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Stochastic Gilpin-Ayala Model with Environmental Pollution and Markovian Switching

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Abstract: This study examines the stochastic Gilpin-Ayala model in a polluted environment, incorporating Markovian switching and white noise effects. The analysis establishes conditions under which the trivial equilibrium remains globally stable. Additionally, explicit criteria are derived to determine whether the population persists or faces extinction within this stochastic framework. The study also demonstrates the existence of a stationary distribution under specific conditions. Numerical simulations are carried out to validate the theoretical findings, providing empirical evidence and practical insights.

Keywords: Stochastic modeling, Markovian switching dynamics, Stability analysis, Population extinction, Long-term persistence, Polluted environments, Stationary distribution properties.

1 Introduction

Studying ecological systems impacted by environmental pollutants has become an essential and highly valued area of scientific inquiry. Growing concerns over the effects of pollution on biodiversity and ecosystem health underscore the urgent need to understand population dynamics in pollutant-affected environments. The intersection of environmental toxicology and mathematical modeling has emerged as a critical framework for predicting and mitigating the adverse effects of anthropogenic stressors on natural populations. Recent decades have witnessed an exponential increase in research efforts aimed at quantifying the complex relationships between pollutant exposure and population viability, particularly in the context of climate change and intensifying industrial activities. In this context, the Gilpin-Ayala model is a robust framework for examining the effects of pollutants on population behaviors. Originally proposed as a generalization of the classical logistic growth equation [1], this model has proven remarkably versatile in capturing density-dependent phenomena that deviate from linear assumptions. This model provides essential insights into the complex interactions between species and the environmental stressors they face. As a specialized subfield, population ecology focuses on understanding the intricate dynamics of species populations and their responses to environmental factors. The mathematical formulation of ecological processes has a rich history, dating back to the pioneering works of Lotka [2] and Volterra [3], and has since evolved to incorporate increasingly sophisticated representations of biological realism and environmental complexity. However, the accelerated growth of industrial and agricultural activities has resulted in the widespread release of toxic substances into the environment, creating significant threats to the survival of affected organisms. Heavy metals, persistent organic pollutants, pesticides, and pharmaceutical residues now pervade terrestrial and aquatic ecosystems at concentrations that can disrupt physiological processes, reproductive success, and community structure. This growing concern has driven scientific researchers to investigate the resilience of populations in contaminated habitats and to identify critical thresholds that determine whether populations can persist or face potential extinction. Understanding these thresholds is paramount for conservation biology and environmental risk assessment, as they inform regulatory policies and remediation strategies. Species proliferation in natural ecosystems is closely linked to the effects of environmental variability [4]. Stochastic fluctuations in environmental conditions can dramatically alter

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population trajectories and extinction probabilities. As a result, numerous researchers have investigated stochastic population models to explore population dynamics in polluted environments (see, e.g., [5,6]). The incorporation of stochasticity into population models represents a significant advancement over deterministic approaches, as it more accurately reflects the inherent unpredictability of natural systems and provides probabilistic frameworks for risk assessment [7, 8]. The logistic growth framework, expressed through an elementary ordinary differential equation, constitutes one of the most extensively studied and analytically amenable frameworks for characterizing temporal dynamics of single-species populations within equilibrium environments:

$$dN(t) = [r - \lambda N(t)]\,N(t)dt, \tag{1.1}$$

where, $N(t)$ signifies the population size at temporal instant t, where $r > 0$ characterizes the species growth rate, and $\lambda > 0$ quantifies the self-inhibition rate. This fundamental equation has served as the cornerstone for countless theoretical and applied studies in population biology, forming the basis for more complex models that incorporate age structure, spatial heterogeneity, and interspecific interactions [9]. Beyond population ecology, the logistic growth framework has found important applications in mathematical epidemiology [10–12]. For instance, recent work has developed a stochastic SIRI epidemic model that incorporates nonlinear relapse dynamics, logistic population growth, and a bilinear incidence rate [13]. This study establishes the existence and uniqueness of positive global solutions and derives sufficient conditions determining whether the disease will persist or become extinct, demonstrating the versatility of logistic growth in modeling complex biological phenomena including disease dynamics with relapse and population regulation. Nevertheless, in the logistic formulation (1.1), the individual population growth rate is expressed as $r - \lambda N(t)$ which depends linearly on density, thereby excluding numerous ecologically significant phenomena from consideration. These excluded phenomena include Allee effects, overcompensatory dynamics, and threshold-dependent behaviors that are frequently observed in empirical population studies. Consequently, extensive modifications to framework (1.1) have been developed to achieve more ecologically realistic formulations [14]. Among these modifications, the theta-logistic model and its variants have gained particular prominence due to their ability to capture diverse density-dependent patterns through a single shape parameter. In [15], Liu and Wang examine the population dynamics described by (1.1) in a polluted environment, characterized by the following system:

$$\begin{cases} dN(t) = N(t)\,[r - l\mathscr{E}_0(t) - \lambda N(t)]\,dt \\ \qquad\quad +\alpha N(t)dW(t), \\ d\mathscr{E}_0(t) = [k\mathscr{E}_e(t) - (g+m)\mathscr{E}_0(t)]\,dt, \\ d\mathscr{E}_e(t) = [-h\mathscr{E}_e(t) + \tilde{u}(t)]\,dt. \end{cases} \tag{1.2}$$

In this model, $N(t)$ signifies the population size, whereas $r > 0$ indicates the intrinsic growth rate of the toxicant-free population. The parameter $l > 0$ quantifies the population's response to toxicant exposure, representing the per capita reduction in growth rate per unit toxicant concentration—a crucial parameter for ecotoxicological risk assessment. Where $\mathscr{E}_0$ and $\mathscr{E}_e$ represent the internal and external toxicant concentrations, respectively, $W(t)$ denote a standard Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ equipped with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ that satisfies the standard conditions of completeness. This probabilistic framework ensures the mathematical rigor necessary for proving existence, uniqueness, and stability properties of solutions. The coefficient α quantifies the intensity of the white noise. The parameter $k > 0$ describes the net uptake rate of the toxicant by the organism from its environment, while $g > 0$ and $m > 0$ represent the rates of toxicant egestion and depuration within the organism, respectively. These biokinetic parameters are typically estimated through controlled laboratory experiments or field biomonitoring studies. The environmental loss rate of the toxicant, typically due to volatilization, is represented by $h > 0$. The function $\tilde{u}(t)$, defined on the interval $[0, +\infty)$, is non-negative, bounded, and continuous, and models the rate of external toxicant input into the environment. In ecological systems, model parameters frequently deviate from deterministic behavior, exhibiting stochastic fluctuations around mean values. These variations reflect the inherent randomness of environmental conditions, which play a critical role in shaping population dynamics. Empirical evidence suggests that ignoring environmental stochasticity can lead to substantial underestimation of extinction risks and mischaracterization of population responses to perturbations. Moreover, the assumption of constant parameters may be particularly problematic in systems subject to regime shifts, seasonal variations, or episodic disturbances. Recent studies have focused on stochastic generalizations of the Gilpin-Ayala model (see [16, 17]). These investigations have revealed rich dynamical behaviors including noise-induced transitions, stochastic resonance, and the emergence of stationary distributions with multimodal structures. A notable contribution in this direction is the work of Harchaoui et al. [17], who comprehensively examined the impact of incorporating pollution into a stochastic Gilpin-Ayala model with spatial patches. Their critical contribution lies in expressing the conditions under which species extinction or persistence occurs based on pollution parameters, thereby enabling a more accurate assessment of contaminated environments. Their results emphasize the importance of considering pollution as a crucial factor in ecological systems, providing valuable insights into the complexities of polluted environments and the role of spatial dispersal in population dynamics. Jiang et al. [18, 19], proposed a nonautonomous stochastic formulation based on equation (1.1) as follows:

$$dN(t) = \left[r - \lambda N^{\theta}(t)\right] N(t)dt + \alpha N(t)dW(t). \tag{1.3}$$

Here, $\alpha^2(t)$ denotes the intensity of the associated white noise [20–32]. The parameter θ plays a pivotal role in determining the qualitative behavior of the system, with different values corresponding to distinct ecological scenarios ranging from weak Allee effects ($\theta < 1$) to strong overcompensation ($\theta > 1$). Settati and Lahrouz further explored the switching dynamics of this model in [33], introducing the parameter θ to generalize the classical logistic framework. Their work demonstrated that environmental switching can stabilize populations that would otherwise go extinct under constant environmental conditions, a phenomenon with important implications for conservation in variable environments. To incorporate environmental pollution, the population dynamics are extended by coupling equation (1.3) with toxicant concentration dynamics:

$$\begin{cases} dN(t) = \left[r - l\mathscr{E}_0(t) - \lambda N^{\theta}(t)\right] N(t)dt \\ \qquad\qquad +\alpha N(t)dW(t), \\ d\mathscr{E}_0(t) = \left[k\mathscr{E}_e(t) - (g+m)\mathscr{E}_0(t)\right]dt, \\ d\mathscr{E}_e(t) = \left[-h\mathscr{E}_e(t) + \tilde{u}(t)\right]dt. \end{cases} \tag{1.4}$$

