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A Generalized Local Fractional Laplace Transform via a Unified Derivative Operator

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Abstract: Recently, generalized local fractional derivatives have attracted increasing attention due to their ability to model nonlocal and scale-dependent phenomena while preserving locality. In this paper, we introduce a generalized local fractional Laplace transform constructed through a unified derivative operator. This new transform extends the classical Laplace transform to local fractional orders while retaining its fundamental operational properties. The proposed framework provides a consistent analytical tool for the treatment of differential equations involving generalized local fractional derivatives and establishes a natural bridge between classical and local fractional analysis.

Keywords: Generalized local fractional derivative, generalized local fractional integral

1 Introduction

The Laplace transform is one of the most important analytical tools in applied mathematics and engineering, as it allows transforming problems involving differential equations into algebraic problems that are easier to solve. Originally introduced in the context of classical analysis, the Laplace transform has become a fundamental technique for analyzing linear systems, studying stability, and solving initial value problems.

In recent decades, fractional calculus has attracted increasing attention as an extension of differentiation and integration to non-integer orders. This perspective has proven especially valuable for describing systems that exhibit memory effects or behaviors intermediate between discrete and continuous models.

Within this framework, local formulations of fractional derivatives have become particularly relevant due to their computational convenience and their ability to preserve important properties of classical calculus, such as linearity and the product rule.

The development of local fractional operators is closely connected with the historical evolution of fractional calculus itself.

Fractional calculus is almost as old as classical calculus itself. In its early developments, fractional derivatives were mainly defined through global integral formulations, which led to the loss of the local character naturally associated with the classical derivative (see [11], [15]).

Despite these limitations, fractional calculus has progressively gained relevance in several scientific and technological disciplines. Since the 1960s, different formulations of fractional derivatives have emerged in various contexts, and more recently, local fractional derivatives have been introduced in order to overcome some of the drawbacks associated with global operators (see [2], [3], [4], [5], [6], [18], [19], [28], [30], [31], [32], [33], [34]).

However, it was not until 2014 that Khalil introduced the conformable derivative (see [10]), whose definition is recalled below.

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Definition 1.1

Let $\gamma:[0,\infty)\to\mathbb{R}$. The conformable derivative of order α, with $0<\alpha\leq 1$, is defined as:

$$T_\alpha\gamma(t)=\lim_{\varepsilon\to 0}\frac{\gamma(t+\varepsilon t^{1-\alpha})-\gamma(t)}{\varepsilon},\quad t>0.$$

This operator constitutes a local derivative that preserves several important properties not generally satisfied by global fractional derivatives. Numerous studies related to this derivative have been developed in recent years (see [14], [22], [23], [25], [29]).

Moreover, this definition generalizes the classical derivative, since for $\alpha=1$ the ordinary derivative is recovered. In addition, if γ is differentiable on some interval, then $T_\alpha\gamma(0)$ is also well-defined.

Subsequently, in 2017, Priyanka Ahuja introduced the deformable derivative, another local fractional formulation possessing similar structural properties (see [1]).

Definition 1.2

Let $\beta\in[0,1]$. The deformable derivative of a function $F:\mathbb{R}\to\mathbb{R}$ is given by:

$$\mathscr{D}^\beta\mathscr{F}(t)=\lim_{\varepsilon\to 0}\frac{(1+\varepsilon(1-\beta))\mathscr{F}(t+\varepsilon\beta)-\mathscr{F}(t)}{\varepsilon}$$

whenever this limit exists.

Additional contributions concerning local fractional derivatives and related operators can be found in [12], [13], [16], [17], [20], [21], [26].

In Definition 1.3, an even more general extension of the derivative concept is introduced, as presented in [8].

Definition 1.3 (see [8])

Let $\mathscr{F}:[0,+\infty)\to\mathbb{R}$ be a function. The CN-composite derivative of $\mathscr{F}$ of order β is defined as

$$\mathscr{F}^\beta_{CN}(a)=\lim_{t\to a}\frac{\mathscr{F}(N(t,\beta))-\mathscr{F}(N(a,\beta))}{t-a},\quad \beta\in(0,1].$$

The flexibility of this derivative makes it especially suitable for extending the classical notion of local differentiation to more general analytical settings.

With the purpose of further extending the framework of local fractional derivatives, Vivas et al. introduced in May 2025 (see [27]) a new two-parameter formulation that provides a more versatile structure for modeling behaviors influenced by combined effects.

Definition 1.4

Let $g:\mathbb{R}\longrightarrow\mathbb{R}$, with $\alpha\geq 0$ and $\beta>0$. The V-derivative (Biparametric) is defined as

$$V^{\alpha,\beta}(g(x)):=\lim_{h\to 0}\frac{(\beta+h(\beta-\alpha))\,g\left(x+h\dfrac{\alpha}{\beta}\right)-\beta g(x)}{\beta\cdot h}.$$

Based on this formulation, Vivas et al. obtained in November 2025 (see [24]) the biparametric Laplace transform, an associated analytical tool that extends the applicability of the V-derivative in the analysis of fractional differential equations.

More recently, in June 2025, Soon and Elaydi et al. [7] introduced the generalized local fractional derivative, whose definition will be presented later. This development reinforces the necessity of extending the Laplace transform to the setting of generalized local fractional derivatives, thereby providing a coherent analytical framework capable of unifying previous operators while offering greater flexibility for the study of dynamical systems.

Following this line of research, several authors have proposed modified versions of the Laplace transform adapted to different classes of fractional derivatives. However, many of these transforms are specifically designed for particular derivative operators and do not provide a unified analytical framework.

Recently, S. Jain et al. (2025) introduced the extended integral Laplace transform [9], whose formula is

$$F_\alpha^{(\alpha_1,\beta_1)}(f(t))=F_\alpha^{(\alpha_1,\beta_1)}(v)=\frac{e^{-\beta_1 v}}{(\alpha,v)^n}\int_0^\infty f(t)\,a^{-(\alpha_1 v)t}\,dt.$$

Motivated by these developments, the present work introduces the generalized local fractional Laplace transform (UGLT), constructed from the generalized local fractional derivative operator (UG). The proposed transform recovers the classical Laplace transform as a particular case while simultaneously extending its applicability to generalized local fractional settings. Consequently, it provides mathematicians, physicists, and engineers with a broader and more flexible analytical framework for the treatment of fractional differential equations.

The remainder of this paper is organized as follows. In Section 2, we recall the definitions of the generalized local fractional derivative (UG) and the generalized local fractional integral (UGI), together with their fundamental properties, as introduced in [7]. In Section 3, we present the generalized local fractional Laplace transform (UGLT), establish its existence, define the corresponding inverse transform (UGILT), and derive its main analytical properties. Finally, Section 4 presents several fully worked-out examples illustrating the effectiveness of the UGLT in solving generalized local fractional differential equations arising in different applied contexts.

Notation

We summarize below the main notation used throughout this work.

Table 1: List of symbols and notations used throughout the paper

Symbol	Description
α,β	Fractional parameters associated with generalized local fractional operators.
$g(x)$	Function involved in the biparametric V-derivative.
$V^{\alpha,\beta}$	Biparametric V-derivative operator.
$D_{UG}^{(\alpha,A,\psi)}$	Generalized local fractional derivative operator (UG).
$I_{UG}^{(\alpha,A,\psi)}$	Generalized local fractional integral operator (UGI).
$A(\alpha)$	Auxiliary function associated with the UG operator.
$\psi(x,\alpha)$	Kernel-type function in the generalized operator formulation.
UGLT	Generalized local fractional Laplace transform.
UGILT	Inverse generalized local fractional Laplace transform.
$F(s)$	Laplace image of a function under the UGLT framework.

2 Basic Definitions

The unified and generalized local fractional derivative, denoted by $D_{UG}^{(\alpha,\mathscr{A},\psi)}$, is defined as follows:

Definition 2.1

Given a function $f:[0,\infty)\to\mathbb{R}$, the UG derivative of f of order α, denoted by $D_{UG}^{(\alpha,\mathscr{A},\psi)}f(x)$, is defined by:

$$D_{UG}^{(\alpha,\mathscr{A},\psi)}f(x) = \lim_{\lambda\to 0}\frac{(1+\lambda\mathscr{A}(\alpha))f\big(x+\lambda\psi(x,\alpha)\big)-f(x)}{\lambda}.$$

where α is the order of the derivative, $\mathscr{A}(\alpha)$ is a function of the order α, $\psi(x,\alpha)$ is a function of x and α, and λ is a positive real number.

Remarks

–If the limit in the definition exists, then f is said to be a UG–differentiable function (UG$_\alpha$–differentiable function).

