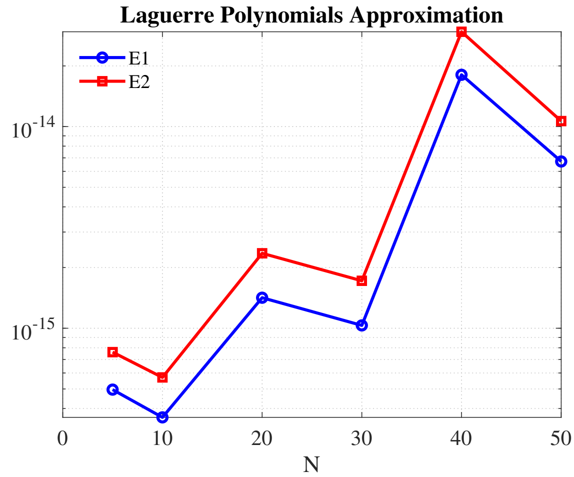

Fig. 3: For Example 1 with $N = 5,10,20,30,40,50$, the two relative errors.

Table 3 presents a comparison at $N = 5,10,20,30,40,50$ between E_1 and E_2 obtained using the method in [11] and the proposed approach. The results clearly indicate that the suggested method outperforms the one reported in [11], as evidenced by the two relative error measures that evaluate the precision of the proposed method in approximating the solution of Example 1.

Example 2.[12] Consider the linear VIE of the formula:

$$\begin{aligned} Q_1(u) &= 1-u^2+\int_0^u Q_2(v)dv, \\ Q_2(u) &= u+\int_0^u Q_1(v)dv, \end{aligned} \tag{16}$$

where the exact solution is given by $Q_1(u) = 1$, $Q_2(u) = 2u$.

Table 4: Our computational AE estimation for Example 2 at $N = 10$ for Q_1 provided by the presented method.

u_i	$Q_1(u_i)$	$Q_{1,N}(u_i)$	AE_1
0	1.00000000	1.00000000	0
0.1	1.00000000	1.00000000	0
0.2	1.00000000	1.00000000	0
0.3	1.00000000	1.00000000	0
0.4	1.00000000	1.00000000	2.2204×10^{-16}
0.5	1.00000000	1.00000000	0
0.6	1.00000000	1.00000000	0
0.7	1.00000000	0.99999999	1.1102×10^{-16}
0.8	1.00000000	1.00000000	2.2204×10^{-16}
0.9	1.00000000	1.00000000	2.2204×10^{-16}
1.0	1.00000000	1.00000000	2.2204×10^{-16}

Table 5: Our computational AE estimation for Example 2 at $N = 10$ for Q_2 provided by the presented method.

u_i	$Q_2(u_i)$	$Q_{2,N}(u_i)$	AE_2
0	0	-6.6895×10^{-17}	6.6895×10^{-17}
0.1	0.20000000	0.20000000	2.7756×10^{-17}
0.2	0.40000000	0.40000000	2.2204×10^{-16}
0.3	0.60000000	0.59999999	1.1102×10^{-16}
0.4	0.80000000	0.80000000	1.1102×10^{-16}
0.5	1.00000000	1.00000000	2.2204×10^{-16}
0.6	1.20000000	1.20000000	2.2204×10^{-16}
0.7	1.40000000	1.40000000	0
0.8	1.60000000	1.60000000	2.2204×10^{-16}
0.9	1.80000000	1.80000000	2.2204×10^{-16}
1.0	2.00000000	2.00000000	0