This coupled system represents a significant advancement in modeling polluted ecosystems, as it explicitly tracks both population abundance and toxicant bioaccumulation, enabling more accurate predictions of long-term ecological outcomes. This study further investigates the stochastic behavior of the system under regime-switching dynamics, leading to the following formulation:

$$\begin{cases} dN(t) = \left[r(\gamma(t)) - l_{\gamma(t)}\mathscr{E}_0(t) - \lambda(\gamma(t))N^{\theta(\gamma(t))}\right] \\ \qquad\qquad \times N(t)dt + \alpha[\gamma(t)]N(t)dW(t), \\ d\mathscr{E}_0(t) = \left[k\mathscr{E}_e(t) - (g+m)\mathscr{E}_0(t)\right]dt, \\ d\mathscr{E}_e(t) = \left[-h\mathscr{E}_e(t) + \tilde{u}(t)\right]dt, \end{cases} \tag{1.5}$$

where $\gamma(t)$ denotes a finite-state Markov process governing the switching regimes. Markov-switching models have proven particularly valuable for representing abrupt environmental changes such as droughts, floods, or sudden pollution events, which are poorly captured by models with continuous parameter variation. The authors demonstrated that equation (1.5) admits a unique, globally positive solution for any positive initial condition. Moreover, under the given assumptions, they proved that the process $(\gamma(t))_{t\geq 0}$ is ergodic and possesses a unique stationary distribution, denoted by $\mathfrak{I} = (\mathfrak{I}_1, \mathfrak{I}_2, \ldots, \mathfrak{I}_m)$. The existence of such a stationary distribution is crucial for understanding the long-term probabilistic behavior of the system and has profound implications for risk assessment and population management. The asymptotic behavior of (1.5) is governed by:

$$\bar{b} = \sum_{j=1}^{m} \mathfrak{I}_j \left[r(j) - l_j \liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathscr{E}_0(s)ds - \frac{\alpha^2(j)}{2} \right] < 0. \tag{1.6}$$

This threshold quantity $\bar{b}$ serves as a stochastic analogue to the classical Malthusian parameter in deterministic theory, delineating the boundary between population persistence and extinction. The presence of the noise intensity term $\alpha^2(j)/2$ reflects the well-known phenomenon that environmental stochasticity reduces the effective growth rate, extending classical results to regime-switching contexts. Based on the theoretical foundations presented above and motivated by the pressing need to understand population resilience, this study provides a comprehensive analysis of the long-term behavior of stochastic population models under regime-switching dynamics. We aim to establish rigorous conditions for the global asymptotic stability of extinction, characterize the conditions for population weak persistence with precise growth rate estimates, prove the existence of a unique ergodic stationary distribution, and validate these findings with numerical illustrations. The manuscript is structured to first demonstrate that the trivial equilibrium is globally asymptotically stable when $\bar{b} < 0$. It then shows that, under stochastic regime-switching, the population exhibits weak persistence when $\bar{b} > 0$, including characterizations of its growth rates and temporal averages. Furthermore, we prove that the system admits a unique ergodic stationary distribution under the same constraint ($\bar{b} > 0$), and finally present numerical simulations to illustrate these principal theoretical findings.

2 Global Stability

We define

$$\langle \mathscr{E}_0(t) \rangle = \int_0^t \mathscr{E}_0(s)ds, \quad \bar{\mathfrak{F}} = \liminf_{t\to\infty} \frac{1}{t}\langle \mathscr{E}_0(t) \rangle,$$
$$\bar{\Psi} = \limsup_{t\to\infty} \frac{1}{t}\langle \mathscr{E}_0(t) \rangle.$$

Note that $\bar{\mathfrak{F}} \leq \bar{\Psi}$ by definition. In extinction, results $\bar{\mathfrak{F}}$ yields the weakest sufficient condition, while in upper-bound estimates, $\bar{\Psi}$ is used; both quantities refer to the time-averaged internal toxicant concentration $\frac{1}{t}\langle \mathscr{E}_0(t) \rangle$. We present the following lemma ([34]).

Lemma 1.*If* $\limsup_{t\to\infty} \tilde{u}(t) \leq h$ *and* $0 < k \leq g+m$, *then* $0 \leq \mathscr{E}_0(t) < 1$, $0 \leq \mathscr{E}_e(t) < 1$ *for each* $t \geqslant 0$.

Here, it $\mathscr{E}_0(t)$ Functions as a stochastic analogue of the basic reproduction number in classical population dynamics. Hence, the condition $\mathscr{E}_0(t) \leq 1$ exhibits a regime in which the net increase rate, after calculating white noise intensity and toxicant focus, is inadequate to maintain the population. signifies a regime in which the net growth rate, after accounting for the intensity of the white noise and the concentration of toxicants, is insufficient to sustain the population.
We assume $0 < k \leq g+m$ and $\limsup_{t\to\infty} \tilde{u}(t) \leq h$. It is important to emphasize that the last two equations of the

model (1.5) are linear for $\mathscr{E}_0(t)$ and $\mathscr{E}_e(t)$, which allows for an efficient and direct derivation of their explicit solutions. Thus, our primary analytical focus can be concentrated on the initial equation in model (1.5).
In this study, we consider $(\gamma(t))_{t\geq 0}$ be a right-continuous Markov chain, defined on the same space, which takes values in the finite state space $\mathscr{S}=\{1,2,\ldots,m\}$. This chain models the stochastic switching dynamics of the system, governed by a generator matrix $\Phi=(\Upsilon_{uv})_{1\leq u,v\leq m}$, where the matrix is specified for $\kappa>0$ as follows:

$$\mathbb{P}\left[\gamma(t+\kappa)=v|\gamma(t)=u\right] = \begin{cases}\Upsilon_{uv}\kappa+o(\kappa), & \text{if } u\neq v,\\ 1+\Upsilon_{uu}\kappa+o(\kappa), & \text{if } u=v.\end{cases}$$

In this context, Υ_{uv} denotes the rate at which transitions occur from state u to state v, while $\Upsilon_{uu}=-\sum_{u\neq v}\Upsilon_{uv}$ ensures that row sums are zero. Throughout this paper, we adopt the assumption that the Markov chain $(\gamma(t))_{t\geq 0}$ is irreducible, implying that the system can transition between any pair of regimes. Under this irreducibility condition, the Markov chain possesses a unique stationary distribution $\Im=(\Im_1,\Im_2,\ldots,\Im_m)$, which is obtained by solving the linear system $\Im\Phi=0$ with the constraints $\sum_{j=1}^{m}\Im_j=1$ and $\Im_j>0$ for all $j\in\mathscr{S}$. It is essential to demonstrate that the density of population $N(t)$ stays non-negative for every $t\geq 0$. Using a methodology similar to the proof of Theorem 1 in [35], we present the following theorem, which establishes that the interval $(0,\infty)$ is positively invariant under equation (1.5).

Theorem 1. *Given $N(0)\in(0,\infty)$, there exists a unique solution $N(t)$ to the stochastic differential equation (SDE) (1.5) that is well-defined for all $t\geq 0$. Furthermore, with probability one, this solution remains confined to interval $(0,\infty)$ for its evolution.*

The process $[N(t),r(t)]$ is governed by a generator $\mathscr{L}$, which is defined as follows: For each $j\in\mathscr{S}$ and any function $V(N,j)$ that is twice continuously differentiable, the following expression holds:

$$\begin{aligned}\mathscr{L}V(N,j) = {} & N\left[r(j)-l_j\mathscr{E}_0(t)-\lambda(j)N^{\theta(j)}(t)\right]\\ & \times\frac{\partial V(N,j)}{\partial N}+\frac{1}{2}\alpha^2(j)N^2\frac{\partial^2 V(N,j)}{\partial N^2}\\ & +\sum_{k\neq j,k\in\mathscr{S}}\Upsilon_{jk}\left[V(N,k)-V(N,j)\right].\end{aligned} \tag{2.1}$$

A crucial qualitative feature of SDE solutions is stability, which is typically studied without explicit solutions. Drawing on the contributions of Khasminskii *et al.* [36] and Yuan and Mao [37], we introduce the following lemma. This result provides a sufficient condition for probabilistic asymptotic stability, presented within the framework of Lyapunov functions.

