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Fractional Order Optimal Control of Tuberculosis Model with Exogenous Reinfection

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Abstract: As a general formulation for the SEIT epidemiological model, an exogenous reinfection system of fractional order TB model is presented in this paper. Control, which stands for the prevention of external reinfection, is used to successfully minimize the number of infectious individuals who are actively contagious and latently infectious. Based on the fractional-order Caputo derivative, this fractional mathematical formulation has been theoretically examined using Pontryagin's maximum principle. The generalized Euler method is used in the proposed Forward-Backward sweep methodology to solve the resulting optimality problem numerically. Numerical data demonstrate the efficacy of the optimization procedures.

Keywords: Optimal control, Tuberculosis, TB model with exogenous reinfection, Mathematical model.

1 Introduction

A chronic infectious disease called tuberculosis (TB) is brought on by the bacillus Mycobacterium tuberculosis. The organs commonly infected are the lungs, but it can also impact other sites. The symptoms include coughing up blood (Haemoptysis), chest discomfort, shortness of breath, fever, loss of appetite, and night sweats. Airborne droplets are the main cause of transmission of TB. When contagious individuals sneeze, cough, talk, laugh, or spit, M. tuberculosis droplets are ejected into the air. As a result, healthy people may inhale the bacterium and become infected [1]. The top 10 causes of death worldwide continue to include tuberculosis (TB), despite technological advancements. According to recent research, TB incidence is increasing globally, and the human immunodeficiency virus's connection to the disease is the primary reason for this (HIV). There were 10 million new cases of TB worldwide in 2017, and the disease claimed the lives of 1.6 million individuals (including 0.3 million with HIV) [2].

The multiplication of bacteria is halted by man's natural immunity after the infection. This results in only 10% of the people acquiring active TB, the rest becoming latent individuals. The typical non-infectious incubation period for TB can last anywhere between a few months and several decades [3]. It is important to appropriately categorize the TB condition because treatment and prevention strategies differ so much. Drugs can, of course, be used to treat tuberculosis [4]. However, the majority of high-incidence individuals also just utilize the current TB vaccinations, like BCG. Additionally, there is debate regarding its ability to prevent TB [5]. Over 90% of patients with typical tuberculosis will recover. However, only 50% of patients with multi-drug resistant tuberculosis are cured, and the course of treatment might last up to 18 to 24 months. [2]. Because of this, TB needs to be managed in other ways in addition to being treated.

The study of numerical techniques and the theoretical analysis of fractional differential equations have received increasing amounts of interest in recent years. The real or complex order [6,7,8] can be obtained from conventional integral order calculus by using fractional calculus. There are currently over six ways to define a fractional derivative,

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with Riemann-Liouville and Caputo derivatives being the most widely applied [9]. The values of the unknown function and its integer derivative, which have a clear physical meaning, explain the initial conditions of the time-dependent fractional Caputo derivative [10]. So, in our work, we shall modify Caputo's concept. In past years, we have seen a significant increase in the use of fractional differential equations in a range of fields, including physics, chemistry, electronics, biology, economics, epidemiology, and others [11,12,13].

Mathematical models are significant in many dynamics and controls, including outbreaks of malaria and tuberculosis [14,15]. Furthermore, the large public health burden of tuberculosis needs mathematical models to have a better understanding of the dynamics of transmission and to establish efficient management methods. Sweilam and Mekhlafi, for instance, studied the optimum control approach for a fractional general nonlinear multi-strain TB model in [16]. The conclusion demonstrates that the fractional derivative model may represent more intricate dynamical processes than the integer model and that it is simple to incorporate real-world memory effects. The objective of this study is to use control, which stands for the prevention of external reinfection, to reduce the infected group by limiting interaction between infectious and exposed individuals. In recent years, an increasing number of academics have explored the best regulation of fractional order, for instance in [17,18,19]. The objective of this study is to use control, which stands for the prevention of external reinfection, to reduce the infected group by limiting interaction between infectious and exposed individuals. In this paper, we review the TB model [20] and describe its dynamics using fractional analysis.

The remaining paper is given as follows. Section 2 describes a few of the fractional calculus's preliminary definitions. The mathematical formulation of the fractional optimal control problem is summarised in Section 3. This section also highlighted the necessary conditions for the models optimally. Section 4 presents numerical simulation results to demonstrate the accuracy and effectiveness of the suggested approach. Finally, the paper finishes with some concluding remarks.