It is important to emphasize that the above definition allows several possible choices for the functions $\mathscr{A}(\alpha)$ and $\psi(x,\alpha)$, which makes it possible to recover different derivative operators as particular cases. Thus, by considering

$$\mathscr{A}(\alpha)=1-\alpha=\beta \qquad \text{and} \qquad \psi(x,\alpha)=\alpha,$$

one obtains, as a particular case, the deformable derivative. Likewise, the biparametric derivative can be viewed as a particular case of the generalized local fractional derivative by taking

$$\mathscr{A}(\alpha)=1-\tfrac{\alpha}{\beta} \qquad \text{and} \qquad \psi(x,\alpha)=\tfrac{\alpha}{\beta}.$$

Theorem 2.1

If $f:[0,\infty)\to\mathbb{R}$ is a UG$_\alpha$–differentiable function for $x>0$, then

$$D_{UG}^{(\alpha,\mathscr{A},\psi)}f(x) = \mathscr{A}(\alpha)f(x)+\psi(x,\alpha)f'(x).$$

Theorem 2.2

For $\alpha\in(0,1)$. Suppose that $f,g:[0,\infty)\to\mathbb{R}$ are UG$_\alpha$–differentiable functions for $x>0$, then the following properties hold:

–**Linearity.**
For any functions $f(x)$ and $g(x)$, and constants a,b, we have:

$$D_{UG}^{(\alpha,\mathscr{A},\psi)}\big[af(x)+bg(x)\big] = aD_{UG}^{(\alpha,\mathscr{A},\psi)}f(x)+bD_{UG}^{(\alpha,\mathscr{A},\psi)}g(x).$$

–**Product rule.**
For the product of two functions $f(x)$ and $g(x)$, we have:

$$D_{UG}^{(\alpha,\mathscr{A},\psi)}\big[f(x)g(x)\big] = g(x)D_{UG}^{(\alpha,\mathscr{A},\psi)}f(x)+\psi(x,\alpha)f(x)g'(x).$$

–**Commutativity.**
For two functions $f(x)$ and $g(x)$, we have:

$$D_{UG}^{(\alpha,\mathscr{A},\psi)}\big[f(x)g(x)\big]=D_{UG}^{(\alpha,\mathscr{A},\psi)}\big[g(x)f(x)\big].$$

–**Quotient rule.**
For the quotient of two functions $f(x)$ and $g(x)$:

$$D_{UG}^{(\alpha,\mathscr{A},\psi)}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)D_{UG}^{(\alpha,\mathscr{A},\psi)}f(x)-f(x)D_{UG}^{(\alpha,\mathscr{A},\psi)}g(x)+\mathscr{A}f(x)g(x)}{g(x)^2}.$$

Now we define the generalized local fractional integral $UGI_{UG}^{(\alpha,\mathscr{A},\psi)}$ as follows:

Definition 2.2

For $x > 0$ and $\alpha \in (0,1)$, if f is a function defined on the interval $(0,x]$, the generalized local fractional integral of f of order α, denoted by $I_{UG}^{(\alpha,\mathscr{A},\psi)}$, is defined by:

$$I_{UG}^{(\alpha,\mathscr{A},\psi)} f(x) = \int_0^x \frac{1}{\psi(t,\alpha)} e^{\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(r,\alpha)} dr} f(t)\, dt.$$

For $I_{UG}^{(\alpha,\mathscr{A},\psi)}$ to be well-defined, the following conditions must be satisfied:

– $\psi(t,\alpha) \neq 0$ for all $t \in (0,x]$ and $\alpha \in (0,1)$.
– $\psi(\cdot,\alpha)$ is continuous (or at least piecewise continuous and nonzero) on $[0,x]$.
– $\int_0^x \frac{\mathscr{A}(\alpha)}{\psi(r,\alpha)} dr$ is finite for $0 < x < \infty$.
– f is continuous on $[0,x]$ (or sufficiently integrable).

Remarks

– If the UGI exists, then f is said to be a UGI–integrable function (UGI_α–integrable function).

Theorem 2.3

For $x > 0$, if f is a UGI_α–integrable function, then

$$D_{UG}^{(\alpha,\mathscr{A},\psi)} \left(I_{UG}^{(\alpha,\mathscr{A},\psi)} f(x) \right) = f(x),$$

and

$$I_{UG}^{(\alpha,\mathscr{A},\psi)} \left(D_{UG}^{(\alpha,\mathscr{A},\psi)} f(x) \right) = f(x) - f(0)\, e^{-\int_0^x \frac{\mathscr{A}(\alpha)}{\psi(r,\alpha)} dr},$$

where $\alpha \in (0,1)$.

3 Generalized Local Fractional Laplace Transform

In this section, we introduce the definitions of the generalized local fractional exponential function and the generalized local fractional Laplace transform, as well as some of their properties.

Definition 3.1

For some points $s,t \in \mathbb{R}$ with $s \leq t$, the exponential function with respect to $D_{UG}^{(\alpha,\mathscr{A},\psi)}$ is defined as

$$e_{UG}^{(\alpha,\mathscr{A},\psi)}(t,s) := e^{-\int_s^t \frac{\mathscr{A}(\alpha)}{\psi(\mu,\alpha)} d\mu},$$

where $\alpha \in (0,1)$.

Definition 3.2

Let $f : [0,\infty) \to \mathbb{R}$. For $\alpha \in (0,1)$, we define the generalized local fractional Laplace transform of f of order α, denoted by $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}$, as:

$$\begin{aligned}\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) &= F^{(\alpha,\mathscr{A},\psi)}(s) \\ &:= \int_0^\infty e^{-st}\, e_{UG}^{(\alpha,\mathscr{A},\psi)}(t,0) \frac{f(t)}{\psi(t,\alpha)} dt \\ &= \int_0^\infty e^{-st-\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(\mu,\alpha)} d\mu} \frac{f(t)}{\psi(t,\alpha)} dt,\end{aligned}$$

where $\alpha \in (0,1)$ and $s \in \mathbb{C}$.

The generalized local fractional Laplace transform will be referred to as the UGLT.

It is observed that, by taking $\psi(t,\alpha) \equiv 1$ and $\mathscr{A}(\alpha) = \frac{\beta - \alpha}{\alpha}$, the biparametric Laplace transform defined in [24] constitutes a particular case of the generalized local fractional Laplace transform. Therefore, the theorems developed for the biparametric Laplace transform can be obtained as direct corollaries of the results associated with the generalized local fractional Laplace transform.
Similarly, the deformable Laplace transform introduced in [1] can also be interpreted as a particular case of the generalized local fractional Laplace transform by considering

$$\mathscr{A}(\alpha) = \beta \quad \text{and} \quad \psi(t,\alpha) = \alpha.$$

Consequently, the theorems corresponding to the deformable Laplace transform can also be derived as corollaries of the results established for the generalized local fractional Laplace transform.

Theorem 3.1 (Existence of $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}$)

Let $f : [0,\infty) \to \mathbb{R}$ be continuous and of exponential order a_*; that is, there exist constants $C > 0$ and $a_* \in \mathbb{R}$ such that

$$|f(t)| \leq Ce^{a_* t}, \quad \text{for all } t \geq 0.$$

Assume moreover that, for a fixed order α, the functions $\psi(\cdot,\alpha)$ and $\mathscr{A}(\alpha)$ satisfy the following conditions:

1) $\psi(t,\alpha)$ is continuous and there exists $\psi_{\min} > 0$ such that

$$\psi(t,\alpha) \geq \psi_{\min}, \quad \text{for all } t \geq 0.$$

2) There exists $a_0 \in \mathbb{R}$ such that

$$\frac{\mathscr{A}(\alpha)}{\psi(t,\alpha)} \geq a_0, \quad \text{for all } t \geq 0.$$

Then, the integral that defines $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)$ converges absolutely for all $\mathrm{Re}(s) > a_* - a_0$.

Moreover, for $\mathrm{Re}(s) > a_* - a_0$, we have:

$$\left|\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)\right| \le \frac{C}{\psi_{\min}}\frac{1}{s-a_*+a_0}.$$

Proof. We have that

$$\left|\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)\right| = \left|\int_0^\infty e^{-st-\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(\mu,\alpha)}d\mu}\frac{f(t)}{\psi(t,\alpha)}dt\right|$$
$$\le \int_0^\infty e^{-st-\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(\mu,\alpha)}d\mu}\frac{|f(t)|}{\psi(t,\alpha)}dt.$$

From condition 2), we have

$$\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(\mu,\alpha)}d\mu \ \ge\ a_0 t,$$

which implies that

$$e^{-\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(\mu,\alpha)}d\mu} \ \le\ e^{-a_0 t}.$$

From condition 1), the exponential order hypothesis on f, and the last result, we obtain that, for all $t \ge 0$:

$$e^{-st-\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(\mu,\alpha)}d\mu}\frac{|f(t)|}{\psi(t,\alpha)} \ \le\ e^{-st}e^{-a_0 t}\frac{Ce^{a_* t}}{\psi_{\min}}$$
$$= \frac{C}{\psi_{\min}}e^{-(s+a_0-a_*)t}.$$

Hence,

$$\left|\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)\right| \ \le\ \frac{C}{\psi_{\min}}\int_0^\infty e^{-(s-a_*+a_0)t}dt$$
$$= \frac{C}{\psi_{\min}}\frac{1}{s-a_*+a_0}.$$

for $\mathrm{Re}(s) > a_* - a_0$.

Now we present the UGLT of some functions.

Theorem 3.2

Let $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}$ be the UGLT defined by

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) := \int_0^\infty e^{-st-\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(\mu,\alpha)}d\mu}\frac{f(t)}{\psi(t,\alpha)}dt.$$

For $z \in \mathbb{C}$ and $k \ge 0$, let us denote:

$$J_z(s) := \int_0^\infty e^{-(s-z)t-I(t)}dt, \quad J_k(s) := \int_0^\infty t^k e^{-st-I(t)}dt,$$

where $I(t) := \int_0^t \frac{\mathscr{A}(\alpha)}{\psi(\mu,\alpha)}d\mu$.

