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On Translations, Contractions, and Dilations of Order Statistics and Record Values from the Extended Erlang–Truncated Exponential Distribution

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Abstract: In this study, a new continuous distribution called the Erlang-truncated exponential (ETE) distribution is introduced and studied. The Erlang-truncated exponential (ETE) distribution is modified and the new lifetime distribution is called the extended Erlang-truncated exponential (EETE) distribution. Some statistical and reliability properties of the new distribution are given and the method of moment generating function estimate was proposed for estimating the model parameters. The usefulness and flexibility of the EETE distribution was illustrated with an uncensored data set and its fit was compared with that of the ETE and three other three-parameter distributions. Order statistics and record values play an important role in statistical inference, reliability analysis, and extreme value studies. In this paper, we investigate the properties of order statistics and record values arising from the Erlang–truncated exponential distribution. Particular attention is given to the effects of linear transformations, namely translations, contractions, and dilations. Explicit expressions for the probability density functions and distribution functions of order statistics and record values are obtained, and several important distributional properties are derived. The behavior of these statistics under the considered transformations is examined in detail. The results provide useful insights into the structural properties of the Erlang–truncated exponential distribution and may have applications in reliability theory, survival analysis, and life-testing experiments.

Keywords: Erlang–truncated exponential distribution, order statistics, record values, translations, contractions, dilations, linear transformations, reliability analysis.

1 Introduction

The Erlang-Truncated Exponential (ETE) distribution was originally introduced by El-Alosey [1] as an extension of the standard one parameter exponential distribution. The ETE distribution results from the mixture of Erlang distribution and the left truncated one-parameter exponential distribution. The cumulative distribution function (cdf) $G(x)$, and probability density function (pdf) $g(x)$ of the ETE distribution are given by:

$$G(x) = 1 - e^{-\beta(1-e^{-\lambda})x}, \quad 0 \le x < \infty,\ \beta, \lambda > 0 \tag{1}$$

and

$$g(x) = \beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}, \quad 0 \le x < \infty,\ \beta, \lambda > 0 \tag{2}$$

respectively, where β is the shape parameter and λ is the scale parameter. The ETE distribution collapses to the classical one-parameter exponential distribution with parameter β when $\lambda \to \infty$.

The ETE distribution shares the same limitation of constant failure rate property with the exponential distribution which makes it unsuitable for modelling many complex lifetime data sets that have nonconstant failure rate characteristics. Generally speaking, research has shown that the standard probability distributions are largely inadequate for modelling complex lifetime data sets and various excellent ways of overcoming this shortcoming have been proposed in the literature; for instance: Beta exponential G distributions, due to Alzaatreh et al. [2]; Beta extended G distributions, due to Cordeiro et al. [3]; Beta G distributions, due to Eugene et al. [4]; Exponentiated exponential Poisson G distributions, due to Ristić

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and Nadarajah [5]; Exponentiated generalized G distributions, due to Cordeiro et al. [6]; Marshall–Olkin G distributions, due to Marshall and Olkin [7]; Transmuted family of distributions, due to Shaw and Buckley [8]; and so on.

Mainly, by introducing extra shape parameter(s) to standard distribution a robust and more flexible distribution is derived. For a comprehensive list of methods of generating new distributions readers are encouraged to see Nadarajah and Rocha [9], AL-Hussaini, Ahsanullah [10], Ali et al. [11], Cordeiro et al. [12], Alzaatreh et al. [13] and Pescim et al. [14].

1.1 Order Statistics: Definitions and Notations

Order statistics and record values play an important role in statistical theory and applications. They are widely used in reliability theory, survival analysis, quality control, and life-testing experiments. Order statistics arise naturally when a random sample is arranged in increasing or decreasing order, while record values occur in sequences of observations when a value exceeds all previous observations. Due to their importance in statistical inference and extreme value analysis, many researchers have investigated their theoretical properties under different probability distributions.

The exponential distribution is one of the most widely used lifetime distributions in reliability and survival studies because of its mathematical simplicity and memoryless property. However, in many practical situations, the standard exponential distribution may not adequately describe truncated or modified lifetime data. To overcome this limitation, several modified forms of the exponential distribution have been proposed. Among them, the Erlang–truncated exponential distribution has received attention for modeling lifetime data where truncation occurs naturally.

Linear transformations such as translations, contractions, and dilations play a significant role in probability theory and statistical analysis. These transformations help in understanding how statistical characteristics of random variables change under scaling and shifting. The study of order statistics and record values under such transformations provides useful insights into their structural and distributional properties.

Although a considerable amount of work has been done on order statistics and record values for many classical distributions, limited attention has been given to their behavior under linear transformations for the Erlang–truncated exponential distribution. Motivated by this gap, the present study investigates translations, contractions, and dilations of order statistics and record values arising from this distribution.