Lemma 2. *Suppose there exist functions $\mathscr{V}\in\mathscr{C}^2(\mathbb{R}\times\mathscr{S};\mathbb{R}^+)$ and $\rho\in\mathscr{C}(\mathbb{R};\mathbb{R}^+)$, where ρ vanishes exclusively at $N=0$, such that the following conditions are satisfied:*

$$\mathscr{L}\mathscr{V}(N,j)\leq-\rho(N),\ \textit{for each}\ \ (N,j)\in\mathbb{R}\times\mathscr{S}, \tag{2.2}$$

and

$$\lim_{|N|\to\infty}\inf_{j\in\mathscr{S}}\mathscr{V}(N,j)=\infty. \tag{2.3}$$

Consequently, for the system (1.5), equilibrium $N=0$ is globally asymptotically stable in probability.

Theorem 2. *For each $N(0)\in(0,\infty)$, if*

$$\sum_{j=1}^{m}\Im_j\left[r(j)-l_j\inf_{t\geq 0}\mathscr{E}_0(t)-\frac{\alpha^2(j)}{2}\right]<0. \tag{2.4}$$

Then, the system's equilibrium $N=0$ is globally asymptotically stable with probability one.

Proof. Let

$$\mathscr{V}_1(N,j)=(\kappa+a_j)N^{\frac{1}{\kappa}},\ \ \text{for each}\ \ j\in\mathscr{S}, \tag{2.5}$$

where κ and a_j are strictly positive constants whose values will be determined later. Using Itô's formula, one can easily have

$$\begin{aligned}\mathscr{L}\mathscr{V}_1 = {} & \frac{1}{\kappa}(\kappa+a_j)\left[r(j)-l_j\mathscr{E}_0(t)+\frac{\alpha^2(j)}{2}\left(\frac{1}{\kappa}-1\right)\right.\\ & \left.-\lambda(j)N^{\theta(j)}\right]N^{\frac{1}{\kappa}}+N^{\frac{1}{\kappa}}\sum_{k\neq j\in\mathbb{S}}\Upsilon_{jk}\left(a_k-a_j\right),\\ = {} & \frac{1}{\kappa}(\kappa+a_j)\left[\left(r(j)-\frac{\alpha^2(j)}{2}+\sum_{k\neq j\in\mathbb{S}}\Upsilon_{jk}\left(a_k-a_j\right)\right)\right.\\ & -\lambda(j)N^{\theta(j)}-l_j\mathscr{E}_0(t)+\frac{\alpha^2(j)}{2\kappa}\\ & \left.-\frac{a_j}{\kappa+a_j}\sum_{k\neq j\in\mathbb{S}}\Upsilon_{jk}(a_k-a_j)\right]N^{\frac{1}{\kappa}}.\end{aligned} \tag{2.6}$$

Given that the matrix generator Φ is irreducible, and for $\xi=(\xi_1,\ldots,\xi_m)^T$, where $\xi_j=r(j)-\frac{\alpha^2(j)}{2}$, there is a solution $A=(a_1,\ldots,a_m)^T$ to the associated system of Poisson [38] that satisfies the following condition:

$$\Phi A=-\xi+\left(\sum_{j=1}^{m}\Im_j\xi_j\right)\mathbf{e}, \tag{2.7}$$

where $\mathbf{e}$ denotes a column vector consisting entirely of ones. Substituting (2.7) into (2.6), and using the fact

$$\begin{aligned}\sum_{k\neq j\in\mathscr{S}}\Upsilon_{jk}(a_k-a_j) = {} & -\left(r(j)-\frac{\alpha^2(j)}{2}\right)\\ & +\sum_{j=1}^{m}\Im_j\left(r(j)-\frac{\alpha^2(j)}{2}\right),\end{aligned}$$

the desired inequality is obtained by collecting the terms in $N^{\frac{1}{\kappa}}$, and bounding $\mathscr{E}_0(t)$ from below by $\inf_{t\geq 0}\mathscr{E}_0(t)$

$$\mathscr{L}V_1 \leq \kappa^{-1}\left(\kappa+a_j\right)\left[C(\kappa,j)-\lambda(j)N^{\theta(j)}\right]N^{\frac{1}{\kappa}}, \tag{2.8}$$

where

$$\begin{aligned} C(\kappa,j) \triangleq {} & \sum_{j=1}^{m}\mathfrak{I}_j\left(r(j)-\frac{\alpha^2(j)}{2}\right)+\frac{\alpha^2(j)}{2\kappa} \\ & -\sum_{j=1}^{m}\mathfrak{I}_j\left(l_j\inf_{t\geq 0}\mathscr{E}_0(t)\right)\right) \\ & +\frac{a_j}{\kappa+a_j}\left[r(j)-\frac{\alpha^2(j)}{2}\right. \\ & \left.-\sum_{j=1}^{m}\mathfrak{I}_j\left(r(j)-\frac{\alpha^2(j)}{2}\right)\right]. \end{aligned}$$

From (2.4), one can easily verify that for a sufficiently large κ, one has

$$\kappa > -\min_{j\in\mathscr{S}}\left\{a_j\right\}, \ \text{ and } \ \max_{j\in\mathscr{S}}\left\{C(\kappa,j)\right\}<0. \tag{2.9}$$

Using (2.8) and (2.9), one obtains

$$\mathscr{L}\mathscr{V}_1 \leq -Q(\kappa)N^{\frac{1}{\kappa}},$$

where

$$Q(\kappa) = -\max_{j\in\mathscr{S}}\left\{\frac{1}{\kappa}(\kappa+a_j)C(\kappa,j)\right\}>0.$$

The proof is thus completed by applying Lemmas 1 and 2.

Theorem 3.*For each $N(0)\in(0,\infty)$, the solution to SDE (1.5) adheres to the following properties:*

$$\limsup_{t\to\infty}\frac{1}{t}\log N(t) \leq \sum_{j=1}^{m}\mathfrak{I}_j\left[r(j)-l_j\bar{\mathfrak{F}}-\frac{\alpha^2(j)}{2}\right], \ a.s..$$

Moreover, if

$$\sum_{j=1}^{m}\mathfrak{I}_j\left[r(j)-l_j\bar{\mathfrak{F}}-\frac{\alpha^2(j)}{2}\right]<0.$$

Then, $N(t)$ converges to zero exponentially with probability one.

*Proof.*The proof adopts a methodology analogous to the proof of theorem 2 in [39].

3 Persistence

We will now examine the persistence of the species under various scenarios, considering different noise intensities and model parameters. To ensure simplicity and clarity in the analysis that follows, we introduce a constant vector $(\theta_j)_{j\in\mathscr{S}}$, where

$$\tilde{\theta}=\min_{j\in\mathscr{S}}\left\{\theta_j\right\}, \quad \widehat{\theta}=\max_{j\in\mathscr{S}}\left\{\theta_j\right\}.$$

Theorem 4.*If $\sum_{j=1}^{m}\mathfrak{I}_j\left[r(j)-l_j\bar{\mathfrak{F}}-\frac{\alpha^2(j)}{2}\right]>0$, then*

$$\liminf_{t\to\infty}N(t)\leq\Lambda\leq\limsup_{t\to\infty}N(t) \ \ a.s.. \tag{3.1}$$

In this context, Λ represents the unique positive solution to the following equation:

$$\sum_{j=1}^{m}\mathfrak{I}_j\left[r(j)-l_j\bar{\mathfrak{F}}-\frac{\alpha^2(j)}{2}-\lambda(j)\Lambda^{\theta(j)}\right]=0. \tag{3.2}$$

*Proof.*Let

$$f(z)=\sum_{j=1}^{m}\mathfrak{I}_j\left[r(j)-l_j\bar{\mathfrak{F}}-\frac{\alpha^2(j)}{2}-\lambda(j)z^{\theta(j)}\right].$$

The function $f(z)$ is continuous and exhibits a strictly decreasing behavior over the interval $(0,\infty)$, with

$$f(0^+)=\sum_{j=1}^{m}\mathfrak{I}_j\left[r(j)-l_j\bar{\mathfrak{F}}-\frac{\alpha^2(j)}{2}\right]>0, \ \ f(\infty)=-\infty.$$

The intermediate value theorem ensures that a solution for Λ exists. Furthermore, this solution is both unique and positive. Examining the right-hand side of (3.1), we assume by contradiction that

$$\mathbb{P}\left\{w\in\Omega,\limsup_{t\to\infty}N(t,w)<\Lambda\right\}>0.$$

Consequently, there exists a constant $A\in\left(\frac{1}{2},1\right)$ such that $\mathbb{P}(\Omega_1)>0$,

$$\Omega_1=\left\{w\in\Omega,\limsup_{t\to\infty}N(t,w)\leq(2A-1)\Lambda\right\}.$$