If $\mathscr{A}(\alpha) \ne 0$ and $\lim_{t\to\infty} e^{-st-I(t)} = 0$,

then the following hold:

1. $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{1\}(s) = \frac{1}{\mathscr{A}(\alpha)} - \frac{s}{\mathscr{A}(\alpha)}J_0(s)$
2. $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{t^m\}(s) = \frac{m}{\mathscr{A}(\alpha)}J_{m-1}(s) - \frac{s}{\mathscr{A}(\alpha)}J_m(s)$
3. $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{e^{at}\}(s) = \frac{1}{\mathscr{A}(\alpha)} - \frac{s-a}{\mathscr{A}(\alpha)}J_a(s),$

   $J_a(s) := \int_0^\infty e^{-(s-a)t-I(t)}dt$
4. $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\sin(at)\}(s) = \frac{1}{2i\mathscr{A}(\alpha)}\left[(s+ia)J_{-ia}(s) - (s-ia)J_{ia}(s)\right]$
5. $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\cos(at)\}(s) = \frac{1}{\mathscr{A}(\alpha)} - \frac{1}{2\mathscr{A}(\alpha)}\left[(s-ia)J_{ia}(s) + (s+ia)J_{-ia}(s)\right]$
6. $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\sinh(at)\}(s) = \frac{1}{2\mathscr{A}(\alpha)}\left[(s+a)J_{-a}(s) - (s-a)J_a(s)\right]$
7. $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\cosh(at)\}(s) = \frac{1}{\mathscr{A}(\alpha)} - \frac{1}{2\mathscr{A}(\alpha)}\left[(s-a)J_a(s) + (s+a)J_{-a}(s)\right]$

Proof. 1. For $f(t) = 1$, we have

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{1\}(s) = \int_0^\infty e^{-st-I(t)}\frac{dt}{\psi(t,\alpha)}.$$

Observe that

$$\frac{d}{dt}\left(e^{-st-I(t)}\right) = -\left(s+I'(t)\right)e^{-st-I(t)}$$
$$= -\left(s+\frac{\mathscr{A}(\alpha)}{\psi(t,\alpha)}\right)e^{-st-I(t)}.$$

hence

$$\frac{e^{-st-I(t)}}{\psi(t,\alpha)} = -\frac{1}{\mathscr{A}(\alpha)}\frac{d}{dt}\left(e^{-st-I(t)}\right) - \frac{s}{\mathscr{A}(\alpha)}e^{-st-I(t)}.$$

Therefore,

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{1\}(s) = \int_0^\infty \frac{e^{-st-I(t)}}{\psi(t,\alpha)}dt$$
$$= -\frac{1}{\mathscr{A}(\alpha)}\int_0^\infty \frac{d}{dt}\left(e^{-st-I(t)}\right)dt - \frac{s}{\mathscr{A}(\alpha)}\int_0^\infty e^{-st-I(t)}dt$$
$$= -\frac{1}{\mathscr{A}(\alpha)}\left[e^{-st-I(t)}\right]_0^\infty - \frac{s}{\mathscr{A}(\alpha)}\int_0^\infty e^{-st-I(t)}dt$$
$$= \frac{1}{\mathscr{A}(\alpha)} - \frac{s}{\mathscr{A}(\alpha)}J_0(s).$$

Also,

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{1\}(s) = \frac{1}{\mathscr{A}(\alpha)} - \frac{s}{\mathscr{A}(\alpha)}\mathscr{L}\{e^{-I(t)}\}(s).$$

2. For $f(t)=t^m$, we have

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{t^m\}(s)=\int_0^\infty t^m e^{-st-I(t)}\,\frac{dt}{\psi(t,\alpha)}.$$

Since

$$\frac{e^{-st-I(t)}}{\psi(t,\alpha)}=-\frac{1}{\mathscr{A}(\alpha)}\frac{d}{dt}\left(e^{-st-I(t)}\right)-\frac{s}{\mathscr{A}(\alpha)}e^{-st-I(t)},$$

we obtain

$$\begin{aligned}\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{t^m\}(s)&=\int_0^\infty t^m e^{-st-I(t)}\,\frac{dt}{\psi(t,\alpha)}\\&=-\frac{1}{\mathscr{A}(\alpha)}\int_0^\infty t^m\frac{d}{dt}\left(e^{-st-I(t)}\right)dt\\&\quad-\frac{s}{\mathscr{A}(\alpha)}\int_0^\infty t^m e^{-st-I(t)}\,dt\\&=-\frac{1}{\mathscr{A}(\alpha)}\left[\left(t^m e^{-st-I(t)}\right)\right]_0^\infty\\&\quad+\frac{m}{\mathscr{A}(\alpha)}\int_0^\infty t^{m-1}e^{-st-I(t)}\,dt\\&\quad-\frac{s}{\mathscr{A}(\alpha)}\int_0^\infty t^m e^{-st-I(t)}\,dt\\&=\frac{m}{\mathscr{A}(\alpha)}J_{m-1}(s)-\frac{s}{\mathscr{A}(\alpha)}J_m(s).\end{aligned}$$

where $J_k(s):=\int_0^\infty t^k e^{-st-I(t)}\,dt=\mathscr{L}\{t^k e^{-I(t)}\}(s)$.

3. If $f(t)=e^{at}$, we have

$$\begin{aligned}\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{e^{at}\}(s)&=\int_0^\infty e^{-st-I(t)}\,\frac{e^{at}}{\psi(t,\alpha)}\,dt\\&=\int_0^\infty\frac{e^{-(s-a)t-I(t)}}{\psi(t,\alpha)}\,dt.\end{aligned}$$

Observe that

$$\begin{aligned}\frac{d}{dt}\left(e^{-(s-a)t-I(t)}\right)&=-(s-a+I'(t))e^{-(s-a)t-I(t)}\\&=-\left(s-a+\tfrac{\mathscr{A}(\alpha)}{\psi(t,\alpha)}\right)e^{-(s-a)t-I(t)}.\end{aligned}$$

Then we obtain

$$\begin{aligned}\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{e^{at}\}(s)&=\int_0^\infty\frac{e^{-(s-a)t-I(t)}}{\psi(t,\alpha)}\,dt\\&=-\frac{1}{\mathscr{A}(\alpha)}\left[e^{-(s-a)t-I(t)}\right]_0^\infty\\&\quad-\frac{(s-a)}{\mathscr{A}(\alpha)}\int_0^\infty e^{-(s-a)t-I(t)}\,dt\\&=-\frac{1}{\mathscr{A}(\alpha)}-\frac{(s-a)}{\mathscr{A}(\alpha)}J_a(s).\end{aligned}$$

where $J_a(s)=\int_0^\infty e^{-(s-a)t-I(t)}\,dt=\mathscr{L}\{e^{-I(t)}\}(s-a)$.

4. For $f(t)=\sin(at)$ we use the exponential decomposition: $\sin(at)=\frac{e^{iat}-e^{-iat}}{2i}$.
Using the formula obtained in 3, we get

$$\begin{aligned}&\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\sin(at)\}(s)\\&=\frac{1}{2i}\left(\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{e^{iat}\}(s)-\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{e^{-iat}\}(s)\right)\\&\qquad=\frac{1}{2i\mathscr{A}(\alpha)}\Big[(s+ia)J_{-ia}(s)-(s-ia)J_{ia}(s)\Big].\end{aligned}$$

where
$J_{\pm ia}(s)=\int_0^\infty e^{-(s\mp ia)t-I(t)}\,dt=\mathscr{L}\{e^{-I(t)}\}(s\mp ia)$.

The results of 5, 6, and 7 are obtained similarly as in 4.

Table 2: Generalized local fractional Laplace transforms (UGLT)

Function $f(t)$	$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)$
1	$\frac{1}{\mathscr{A}(\alpha)}-\frac{s}{\mathscr{A}(\alpha)}J_0(s)$
t^m	$\frac{m}{\mathscr{A}(\alpha)}J_{m-1}(s)-\frac{s}{\mathscr{A}(\alpha)}J_m(s)$
e^{at}	$\frac{1}{\mathscr{A}(\alpha)}-\frac{s-a}{\mathscr{A}(\alpha)}J_a(s),\qquad J_a(s)=\int_0^\infty e^{-(s-a)t-I(t)}\,dt$
$\sin(at)$	$\frac{1}{2i\mathscr{A}(\alpha)}[(s+ia)J_{-ia}(s)-(s-ia)J_{ia}(s)]$
$\cos(at)$	$\frac{1}{\mathscr{A}(\alpha)}-\frac{1}{2\mathscr{A}(\alpha)}[(s-ia)J_{ia}(s)+(s+ia)J_{-ia}(s)]$
$\sinh(at)$	$\frac{1}{2\mathscr{A}(\alpha)}[(s+a)J_{-a}(s)-(s-a)J_a(s)]$
$\cosh(at)$	$\frac{1}{\mathscr{A}(\alpha)}-\frac{1}{2\mathscr{A}(\alpha)}[(s-a)J_a(s)+(s+a)J_{-a}(s)]$

From Theorem 3.2, one can explicitly obtain the generalized local fractional Laplace transforms (UGLT) for some elementary functions, which are presented in Table 2.

In the following theorem, we analyze the fundamental properties of the transform $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}$, such as linearity, scaling, and shifting properties.

Theorem 3.3

The operator $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}$ possesses the following properties:

1.**Linearity:**

$$\begin{aligned}&\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{af(t)+bg(t)\}(s)\\&=a\,\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)+b\,\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{g(t)\}(s).\end{aligned}$$

2.**First Shifting Property:**
If $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)=F^{(\alpha,\mathscr{A},\psi)}(s)$, then

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{e^{at}f(t)\}(s)=F^{(\alpha,\mathscr{A},\psi)}(s-a).$$

3.**Second Shifting Property:**
If $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)=F^{(\alpha,\mathscr{A},\psi)}(s)$ and

$$G(t)=\begin{cases} f(t-a), & t>a,\\ 0, & t<a,\end{cases}$$

then

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{G(t)\}(s) = e^{-as-\int_0^a \frac{\mathscr{A}(\alpha)}{\psi(\mu,\alpha)}d\mu} F^{(\alpha,\mathscr{A},\psi(\cdot+a))}(s).$$

4.**Scaling Property:**
If $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)=F^{(\alpha,\mathscr{A},\psi)}(s)$, then

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(at)\}(s)=\frac{1}{a}F^{(\alpha,\frac{\mathscr{A}}{a},\psi(\frac{\cdot}{a},\alpha))}\left(\frac{s}{a}\right).$$

*Proof.*1 Linearity follows directly from the definition.
2 For the first shifting property, we have:

$$\begin{aligned}&\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{e^{at}f(t)\}(s)\\ &=\int_0^\infty e^{-st-\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}du} e^{at} f(t)\frac{dt}{\psi(t,\alpha)}\\ &=\int_0^\infty e^{-(s-a)t-\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}du} f(t)\frac{dt}{\psi(t,\alpha)}\\ &=\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s-a).\end{aligned}$$