The main objective of this paper is to derive the probability density functions and distributional properties of order statistics and record values for the Erlang–truncated exponential distribution under the considered transformations. The obtained results contribute to a better understanding of the structural behaviour of these statistics and may be useful in reliability analysis, survival studies, and other related applications.

Suppose that (X_r), $r = 1,2,\ldots,n$ is a sequence of n independent and identically distributed (iid) random variables (rv's), each with cdf $F(x)$. If these are rearranged in ascending order of magnitude and written as $X_{(1:n)} \leq X_{(2:n)} \leq \cdots \leq X_{(n:n)}$, then $X_{r:n}$, $(r = 1,2,\ldots,n)$, is called the rth order statistic from a sample of size n. Further $X_{(1:n)}$, $X_{(n:n)}$ are called extreme order statistics, and $R = X_{(n:n)} - X_{(1:n)}$ is called the range. If $F_{X_{r:n}}(x)$ and $f_{X_{r:n}}(x)$, $(r = 1,2,\ldots,n)$, denote the cumulative distribution function (cdf) and probability density function (pdf) of the rth order statistic $X_{(r:n)}$ respectively, then the df and pdf of $X_{(r:n)}$, the rth order statistic from a sample of size n is given as [Arnold et al. (1992), David and Nagaraja (2003)]

$$F_{X_{(r:n)}}(x) = \sum_{j=r}^{n} \binom{n}{j} [F(x)]^j [1-F(x)]^{n-j} = I_{F(x)}(r, n-r+1) \tag{3}$$

and

$$f_{X_{(r:n)}}(x) = \frac{1}{B(r,n-r+1)} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x) \tag{4}$$

where $I_p(a,b) = \frac{1}{B(a,b)} \int_0^p t^{a-1}(1-t)^{b-1} dt$ denotes the incomplete beta function.

Many authors and researchers have studied the exact distributions of order statistics in samples of n observations from various classes of distributions, among them, Sarhan and Greenberg (1962), Arnold et al. (1992), Ahsanullah and Nevzorov (2001) and David and Nagaraja (2003) are notable.

1.2 Record Values: Definitions and Notations

Here we provide the distributions of record values. For detailed treatment of record values, the interested readers are referred to Ahsanullah (2004), and references therein.

Record Values: Suppose that $(X_n)_{n\geq 1}$ is a sequence of independent and identically distributed (iid) random variables (rv's) with cumulative distribution function (cdf) $F(x)$. Let $Y_n = \max(\min)\{X_j \mid 1 \leq j \leq n\}$ for $n \geq 1$. We say X_j is an upper (lower) record value of $\{X_n \mid n \geq 1\}$, if $Y_j > (<) Y_{j-1}$, $j > 1$. By definition X_1 is an upper as well as a lower record value.

Upper Record Values: The indices at which the upper record values occur are given by the record times $\{U(n), n \geq 1\}$, where $U(n) = \min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$ and $U(1) = 1$. Many properties of the upper record value sequence can be expressed in terms of the cumulative hazard rate function $R(x) = -\ln \bar{F}(x)$, where $\bar{F}(x) = 1 - F(x)$, $0 < \bar{F}(x) < 1$. Then the pdf of $X_{U(r)}$, the rth upper record is [Ahsanullah (1995), Arnold et al. (1998)]

$$f_{X_{U(r)}}(x) = \frac{1}{(r-1)!}[R(x)]^{r-1} f(x), \quad -\infty < x < \infty \tag{5}$$

and

$$\bar{F}_{X_{U(r)}}(x) = 1 - F_{X_{U(r)}}(x) = e^{-R(x)} \sum_{j=0}^{r-1} \frac{[R(x)]^j}{j!} \tag{6}$$

where

$$R(x) = -\ln \bar{F}(x), \quad \bar{F}(x) = 1 - F(x) \tag{7}$$

If the support of the distribution $F(x)$ is over (α, β), then by convention, we will write

$$X_{0:n} = \alpha, \quad X_{n:n-1} = \beta, \quad X_{U(0)} = \alpha, \quad X_{L(0)} = \beta \tag{8}$$

The distributions have been characterized through equality of distributions

$$Y \stackrel{d}{=} VX$$

between random variables (r.v's) X and Y scaled by another random variable (r.v) V with known distribution. If $0 < V < 1$, we say that Y is a random contraction of X and if $V > 1$, Y is a random dilation of X. Also if

$$Y \stackrel{d}{=} X + V$$

then for $V > 0$, Y is a random translation of X [Beutner and Kamps (2008), Castaño-Martínez et al. (2010)]. The equality of distributions for adjacent order statistics

$$\begin{aligned} X_{r+1:n} &\stackrel{d}{=} X_{r:n} \cdot V \\ X_{r:n} &\stackrel{d}{=} X_{r-1:n-1} \cdot V \\ X_{r:n} &\stackrel{d}{=} X_{r+1:n} \cdot W \\ X_{r:n} &\stackrel{d}{=} X_{r:n-1} \cdot W \end{aligned}$$

where $V \sim \text{Par}(\alpha)$ and $W \sim \text{pow}(\alpha)$ have been characterized among others by Alzaid and Ahsanullah (2003), Wesolowski and Ahsanullah (2004) and Oncel et al. (2005). The results have been extended for non adjacent order statistics by Castaño-Martínez et al. (2010) and Khan and Shah (2011).