Therefore, for any $w\in\Omega_1$, there exists a $T(w)>0$ such that for each $t\geq T(w)$,

$$N(t)\leq(2A-1)\Lambda+(1-A)\Lambda=A\Lambda. \tag{3.3}$$

By using (1.5) in conjunction with the Itô formula, we derive

$$\begin{aligned} d\log\left[N(t)\right] = {} & \left[r(\gamma(t))-l_{\gamma(t)}\mathscr{E}_0(t)-\frac{1}{2}\alpha^2(\gamma(t))\right. \\ & \left.-\lambda(\gamma(t))N^{\theta(\gamma(t))}(t)\right]dt+\alpha(\gamma(t))dW(t). \end{aligned} \tag{3.4}$$

We integrate and apply (3.3) to obtain

$$\begin{aligned} \log\left[N(t)\right] \geq {} & \int_0^t\left[r(\gamma(s))-\frac{1}{2}\alpha^2(\gamma(s))\right]ds \\ & -\int_0^t l_{\gamma(t)}\mathscr{E}_0(s)ds \\ & -\int_0^T\lambda\left[\gamma(s)\right]N^{\theta(\gamma(s))}(s)ds \\ & -\int_T^t\lambda\left[\gamma(s)\right](A\Lambda)^{\theta(\gamma(s))}ds \\ & +\int_0^t\alpha\left[\gamma(s)\right]dW(s)+\log[N(0)]. \end{aligned} \tag{3.5}$$

Given that $A < 1$, it can be concluded that

$$\begin{aligned}\log[N(t)] \geq & \int_0^t \left[r(\gamma(s)) - \frac{1}{2}\alpha^2(\gamma(s))\right] ds \\ & - \int_0^t l_{\gamma(t)} \mathscr{E}_0(s) ds \\ & - A^{\tilde{\theta}} \int_T^t \lambda[\gamma(s)] \Lambda^{\theta(\gamma(s))} ds \\ & - \int_0^T \lambda[\gamma(s)] N^{\theta(\gamma(s))}(s) ds \\ & + \int_0^t \alpha[\gamma(s)] dW(s) + \log[N(0)], \end{aligned} \tag{3.6}$$

with, $M_t = \int_0^t \alpha(\gamma(s)) dW(s)$ represents a real-valued continuous martingale with a quadratic variation given by $\langle M_t, M_t \rangle = \int_0^t \alpha(\gamma(s))^2 ds$. Employing the strong law of large numbers for martingales ([40]) combined with ergodic properties of Markov chains, we deduce from (3.6) the existence of a subset $\Omega_1' \subset \Omega_1$ satisfying $\mathbb{P}(\Omega_1') = 1$. Furthermore, for any $w \in \Omega_1'$, the following holds:

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t} \log[N(t)] \geq & \sum_{j=1}^m \mathfrak{I}_j \left[r(j) - \frac{\alpha^2(j)}{2}\right] \\ & - A^{\tilde{\theta}} \sum_{j=1}^m \mathfrak{I}_j \lambda(j) \Lambda^{\theta(j)} - \sum_{j=1}^m \mathfrak{I}_j l_j \bar{\mathfrak{F}}, \end{aligned} \tag{3.7}$$

thus

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t} \log[N(t)] \geq & \sum_{j=1}^m \mathfrak{I}_j \left[r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2}\right] \\ & - A^{\tilde{\theta}} \sum_{j=1}^m \mathfrak{I}_j \lambda(j) \Lambda^{\theta(j)}. \end{aligned} \tag{3.8}$$

From the Λ-equation (3.2) and the condition $A^{\tilde{\theta}} < 1$, one obtains

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t} \log[N(t)] \geq & \left(1 - A^{\tilde{\theta}}\right) \sum_{j=1}^m \mathfrak{I}_j \left[r(j) - l_j \bar{\mathfrak{F}} \right. \\ & \left. - \frac{\alpha^2(j)}{2}\right] \; > \; 0. \end{aligned} \tag{3.9}$$

Hence, $\lim\limits_{t\to\infty} N(t) = \infty$, which contradicts (3.3). Consequently, the claimed assertion is established. Similarly, assuming the left side of (3.1) fails to hold, there would exist $B > 1$ satisfying $\mathbb{P}(\mathscr{D}) > 0$,

$$\mathscr{D} = \left\{ w \in \Omega, \liminf_{t\to\infty} N(t,w) \geq (2B-1)\Lambda \right\}. \tag{3.10}$$

Therefore, for any $w \in \mathscr{D}$, there exists a $T'(w) > 0$ such that for any $t \geq T'(w)$,

$$N(t) \geq (2B-1)\Lambda - (B-1)\Lambda = B\Lambda. \tag{3.11}$$

Using (3.4), (3.11), and the condition $B > 1$, one obtains

$$\begin{aligned}\log[N(t)] \leq & \int_0^t \left[r(\gamma(s)) - \frac{1}{2}\alpha^2(\gamma(s))\right] ds \\ & - \int_0^t l_{\gamma(t)} \mathscr{E}_0(s) ds \\ & - \int_{T'}^t \lambda[\gamma(s)] (B\Lambda)^{\theta(\gamma(s))} ds \\ & - \int_0^{T'} \lambda[\gamma(s)] N^{\theta(\gamma(s))}(s) ds \\ & + \int_0^t \alpha[\gamma(s)] dW(s) + \log[N(0)], \\ \leq & \int_0^t \left[r(\gamma(s)) - \frac{1}{2}\alpha^2(\gamma(s))\right] ds \\ & - \int_0^t l_{\gamma(t)} \mathscr{E}_0(s) ds - B^{\tilde{\theta}} \int_{T'}^t \lambda[\gamma(s)] \Lambda^{\theta(\gamma(s))} ds \\ & - \int_0^{T'} \lambda[\gamma(s)] N^{\theta(\gamma(s))}(s) ds \\ & + \int_0^t \alpha[\gamma(s)] dW(s) + \log[N(0)]. \end{aligned} \tag{3.12}$$

Following the approach used in proving the right side of (3.1), and by applying the law of large numbers for martingales alongside the ergodic theory of Markov chains, we conclude that there exists a subset $\mathscr{D}' \subset \mathscr{D}$ with $\mathbb{P}(\mathscr{D}') = 1$ such that, for any $w \in \mathscr{D}'$, one has

$$\begin{aligned} & \limsup_{t\to\infty} \frac{1}{t} \log[N(t)] \\ \leq & \sum_{j=1}^m \mathfrak{I}_j \left[r(j) - l_j \bar{\Psi} - \frac{\alpha^2(j)}{2}\right] - B^{\tilde{\theta}} \sum_{j=1}^m \mathfrak{I}_j \lambda(j) \Lambda^{\theta(j)}, \\ \leq & \sum_{j=1}^m \mathfrak{I}_j \left[r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2}\right] - B^{\tilde{\theta}} \sum_{j=1}^m \mathfrak{I}_j \lambda(j) \Lambda^{\theta(j)}, \\ = & \left(1 - B^{\tilde{\theta}}\right) \sum_{j=1}^m \mathfrak{I}_j \left[r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2}\right] < 0. \end{aligned}$$

Therefore, $\lim\limits_{t\to\infty} N(t) = 0$, which stands in contradiction to (3.11). This concludes the proof of assertion (3.1).

Theorem 4 demonstrates that the solutions of equation (1.5) exhibit oscillatory behavior around Λ infinitely often with probability one. Consequently, it is essential to derive further insights about Λ. One approach to achieve this is examining the sign of $f(1)$, which is critical in this analysis. Specifically,

- (•) $f(1) = \sum_{j=1}^m \mathfrak{I}_j \left[r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2} - \lambda(j)\right] = 0$ implies $\Lambda = 1$.
- (•) $f(1) > 0$, applying the Intermediate Value Theorem, it follows that $\Lambda > 1$, which leads to the conclusion that for any $j \in \mathscr{S}$, the inequality $\Lambda^{\tilde{\theta}} \leq \Lambda^{\theta(j)} \leq \Lambda^{\hat{\theta}}$ holds. When combined with the Λ-equation, this result

establishes the bounds $\Lambda_1 \leq \Lambda \leq \Lambda_2$, where

$$\Lambda_1 = \left(\frac{\sum_{j=1}^{m} \mathfrak{I}_j \left(r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2} \right)}{\sum_{j=1}^{m} \mathfrak{I}_j \lambda(j)} \right)^{\frac{1}{\hat{\theta}}},$$

$$\Lambda_2 = \left(\frac{\sum_{j=1}^{m} \mathfrak{I}_j \left(r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2} \right)}{\sum_{j=1}^{m} \mathfrak{I}_j \lambda(j)} \right)^{\frac{1}{\tilde{\theta}}}. \tag{3.13}$$

As a result, the function $N(t)$ will lie within the interval $[\Lambda_1, \Lambda_2]$ infinitely often with probability one.