2 Optimal Control Problem with Fractional Derivatives

The mathematical formulation of the fractional optimal control model is briefly reviewed in this section. Following that, in section 3, we will put the modeling approach to use to create our new TB model. There are many other types of fractional derivatives, but the Riemann-Liouville derivative and the Caputo derivative are the two that are most frequently employed in engineering and mathematical modeling [21]. With Caputo fractional derivatives, we create the optimization model for this work. Some basic definitions are

Definition 2.1: The Left R-L fractional derivative is given as,

$$ {}_{a}^{L}D_{t}^{\gamma}f(t) = \frac{1}{\Gamma(r-\gamma)}\frac{d^{r}}{dt^{r}}\left[\int_{a}^{t}(t-p)^{r-\gamma-1}f(p)\right]dp, \tag{1} $$

Definition 2.2: The right R-L fractional derivative is given as,

$$ {}_{t}^{R}D_{b}^{\gamma}f(t) = \frac{1}{\Gamma(r-\gamma)}(-1)^{r}\frac{d^{r}}{dt^{r}}\left[\int_{t}^{b}(p-t)^{r-\gamma-1}f(p)\right]dp \tag{2} $$

where γ is a positive real number, r is the integer satisfying $r-1 \leq \gamma < r$, and Γ is the Euler's Gamma function.

Definition2.3: Left Caputo fractional derivative termed as,

$$ {}_{a}^{C}D_{t}^{\gamma}f(t) = \frac{1}{\Gamma(\gamma-r)}\left[\int_{a}^{t}(t-p)^{r-\gamma-1}(\frac{d}{dt})^{r}f(p)\right]dp \tag{3} $$

Definition2.4: Right Caputo fractional derivative termed as,

$$ {}_{t}^{C}D_{b}^{\gamma}f(t) = \frac{1}{\Gamma(r-\gamma)}\left[\int_{t}^{b}(-1)^{r}(p-t)^{r-\gamma-1}(\frac{d}{dt})^{r}f(p)\right]dp \tag{4} $$

where γ is a positive real number, r is the integer satisfying $r-1 < \gamma < r$, and Γ is the Euler Gamma function

3 Fractional Optimal Control Formulation

The optimal control problem with Caputo fractional derivatives has the following formulation, from Agrawal's works in [22]. The main goal is to identify an optimum control u^* which minimises objective functional:

$$J(u) = \int_0^1 M(x,u,t)dt \tag{5}$$

subject to the fractional dynamical constraint

$$ {}_0^C D_t^\alpha x(t) = V(x,u,t) \tag{6}$$

with the initial condition

$$x(0) = x_0 \tag{7}$$

where M(x, u, t) and V(x, u, t) will be the two arbitrary functions and x(t) is our state variable. Taking into account the author's comparable derivation in [22] utilizing the Lagrange multiplier technique, variational calculus, and integration by parts formula for the derivatives of fractional order to produce the necessary conditions or equations (8-10) for the fractional order controlled problem.

$$ {}_0^C D_t^\alpha x(t) = V(x,u,t) \tag{8}$$

$$ {}_0^C D_t^\alpha x(t) = \frac{\partial M}{\partial x} + \lambda \frac{\partial V}{\partial x} \tag{9}$$

$$\frac{\partial M}{\partial x} + \lambda \frac{\partial V}{\partial x} = 0 \tag{10}$$

with

$$x(0) = x_0 \;\; and \;\; \lambda(1) = 0 \tag{11}$$

where λ is a costate variable referred to as the Lagrange multiplier.

4 Controlled TB Model With Fractional Derivatives

In this section, we will develop the necessary conditions or equations for the optimality of the optimal control problem for the TB model using the information acquired from the formulation of the optimum control problem, which was briefly discussed in Section 2. We employ control, which stands for the prevention of exogenous reinfection, to decrease the infected group by reducing the interaction between infectious and exposed persons. We have formulated a fractional optimal control problem. Finding an optimal control u^* that minimizes the objective function is the main objective.

$$J(u) = \int_0^1 M(X,u,t)dt \tag{12}$$

subject to the fractional dynamics constraints

$$ {}_0^C D_t^\alpha X(t) = V(X,u,t) \tag{13}$$

with the initial condition

$$X(0) = X_0 \tag{14}$$

where

$$X(t) = (S,E,I,T)^T$$

$$X(0) = (S(0),E(0),I(0),T(0))^T$$

$$M(X,u,t) = I(t) + Au^2(t)$$

where V(X,u,t) represents equation (15-18),

Table 1: Variable and control functions

Variables	Description
S(t)	Susceptible Individuals
E(t)	Exposed Individuals
I(t)	Infectious Individuals
T(t)	Treated Individuals
u(t)	Control variable represents the distance between infected and exposed individuals