Arnold et al. (2008), Nevzorov (2001) and Navarro (2008) have set up equality

$$X_{i:m} \stackrel{d}{=} X_{j:n} \cdot Y, \quad i \leq m, \; j \leq n \tag{9}$$

to characterize the distribution of X which they claim is still, in general, an open problem. We have made here an attempt to characterize distributions for varying conditions on the relation (9).

We assume that the df is differentiable w.r.t its arguments. Further, it is noted that if Y is a measurable function of X with the relation

$$Y = h(X), \tag{10}$$

then

$$(i) \qquad Y_{r:n} = h(X_{r:n}) \tag{11}$$

$$(ii) \qquad Y_{U(r)} = h(X_{U(r)}) \tag{12}$$

$$(iii) \qquad Y_{L(r)} = h(X_{L(r)}) \tag{13}$$

if h is an increasing function and

$$(i) \qquad Y_{n-r+1:n} = h(X_{r:n}) \tag{14}$$
$$(ii) \qquad Y_{U(r)} = h(X_{L(r)}) \tag{15}$$
$$(iii) \qquad Y_{L(r)} = h(X_{U(r)}) \tag{16}$$

if h is a decreasing function. Also for simplicity, we shall denote

$$(i) \qquad X \sim \text{Erlang-truncated}\exp(\beta(\alpha_\lambda))$$

if X has an Erlang-truncated exponential distribution with the df

$$F(x) = [1 - e^{-\beta(\alpha_\lambda)x}], \quad 0 \le x < \infty,\ \beta > 0,\ \lambda > 0 \tag{17}$$

where $\alpha_\lambda = (1 - e^{-\lambda})$.

$$(ii) \qquad X \sim \text{Par}(\beta(\alpha_\lambda))$$

if X has a Pareto distribution with the df

$$F(x) = [1 - x^{-\beta(\alpha_\lambda)}], \quad 1 < x < \infty,\ \beta > 0,\ \lambda > 0 \tag{18}$$

$$(iii) \qquad X \sim \text{pow}(\beta(\alpha_\lambda))$$

if X has a power function distribution with the df

$$F(x) = x^{\beta(\alpha_\lambda)}, \quad 0 < x < 1,\ \beta > 0,\ \lambda > 0 \tag{19}$$

It may further be noted that

$$\text{if } \log X \sim \text{Erlang-truncated}\exp(\beta(\alpha_\lambda)) \text{ then } X \sim \text{Par}(\beta(\alpha_\lambda)) \tag{20}$$
$$\text{if } -\log X \sim \text{Erlang-truncated}\exp(\beta(\alpha_\lambda)) \text{ then } X \sim \text{pow}(\beta(\alpha_\lambda)) \tag{21}$$

It has been assumed here throughout that the df is differentiable w.r.t. its argument.

The remaining part of this paper is organized as follows. In Section 2, we present the closed form mathematical expression for the pdf and cdf of the new probability distribution EETE and its statistical and reliability properties. In Section 3, the parameters of the EETE distribution are estimated through the method of maximum likelihood estimation. In Section 4, we perform a Monte-Carlo simulation study to assess the stability of the maximum likelihood estimates of the parameters of the EETE distribution.

2 Model

The cdf $F(x)$ and pdf $f(x)$ of the EETE distribution are given by:

$$F(x) = \left(1 - e^{-\beta(\alpha_\lambda)x}\right)^{\alpha}, \quad 0 \le x < \infty,\ \alpha, \beta, \lambda > 0 \tag{22}$$

and

$$f(x) = \alpha\beta(\alpha_\lambda)e^{-\beta(\alpha_\lambda)x}\left(1 - e^{-\beta(\alpha_\lambda)x}\right)^{\alpha-1}, \quad 0 \le x < \infty,\ \beta, \lambda > 0 \tag{23}$$

where α and β are shape parameters and λ is the scale parameter.

The EETE distribution reduces to the ETE distribution when $\alpha = 1$. The plots of the cdf and pdf are shown in Figure 1.