(•) In a similar manner, if $f(1) < 0$, it follows that $\Lambda < 1$, leading to the conclusion

$$\Lambda_2 \leq \Lambda \leq \Lambda_1.$$

The preceding analysis is encapsulated in the following result.

Corollary 1*If* $\sum_{j=1}^{m} \mathfrak{I}_j \left[r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2} \right] > 0$, *then*

(i)*If* $\sum_{j=1}^{m} \mathfrak{I}_j \left[r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2} - \lambda(j) \right] = 0$, *then*

$$\liminf_{t\to\infty} N(t) \leq 1 \leq \limsup_{t\to\infty} N(t) \quad a.s..$$

(ii)*If* $\sum_{j=1}^{m} \mathfrak{I}_j \left[r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2} - \lambda(j) \right] > 0$, *then*

$$\liminf_{t\to\infty} N(t) \leq \Lambda_2 \ \textit{and} \ \limsup_{t\to\infty} N(t) \geq \Lambda_1 \quad a.s.$$

(iii)*Otherwise, the following holds:*

$$\liminf_{t\to\infty} N(t) \leq \Lambda_1 \ \textit{and} \ \limsup_{t\to\infty} N(t) \geq \Lambda_2 \ \ a.s.,$$

where Λ_1 *and* Λ_2 *are defined by (3.13).*

Remark 1*The persistence threshold* $\sum_{j=1}^{m} \mathfrak{I}_j \left[r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2} - \lambda(j) \right]$ *depends on the stationary weights* $(\mathfrak{I}_j)$*, which are determined by the generator matrix* Φ*. Increasing the off-diagonal entries* Υ_{uv} *derives fast switching, pushing* $\mathfrak{I}_j$ *toward the uniform distribution* $\frac{1}{m}$*. Consequently, fast switching promotes persistence when favorable regimes carry higher stationary weight and accelerates extinction otherwise.*

We now investigate the mean persistence of the system described by equation (1.5). To begin, we introduce the following lemma (see Lemma 17 in [41] or Lemma 4 in [42]).

Lemma 3.*Let* $\mathscr{Y} \in \mathscr{C}(\mathbb{R}^+ \times \Omega, \mathbb{R}^+)$ *and* $\mathscr{Z} \in \mathscr{C}(\mathbb{R}^+ \times \Omega, \mathbb{R})$ *such that* $\lim_{t\to\infty} \frac{\mathscr{Z}(t)}{t} = 0 a.s.$*. If for any* $t \geq 0$,

$$\ln \mathscr{Y}(t) \geq \nu_0 t - \nu \int_0^t \mathscr{Y}(s)ds + \mathscr{Z}(t), \tag{3.14}$$

then

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \mathscr{Y}(s)ds \geq \frac{\nu_0}{\nu} \ \ a.s., \tag{3.15}$$

and if

$$\ln \mathscr{Y}(t) \leq \nu_0 t - \nu \int_0^t \mathscr{Y}(s)ds + \mathscr{Z}(t),$$

then

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathscr{Y}(s)ds \leq \frac{\nu_0}{\nu} \ \ a.s., \tag{3.16}$$

where $\nu_0 \geq 0$ *and* $\nu > 0$ *are two positive real values.*

In the following theorem, we will demonstrate the species' long-term survival under various conditions determined by the noise intensities and the parameters of the model. To proceed, we define constants vectors $(\mathscr{E}_j)_{j\in\mathscr{S}}$ and $(d_j)_{j\in\mathscr{S}}$, where

$$\widetilde{cd} = \min_{j\in\mathbb{S}} \{c(j)d(j)\}, \quad \widehat{cd} = \max_{j\in\mathbb{S}} \{c(j)d(j)\}. \tag{3.17}$$

Theorem 5.*Let us consider the stochastic system given by (1.5), where the initial condition is specified within the interval* $(0, \infty)$.

(i)*If*

$$\sum_{j=1}^{m} \mathfrak{I}_j \left[r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2} - \lambda(j) + \frac{\lambda(j)\theta(j)}{\widehat{\theta}} \right] > 0,$$

then

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t N^{\widehat{\theta}}(s)ds \geq \frac{\widehat{\theta}}{\widehat{\lambda\theta}} \sum_{j=1}^{m} \mathfrak{I}_j \Big[r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2} - \lambda(j) + \frac{\lambda(j)\theta(j)}{\widehat{\theta}} \Big].$$

Thus, the species exhibits strong persistence on average.

(ii)*If*

$$\sum_{j=1}^{m} \mathfrak{I}_j \left[r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2} - \lambda(j) + \frac{\lambda(j)\theta(j)}{\widetilde{\theta}} \right] > 0,$$

then

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t N^{\widetilde{\theta}}(s)ds \leq \frac{\widetilde{\theta}}{\widetilde{\lambda\theta}} \sum_{j=1}^{m} \mathfrak{I}_j \Big[r(j) - l_j \bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2} - \lambda(j) + \frac{\lambda(j)\theta(j)}{\widetilde{\theta}} \Big].$$

(iii)If

$$\sum_{j=1}^{m}\Im_j\left[r(j)-l_j\bar{\mathfrak{F}}-\frac{\alpha^2(j)}{2}-\lambda(j)+\frac{\lambda(j)\theta(j)}{\widetilde{\theta}}\right]=0,$$

then

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t N^{\widetilde{\theta}}(s)ds=0.$$

This indicates that the population does not exhibit persistence on average.

Proof.(i) Using (3.4) and the following inequality:

$$z^{\varpi}\le 1+\varpi(z-1)\quad\text{for } z=[N(t)]^{\widehat{\theta}}\ge 0,$$

and

$$\varpi=\frac{\theta\,[\gamma(t)]}{\widehat{\theta}}\le 1.$$

Thus,

$$\begin{aligned}d\log[N(t)]=&\Bigg[r(\gamma(t))-l_{\gamma(t)}\mathscr{E}_0(t)-\frac{1}{2}\alpha^2(\gamma(t))\\&-\lambda(\gamma(t))\left((N(t))^{\widehat{\theta}}\right)^{\frac{\theta(\gamma(t))}{\widehat{\theta}}}\Bigg]dt\\&+\alpha(\gamma(t))dW(t),\\\ge&\Bigg[r(\gamma(t))-l_{\gamma(t)}\mathscr{E}_0(t)-\frac{1}{2}\alpha^2(\gamma(t))\\&-\lambda(\gamma(t))+\frac{\lambda(\gamma(t))\theta(\gamma(t))}{\widehat{\theta}}\\&-\frac{\widehat{\lambda\theta}}{\widehat{\theta}}N(t)^{\widehat{\theta}}\Bigg]dt+\alpha(\gamma(t))dW(t).\end{aligned}\tag{3.18}$$

Based on the principles of ergodic theory for Markov chains, for any $\varepsilon>0$ sufficiently small and t sufficiently large, the following holds:

$$\begin{aligned}&\int_0^t\Bigg[r(\gamma(s))-l_{\gamma(s)}\mathscr{E}_0(s)-\frac{1}{2}\alpha^2(\gamma(s))-\lambda(\gamma(s))\\&+\frac{\lambda(\gamma(s))\theta(\gamma(s))}{\widehat{\theta}}\Bigg]ds\\\ge&\left\{\sum_{j=1}^{m}\Im_j\left[r(j)-l_j\bar{\mathfrak{F}}-\frac{\alpha^2(j)}{2}-\lambda(j)+\frac{\lambda(j)\theta(j)}{\widehat{\theta}}\right]-\varepsilon\right\}t.\end{aligned}\tag{3.19}$$

Integrating (3.18) and using (3.19), one obtains

$$\begin{aligned}\log\left[N^{\widehat{\theta}}(t)\right]\ge\widehat{\theta}\Bigg\{&\sum_{j=1}^{m}\Im_j\Bigg[r(j)-l_j\bar{\mathfrak{F}}-\frac{\alpha^2(j)}{2}-\lambda(j)\\&+\frac{\lambda(j)\theta(j)}{\widehat{\theta}}\Bigg]-\varepsilon\Bigg\}t-\widehat{\lambda\theta}\int_0^t N^{\widehat{\theta}}(s)ds\\&+Z(t),\end{aligned}$$

with, $Z(t)=\widehat{\theta}\int_0^t\alpha[\gamma(s)]dW(s)+\widehat{\theta}\log[N(0)]$.
By (3.15), one obtains

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\int_0^t N^{\widehat{\theta}}(s)ds\\\ge&\frac{\widehat{\theta}}{\widehat{\lambda\theta}}\Bigg\{\sum_{j=1}^{m}\Im_j\Bigg[r(j)-l_j\bar{\mathfrak{F}}-\frac{\alpha^2(j)}{2}\\&-\lambda(j)+\frac{\lambda(j)\theta(j)}{\widehat{\theta}}\Bigg]-\varepsilon\Bigg\}a.s..\end{aligned}\tag{3.20}$$