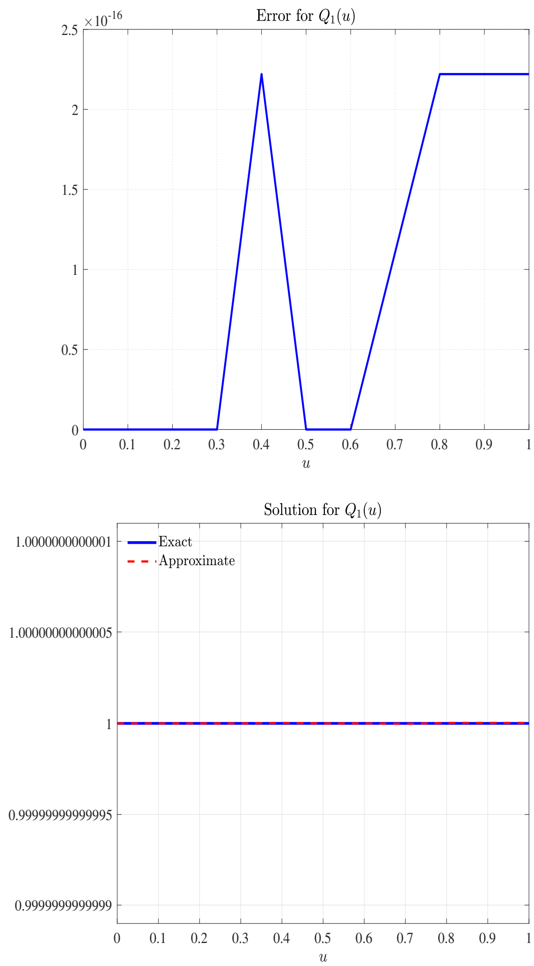

Fig. 4: For Example 2 with $N = 10$, the exact, approximate solutions, and the AE for Q_1.

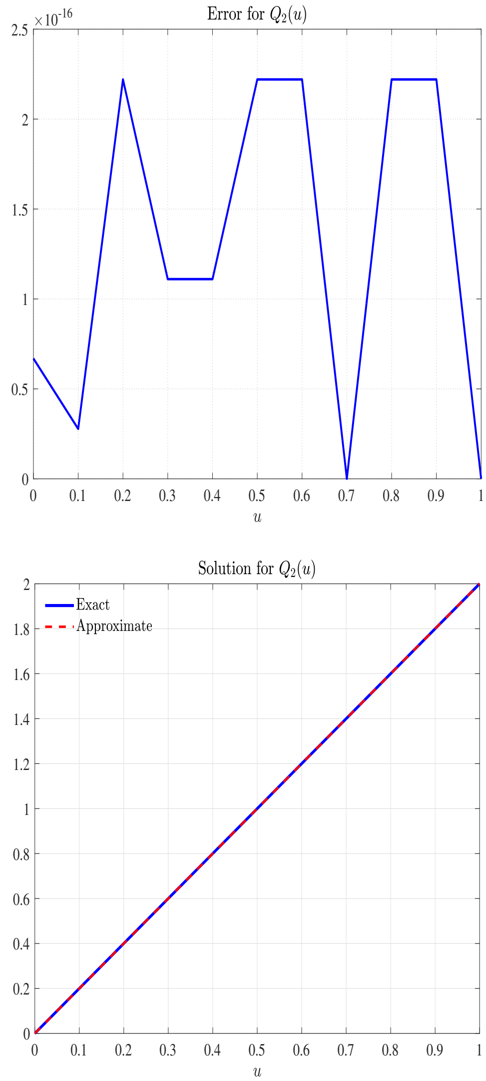

Fig. 5: For Example 2 with $N = 10$, the exact, approximate solutions, and the AE for Q_2.

The error norms produced by the proposed method for various values of $N+1$ are presented in Table 4 for Q_1 and Table 4 for Q_2. The AE outcomes for Example 2 for $N = 10$ are shown in Table 4 and Table 5. The AE_1 and the analytical and numerical solutions for Q_1 are plotted in Figure 4 at $N = 10$, where the AE_2 and the analytical and numerical solutions for Q_2 are plotted in Figure 5. It is clear that the numerical solution is almost similar to the exact solution.

Where $f_1^*(t)$ and $f_2^*(t)$ in Table 6 refers to the approximate solutions in [12]. Table 6 presents a comparison at $N = 40$ between the exact solutions and the approximate solutions obtained using the method in [12] and the proposed approach. The results clearly indicate

Table 6: Our computational the exact and approximate solutions for Example 2 at $N = 40$ for $Q_1(u)$ and $Q_2(u)$ provided by the presented method are compared with the exact and approximate solutions provided by the method in [12].