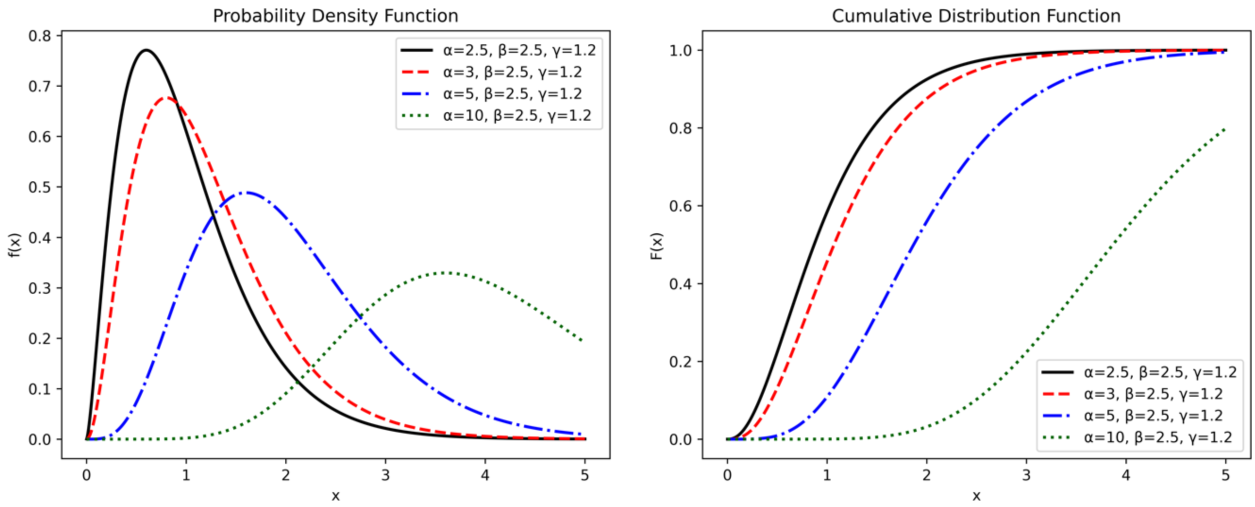

Fig. 1: Possible shapes of the (left) pdf $f(x)$ and (right) cdf $F(x)$ of the EETE distribution for fixed parameter values of β and λ.

2.1 *Reliability Characteristics*

The estimation of reliability is important in stress-strength models. If X_1 is the strength of a component and X_2 is the stress, then the component fails when $X_2 > X_1$. Then the estimation of the reliability of the component R is $P(X_2 < X_1)$, when X_1 and X_2 are distributed independently. The reliability function $R(x)$ is an important tool for characterizing life phenomenon. $R(x)$ is analytically expressed as $R(x) = 1 - F(x)$. Under certain predefined conditions, the reliability function $R(x)$ gives the probability that a system will operate without failure until a specified time x. The reliability function of the EETE distribution is given by:

$$R(x) = 1 - \left(1 - e^{-\beta(\alpha_\lambda)x}\right)^{\alpha}, \quad 0 \leq x < \infty,\ \alpha, \beta, \lambda > 0. \tag{24}$$

Another important reliability characteristic is the failure rate function. The frf gives the probability of failure for a system that has survived up to time x. The frf $h(x)$ is mathematically expressed as $h(x) = f(x)/R(x)$. The frf of the EETE distribution is given by:

$$h(x) = \frac{\alpha\beta(\alpha_\lambda)e^{-\beta(\alpha_\lambda)x}\left(1 - e^{-\beta(\alpha_\lambda)x}\right)^{\alpha-1}}{1 - \left(1 - e^{-\beta(\alpha_\lambda)x}\right)^{\alpha}}, \quad 0 \leq x < \infty,\ \alpha, \beta, \lambda > 0. \tag{25}$$

In this section we present the asymptotic and shape characteristics of the pdf and frf of the EETE distribution. The following asymptotic behaviours are observed:

$$f(0) = h(0) = h(\infty) = \begin{cases} \infty, & \text{if } \alpha < 1 \\ \beta(\alpha_\lambda), & \text{if } \alpha = 1; \\ 0, & \text{if } \alpha > 0 \end{cases} \quad \text{and } f(\infty) = 0;\ \forall \alpha > 0$$

2.1.1 **Characterization results based on upper records**

Theorem 2.1 Let $X_{U(r)}$ be the rth upper record statistic from a continuous population with the pdf $f(x)$ and the df $F(x)$, then

$$X_{U(r+j)} \stackrel{d}{=} X_{U(r)} + Y_j, \quad j = s-r-1, s-r;\ 1 \leq r < s \tag{26}$$

where $Y_j \stackrel{d}{=} X_{U(j)}$ is independent of $X_{U(r)}$ if and only if $X_1 \sim$ Erlang-truncated $\exp(\beta(\alpha_\lambda))$.