We arrive at the desired estimate by taking the limit as $\varepsilon\to 0$.
(ii) Using (3.4) and the corresponding inequality,

$$1+\varpi(y-1)<y^{\varpi},\quad\text{for } y=[N(t)]^{\widetilde{\theta}}\ge 0,\tag{3.21}$$

and

$$\varpi=\frac{\theta[\gamma(t)]}{\widetilde{\theta}}\ge 1.$$

So,

$$\begin{aligned}d\log[N(t)]=&\Bigg\{r(\gamma(t))-l_{\gamma(t)}\mathscr{E}_0(t)-\frac{1}{2}\alpha^2(\gamma(t))\\&-\lambda(\gamma(t))\left((N(t))^{\widetilde{\theta}}\right)^{\frac{\theta(\gamma(t))}{\widetilde{\theta}}}\Bigg\}dt\\&+\alpha(\gamma(t))dW(t)\\\le&\Bigg\{r(\gamma(t))-l_{\gamma(t)}\mathscr{E}_0(t)-\frac{1}{2}\alpha^2(\gamma(t))\\&-\lambda(\gamma(t))+\frac{\lambda(\gamma(t))\theta(\gamma(t))}{\widetilde{\theta}}-\frac{\widetilde{\lambda\theta}}{\widetilde{\theta}}N(t)^{\widetilde{\theta}}\Bigg\}dt\\&+\alpha(\gamma(t))dW(t),\end{aligned}\tag{3.22}$$

From (3.16) and (3.22), and applying a similar technique as in (3.20), we can easily deduce

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\int_0^t N^{\widetilde{\theta}}(s)ds\\\le&\frac{\widetilde{\theta}}{\widetilde{k\theta}}\Bigg[\sum_{j=1}^{m}\Im_j\left(r(j)-l_j\bar{\psi}-\frac{\alpha^2(j)}{2}-\lambda(j)+\frac{\lambda(j)\theta(j)}{\widetilde{\theta}}\right)\\&+\varepsilon\Bigg]a.s.,\\\le&\frac{\widetilde{\theta}}{\widetilde{k\theta}}\Bigg[\sum_{j=1}^{m}\Im_j\left(r(j)-l_j\bar{\mathfrak{F}}-\frac{\alpha^2(j)}{2}-\lambda(j)+\frac{\lambda(j)\theta(j)}{\widetilde{\theta}}\right)\\&+\varepsilon\Bigg]a.s..\end{aligned}$$

We obtain the wanted result by taking the limit as $\varepsilon\to 0$.
(iii) This result directly follows from assertion (ii).

4 Stationary Distribution

We investigate whether the solution of SDE (1.5) admits an asymptotically invariant distribution, which, if established, would indicate stochastic stability. Recall that the process $N_t^a = [N(t),\gamma(t)]$, defined on the state space $(0,\infty)\times\mathcal{S}$ with the initial condition $N_0 = a$, is considered recurrent with respect to a bounded set $\mathcal{U}\subset(0,\infty)\times\mathcal{S}$ if

$$\mathbb{P}(\tau^a<\infty)=1,\ \ a\in\mathcal{U}^c.$$

In this context, τ^a denotes the stopping time associated with the set $\mathcal{U}$ for the process N_t^a, and is formally defined as:

$$\tau^N = \inf\{t>0, N_t^a\in\mathcal{U}\}.$$

The process N_t^a is considered positively recurrent with respect to the set $\mathcal{U}$ if

$$\mathbb{E}(\tau^a)<\infty,\ \text{ for each }\ a\notin\mathcal{U}.$$

To establish ergodicity and positive recurrence, we impose the condition that $\Upsilon_{ij}>0$ for $i\neq j$. Given this assumption, condition (A) from [43] holds for any bounded open subset of $(0,\infty)$. The subsequent theorem establishes a positive recurrence criterion, leveraging a Lyapunov function as detailed in [43].

Theorem 6.*A necessary and sufficient condition for the positive recurrence of set $\mathcal{U}=\mathcal{D}\times\{j\}\subset(0,\infty)\times\mathcal{S}$ is that for each $j\in\mathcal{S}$, there exists a nonnegative function $V(\cdot,j):\mathcal{D}^c\to\mathbb{R}$ that satisfies the following conditions: $V(\cdot,j)$ is twice continuously differentiable, and there exists a constant $\mathfrak{M}>0$ such that*

$$\mathcal{L}V(N,j)\le-\mathfrak{M},\ \ \textit{for every}\ \ (N,j)\in\mathcal{D}^c\times\mathcal{S}. \tag{4.1}$$

Furthermore, the Markov process $N(t)$ is linked to a unique ergodic stationary distribution ν. In particular, if $\mathfrak{F}$ is a function integrable with respect to the distribution ν, then

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{1}{t}\int_0^t\mathfrak{F}(N(t))\,du=\int_0^\infty\mathfrak{F}(N)\nu(dN)\right]=1.$$

The following theorem provides a sufficient condition for a stationary distribution in the system defined by equation (1.5).

Theorem 7.*Consider the stochastic system (1.5), with the initial condition taken from the interval $(0,\infty)$. If*

$$\sum_{j=1}^{m}\mathfrak{I}_j\left[r(j)-l_j\inf_{t\ge0}\mathcal{E}_0(t)-\frac{\alpha^2(j)}{2}\right]>0. \tag{4.2}$$

Thus, the process $N(t)$ is positive recurrent and has a unique ergodic distribution ν, which is asymptotically invariant.

*Proof.*Let $\mathcal{D}=\left(d^{-1},d\right)\subset(0,\infty)$, where d is a suitably large constant. Next, define the following positive functions:

$$V(N,j)=[1-\zeta\vartheta_j]N^{-\zeta}+N. \tag{4.3}$$

Here, ζ denotes a sufficiently small positive constant fulfilling $\zeta<\min\limits_{j=1,\dots,m}\left\{\frac{1}{\vartheta_j}\right\}$, where the vector $\vartheta=(\vartheta_1,\dots,\vartheta_m)$ will be defined in the forthcoming steps of the demonstration. Upon applying the operator $\mathcal{L}$ to V, we derive

$$\begin{aligned}\mathcal{L}V(N,j)&=-\zeta(1-\zeta\vartheta_j)\left[r(j)-l_j\mathcal{E}_0(t)-\lambda(j)N^{\theta(j)}\right]\\&\quad\times N^{-\zeta}+\frac{1}{2}\zeta(\zeta+1)(1-\zeta\vartheta_j)\alpha^2(j)N^{-\zeta}\\&\quad-\zeta N^{-\zeta}\sum_{l\neq j\in\mathcal{S}}\Upsilon_{jl}(\vartheta_l-\vartheta_j)\\&\quad+N\left[r(j)-l_j\mathcal{E}_0(t)-\lambda(j)N^{\theta(j)}\right].\end{aligned}$$

This means

$$\begin{aligned}\mathcal{L}V(N,j)&=-\zeta(1-\zeta\vartheta_j)\left[r(j)-\frac{1}{2}\alpha^2(j)-l_j\mathcal{E}_0(t)\right.\\&\quad-\frac{1}{2}\zeta\alpha^2(j)+\frac{\zeta\vartheta_j}{1-\zeta\vartheta_j}\sum_{l\neq j\in\mathcal{S}}\Upsilon_{jl}(\vartheta_l-\vartheta_j)\\&\quad\left.+\sum_{l\neq j\in\mathcal{S}}\Upsilon_{jl}(\vartheta_l-\vartheta_j)\right]N^{-\zeta}\\&\quad+\zeta(1-\zeta\vartheta_j)\lambda(j)N^{\theta(j)-\zeta}\\&\quad+N\left[r(j)-l_j\mathcal{E}_0(t)-\lambda(j)N^{\theta(j)}\right].\end{aligned} \tag{4.4}$$

Now, define the vector $\psi=(\psi_1,\dots,\psi_m)$, where

$$\psi_j=r(j)-\frac{1}{2}\alpha^2(j).$$

Alternatively, if the generator matrix Φ is irreducible, then for $\psi=(\psi_1,\dots,\psi_m)^T$, there exists a solution $\vartheta^T=(\vartheta_1,\dots,\vartheta_m)^T$ to the Poisson system (see Lemma 2.3 in [36]), such that

$$\Phi\vartheta=\psi-\left(\sum_{j=1}^{m}\mathfrak{I}_j\psi_j\right)(1,1,\dots,1)^T.$$