u_i	Exact		Method in [12]		Our proposed method	
	$Q_1(u_i)$	$Q_2(u_i)$	$f_1^*(t)$	$f_2^*(t)$	$Q_{1,N}(u_i)$	$Q_{2,N}(u_i)$
0.1	1.00000	0.20000	1.00000	0.20001	1.00000	0.20000
0.2	1.00000	0.40000	1.00000	0.40002	1.00000	0.40000
0.3	1.00000	0.60000	1.00000	0.60003	1.00000	0.60000
0.4	1.00000	0.80000	1.00001	0.80004	1.00000	0.80000
0.5	1.00000	1.00000	1.00001	0.80005	1.00000	1.00000
0.6	1.00000	1.20000	1.00002	1.20007	1.00000	1.20000
0.7	1.00000	1.40000	1.00003	1.40008	1.00000	1.40000
0.8	1.00000	1.60000	1.00004	1.60009	1.00000	1.60000
0.9	1.00000	1.80000	1.00005	1.80011	1.00000	1.80000
1.0	1.00000	2.00000	1.00006	2.00012	1.00000	2.00000

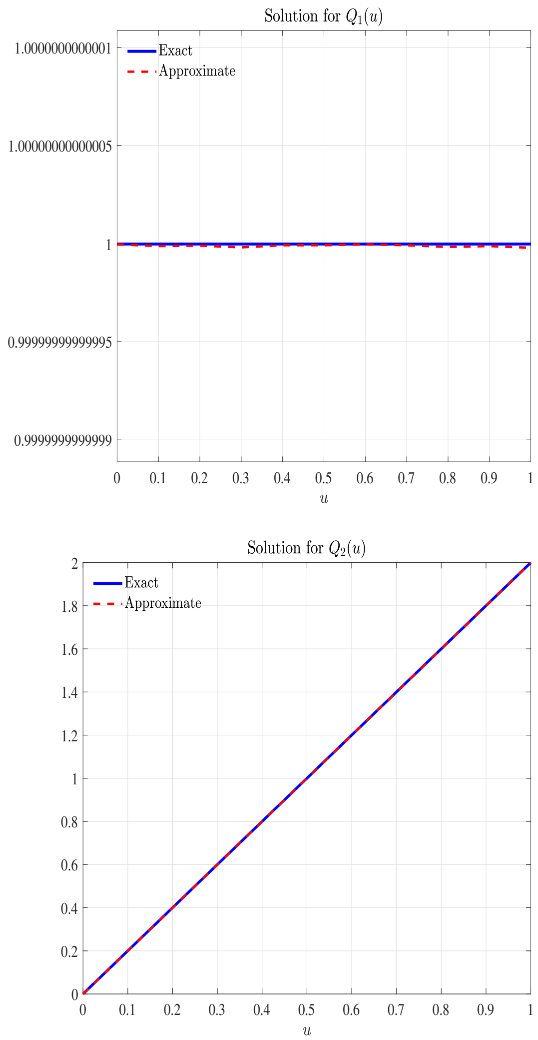

Fig. 6: For Example 2 with $N = 40$, the exact and the approximate solutions for Q_1 and Q_2.

Table 7: Our computational AE estimation for Example 3 at $N = 10$ for Q_1 provided by the presented method.

u_i	$Q_1(u_i)$	$Q_{1,N}(u_i)$	AE_1
0	1.00000000	1.00000000	1.2212×10^{-14}
0.1	1.10517092	1.10517092	1.3101×10^{-14}
0.2	1.22140276	1.22140276	1.5987×10^{-14}
0.3	1.34985881	1.34985881	1.1902×10^{-13}
0.4	1.49182470	1.49182470	1.4433×10^{-14}
0.5	1.64872127	1.64872127	1.2657×10^{-14}
0.6	1.82211880	1.82211880	3.4417×10^{-14}
0.7	2.01375271	2.01375271	5.0626×10^{-14}
0.8	2.22554093	2.22554093	8.3489×10^{-14}
0.9	2.45960311	2.45960311	6.6169×10^{-14}
1.0	2.71828183	2.71828183	3.1086×10^{-15}

that the suggested method outperforms the one reported in [12], with the numerical solution closely matching the exact solution.

Example 3.[12] Consider the linear VIE of the formula:

$$
\begin{aligned}
Q_1(u) &= 1 - \frac{u^2}{2} + \int_0^u Q_1(v)dv + \int_0^u ve^v Q_2(v)dv, \\
Q_2(u) &= 1 + \frac{u^2}{2} + \int_0^u -ve^{-v} Q_1(v)dv - \int_0^u Q_2(v))dv,
\end{aligned} \tag{17}
$$

where the exact solution is given by $Q_1(u) = e^u$, $Q_2(u) = e^{-u}$.