3 Regarding the second shifting property, we obtain:

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{G(t)\}(s)=\int_0^\infty e^{-st-\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}du} G(t)\frac{dt}{\psi(t,\alpha)}$$

$$=\int_0^a 0\,dt+\int_a^\infty e^{-st-\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}du} f(t-a)\frac{dt}{\psi(t,\alpha)}.$$

Using $y=t-a$, $dt=dy$:

$$\begin{aligned}&\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{G(t)\}(s)\\ &=\int_0^\infty e^{-s(y+a)-\int_0^{y+a} \frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}du} f(y)\frac{dy}{\psi(y+a,\alpha)}\\ &=e^{-as-\int_0^a \frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}du}\int_0^\infty e^{-sy-\int_0^y \frac{\mathscr{A}(\alpha)}{\psi(v+a,\alpha)}dv} f(y)\frac{dy}{\psi(y+a,\alpha)}\\ &=e^{-as-\int_0^a \frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}du} F^{(\alpha,\mathscr{A},\psi(\cdot+a))}(s).\end{aligned}$$

4 For the scaling property:

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(at)\}(s)=\int_0^\infty e^{-st-\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}du} f(at)\frac{dt}{\psi(t,\alpha)}.$$

Let $y=at$, $dt=\frac{dy}{a}$:

$$\begin{aligned}&\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(at)\}(s)\\ &\quad=\frac{1}{a}\int_0^\infty e^{-\frac{s}{a}y-\int_0^{y/a} \frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}du} f(y)\frac{dy}{\psi(y/a,\alpha)}.\end{aligned}$$

Define $\tilde{\psi}(y,\alpha)=\psi(\frac{y}{a},\alpha)$:

$$\begin{aligned}&\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(at)\}(s)\\ &\quad=\frac{1}{a}\int_0^\infty e^{-\frac{s}{a}y-\int_0^{y} \frac{\mathscr{A}(\alpha)}{a\tilde{\psi}(v,\alpha)}dv} f(y)\frac{dy}{\tilde{\psi}(y,\alpha)}.\end{aligned}$$

Define $\tilde{\mathscr{A}}(\alpha)=\frac{\mathscr{A}(\alpha)}{a}$:

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(at)\}(s)=\frac{1}{a}F^{\left(\alpha,\frac{\mathscr{A}}{a},\psi(\frac{\cdot}{a},\alpha)\right)}\left(\frac{s}{a}\right).$$

Theorem 3.4

If $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)=F^{(\alpha,\mathscr{A},\psi)}(s)$, then the UGLT of the UG derivative

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{D_{UG}^{(\alpha,\mathscr{A},\psi)}f(t)\}(s)=sR(s)+2\mathscr{A}(\alpha)H(s)-f(0),$$

where

$$R(s)=\int_0^\infty W(t)\,f(t)\,dt=\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\psi(t,\alpha)\,f(t)\}(s),$$

$$H(s)=\int_0^\infty W(t)\,\frac{f(t)}{\psi(t,\alpha)}\,dt=\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s).$$

and

$$W(t)=e^{-st-\int_0^t \frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}du}.$$

*Proof.*By definition,

$$\begin{aligned}&\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{D_{UG}^{(\alpha,\mathscr{A},\psi)}f(t)\}(s)\\ &=\int_0^\infty W(t)D_{UG}^{(\alpha,\mathscr{A},\psi)}f(t)\frac{dt}{\psi(t,\alpha)}\\ &=\int_0^\infty W(t)\frac{\mathscr{A}(\alpha)f(t)+\psi(t,\alpha)f'(t)}{\psi(t,\alpha)}dt\\ &=\int_0^\infty W(t)\frac{\mathscr{A}(\alpha)f(t)+\psi(t,\alpha)f'(t)}{\psi(t,\alpha)}dt\\ &=\int_0^\infty W(t)\left(\frac{\mathscr{A}(\alpha)}{\psi(t,\alpha)}f(t)+f'(t)\right)dt\\ &=\int_0^\infty W(t)f'(t)\,dt\,+\int_0^\infty W(t)\frac{\mathscr{A}(\alpha)}{\psi(t,\alpha)}f(t)\,dt\\ &=\int_0^\infty W(t)f'(t)\,dt\,+\mathscr{A}(\alpha)\int_0^\infty W(t)\frac{f(t)}{\psi(t,\alpha)}dt\\ &=\int_0^\infty W(t)f'(t)\,dt\,+\mathscr{A}(\alpha)\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s).\end{aligned}$$

Applying integration by parts to $\int_0^\infty W(t)f'(t)\,dt$, we obtain:

$$\begin{aligned}&\int_0^\infty W(t)f'(t)\,dt\\&=\left[W(t)f(t)\right]_0^\infty+\int_0^\infty\left(s+\tfrac{\mathscr{A}(\alpha)}{\psi(t,\alpha)}\right)W(t)f(t)\,dt\\&=-f(0)+s\int_0^\infty W(t)f(t)\,dt+\mathscr{A}(\alpha)\int_0^\infty W(t)\,\frac{f(t)}{\psi(t,\alpha)}\,dt.\end{aligned}$$

Thus,

$$\begin{aligned}&\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{D_{UG}^{(\alpha,\mathscr{A},\psi)}f(t)\}(s)\\&=-f(0)+s\int_0^\infty W(t)f(t)\,dt\\&+2\mathscr{A}(\alpha)\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)\\&=-f(0)+s\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\psi(t,\alpha)f(t)\}(s)\\&+2\mathscr{A}(\alpha)\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s).\end{aligned}$$

Theorem 3.5

If $\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)=F^{(\alpha,\mathscr{A},\psi)}(s)$, then

$$\begin{aligned}&\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{D_{UG}^{2(\alpha,\mathscr{A},\psi)}f(t)\}(s)\\&=-\psi(0)f'(0)-(s\psi(0)+3\mathscr{A}(\alpha))f(0)\\&\quad+s^2\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\psi^2(t,\alpha)f(t)\}(s)\\&\quad+4s\mathscr{A}(\alpha)\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\psi(t,\alpha)f(t)\}(s)\\&\quad+4\mathscr{A}^2(\alpha)\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)\\&\quad-s\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\psi'(t,\alpha)\psi(t,\alpha)f(t)\}(s),\end{aligned}$$

where

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s)=\int_0^\infty W(t)\frac{f(t)}{\psi(t,\alpha)}\,dt,$$

and $W(t)=e^{-st-\int_0^t\frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}du}$.

Proof. We have

$$\begin{aligned}&D_{UG}^{2(\alpha,\mathscr{A},\psi)}f(t)\\&=\mathscr{A}^2(\alpha)f(t)+\left(2\mathscr{A}(\alpha)\psi(t,\alpha)+\psi(t,\alpha)\psi'(t,\alpha)\right)f'(t)\\&+\psi^2(t,\alpha)f''(t).\end{aligned}$$

The transform of the second derivative is obtained as follows:

$$\begin{aligned}&\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{D_{UG}^{2(\alpha,\mathscr{A},\psi)}f(t)\}(s)\\&=\mathscr{A}^2(\alpha)\int_0^\infty W(t)\frac{f(t)}{\psi(t,\alpha)}\,dt\\&\quad+\int_0^\infty W(t)\frac{2\mathscr{A}(\alpha)\psi(t,\alpha)+\psi(t,\alpha)\psi'(t,\alpha)}{\psi(t,\alpha)}f'(t)\,dt\\&\quad+\int_0^\infty W(t)\frac{\psi^2(t,\alpha)}{\psi(t,\alpha)}f''(t)\,dt\\&=\mathscr{A}^2(\alpha)\int_0^\infty W(t)\frac{f(t)}{\psi(t,\alpha)}\,dt\\&\quad+\int_0^\infty W(t)\left(2\mathscr{A}(\alpha)+\psi'(t,\alpha)\right)f'(t)\,dt\\&\quad+\int_0^\infty W(t)\psi(t,\alpha)f''(t)\,dt.\end{aligned}$$

Applying integration by parts to the last integral, we get:

$$\begin{aligned}&\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{D_{UG}^{2(\alpha,\mathscr{A},\psi)}f(t)\}(s)\\&\quad=\mathscr{A}^2(\alpha)\int_0^\infty W(t)\frac{f(t)}{\psi(t,\alpha)}\,dt+\left[W(t)\psi(t,\alpha)f'(t)\right]_0^\infty\\&\qquad+\int_0^\infty\left(s\psi(t,\alpha)+3\mathscr{A}(\alpha)\right)W(t)f(t)\,dt.\end{aligned}$$

Now, applying integration by parts to the last integral, we obtain

$$\begin{aligned}&\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{D_{UG}^{2(\alpha,\mathscr{A},\psi)}f(t)\}(s)\\&=\mathscr{A}^2(\alpha)\int_0^\infty W(t)\frac{f(t)}{\psi(t,\alpha)}\,dt\\&\quad+\left[W(t)\psi(t,\alpha)f'(t)\right]_0^\infty+\left[W(t)(s\psi(t,\alpha)+3\mathscr{A}(\alpha))f(t)\right]_0^\infty\\&\quad-\int_0^\infty W(t)\Big(-(s+\tfrac{\mathscr{A}(\alpha)}{\psi(t,\alpha)})(s\psi(t,\alpha)+\\&\qquad\qquad 3\mathscr{A}(\alpha))+s\psi'(t,\alpha)\Big)f(t)\,dt\\&=\mathscr{A}^2(\alpha)\int_0^\infty W(t)\frac{f(t)}{\psi(t,\alpha)}\,dt\\&\quad+\left[W(t)\psi(t,\alpha)f'(t)\right]_0^\infty+\left[W(t)(s\psi(t,\alpha)+3\mathscr{A}(\alpha))f(t)\right]_0^\infty\\&\quad+\int_0^\infty W(t)\left(s^2\psi(t,\alpha)+4s\mathscr{A}(\alpha)+\tfrac{3\mathscr{A}^2(\alpha)}{\psi(t,\alpha)}\right)f(t)\,dt\\&\quad-\int_0^\infty W(t)\,s\,\psi'(t,\alpha)f(t)\,dt\\&=4\mathscr{A}^2(\alpha)\int_0^\infty W(t)\frac{f(t)}{\psi(t,\alpha)}\,dt-\psi(0)f'(0)\\&\quad-(s\psi(0)+3\mathscr{A}(\alpha))f(0)\\&\quad+s^2\int_0^\infty W(t)\psi(t,\alpha)f(t)\,dt+4s\mathscr{A}(\alpha)\int_0^\infty W(t)f(t)\,dt\\&\quad-s\int_0^\infty W(t)\psi'(t,\alpha)f(t)\,dt\end{aligned}$$

$$
\begin{aligned}
&= -\psi(0)f'(0) - (s\psi(0) + 3\mathscr{A}(\alpha))f(0) \\
&\quad + s^2 \mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\psi^2(t,\alpha)f(t)\}(s) \\
&\quad + 4s\mathscr{A}(\alpha)\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\psi(t,\alpha)f(t)\}(s) \\
&\quad + 4\mathscr{A}^2(\alpha)\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) \\
&\quad - s\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\psi'(t,\alpha)\psi(t,\alpha)f(t)\}(s).
\end{aligned}
$$