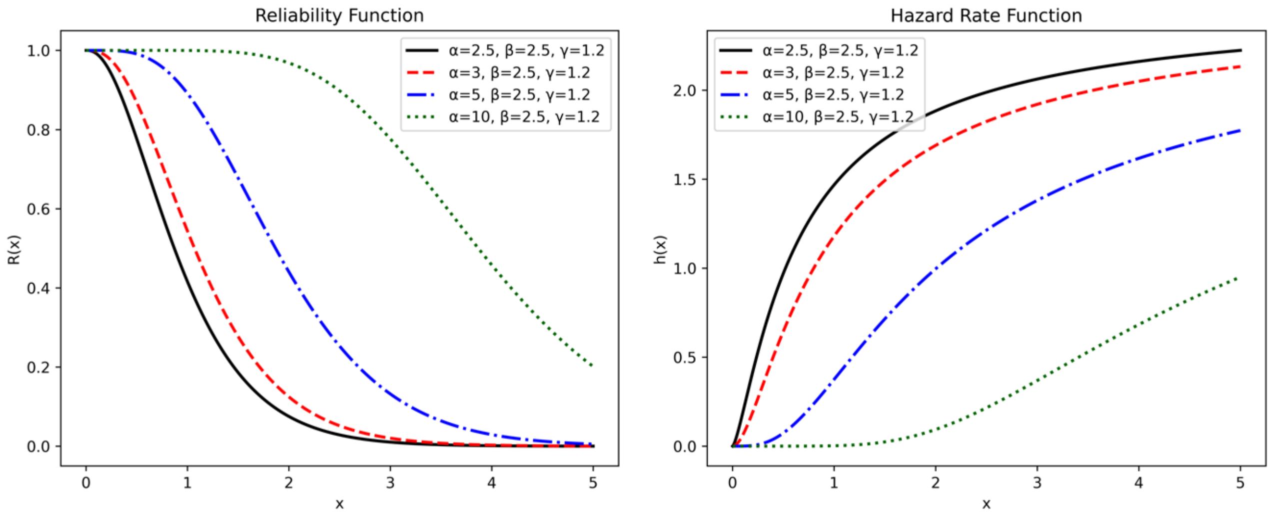

Fig. 2: Possible shapes of the (left) reliability function $R(x)$ and (right) failure rate function $h(x)$ of the EETE distribution for fixed parameter values of β and λ.

*Proof.*To prove the necessary part, let the moment generating function (mgf) of $X_{U(s)}$ be $M_{X_{U(s)}}(t)$, then

$$X_{U(s)} \overset{d}{=} X_{U(r)} + Y$$

implies

$$M_{X_{U(s)}}(t) = M_{X_{U(r)}}(t) \cdot M_Y(t)$$

Since for the Erlang-truncated $\exp(\beta(\alpha_\lambda))$ distribution,

$$M_{X_{U(s)}}(t) = \left(\frac{\beta(\alpha_\lambda)}{\beta(\alpha_\lambda) - t}\right)^s$$

Therefore,

$$M_Y(t) = \frac{M_{X_{U(s)}}(t)}{M_{X_{U(r)}}(t)} = \left(\frac{\beta(\alpha_\lambda)}{\beta(\alpha_\lambda) - t}\right)^{s-r}$$

But this is the mgf of $X_{U(s-r)}$, the $(s-r)$th upper record statistics and is independent of $X_{U(r)}$ if and only if $X_1 \sim$ Erlang-truncated $\exp(\beta(\alpha_\lambda))$ and the result follows. To prove the sufficiency part, note that the pdf of $X_{U(s)}$ by the convolution method is

$$f_{X_{U(s)}}(y) = \int_0^y f_{X_{U(r)}}(x) f_{V_{(s-r)}}(y-x)dx = \frac{(\beta\alpha_\lambda)^{s-r}}{\Gamma(s-r)} \int_0^y (y-x)^{s-r-1}[e^{-\beta(\alpha_\lambda)(y-x)}] f_{X_{U(r)}}(x)dx \tag{27}$$

Differentiating both sides of (27) w.r.t. y, we get

$$\frac{d}{dy} f_{X_{U(s)}}(y) = \frac{(\beta\alpha_\lambda)^{s-r}}{\Gamma(s-r-1)} \int_0^y (y-x)^{s-r-2}[e^{-\beta(\alpha_\lambda)(y-x)}] f_{X_{U(r)}}(x)dx - \frac{(\beta\alpha_\lambda)^{s-r+1}}{\Gamma(s-r)} \int_0^y (y-x)^{s-r-1}[e^{-\beta(\alpha_\lambda)(y-x)}] f_{X_{U(r)}}(x)dx$$

This leads to

$$\frac{d}{dy} f_{X_{U(s)}}(y) = \beta(\alpha_\lambda)[f_{X_{U(s-1)}}(y) - f_{X_{U(s)}}(y)] \tag{28}$$

or,

$$f_{X_{U(s)}}(y) = \beta\alpha_\lambda[F_{X_{U(s-1)}}(y) - F_{X_{U(s)}}(y)] \tag{29}$$

Therefore, in view of (5), (6) and (29), we have

$$F(y) = 1 - e^{-\beta(\alpha_\lambda)y}, \quad 0 \le x < \infty,\ \beta > 0,\ \lambda > 0$$

and hence the proof.