Table 2: Description of Model parameters

Variables	Description
Parameters	Definition
$\sigma\beta$ $(0 \leq \sigma \leq 1)$	The average number of treated people infected by one infectious person
β	The average number of susceptible individuals infected by one infectious person
$\Lambda = \mu * N$	Rate of recruitment
μ	Rate of natural mortality
c	Per-capita contact rate
r	Per-capita rate of treatment
d	Rate of TB induced mortality
k	Rate of progression to active TB
p	Level of exogenous reinfection
N	Total population (S+E+I+T)

$$ {}_{C}^{0}D_t^{\alpha} S(t) = \Lambda - S\mu - \frac{I}{N} S\beta c \tag{15} $$

$$ {}_{C}^{0}D_t^{\alpha} E(t) = \frac{I}{N} S\beta c - \frac{I}{N} E(1-u)\beta pc - E(\mu + k) + \frac{I}{N} T\beta\sigma c \tag{16} $$

$$ {}_{C}^{0}D_t^{\alpha} I(t) = \frac{I}{N} E(1-u)\beta pc + Ek - I(\mu + r + d) \tag{17} $$

$$ {}_{C}^{0}D_t^{\alpha} T(t) = Ir - T\mu - \frac{I}{N} T\beta\sigma c \tag{18} $$

$X(t)$ represents a state variable while u(t) represents a control variable and initial conditions are $S(0) = S_0$, $E(0) = E_0$, $I(0) = I_0$ and $T(0) = T_0$ where the parameter A represents the weight on the benefits and cost (A balance the size of the terms). For the uncontrolled TB model presented and examined by K. Hattaf (2009), the defined TB-controlled dynamical model becomes a conventional optimal control issue when $\alpha = 1$. Also, with the above initial conditions, the dynamic constraint equation(15-18) transforms into the TB infectious model with the control functions u=0.

It is noteworthy to note that several fractional dynamical systems have rigorous demonstrations of the derivation of the necessary conditions in the literature [24,25]. As a result, in this study, we present the necessary conditions (19-21) for our optimization problem utilizing the authors' theories [23,24,25].

$$ {}_{0}^{C}D_t^{\alpha} X(t) = V(X, u, t) \tag{19} $$

$$ {}_{t}^{C}D_b^{\alpha} \lambda(t) = \frac{\partial M}{\partial X} + \lambda^T \frac{\partial V}{\partial X} \tag{20} $$

$$ \frac{\partial M}{\partial u} + \lambda^T \frac{\partial V}{\partial u} = 0 \tag{21} $$

with

$$ X(0) = X_0 \ \ and \ \ \lambda(1) = 0 \tag{22} $$

where co-state vector is $\lambda(t) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$.

The compact version of the aforementioned conditions is now used to construct the optimally system in an expanded form for the fractional optimal control TB model as follows:

$$ {}_{C}^{0}D_t^{\alpha}S(t) = \Lambda - S\mu - \frac{I}{N}S\beta c \tag{23}$$

$$ {}_{C}^{0}D_t^{\alpha}E(t) = \frac{I}{N}S\beta c - \frac{I}{N}E[1-u]\beta pc - E[\mu+k] + \frac{I}{N}T\beta\sigma c \tag{24}$$

$$ {}_{C}^{0}D_t^{\alpha}I(t) = \frac{I}{N}E[1-u]\beta pc + Ek - I[\mu+r+d] \tag{25}$$

$$ {}_{C}^{0}D_t^{\alpha}T(t) = Ir - T\mu - \frac{I}{N}T\beta\sigma c \tag{26}$$

$$ {}_{t}^{C}D_b^{\alpha}\lambda_1 = \lambda_1\mu + (\lambda_1-\lambda_2)\beta c\frac{I}{N} \tag{27}$$

$$ {}_{t}^{C}D_b^{\alpha}\lambda_2 = \lambda_2\mu + (\lambda_2-\lambda_3)(k+\frac{I}{N}(1-u)\beta pc) \tag{28}$$

$$ {}_{t}^{C}D_b^{\alpha}\lambda_3 = -1 + \lambda_3(\mu+d) + (\lambda_1-\lambda_2)\beta c\tfrac{S}{N} + (\lambda_2-\lambda_3)$$
$$ p(1-u)\beta c\tfrac{E}{N} + (\lambda_4-\lambda_3)\sigma\beta c\tfrac{T}{N} + (\lambda_3-\lambda_4)r$$

$$ {}_{t}^{C}D_b^{\alpha}\lambda_4 = \lambda_4\mu + (\lambda_4-\lambda_2)\sigma\beta c\frac{I}{N}) \tag{29}$$