Theorem 3.6

Let

$$
\begin{aligned}
F^{(\alpha,\mathscr{A},\psi)}(s) &:= \mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) \\
&= \int_0^\infty e^{-st-I(t)} \frac{f(t)}{\psi(t,\alpha)}\,dt,
\end{aligned}
$$

where

$$I(t) = \int_0^t \frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}\,du.$$

If there exists a half-plane $\mathrm{Re}(s) > \sigma_c$ and an integer $n \geq 0$ such that, for all s with $\mathrm{Re}(s) > \sigma_0$, the function

$$t^n \frac{f(t)}{\psi(t,\alpha)} e^{-st-I(t)}$$

is integrable, and moreover the derivatives ∂_s^k of the integrand are dominated by an integrable function independent of s for $k = 0,\ldots,n$, then for all s with $\mathrm{Re}(s) > \sigma_0$ we have:

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n} F^{(\alpha,\mathscr{A},\psi)}(s).$$

*Proof.*Consider

$$F^{(\alpha,\mathscr{A},\psi)}(s) = \int_0^\infty e^{-st-I(t)} \frac{f(t)}{\psi(t,\alpha)}\,dt.$$

We compute the derivative of $F^{(\alpha,\mathscr{A},\psi)}(s)$ with respect to s, and obtain

$$
\begin{aligned}
\frac{d}{ds}F^{(\alpha,\mathscr{A},\psi)}(s) &= \frac{d}{ds}\int_0^\infty e^{-st-I(t)} \frac{f(t)}{\psi(t,\alpha)}\,dt \\
&= \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st-I(t)}\right) \frac{f(t)}{\psi(t,\alpha)}\,dt \\
&= -\int_0^\infty t\,e^{-st-I(t)} \frac{f(t)}{\psi(t,\alpha)}\,dt \\
&= -\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{tf(t)\}(s).
\end{aligned}
$$

Thus,

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{tf(t)\}(s) = (-1)\frac{d}{ds}F^{(\alpha,\mathscr{A},\psi)}(s).$$

Similarly,

$$
\begin{aligned}
\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{t^2 f(t)\}(s) &= (-1)^2 \frac{d^2}{ds^2}F^{(\alpha,\mathscr{A},\psi)}(s), \\
\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{t^3 f(t)\}(s) &= (-1)^3 \frac{d^3}{ds^3}F^{(\alpha,\mathscr{A},\psi)}(s), \\
\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{t^n f(t)\}(s) &= (-1)^n \frac{d^n}{ds^n}F^{(\alpha,\mathscr{A},\psi)}(s).
\end{aligned}
$$

Theorem 3.7

Let

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) = \int_0^\infty e^{-st-I(t)} \frac{f(t)}{\psi(t,\alpha)}\,dt,$$

where

$$I(t) = \int_0^t \frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}\,du.$$

Suppose there exists $\sigma_0 \in \mathbb{R}$ such that for all s with $\mathrm{Re}(s) > \sigma_0$ the integral defining $F^{(\alpha,\mathscr{A},\psi)}(s)$ converges, and furthermore

$$\int_s^\infty \int_0^\infty \left| e^{pt-I(t)} \frac{f(t)}{\psi(t,\alpha)} \right| dt\,dp < \infty.$$

Then, for all s with $\mathrm{Re}(s) > \sigma_0$, we have

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F^{(\alpha,\mathscr{A},\psi)}(p)\,dp.$$

*Proof.*Consider

$$F^{(\alpha,\mathscr{A},\psi)}(p) = \int_0^\infty e^{-pt-I(t)} \frac{f(t)}{\psi(t,\alpha)}\,dt,$$

then

$$
\begin{aligned}
\int_s^\infty F^{(\alpha,\mathscr{A},\psi)}(p)\,dp &= \int_s^\infty \left(\int_0^\infty e^{-pt-I(t)} \frac{f(t)}{\psi(t,\alpha)}\,dt\right)dp \\
&= \int_0^\infty \left(\int_s^\infty e^{-pt}\,dp\right) e^{-I(t)} \frac{f(t)}{\psi(t,\alpha)}\,dt.
\end{aligned}
$$

But

$$\int_s^\infty e^{-pt}\,dp = \left[\frac{-1}{t}e^{-pt}\right]_s^\infty = \frac{e^{-st}}{t}.$$

Hence,

$$
\begin{aligned}
\int_s^\infty F^{(\alpha,\mathscr{A},\psi)}(p)\,dp &= \int_0^\infty e^{-st-I(t)} \frac{f(t)}{t\,\psi(t,\alpha)}\,dt \\
&= \mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\left\{\frac{f(t)}{t}\right\}(s).
\end{aligned}
$$

If $F^{(\alpha,\mathscr{A},\psi)}(s)$ is the generalized local fractional Laplace transform of $f(t)$, then $f(t)$ is called the inverse generalized local fractional Laplace transform (UGLTI), whose operator is denoted by $\left(\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\right)^{-1}$.

Now we present the Convolution Theorem for the UGLT.

Theorem 3.8

Given

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) = \int_0^\infty e^{-st-I(t)} \frac{f(t)}{\psi(t,\alpha)}\,dt,$$

where

$$I(t) = \int_0^t \frac{\mathscr{A}(\alpha)}{\psi(u,\alpha)}\,du.$$

We define the functions

$$G_i(t) := e^{-I(t)} \frac{f_i(t)}{\psi(t,\alpha)}, \qquad (i=1,2).$$

Then, for all $s \in \mathbb{C}$ we have

$$\left(\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\right)^{-1}\left\{F_1^{(\alpha,\mathscr{A},\psi)}(s)\,F_2^{(\alpha,\mathscr{A},\psi)}(s)\right\}(t) = (f_1 *_{\mathrm{UG}} f_2)(t),$$

where the UG-convolution, $*_{UG}$, is given by

$$(f_1 *_{UG} f_2)(t) = \psi(t,\alpha)\,e^{I(t)} \int_0^t \left[e^{-I(x)-I(t-x)} \frac{f_1(x)}{\psi(x,\alpha)} \frac{f_2(t-x)}{\psi(t-x,\alpha)}\right] dx.$$

Proof. By definition, we have

$$F_i^{(\alpha,\mathscr{A},\psi)}(s) = \int_0^\infty e^{-st} G_i(t)\,dt.$$

By the classical convolution property, it follows that

$$F_1^{(\alpha,\mathscr{A},\psi)}(s)\,F_2^{(\alpha,\mathscr{A},\psi)}(s) = \mathscr{L}\{G_1(t)\}(s)\,\mathscr{L}\{G_2(t)\}(s) = \mathscr{L}\{(G_1 * G_2)(t)\}(s),$$

where

$$(G_1 * G_2)(t) := \int_0^t G_1(x)\,G_2(t-x)\,dx.$$

Thus,

$$\mathscr{L}^{-1}\left\{F_1^{(\alpha,\mathscr{A},\psi)}(s)\,F_2^{(\alpha,\mathscr{A},\psi)}(s)\right\}(t) = (G_1 * G_2)(t).$$

Moreover,

$$\begin{aligned}(G_1 * G_2)(t) &= \int_0^t G_1(x)\,G_2(t-x)\,dx \\ &= \int_0^t e^{-I(x)} \frac{f_1(x)}{\psi(x,\alpha)} e^{-I(t-x)} \frac{f_2(t-x)}{\psi(t-x,\alpha)}\,dx \\ &= \int_0^t e^{-I(x)-I(t-x)} \frac{f_1(x)}{\psi(x,\alpha)} \frac{f_2(t-x)}{\psi(t-x,\alpha)}\,dx.\end{aligned}$$

However,

$$(G_1 * G_2)(t) = e^{-I(t)} \frac{(f_1 *_{UG} f_2)(t)}{\psi(t,\alpha)}.$$

Therefore,

$$(f_1 *_{UG} f_2)(t) = \psi(t,\alpha)\,e^{I(t)} \int_0^t e^{-I(x)-I(t-x)} \frac{f_1(x)}{\psi(x,\alpha)} \frac{f_2(t-x)}{\psi(t-x,\alpha)}\,dx.$$

4 Examples and Applications

Example 1

To illustrate the application of the UGLT, let us consider Example 3.2.1 from [7].