Remark 2.1 For $\alpha_\lambda = 1$, Alzaid and Ahsanullah (2003, Remark 1) have shown that for two adjacent upper records

$$X_{U(2)} \stackrel{d}{=} X_{U(1)} + V$$

where $V \sim \exp(1)$ if and only if $X_1 \sim \exp(1)$.
Remark 2.2 For $\alpha_\lambda = 1$, Oncel et al. (2005) and Ahsanullah (2006) have shown that

$$X_{U(r+1)} \stackrel{d}{=} X_{U(r)} + V$$

where $V \sim \exp(1)$ if and only if $X_1 \sim \exp(1)$.
Remark 2.3 For $\alpha_\lambda = 1$, Castaño-Martínez et al. (2010) have shown that

$$X_{U(s)} \stackrel{d}{=} X_{U(r)} + V$$

where $V \sim \mathrm{Ga}(s-r,1)$ if and only if $X_1 \sim \exp(1)$.
Corollary 2.1 Let $X_{U(r)}$ be the rth upper record from a continuous population with the pdf $f(x)$ and the df $F(x)$, then

$$X_{U(r+j)} \stackrel{d}{=} X_{U(r)} \cdot Y_j, \quad j = s-r-1, s-r;\ 1 \le r < s \tag{30}$$

where $Y_j \stackrel{d}{=} X_{U(j)}$ is independent of $X_{U(r)}$ if and only if $X_1 \sim \mathrm{Par}(\beta(\alpha_\lambda))$.

Proof. This can be proved by noting that

$$\log X_{U(s)} \stackrel{d}{=} \log X_{U(r)} + \log Y_{(s-r)}$$

implies

$$X_{U(s)} \stackrel{d}{=} X_{U(r)} \cdot Y_{(s-r)}$$

in view of (12) and (20), where Y_j in product (30) is called random dilation of $X_{U(r)}$.

Corollary 2.2 Let $X_{L(r)}$ be the rth lower record from a continuous population with the pdf $f(x)$ and the df $F(x)$, then

$$X_{L(r+j)} \stackrel{d}{=} X_{L(r)} \cdot Y_j, \quad j = s-r-1, s-r;\ 1 \le r < s \tag{31}$$

where $Y_j \stackrel{d}{=} X_{L(j)}$ is independent of $X_{L(r)}$ if and only if $X_1 \sim \mathrm{pow}(\beta(\alpha_\lambda))$.

Proof. Here the product $X_{L(r)} \cdot Y_j$ in (31) is called random contraction of $X_{L(r)}$. The Corollary can be proved by considering

$$-\log X_{U(s)} \stackrel{d}{=} -\log X_{U(r)} - \log Y_{(s-r)}$$

which implies

$$X_{L(s)} \stackrel{d}{=} X_{L(r)} \cdot Y_{(s-r)}$$

with an appeal to (16) and (21).

3 Results based on order statistics

Theorem 3.1 Let $X_{r:n}$ be the rth order statistic from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then

$$X_{r+j:n} \stackrel{d}{=} X_{r:n} + Y_{j:n-r}, \quad j = s-r-1, s-r;\ 1 \le r < s \le n \tag{32}$$

where $Y_{j:n-r} \stackrel{d}{=} X_{j:n-r}$ is independent of $X_{r:n}$ if and only if $X_1 \sim$ Erlang-truncated $\exp(\beta(\alpha_\lambda))$.
Proof. The proof of the necessary part is easy. To prove the sufficiency part, we have

$$f_{X_{s:n}}(y) = \frac{\beta(\alpha_\lambda)}{B(n-s+1,s-r)} \int_0^y e^{-\beta(\alpha_\lambda)(y-x)(n-s+1)} [1-e^{-\beta(\alpha_\lambda)(y-x)}]^{s-r-1} f_{X_{r:n}}(x)dx \tag{33}$$

Differentiating both sides of (33) w.r.t. y, we get for $s \geq r+1$:

$$\frac{d}{dy} f_{X_{s:n}}(y) = \beta(\alpha_\lambda)(n-s+1)[f_{X_{s-1:n}}(y) - f_{X_{s:n}}(y)]$$

and

$$f_{X_{s:n}}(y) = \beta(\alpha_\lambda)(n-s+1)[F_{X_{s-1:n}}(y) - F_{X_{s:n}}(y)]$$

and thus (David and Nagaraja, 2003)

$$F(y) = 1 - e^{-\beta(\alpha_\lambda)y}, \quad 0 \leq x < \infty,\ \beta > 0,\ \lambda > 0$$