Furthermore,

$$ u^* = \min(1, \max(0, \frac{1}{2AN}(\lambda_3-\lambda_2)p\beta cEI)) \tag{30}$$

where, $S(0) = S_0$, $E(0) = E_0$, $I(0) = I_0$, $T(0) = T_0$ and $\lambda_1(t_f)$=0, $\lambda_2(t_f)$=0, $\lambda_3(t_f)$=0, $\lambda_4(t_f)$=0

5 Numerical Simulations and Discussion

Lenhart and Workman provided numerical perspectives on optimal control models for biological systems in their book published in 2007. They developed the forward-backward sweep method utilizing the Runge-Kutta approach to obtain a solution for the biological models' optimality system. In this study, the forward-backward algorithm and modified technique are utilized to resolve the optimality system numerically. One of the numerical techniques for fractional order models, the TB model, recently employed the forward-backward algorithm with a modified Euler method [26]. For the numerical simulations, the initial circumstances and model parameter values listed below are used: S_0=7600, E_0= 3800, I_0= 500, T_0= 100, A= 400, N= 12000, $\Lambda = \mu * N$= 192, d=0.1, c=1, r=2, β=13, μ= 0.016, p=0.4, σ=0.9, k=0.005, h = 0.00008333, T=nh= 1 year.

The figures show that the infected group is reduced by reducing contact between infectious and exposed people when α = 1 (integer order) with time-dependent controls, but that non-integer orders (α = 0.9, 0.89, 0.88, 0.87, 0.86) with time-dependent controls are much more significant. We demonstrate the control variable and model parameters in Tables (1) and (2) respectively.

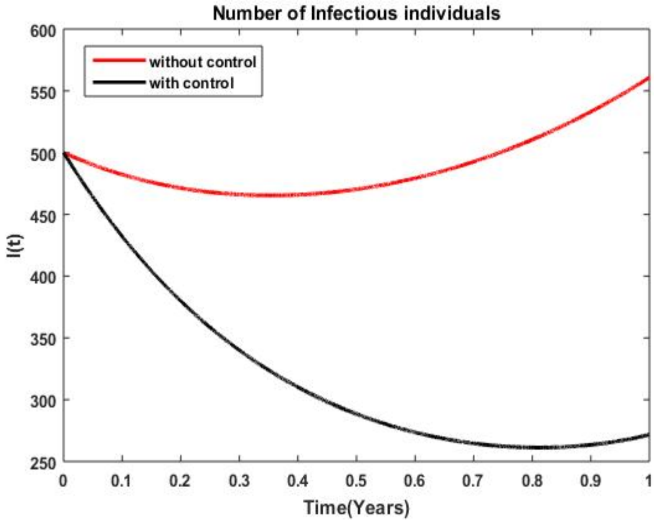

Fig. 1: Function I with and without control

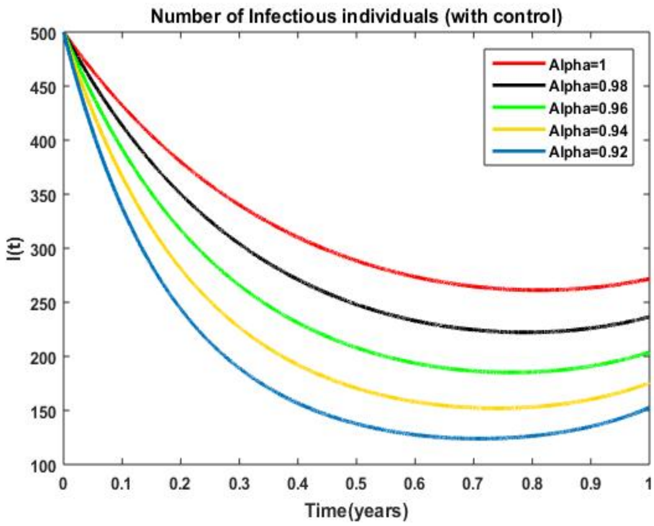

Fig. 2: Number of Infectious individuals under fractional optimal control with various values of Alpha

6 Conclusions

A mathematical model for TB has been presented in this work. By using the fractional optimal control theory, the interaction between infectious and exposed individuals can be limited, thus preventing exogenous reinfection. The optimal system was numerically solved by using the forward-backward sweep method with the generalized Euler method. We have substantially more accurate numerical solutions to optimal control problems with fractional orders than with integer orders.

Data Availability

All the data are available within the article.

Declaration of Competing Interest

The authors declare that they have no competing interests.

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