We start from the following UG differential equation:

$$D_{UG}^{(\alpha,\mathscr{A},\psi)} f(t) + P(t) f(t) = Q(t).$$

In Case 1, we take

$$P(t) = \lambda,\ Q(t) = 0,\ \psi(t,\alpha) = \sin\left(\frac{\pi\alpha}{2}\right),\ \mathscr{A}(\alpha) = \cos\left(\frac{\pi\alpha}{2}\right),$$

which allows us to work with the homogeneous UG differential equation:

$$D_{UG}^{(\alpha,\mathscr{A},\psi)} f(t) + \lambda f(t) = 0.$$

By applying the UGLT to the homogeneous UG differential equation, we obtain

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{D_{UG}^{(\alpha,\mathscr{A},\psi)} f(t)\}(s) + \lambda \mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) = 0. \tag{1}$$

Since $\psi(t,\alpha) = \psi_0$ and $A(\alpha) = A$ are constants, we have

$$F^{(\alpha,\mathscr{A},\psi)}(s) = \mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) = \frac{1}{\psi_0} F\left(s + \frac{A}{\psi_0}\right),$$

and

$$G^{(\alpha,\mathscr{A},\psi)}(s) = \mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\psi_0 f(t)\}(s) = F\left(s + \frac{A}{\psi_0}\right).$$

We know that

$$\begin{aligned}\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}&\{D_{UG}^{(\alpha,\mathscr{A},\psi)} f(t)\}(s) \\ &= s\psi_0 \mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) \\ &\quad + 2A\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) - f(0).\end{aligned}$$

Then, equation (1) becomes

$$s\psi_0 F^{(\alpha,\mathscr{A},\psi)}(s) + (2A+\lambda) F^{(\alpha,\mathscr{A},\psi)}(s) = f_0,$$

from which it follows that

$$F^{(\alpha,\mathscr{A},\psi)}(s) = \frac{f_0}{s\psi_0 + 2A + \lambda}.$$

which implies

$$\frac{1}{\psi_0} F\left(s + \frac{A}{\psi_0}\right) = \frac{f_0}{\psi_0\,(s + \frac{2A+\lambda}{\psi_0})}.$$

Thus,

$$F\left(s + \frac{A}{\psi_0}\right) = \frac{f_0}{s + \frac{2A+\lambda}{\psi_0}},$$

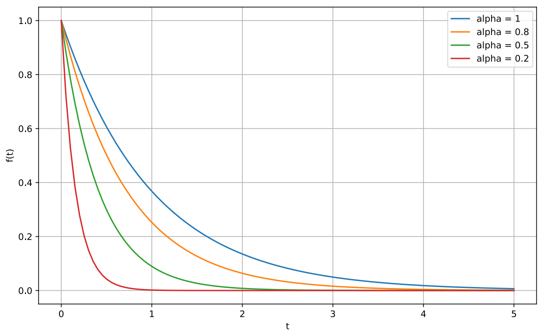
if $p = s + \frac{A}{\psi_0}$, then

$$F(p) = \frac{f_0}{p + \frac{A+\lambda}{\psi_0}},$$

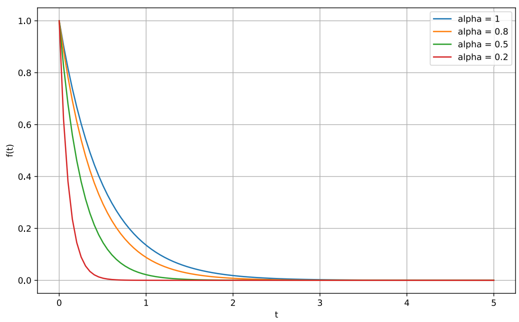
and the solution is given by

$$f(t) = f_0 \, e^{-\frac{A+\lambda}{\psi_0} t} = f_0 \, e^{-\frac{\cos\left(\frac{\pi\alpha}{2}\right)+\lambda}{\sin\left(\frac{\pi\alpha}{2}\right)} t}.$$

This solution coincides with the one obtained in [7], and its graph, shown below, also coincides with that of Case 1 in Example 3.2.1.



Graph of $f(t)$ for $\lambda = 1$ and $\alpha = 1, 0.8, 0.5, 0.2$.



Graph of $f(t)$ for $\lambda = 2$ and $\alpha = 1, 0.8, 0.5, 0.2$.

Fig. 1: Comparison of the graphs of the function $f(t) = f_0 \, e^{-\frac{\cos\left(\frac{\pi\alpha}{2}\right)+\lambda}{\sin\left(\frac{\pi\alpha}{2}\right)} t}$, $f_0 = 1$, for different values of α and λ.

Example 2

If we vary $Q(t)$ in the previous example and now it is a nonzero constant, then Case 1 of Example 3.2.1 in [7] becomes:

$$D_{UG}^{(\alpha,\mathscr{A},\psi)} f(t) + P(t) f(t) = Q(t),$$

with $P(t) = \lambda$, $Q(t) = \sin\left(\frac{\pi\alpha}{2}\right)$, $f(0) = f_0$, $\mathscr{A}(\alpha) = \cos\left(\frac{\pi\alpha}{2}\right)$, and $\psi(t,\alpha) = \sin\left(\frac{\pi\alpha}{2}\right)$.

Now, setting $Q(t) = \sin\left(\frac{\pi\alpha}{2}\right) = c$, $\mathscr{A}(\alpha) = \cos\left(\frac{\pi\alpha}{2}\right) = A$, $\psi(t,\alpha) = \sin\left(\frac{\pi\alpha}{2}\right) = c$, it is possible to work with the non-homogeneous UG differential equation

$$D_{UG}^{(\alpha,\mathscr{A},\psi)} f(t) + \lambda f(t) = c.$$

Applying the UGLT to the non-homogeneous UG differential equation, we obtain

$$s \, \mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{\psi(t,\alpha) f(t)\}(s) + 2\mathscr{A}(\alpha) \, \mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) - f_0 + \lambda \, \mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) = \mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{c\}(s). \tag{2}$$

Since $\psi(t,\alpha) = c$ is constant, we have

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) = \frac{1}{c} \mathscr{L}\{f(t)\}(p), \text{ where } p = s + \tfrac{A}{c}.$$

Then, equation (2) becomes

$$sc \, F^{(\alpha,\mathscr{A},\psi)}(s) + 2A F^{(\alpha,\mathscr{A},\psi)}(s) - f_0 + \lambda F^{(\alpha,\mathscr{A},\psi)}(s) = \frac{1}{s + \frac{A}{c}}.$$

which implies

$$F^{(\alpha,\mathscr{A},\psi)}(s) = \frac{f_0}{cs + 2A + \lambda} + \frac{1}{(cs + 2A + \lambda)\left(s + \frac{A}{c}\right)}.$$

Since

$$F(p) = c \, F^{(\alpha,\mathscr{A},\psi)}(s),$$

we have

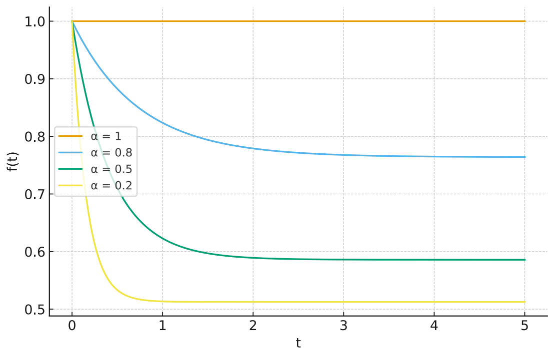
$$F(p) = \frac{c f_0}{cp + A + \lambda} - \frac{c}{(A+\lambda)(cp + A + \lambda)} + \frac{1}{(A+\lambda)p}.$$

Hence, the solution is given by

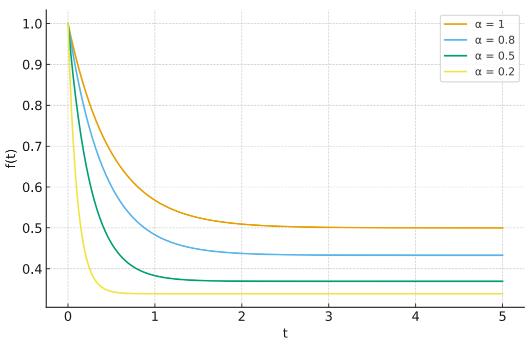
$$f(t) = \left(f_0 - \tfrac{1}{A+\lambda}\right) e^{-\frac{A+\lambda}{c} t} + \frac{1}{A+\lambda}.$$

Finally, replacing $c = \sin\left(\frac{\pi\alpha}{2}\right)$ and $A = \cos\left(\frac{\pi\alpha}{2}\right)$:

$$f(t) = \left(f_0 - \frac{1}{\cos\left(\frac{\pi\alpha}{2}\right)+\lambda}\right) e^{-\frac{\cos\left(\frac{\pi\alpha}{2}\right)+\lambda}{\sin\left(\frac{\pi\alpha}{2}\right)} t} + \frac{1}{\cos\left(\frac{\pi\alpha}{2}\right)+\lambda}.$$



Graph of $f(t)$ for $\lambda = 1$ and $\alpha = 1, 0.8, 0.5, 0.2$.



Graph of $f(t)$ for $\lambda = 2$ and $\alpha = 1, 0.8, 0.5, 0.2$.

Fig. 2: Comparison of the graphs of the function $f(t) = \left(f_0 - \frac{1}{\cos\left(\frac{\pi\alpha}{2}\right)+\lambda}\right) e^{-\frac{\cos\left(\frac{\pi\alpha}{2}\right)+\lambda}{\sin\left(\frac{\pi\alpha}{2}\right)}t} + \frac{1}{\cos\left(\frac{\pi\alpha}{2}\right)+\lambda}, f_0 = 1$ for different values of α and λ.