Thus, we have

$$\begin{aligned}\sum_{l\neq j\in\mathcal{S}}\Upsilon_{jl}(\vartheta_l-\vartheta_j)&=-\left[r(j)-\frac{\alpha^2(j)}{2}\right]\\&\quad-\sum_{j=1}^{m}\mathfrak{I}_j\left[r(j)-\frac{\alpha^2(j)}{2}\right].\end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), we get

$$\begin{aligned}\mathcal{L}V(N,j)&\le-\zeta(1-\zeta\vartheta_j)\left\{\sum_{j=1}^{m}\mathfrak{I}_j\left[r(j)-l_j\inf_{t\ge0}\mathcal{E}_0(t)\right.\right.\\&\quad\left.\left.-\frac{\alpha^2(j)}{2}\right]+o(\zeta)\right\}N^{-\zeta}\\&\quad+\zeta(1-\zeta\vartheta_j)k\lambda(j)N^{\theta(j)-\zeta}+\left[r(j)\right.\\&\quad\left.-l_j\inf_{t\ge0}\mathcal{E}_0(t)-\lambda(j)N^{\theta(j)}\right]N,\\&=F(N,j).\end{aligned} \tag{4.6}$$

Then

$$F(N,j) \underset{N\longrightarrow 0}{\sim} -\zeta(1-\zeta\vartheta_j)\times\left\{\sum_{j=1}^{m}\mathfrak{J}_j\left[r(j) - l_j\inf_{t\geq 0}\mathscr{E}_0(t)-\frac{\alpha^2(j)}{2}\right]+o(\zeta)\right\}N^{-\zeta}, \tag{4.7}$$

and

$$F(N,j) \underset{N\longrightarrow +\infty}{\sim} -\lambda(j)N^{\theta(j)+1}. \tag{4.8}$$

Select a sufficiently small value for ζ such that

$$\sum_{j=1}^{m}\mathfrak{J}_j\left[r(j)-l_j\inf_{t\geq 0}\mathscr{E}_0(t)-\frac{\alpha^2(j)}{2}\right]+o(\zeta)>0, \tag{4.9}$$

which is allowed by the assumption (4.2). By (4.7), (4.8) and (4.9), it is easy to see that for a sufficiently large γ,

$$\mathscr{L}V(N,j)\leq -1, \quad \text{for all} \quad (N,j)\in\mathscr{D}^c\times\mathscr{S}.$$

The proof is completed according to Theorem 6.

Based on Theorem 7, we derive the following result, which demonstrates the stochastic permanence of the population model defined by equation (1.5).

Corollary 2 *Consider the stochastic system described by equation (1.5) with an initial condition within the interval* $(0,\infty)$. *If*

$$\sum_{j=1}^{m}\mathfrak{J}_j\left[r(j)-l_j\inf_{t\geq 0}\mathscr{E}_0(t)-\frac{\alpha^2(j)}{2}\right]>0, \tag{4.10}$$

then, the system described by equation (1.5) exhibits stochastic permanence.

Proof. To begin, note that $\hat{\mu}\left[(0,\infty)\right]=1$. This outcome follows from the ergodic properties of $N(t)$, implying that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{I}_{\{N(s)\in(0,\infty)\}}ds &= \int_0^\infty\mathbb{I}_{(0,\infty)}(N)\mathfrak{J}(dN),\\ &= \hat{\mu}\left[(0,\infty)\right].\end{aligned} \tag{4.11}$$

In this context, $\mathbb{I}_A$ denotes the characteristic function of the set A. Given that $N(t)\in(0,\infty)$ for all t, it follows that $\hat{\mu}\left[(0,\infty)\right]=1$. Consequently, $\hat{\mu}$ represents the stationary distribution of $N(t)$ in the asymptotic sense. Therefore, for any positive constants c and d, we obtain the following relation:

$$\liminf_{t\to\infty}\mathbb{P}\left[N(t)\geq c\right]=\hat{\mu}\left([c,\infty)\right), \tag{4.12}$$

and

$$\liminf_{t\to\infty}\mathbb{P}\left[N(t)\leq d\right]=\hat{\mu}\left[(0,d)\right].$$

In addition

$$\lim_{c\to 0}\hat{\mu}\left([c,\infty)\right)=\lim_{d\to\infty}\hat{\mu}\left[(0,d)\right]=\hat{\mu}\left[(0,\infty)\right]=1.$$

Thus, for $\varepsilon\in(0,1)$, there exist constants $\beta_1>0$ and $\beta_2>0$, with β_1 being sufficiently small and β_2 sufficiently large,

$$\hat{\mu}\left([\beta_1,\infty)\right)\geq 1-\varepsilon, \text{ and } \hat{\mu}\left((0,\beta_2]\right)\geq 1-\varepsilon,$$

This, together with equation (4.13), implies that

$$\liminf_{t\to\infty}\mathbb{P}\left[N(t)\geq\beta_1\right]\geq 1-\varepsilon,$$

and

$$\liminf_{t\to\infty}\mathbb{P}\left[N(t)\leq\beta_2\right]\geq 1-\varepsilon.$$

The proof is finished.

5 Computer Simulations

In this section, we present numerical examples that illustratethe theoretical findings. We apply the Euler scheme (refer to [44] for further details and the references therein) to demonstrate our results. Let $(\gamma(t))_{t\geq 0}$ be a right-continuous Markov chain defined on the state space $\mathscr{S}=\{1,2,3\}$, with its generator specified as:

$$\Theta=\begin{pmatrix}-2 & 1 & 1\\ 3 & -4 & 1\\ 1 & 1 & -2\end{pmatrix}.$$

The Markov chain can be simulated with a given step size $\Delta>0$ by computing the one-step transition probability matrix, which is expressed as $\mathscr{P}=\exp(\Delta\Theta)$. For an in-depth explanation, the reader is encouraged to consult Anderson [45]. Thus, for $\Delta=0.0001$, the transition probability matrix [46–49] and the corresponding stationary distribution [50–54] are given as follows:

$$\mathscr{P}=\begin{pmatrix}0.9998 & 0.0001 & 0.0001\\ 0.0003 & 0.9996 & 0.0001\\ 0.0001 & 0.0001 & 0.9998\end{pmatrix},$$

and

$$(\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3)=(0.4667,\,0.2000,\,0.3333).$$

In the subsequent figures, we select

$$\mathscr{E}_0(t)=0.1+0.05sin(t). \tag{5.1}$$

5.1 Extinction

Example 1 *We now provide numerical simulations to illustrate Theorem 2, which establishes the global asymptotic stability of the trivial solution for equation (1.5). The simulations employ parameter values from Example 8 and examine system behavior across different environmental regimes characterized by the discrete state* $(\gamma(t))$. *The results demonstrate how colored noise strengthens probabilistic stability and reveal the thresholds separating extinction from persistence.*
Set

$$N(0)=0.5,\ \ r=(0.4,0.5,0.3),\ \ \lambda=(1,1.1,1.1),\quad l=(0,1,2),$$

and

$$\alpha=\left(\sqrt{0.64},\sqrt{0.6},\sqrt{0.6}\right),\ \ \theta=(0.5,0.4,0.6).$$

This gives

$$\left[r(1)-\frac{\alpha^2(1)}{2}-l_1\bar{\mathfrak{F}}\right]=0.08,$$
$$\left[r(2)-\frac{\alpha^2(2)}{2}-l_2\bar{\mathfrak{F}}\right]=0.1,$$
$$\left[r(3)-\frac{\alpha^2(3)}{2}-l_3\bar{\mathfrak{F}}\right]=-0.2.$$

Or,

$$\left[r(1)-\frac{\alpha^2(1)}{2}-l_1\bar{\mathfrak{F}}\right]>0,$$
$$\left[r(2)-\frac{\alpha^2(2)}{2}-l_2\bar{\mathfrak{F}}\right]>0,$$

thus, the stability of the systems $[N(t),\gamma=1]$ *and* $[N(t),\gamma=2]$ *remains undetermined. Nonetheless, for the model defined in equation (1.5), which is subject to Markovian noise, the stability characteristics can be further analyzed by:*

$$\sum_{j=1}^{3}\mathfrak{I}_j\left[r(j)-\frac{\alpha^2(j)}{2}-l_j\bar{\mathfrak{F}}\right]=-0.009324<0,$$

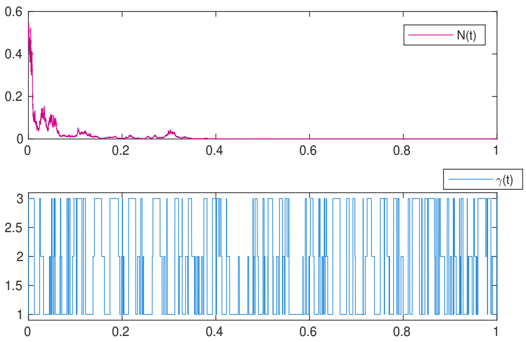

Fig. 1. This figure displays the trajectory of $N(t)$ for the full system.