Table 8: Our computational AE estimation for Example 3 at $N = 10$ for Q_2 provided by the presented method.

u_i	$Q_2(u_i)$	$Q_{2,N}(u_i)$	AE_2
0	1.00000000	1.00000000	3.9968×10^{-15}
0.1	0.90483742	0.90483742	4.6629×10^{-15}
0.2	0.81873075	0.81873075	2.2204×10^{-14}
0.3	0.74081822	0.74081822	1.7764×10^{-15}
0.4	0.67032005	0.67032005	1.4211×10^{-14}
0.5	0.60653066	0.60653066	5.8842×10^{-15}
0.6	0.54881164	0.54881164	3.7748×10^{-15}
0.7	0.49658530	0.49658530	8.8818×10^{-15}
0.8	0.44932896	0.44932896	3.8025×10^{-14}
0.9	0.40656966	0.40656966	2.3204×10^{-14}
1.0	0.36787944	0.36787944	1.7653×10^{-14}

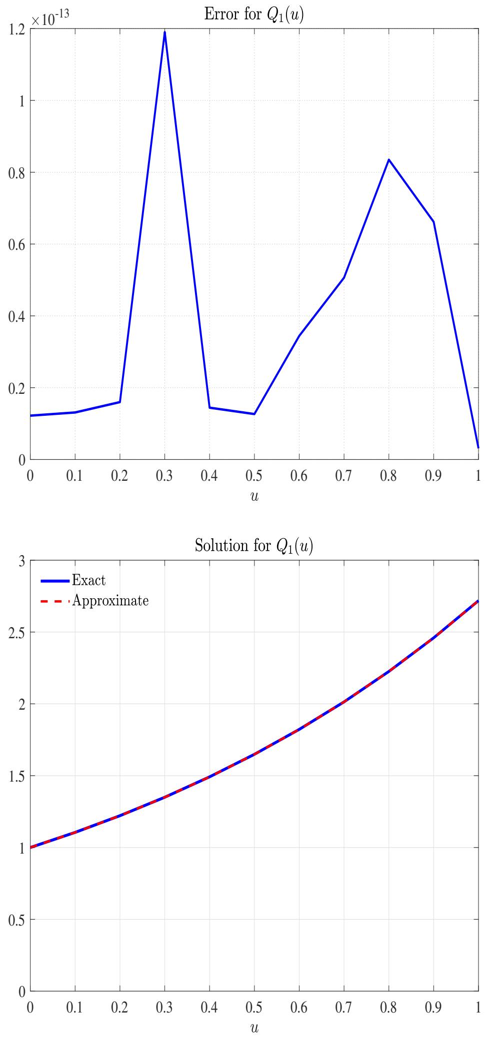

Fig. 7: For Example 3 at $N = 10$, the exact, approximate solutions, and the AE for Q_1.

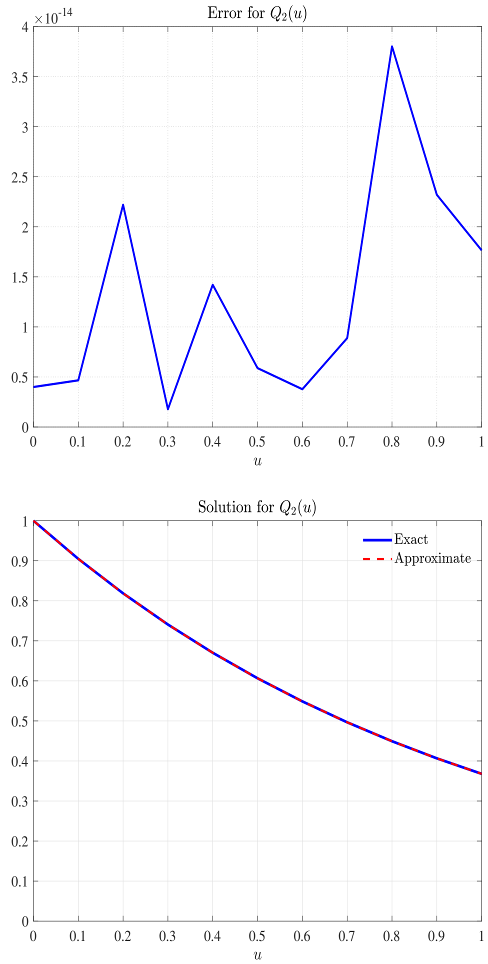

Fig. 8: For Example 3 at $N = 10$, the exact, approximate solutions, and the AE for Q_2.