Remark 3.1 For $\alpha_\lambda = 1$, Alzaid and Ahsanullah (2003) have proved that

$$X_{r:n} \stackrel{d}{=} X_{r-1:n} + V$$

where $V \sim \exp(n-r+1)$ if and only if $X_1 \sim \exp(1)$.
Remark 3.2 For $\alpha_\lambda = 1$, Castaño-Martínez et al. (2010) have shown that

$$X_{s:n} \stackrel{d}{=} X_{r:n} + V$$

where $V \stackrel{d}{=} -\log W$ with $W \sim \text{Be}(n-s+1, s-r)$ if and only if $X_1 \sim \exp(1)$.
Corollary 3.1 Let $X_{r:n}$ be the rth order statistic from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then

$$X_{r+j:n} \stackrel{d}{=} X_{r:n} \cdot Y_{j:n-r}, \quad j = s-r-1, s-r;\ 1 \leq r < s \leq n \tag{34}$$

where $Y_{j:n-r} \stackrel{d}{=} X_{j:n-r}$ is independent of $X_{r:n}$ if and only if $X_1 \sim \text{Par}(\beta(\alpha_\lambda))$.

Proof. To prove, consider

$$\log X_{s:n} \stackrel{d}{=} \log X_{r:n} + \log Y_{s-r:n-r} \tag{35}$$

and the result follows in view of (11) and (20), where $Y_{j:n-r}$ in (34) is called random dilation of $X_{r:n}$.

Corollary 3.2 Let $X_{r:n}$ be the rth order statistic from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then

$$X_{s-j:n} \stackrel{d}{=} X_{s:n} \cdot Y_{s-j:s-1}, \quad j = s-r-1, s-r;\ 1 \leq r < s \leq n \tag{36}$$

where $Y_{s-j:s-1} \stackrel{d}{=} X_{s-j:s-1}$ is independent of $X_{s:n}$ if and only if $X_1 \sim \text{pow}(\beta(\alpha_\lambda))$.

Proof. To prove the Corollary, we note that

$$-\log X_{s:n} \stackrel{d}{=} -\log X_{r:n} - \log Y_{s-r:n-r}$$

implies

$$X_{n-s+1:n} \stackrel{d}{=} X_{n-r+1:n} \cdot Y_{n-s+1:n-r}$$

or,

$$X_{r:n} \stackrel{d}{=} X_{s:n} \cdot Y_{r:s-1}$$

and the result follows in view of (14) and (21). This is a case of random contraction. For $\alpha_\lambda = j = 1$ $(s = r+1)$, Corollary 4.2 was proved by Wesolowski and Ahsanullah (2004).

Theorem 3.2 Let $X_{r:n}$ be the rth order statistic from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then

$$X_{s-j:n-j} \stackrel{d}{=} X_{s-r:n-r} + Y_{r-j:n-j}, \quad j = 0, 1 \tag{37}$$

where $Y_{r-j:n-j} \stackrel{d}{=} X_{r-j:n-j}$ is independent of $X_{s-r:n-r}$ if and only if $X_1 \sim$ Erlang-truncated $\exp(\beta(\alpha_\lambda))$.

Proof. The proof of the necessary part is easy. To prove the sufficiency part, we have

$$f_{X_{s:n}}(y) = \frac{\beta(\alpha_\lambda)}{B(n-r+1,r)} \int_0^y [e^{-\beta(\alpha_\lambda)(y-x)}]^{n-r+1} [1-e^{-\beta(\alpha_\lambda)(y-x)}]^{r-1} f_{X_{s-r:n-r}}(x)dx$$

Therefore,

$$\frac{d}{dy} f_{X_{s:n}}(y) = \frac{\beta(\alpha_\lambda)(r-1)}{B(n-r+1,r)} \int_0^y \alpha [e^{-\beta(\alpha_\lambda)(y-x)}]^{n-r+2} [1-e^{-\beta(\alpha_\lambda)(y-x)}]^{r-2} f_{X_{s-r:n-r}}(x)dx$$
$$-\frac{\beta(\alpha_\lambda)(n-r+1)}{B(n-r+1,r)} \int_0^y \alpha [e^{-\beta(\alpha_\lambda)(y-x)}]^{n-r+1} [1-e^{-\beta(\alpha_\lambda)(y-x)}]^{r-1} f_{X_{s-r:n-r}}(x)dx$$