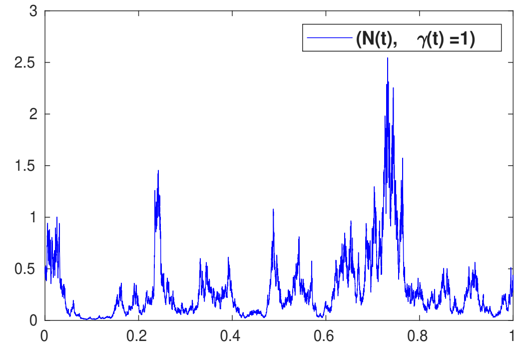

Fig. 2. This plot shows the dynamics of $N(t)$ specifically when the system is locked in an environmental regime $\gamma(t)=1$.

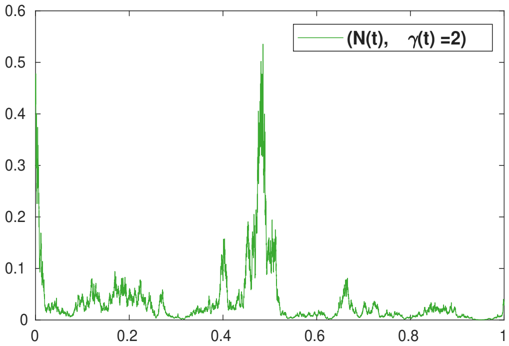

Fig. 3. Here, the simulation depicts the behavior of $N(t)$ under the fixed regime $\gamma(t)=2$. Contrary to Regime 1.

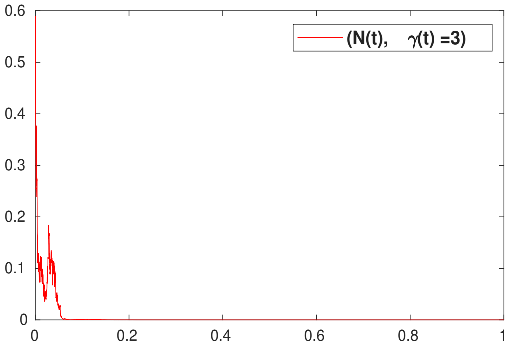

Fig. 4. This figure illustrates the trajectory of $N(t)$ within the fixed environmental regime $\gamma(t)=3$.

5.2 Persistence

Example 2*To confirm the persistence threshold given in Theorem 4, we set*

$$N(0) = 0.5,\ \ r = (0.7, 0.85, 0.96),\ \ \lambda = (0.2, 0.2, 0.1),\\ l = (3, 1, 0.1),$$

and

$$\alpha = \left(\sqrt{0.62}, \sqrt{0.5}, \sqrt{0.4}\right),\ \ \theta = (0.01, 0.02, 0.09).$$

This implies

$$\left[r(1) - l_1\bar{\mathfrak{F}} - \frac{\alpha^2(1)}{2} - \lambda(1)\right] = -0.110,$$
$$\left[r(2) - l_2\bar{\mathfrak{F}} - \frac{\alpha^2(2)}{2} - \lambda(2)\right] = 0.300,$$
$$\left[r(3) - l_3\bar{\mathfrak{F}} - \frac{\alpha^2(3)}{2} - \lambda(3)\right] = 0.650.$$

Consequently, the persistence of the system $[N(t), \gamma = 1]$ *cannot be confirmed. Still,*

$$\sum_{j=1}^{3} \mathfrak{I}_j \left[r(j) - l_j\bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2} - \lambda(j)\right] = 0.225308 > 0.$$

Theorem 4 subsequently guarantees the persistence of the pair $[N(t), \gamma(t)]$*, as stated by:*

$$\limsup_{t\to\infty} N(t) \geq \Lambda_1 = \left[\frac{\sum_{j=1}^{m} \mathfrak{I}_j \left(r(j) - l_j\bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2}\right)}{\sum_{j=1}^{m} \mathfrak{I}_j \lambda(j)}\right]^{\frac{1}{\theta}}\\ = 121.3334,$$

and

$$\liminf_{t\to\infty} N(t) \leq \Lambda_2 = \left[\frac{\sum_{j=1}^{m} \mathfrak{I}_j \left(r(j) - l_j\bar{\mathfrak{F}} - \frac{\alpha^2(j)}{2}\right)}{\sum_{j=1}^{m} \mathfrak{I}_j \lambda(j)}\right]^{\frac{1}{\theta}}\\ = 2120.8.$$

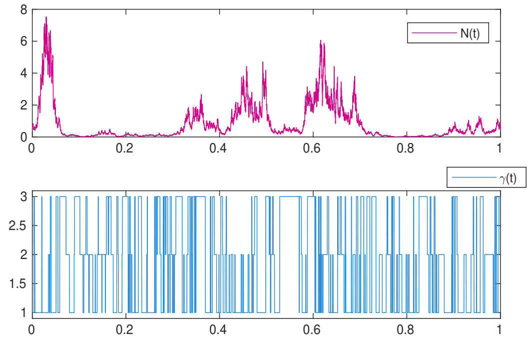

Fig. 5. This figure displays the trajectory of $N(t)$ for the full system.

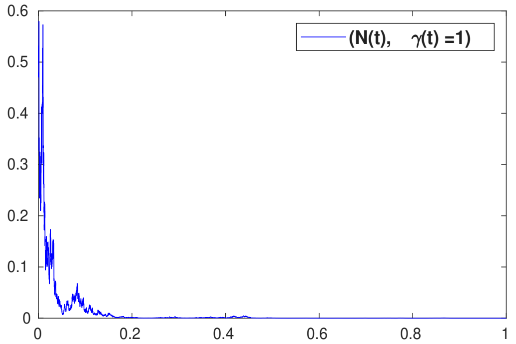

Fig. 6. This plot shows the dynamics of $N(t)$ specifically when the system is locked in environmental regime $\gamma(t) = 1$.

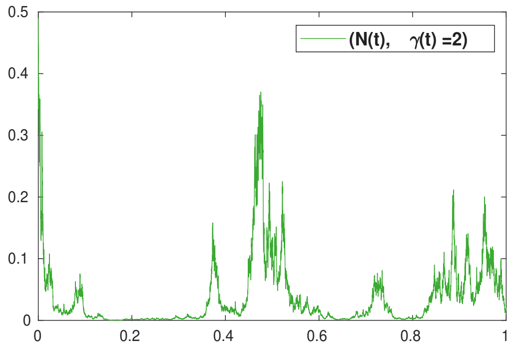

Fig. 7. Here, the simulation depicts the behavior of $N(t)$ under the fixed regime $\gamma(t) = 2$. Contrary to Regime 1.

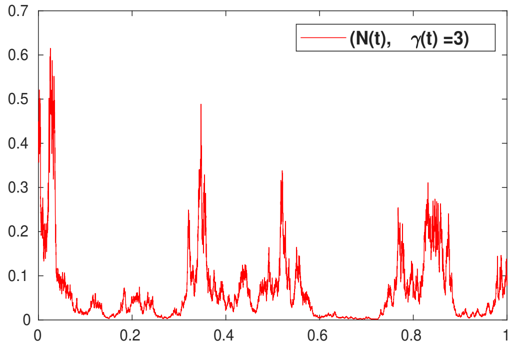

Fig. 8. This figure illustrates the trajectory of $N(t)$ within the fixed environmental regime $\gamma(t) = 3$.

6 Conclusion

This study has provided a rigorous analysis of the long-term dynamics of a stochastic Gilpin-Ayala population model operating in a polluted environment under Markovian regime switching. By integrating toxicant concentration dynamics, environmental stochasticity, and random regime shifts, we have established a mathematically precise framework for predicting population fate, emphasizing the critical role of pollution as a determinant of ecological outcomes. Our central theoretical contribution is the derivation of explicit, threshold-like criteria that definitively separate the regimes of population extinction and persistence. Specifically, we have shown that a species faces inevitable extinction when a weighted average of net growth rates across environmental regimes—adjusted for pollution load and stochastic noise—falls below a critical threshold. Conversely, when this key parameter is positive, the population is proven to exhibit weak persistence and possesses a unique ergodic stationary distribution, guaranteeing its long-term survival within a statistical equilibrium. The power of this approach was demonstrated through computational simulations, which validated the theoretical thresholds and confirmed that population viability is determined by a compound measure across all fluctuating regimes, rather than conditions in a single state. These findings underscore the necessity of incorporating pollution and environmental stochasticity as key components of ecological models. The identified threshold provides a quantifiable target for conservation efforts, opening the door to more accurate predictions and effective strategies. As environmental pollutants exert an increasingly greater impact globally, this holistic framework offers a vital tool for risk assessment and informed ecosystem management.

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