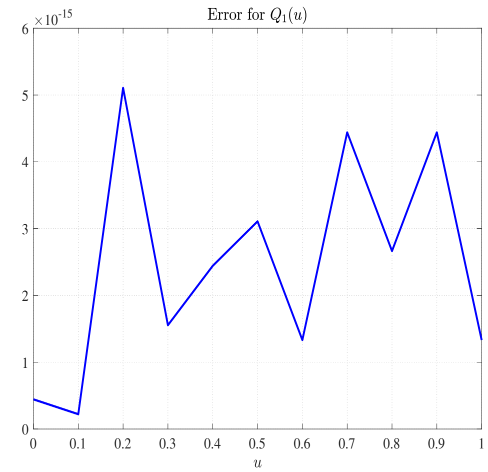

Fig. 9: For Example 3 at $N = 25$, the AE for Q_1.

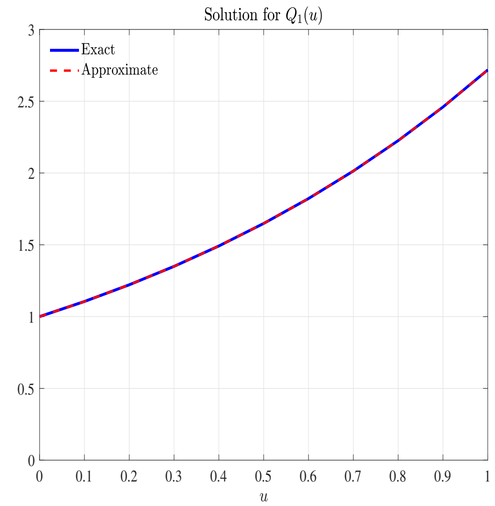

Fig. 10: For Example 3 at $N = 25$, the exact and the approximate solutions for Q_1.

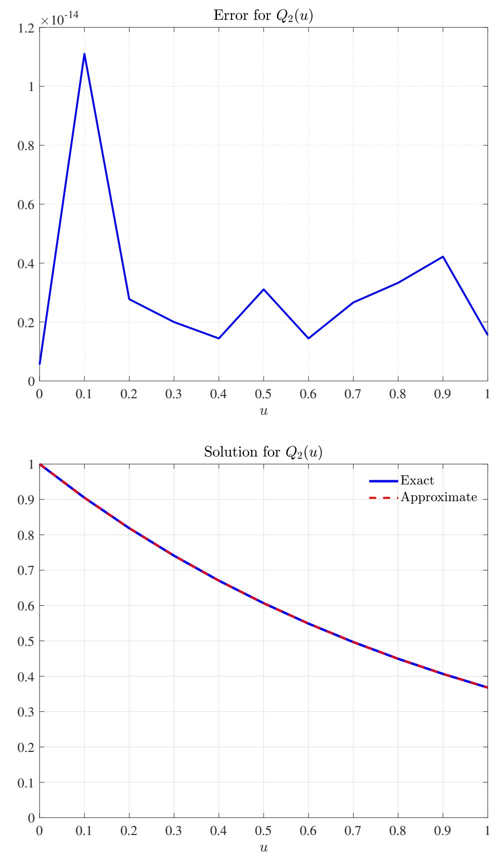

Fig. 11: For Example 3 at $N = 25$, the exact, approximate solutions, and the AE for Q_2.

The error norms produced by the presented method for various values of $N+1$ are presented in Table 7 for Q_1 and Table 8 for Q_2. The AE outcomes for Example 3 at $N = 10$ are shown in Table 7 and Table 8. The AE_1, the exact and the approximate solutions for Q_1 are plotted in Figure 7 at $N = 10$, and Figures 9, 10 at $N = 25$. The AE_2, the exact and the approximate solutions for Q_2 are plotted in Figure 8 at $N = 10$ and Figure 11 at $N = 25$. It is clear that the numerical solution is almost similar to the exact solution.