Now since,

$$f_{X_{r:n}}(x) = \frac{\beta(\alpha_\lambda)}{B(n-r+1,r)} [e^{-\beta(\alpha_\lambda)x}]^{n-r+1} [1-e^{-\beta(\alpha_\lambda)x}]^{r-2} [1-e^{-\beta(\alpha_\lambda)x}]$$

$$= \frac{\beta(\alpha_\lambda)}{B(n-r+1,r)} [e^{-\beta(\alpha_\lambda)x}]^{n-r+1} [1-e^{-\beta(\alpha_\lambda)x}]^{r-2} - \frac{\beta(\alpha_\lambda)}{B(n-r+1,r)} [e^{-\beta(\alpha_\lambda)x}]^{n-r+2} [1-e^{-\beta(\alpha_\lambda)x}]^{r-2}$$

implying

$$\frac{\beta(\alpha_\lambda)}{B(n-r+1,r)} [e^{-\beta(\alpha_\lambda)x}]^{n-r+2} [1-e^{-\beta(\alpha_\lambda)x}]^{r-2} = \frac{n}{r-1} f_{X_{r-1:n-1}}(x) - f_{X_{r:n}}(x)$$

which leads to

$$\frac{d}{dy} f_{X_{s:n}}(y) = \beta(\alpha_\lambda) n [f_{X_{s-1:n-1}}(y) - f_{X_{s:n}}(y)]$$

and

$$f_{X_{s:n}}(y) = \beta(\alpha_\lambda) n [F_{X_{s-1:n-1}}(y) - F_{X_{s:n}}(y)] \tag{38}$$

Further, since (David and Nagaraja, 2003)

$$[F_{X_{s-1:n-1}}(y) - F_{X_{s:n}}(y)] = \binom{n-1}{s-1} [F(y)]^{s-1} [1-F(y)]^{n-s+1}$$

Thus (38) reduces to

$$f(y) = \beta(\alpha_\lambda)[1-F(y)]$$

implying that

$$\bar{F}(y) = e^{-\beta(\alpha_\lambda)y}$$

and the Theorem is proved.

Corollary 3.3 Let $X_{r:n}$ be the rth order statistic from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then

$$X_{s-j:n-j} \stackrel{d}{=} X_{s-r:n-r} \cdot Y_{r-j:n-j}, \quad j = 0, 1 \tag{39}$$

where $Y_{r-j:n-j} \stackrel{d}{=} X_{r-j:n-j}$ is independent of $X_{s-r:n-r}$ if and only if $X_1 \sim \text{Par}(\beta(\alpha_\lambda))$.
Remark 3.3 For $\alpha_\lambda = 1$, $j = 0$ and $r = 1$, we get

$$X_{s:n} \stackrel{d}{=} X_{s-1:n-1} \cdot Y_{1:n}$$

Corollary 3.4 Let $X_{r:n}$ be the rth order statistic from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then

$$X_{r:n-j} \stackrel{d}{=} X_{r:s-1} \cdot Y_{s:n-j}, \quad j = 0, 1 \tag{40}$$

where $Y_{s:n-j} \stackrel{d}{=} X_{s:n-j}$ is independent of $X_{r:s-1}$ if and only if $X_1 \sim \text{pow}(\beta(\alpha_\lambda))$. The Corollary is proved by an appeal of (14) and (21).
Remark 3.4 For $\alpha_\lambda = 1$, $j = 0$ and $s = n$, we get

$$X_{r:n} \stackrel{d}{=} X_{r:n-1} \cdot Y_{n:n}$$

where $Y_{n:n} \stackrel{d}{=} X_{n:n}$ as given in Wesolowski and Ahsanullah (2004).

4 Applications to the Prediction Problem

Many authors have considered prediction problems based on samples of random sizes. The importance of order statistics in reliability theory is attributed to the fact that the rth order statistic $(n-r+1)$ out-of-n system made up of n identical components with independent life lengths. On the other hand, in dealing with censored samples, where the life-test is terminated after observing the rth failure (Type II censoring), or the termination of the test occurs after a given time lapse (Type I censoring), the complete survival times cannot usually be observed (due to time or cost). In many biological and agriculture problems, we often come across a situation where the sample size is not deterministic because either some observations get lost for various reasons, or the size of the target population and its representative sample cannot be determined well. For example, assume that the inhabitants of a populous town are exposed to a dose of radiation resulting from an atomic accident, or exposed to an infection of an unknown epidemic. Furthermore, assume that our interest focuses on the time at which r persons would die among a big random sample of size n that is drawn from the residents of this town. Since the number of infected people in this town is unknown and changes randomly with time, the drawn sample contains a random number of infected and non-infected people. Accordingly, the sample size of the sub-sample of the infected people will be a non-negative integer valued RV, e.g. N, and it will be described by a sequence of independent and identically distributed RVs $X_1, X_2, \ldots, X_N$. Therefore, the rth smallest order statistic will be denoted by $X_{r:N}$, which represents the time at which r persons will die.

5 Conclusion

We introduced and studied a new lifetime model called the Erlang-truncated exponential distribution. The structural properties of this new model, including the expressions for the moment generating functions, order statistics and record statistics were derived. We demonstrated the application of the new model using real data set. The new model provided a better fit than its sub-models and other competing models. It is our hope that the new model will attract wider application in different areas such as engineering and economics.

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