Example 3

If in Example 2 we change to $\mathscr{A}(\alpha) = 1-\alpha, \quad \psi(t,\alpha) = \alpha$, then Case 1 of Example 3.2.1 in [7]

$$D_{UG}^{(\alpha,\mathscr{A},\psi)} f(t) + P(t) f(t) = Q(t).$$

with: $P(t) = \lambda, \quad Q(t) = \sin\left(\frac{\pi\alpha}{2}\right), \quad f(0) = f_0, \mathscr{A}(\alpha) = 1-\alpha, \quad \psi(t,\alpha) = \alpha.$

Setting $\psi(t,\alpha) = \alpha = c, \quad \mathscr{A}(\alpha) = 1-\alpha = A$, $Q(t) = \sin\left(\frac{\pi\alpha}{2}\right) = q$, one can write the following non-homogeneous UG differential equation:

$$D_{UG}^{(\alpha,\mathscr{A},\psi)} f(t) + \lambda f(t) = q. \tag{3}$$

Applying the UGLT to equation (3) and taking into account that

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{f(t)\}(s) = \frac{1}{c}\mathscr{L}\{f(t)\}(p),$$

where, $p = s + \frac{A}{c}$, we obtain

$$(cs + 2A + \lambda) F^{(\alpha,\mathscr{A},\psi)}(s) = f_0 + \frac{q}{cs+A}.$$

Then,

$$c(cs + 2A + \lambda) F^{(\alpha,\mathscr{A},\psi)}(s) = cf_0 + \frac{cq}{cs+A} = cf_0 + \frac{q}{s+\frac{A}{c}}.$$

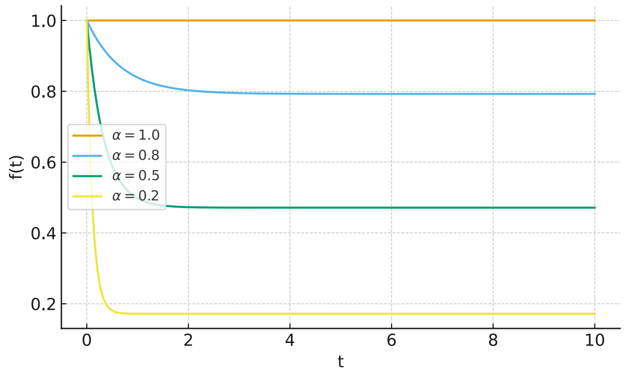
Hence, $c(cp + A + \lambda) F^{(\alpha,\mathscr{A},\psi)}(s) = cf_0 + \frac{q}{p}$, which implies $(cp + A + \lambda)F(p) = cf_0 + \frac{q}{p}$,

therefore,

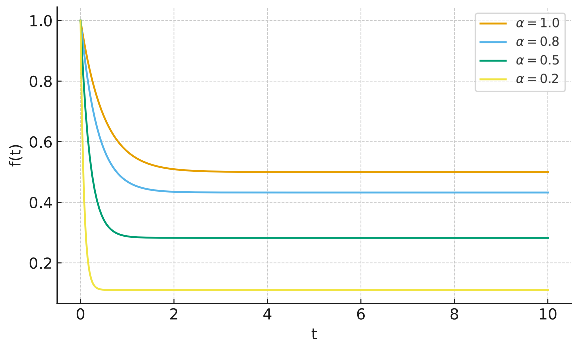
$$F(p) = \frac{cf_0}{cp + A + \lambda} + \frac{q}{(A+\lambda)}\frac{1}{p} - \frac{qc}{(A+\lambda)}\frac{1}{cp + A + \lambda}.$$

Thus,

$$f(t) = \frac{\sin\left(\frac{\pi\alpha}{2}\right)}{1-\alpha+\lambda} + \left(f_0 - \frac{\sin\left(\frac{\pi\alpha}{2}\right)}{1-\alpha+\lambda}\right) e^{-\frac{(1-\alpha+\lambda)}{\alpha}t}.$$



Graph of $f(t)$ for $\lambda = 1$ and $\alpha = 1, 0.8, 0.5, 0.2$.



Graph of $f(t)$ for $\lambda = 2$ and $\alpha = 1, 0.8, 0.5, 0.2$.

Fig. 3: Comparison of the graphs of the function $f(t) = \frac{\sin\left(\frac{\pi\alpha}{2}\right)}{1-\alpha+\lambda} + \left(f_0 - \frac{\sin\left(\frac{\pi\alpha}{2}\right)}{1-\alpha+\lambda}\right) e^{-\frac{(1-\alpha+\lambda)}{\alpha}t}, \quad f_0 = 1$ for different values of α and λ.

Example 4

Metabolite concentration with decreasing stimulus

Let $f(t)$ be the concentration of a metabolite responding to an external stimulus whose intensity decreases exponentially, $e^{-\mu t}$, as occurs in the absorption of a drug whose blood concentration decreases over time. The metabolite dynamics is modeled by the UG differential equation:

$$D_{UG}^{(\alpha,\mathscr{A},\psi)} f(t) + \lambda f(t) = e^{-\mu t}, \qquad \mu > 0, \tag{4}$$

where $\lambda > 0$ measures the natural elimination of the metabolite, $\alpha \in (0,1)$ modulates the strength of exponentially weighted memory effects associated with the stimulus history, $\mathscr{A}(\alpha)$ regulates the weight of this memory, and $\psi(t,\alpha)$ adjusts the timescale at which past effects are perceived.

We have:

$$\mathscr{A}(\alpha) = A = \alpha, \qquad \psi(t,\alpha) = \psi_0.$$

To solve the UG differential equation, we apply the UGLT to the UG derivative.

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{D_{UG}^{(\alpha,\mathscr{A},\psi)} f(t)\}(s) = (s\psi_0 + 2A) F^{(\alpha,\mathscr{A},\psi)}(s) - f_0.$$

Moreover,

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{e^{-\mu t}\}(s) = \frac{1}{A} - \frac{s+\mu}{A} J_\mu(s) = \frac{1}{\psi_0}\cdot\frac{1}{s+\mu+\frac{A}{\psi_0}}.$$

Then, the UG differential equation (4) becomes:

$$(s\psi_0 + 2A + \lambda) F^{(\alpha,\mathscr{A},\psi)}(s) = f_0 + \frac{1}{\psi_0}\cdot\frac{1}{s+\mu+\frac{A}{\psi_0}},$$

from which

$$F^{(\alpha,\mathscr{A},\psi)}(s) = \frac{f_0}{s\psi_0 + 2A + \lambda} + \frac{\frac{1}{\psi_0}}{(s\psi_0 + 2A + \lambda)\left(s + \frac{A}{\psi_0} + \mu\right)}$$

If we set $p = s + \frac{A}{\psi_0}$ and since

$$F(p) = \psi_0 F^{(\alpha,\mathscr{A},\psi)}(s)$$

, we get:

$$F(p) = \frac{\psi_0 f_0}{\psi_0 p + A + \lambda} + \frac{1}{(p+\mu)(\psi_0 p + A + \lambda)} = \frac{\psi_0 f_0}{\psi_0 p + A + \lambda} + \frac{1}{(A+\lambda-\psi_0\mu)(p+\mu)} - \frac{\psi_0}{(A+\lambda-\psi_0\mu)(\psi_0 p + A + \lambda)}$$

Therefore,

$$f(t) = \left(f_0 - \frac{1}{\alpha+\lambda-\psi_0\mu}\right) e^{-\frac{\alpha+\lambda}{\psi_0}t} + \frac{1}{\alpha+\lambda-\psi_0\mu} e^{-\mu t}$$

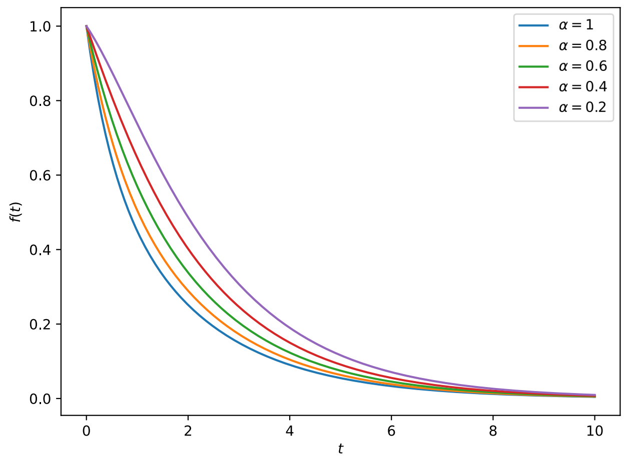


Fig. 4: Graph of the function $f(t) = \left(f_0 - \frac{1}{\alpha+\lambda-\psi_0\mu}\right) \mathrm{e}^{-\frac{\alpha+\lambda}{\psi_0}t} + \frac{1}{\alpha+\lambda-\psi_0\mu} \mathrm{e}^{\mu t}$, with $f_0 = 1$, $\psi_0 = 1$, $\mu = 0.5$ and $\lambda = 1$.

The graph shows that decreasing α enhances the memory-induced attenuation of the response, while larger values of α yield dynamics closer to the classical case with slower decay.