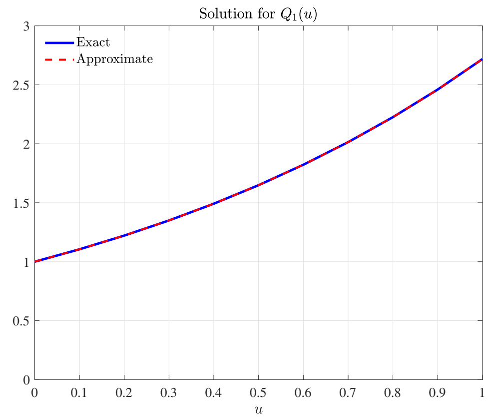

Fig. 12: For Example 3 with $N = 40$, the exact and the approximate solutions for Q_1.

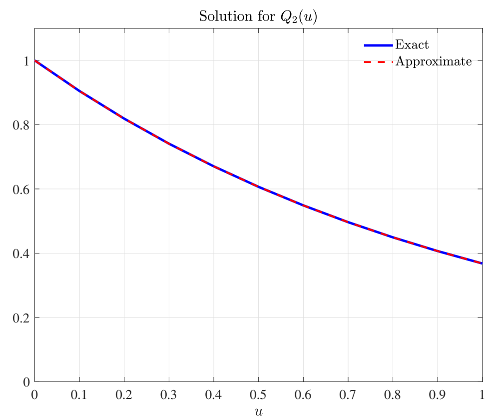

Fig. 13: For Example 3 with $N = 40$, the exact and the approximate solutions for Q_2.

Table 9 presents a comparison at $N = 40$ between the exact solution and the approximate solutions obtained using the method in [12] and the proposed method. The results clearly indicate that the suggested method

Table 9: Our computational the exact and approximate solutions for Example 3 at $N = 40$ for $Q_1(u)$ and $Q_2(u)$ provided by the presented method are compared with the exact and approximate solutions provided by the method in [12].

u_i	Exact		Method in [12]		Our proposed method	
	$Q_1(u_i)$	$Q_2(u_i)$	$Q_{1,N}(u_i)$	$Q_{2,N}(u_i)$	$Q_{1,N}(u_i)$	$Q_{2,N}(u_i)$
0.1	1.10517	0.90484	1.10519	0.90484	1.10517	0.90484
0.2	1.22140	0.81873	1.22144	0.81874	1.22140	0.81873
0.3	1.34986	0.74082	1.34992	0.74083	1.34986	0.74082
0.4	1.49182	0.67032	1.49191	0.67033	1.49182	0.67032
0.5	1.64872	0.60653	1.64884	0.60654	1.64872	0.60653
0.6	1.82212	0.54881	1.82227	0.54881	1.82212	0.54881
0.7	2.01375	0.49659	2.01394	0.49658	2.01375	0.49659
0.8	2.22554	0.44933	2.22577	0.44931	2.22554	0.44933
0.9	2.45960	0.40657	2.45987	0.40655	2.45960	0.40657
1.0	2.71828	0.36788	2.71859	0.36784	2.71828	0.36788

outperforms the one reported in [12], with the numerical solution closely matching the exact solution.

6 Conclusions

In this paper, an efficient numerical method based on LPs is effectively applied to solve systems of linear VIEs of the second kind. Operational matrix method was used to change the VIEs into a system of linear algebraic equations, which were solved using the Gauss elimination method. The numerical experiments performed by MATLAB show that the proposed method is quite accurate and computationally efficient as compared to the methods available in the literature. However, despite its advantages (spectral convergence, natural fit for semi-infinite domains), the present approach is mostly limited to linear systems and may incur increased computational costs when high-order polynomials are needed for non-smooth or highly oscillatory solutions. Future work will be focused on the extension of this method to nonlinear Volterra systems and integro-differential equations. Another purpose of our work is to study the properties and the orthogonality of Laguerre functions with non-integer order, which is an interesting perspective for the modeling of anomalous diffusion and the solution of problems in the frame of fractional calculus.

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