Example 5

Physiological periodic response with memory

Let $f(t)$ be a physiological variable, such as instantaneous blood pressure, modulated by a periodic stimulus $\sin t$ representing biological rhythms, e.g., circadian or respiratory. The dynamics of f are modeled using the generalized local fractional derivative of UG type:

$$D_{UG}^{(\alpha,\mathscr{A},\psi)} f(t) = \sin t, \qquad f(0) = 0, \tag{5}$$

where $\alpha \in (0,1)$ is the parameter controlling the strength of exponentially weighted memory effects, $\mathscr{A}(\alpha)$ regulates the intensity of such memory, and $\psi(t,\alpha)$ describes the time scale in which the effects of biological rhythms manifest. In this case, we consider:

$$\mathscr{A}(\alpha) = A = 1-\alpha, \qquad \psi(t,\alpha) = \alpha.$$

Applying the UGLT to both sides of the UG differential equation (5), we have:

$$(s\alpha + 2A)\,F^{(\alpha,\mathscr{A},\psi)}(s) = \frac{1}{2i\alpha}\Big[(s+i)\,J_{-i}(s) - (s-i)\,J_i(s)\Big],$$

where

$$J_{\pm i}(s) = \int_0^\infty e^{-(s\mp i)t - bt}\,dt = \frac{1}{s \pm i + b}, \qquad b = \frac{1-\alpha}{\alpha}.$$

Then,

$$\begin{aligned}(s\alpha + 2(1-\alpha))\,F^{(\alpha,\mathscr{A},\psi)}(s) \\ = \frac{1}{2i(1-\alpha)}\left[\frac{s+i}{s+i+b} - \frac{s-i}{s-i+b}\right] \\ = \frac{1}{\alpha\,[(s+b)^2+1]}.\end{aligned}$$

Hence:

$$F^{(\alpha,\mathscr{A},\psi)}(s) = \frac{1}{\alpha(s\alpha + 2(1-\alpha))\,[(s+b)^2+1]}.$$

If $p = s+b$, recalling that

$$F^{(\alpha,\mathscr{A},\psi)}(s) = \frac{1}{\alpha}\mathscr{L}\{f(t)\}(s+b),$$

we obtain:

$$\mathscr{L}\{f(t)\}(p) = \frac{1}{(p^2+1)(\alpha p + (1-\alpha))}.$$

Therefore,

$$\mathscr{L}\{f(t)\}(p) = \frac{\frac{\alpha^2}{\alpha^2+(1-\alpha)^2}}{(\alpha p + (1-\alpha))} + \frac{-\frac{\alpha}{\alpha^2+(1-\alpha)^2}p + \frac{1-\alpha}{\alpha^2+(1-\alpha)^2}}{p^2+1}.$$

Hence, applying the inverse transform, we obtain:

$$f(t) = \frac{\alpha}{\alpha^2+(1-\alpha)^2}e^{-\frac{1-\alpha}{\alpha}t} - \frac{\alpha}{\alpha^2+(1-\alpha)^2}\cos(t) + \frac{1-\alpha}{\alpha^2+(1-\alpha)^2}\sin(t).$$

This solution is the same as obtained in Example 6.2 of [1].

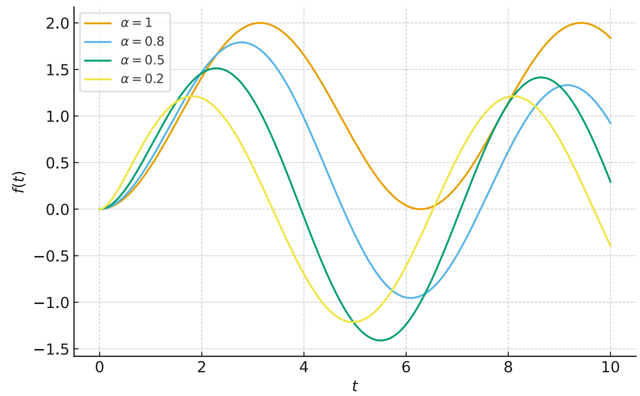


Fig. 5: Graph of the function $f(t) = \frac{\alpha}{\alpha^2+(1-\alpha)^2}e^{-\frac{1-\alpha}{\alpha}t} - \frac{\alpha}{\alpha^2+(1-\alpha)^2}\cos(t) + \frac{1-\alpha}{\alpha^2+(1-\alpha)^2}\sin(t)$.

The solution describes a physiological response modulated by a periodic stimulus under the presence of memory effects. When $\alpha = 1$, the model reduces to the classical case without memory, exhibiting regular oscillatory behavior. As α decreases (e.g., $\alpha = 0.8, 0.5, 0.2$), exponentially weighted memory effects become more pronounced: the oscillations decay more rapidly and the response stabilizes earlier, indicating a progressive reduction in temporal persistence.

Example 6

Memory effect on cell density

In a biological tissue, the response of a cell population to an external stimulus depends not only on the current state but also on the memory of the process. To describe this behavior, a generalized local fractional differential equation (UG) is used.

Let $f(t)$ be the cell density at time t. The term e^{3t} represents an increasing external stimulus, such as the action of a drug.

The mathematical model describing this dynamics is the UG-differential equation:

$$D_{UG}^{\left(\frac{1}{4},\mathscr{A},\psi\right)} D_{UG}^{\left(\frac{1}{4},\mathscr{A},\psi\right)} f(t) = e^{3t}, \tag{6}$$

with initial conditions

$$f(0) = 1, \qquad D_{UG}^{(\alpha,\mathscr{A},\psi)} f(0) = 0,$$

where

$$\alpha = \tfrac{1}{4}, \qquad A(\alpha) = 1-\alpha, \qquad \psi(t,\alpha) = \alpha.$$

The function $f(t)$ describing cell growth under these memory effects is obtained as follows.

From the data, we have:

$$\alpha=\frac{1}{4}, A(\alpha)=A=\frac{3}{4}, \psi(t,\alpha)=C=\frac{1}{4}, f'(0)=-\frac{A(\alpha)}{\psi(t,\alpha)}=-3.$$

Applying the UGLT to the left-hand side of the UG differential equation (6) :

$$\begin{aligned}&\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\left\{D_{UG}^{(\frac{1}{4},\lambda,\psi)}D_{UG}^{(\frac{1}{4},\lambda,\psi)}f(t)\right\}(s)\\&=-Cf'(0)-(sC+3A)f(0)\\&+(Cs^2+4CAs+4A^2)F^{(\alpha,\mathscr{A},\psi)}(s)\\&=(Cs^2+4CAs+4A^2)F^{(\alpha,\mathscr{A},\psi)}(s)-(sC+2A).\end{aligned}$$

On the other hand, to apply the UGLT to the right-hand side of the UG differential equation (6), first we obtain:

$$J_3(s)=\int_0^\infty e^{-(s-3)t-I(t)}dt=\int_0^\infty e^{-st}dt=\frac{1}{s},$$

then:

$$\mathscr{L}_{UG}^{(\alpha,\mathscr{A},\psi)}\{e^{3t}\}(s)=\frac{1}{A}-\frac{s-3}{A}\cdot\frac{1}{s}=\frac{3}{As}.$$

Thus:

$$(Cs+2A)^2F^{(\alpha,\mathscr{A},\psi)}(s)=\frac{3}{As}+(sC+2A).$$

Hence:

$$F^{(\alpha,\mathscr{A},\psi)}(s)=\frac{\frac{4}{s}+\frac{s}{4}+\frac{3}{2}}{\left(\frac{s}{4}+\frac{3}{2}\right)^2}=\frac{\frac{64}{s}+4s+24}{(s+6)^2}.$$

If we set $p=s+\frac{A}{C}=s+3$ and since $F(p)=CF^{(\alpha,\mathscr{A},\psi)}(s)$, we obtain:

$$F(p)=\frac{p^2+7}{(p-3)(p+3)^2}=\frac{\frac{4}{9}}{p-3}+\frac{\frac{5}{9}}{p+3}+\frac{-\frac{8}{3}}{(p+3)^2}.$$

Therefore, applying the inverse transform, the solution is:

$$f(t)=\frac{4}{9}e^{3t}+\frac{5}{9}e^{-3t}-\frac{8}{3}te^{-3t}.$$

It should be noted that this solution is the same as the one obtained in Example 6.3 of [1].

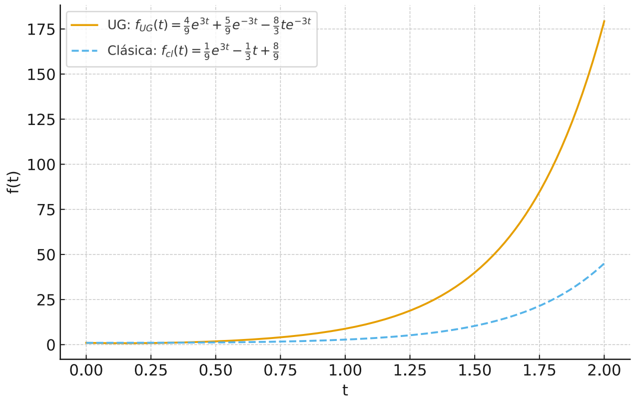


Fig. 6: Graphs of the functions $f_{UG}(t)=\frac{4}{9}e^{3t}+\frac{5}{9}e^{-3t}-\frac{8}{3}te^{-3t}$ and $f_{\text{cl}}(t)=\frac{1}{9}e^{3t}-\frac{1}{3}t+\frac{8}{9}$.

The classical differential equation

$$f''(t)=e^{3t},\qquad f(0)=1,\quad f'(0)=0,$$

has the solution

$$f_{\text{cl}}(t)=\frac{1}{9}e^{3t}-\frac{1}{3}t+\frac{8}{9}.$$

Using the generalized local fractional derivative, a solution is obtained that incorporates local memory effects through the parameters $\mathscr{A}$ and ψ. Consequently, the system response to the external stimulus differs from the classical model and depends explicitly on the underlying memory structure.

5 Conclusion

The generalized local fractional Laplace transform (UGLT) introduced in this work provides a unified analytical framework for the treatment of generalized local fractional differential equations. By incorporating the parameters α, $\mathscr{A}(\alpha)$, and $\psi(t,\alpha)$, the proposed transform extends the classical Laplace transform while preserving consistency with previously known formulations, including the classical and conformable cases under suitable choices of these parameters.

The results obtained throughout the paper show that the UGLT preserves several fundamental properties of the classical Laplace transform, such as linearity, transform rules for derivatives, while simultaneously adapting them to the generalized local fractional setting. This makes the transform an effective and flexible tool for solving differential equations involving generalized local fractional operators.

In addition, the presence of the weighting function $\psi(t,\alpha)$ and the parameter $\mathscr{A}(\alpha)$ allows the proposed framework to model a broader class of local dynamical behaviors that cannot be adequately represented within the classical context. Consequently, the UGLT may be useful in the analysis of phenomena arising in physics, engineering, biology, and other areas where intermediate memory effects or nonclassical local dynamics appear naturally.

Finally, the present work opens several directions for future research, including the study of nonlinear generalized local fractional differential equations, variable-coefficient models, multidimensional extensions, and comparisons with other existing fractional transform methods.

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