Applied Mathematics & Information Sciences Letters
An International Journal

J_H^η-Proximal mapping for solving variational-like inclusions involving η-cocoercive mappings

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Received: 5 Mar. 2013, Revised: 10 Mar. 2013, Accepted: 14 Mar. 2013
Published online: 1 May. 2013

Abstract: In this paper, we introduce the new notion of J_H^η-proximal mapping for a nonconvex, lower semicontinuous, η-subdifferentiable, proper (may not be convex) functional in Banach spaces. The existence and Lipschitz continuity of J_H^η-proximal mapping are prove. By applying this notion, we study a variational-like inclusion problem in reflexive Banach spaces involving η-cocoercive mappings and propose a proximal point algorithm for finding the approximate solutions of a variational-like inclusion problem. The convergence criteria of the iterative sequences generated by the proposed algorithm is also discuss. Some examples are given.

Keywords: Variational-like inclusion, η-subdifferentiability, proximal mapping, reflexive Banach space, algorithm, η-cocoercive

1 Introduction

In recent past, variational inequality theory has appeared as an elegant and fascinating branch of applicable mathematics. This theory provides us effective and powerful tools for studying a wide class of nonlinear problems arising in many diverse fields of pure and applied sciences, such as mathematical programming, optimization theory, engineering, elasticity theory, and equilibrium theory of mathematical economy and game theory etc., for example, see, [1,2,3,8,9,10,11,12,13,15,16,17,18,19,20,21,22,23,24,30,31,32,36,37,38,39,41].

In 1994, Hassouni and Moudafi [30] introduced a perturbed method for solving a new class of variational inequalities, known as variational inclusions. A useful and important generalization of variational inclusion is called variational-like inclusion studied by several authors. A considerable interest has been shown in developing various extensions and generalizations of variational inequalities related to multi-valued operators, nonconvex optimization, nonmonotone operators and structural analysis.

In order to study various variational inequalities and variational inclusions, many authors investigated many generalized operators such as H-monotone [26], H-accretive [27], (H,η)-accretive [28], (H,η)-monotone [29], (A,η)-accretive [35], $H(\cdot,\cdot)$-accretive [42]. Very recently Ahmad et al.[5,6] introduced and studied $H(\cdot,\cdot)$-cocoercive and $H(\cdot,\cdot)$-η-cocoercive operators and applied them to solve some variational inclusion problems.

In this paper, we introduce the new notion of J_H^η-proximal mapping for a lower semicontinuous, η-subdifferentiable, proper (may not be convex) functional in Banach spaces. J_H^η-proximal mapping includes J-proximal mapping [25], J^η-proximal mapping [4], M-proximal mapping [34] as special cases. The existence and Lipschitz continuity of J_H^η-proximal mapping are prove under suitable conditions in reflexive Banach spaces and we propose a proximal point algorithm for finding the approximate solutions of a variational-like inclusion problem. The convergence of the iterative sequences generated by algorithm is discuss. Some examples are given.

2 Preliminaries

Let E be a real Banach space with the dual space E^*, $\langle u,x\rangle$ be the dual pairing between $u\in E^*$ and $x\in E$ and $CB(E^*)$ be the family of all nonempty closed bounded subsets of E^*. Let $\mathscr{D}(\cdot,\cdot)$ be the Hausdorff metric on $CB(E^*)$ defined

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by

$$\mathscr{D}(A,B)=\max\{\sup_{u\in A} d(u,B),\ \sup_{v\in B} d(A,v)\},\ \forall A,B\in CB(E^*),$$

where $d(u,B)=\inf_{v\in B} d(u,v)$ and $d(A,v)=\inf_{u\in A} d(u,v)$.

Let $\eta: E\times E\to E$ and $\phi: E\to R\cup\{+\infty\}$. A vector $w^*\in E^*$ is called η-subgradient of ϕ at $x\in \mathrm{dom}\phi$ if

$$\langle w^*,\eta(y,x)\rangle \le \phi(y)-\phi(x),\ \forall y\in E.$$

Each ϕ can be associated with the following η-subdifferential map $\partial_\eta\phi$ defined by

$$\partial_\eta\phi(x)=\begin{cases}\{w^*\in E^*:\langle w^*,\eta(y,x)\rangle\le\phi(y)-\phi(x)\}, & \\ \qquad\qquad\qquad \forall y\in E, x\in \mathrm{dom}\phi, & \\ \phi, & x\notin \mathrm{dom}\phi.\end{cases}$$

Example 2.1. let $E=\mathbb{R}^2$ and $\phi:\mathbb{R}^2\to\mathbb{R}\cup\{+\infty\}$ be a proper functional defined by

$$\phi:(x_1,y_1)\to\sqrt{x_1^2+y_1^2}.$$

Let $\eta:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2$ be defined by

$$\eta(x,y)\to(x-y),\ \text{where } x=(x_1,y_1), y=(x_2,y_2)\in\mathbb{R}^2,$$

then $\partial_\eta\phi(x)=\{v\in\mathbb{R}^2/\phi(y)\ge\phi(x)+v(y-x),\ \forall y\in\mathbb{R}^2\}$ is a maximal η-monotone set-valued mapping.

We recall the following definitions and results which are needed in the sequel.

Definition 2.1. A mapping $g: E\to E$ is said to be

(i) Lipschitz continuous, if there exists a constant $\lambda_g>0$ such that

$$\|g(x)-g(y)\|\le\lambda_g\|x-y\|,\ \forall x,y\in E.$$

(ii) k-strongly accretive $(k\in(0,1))$, if for any $x,y\in E$, there exists $j(x-y)\in\mathscr{F}(x-y)$ such that

$$\langle g(x)-g(y), j(x-y)\rangle\ge k\|x-y\|^2,$$

where $\mathscr{F}: E\to 2^{E^*}$ is the normalized duality mapping defined by

$$\mathscr{F}=\{f\in E^*:\langle f,x\rangle=\|x\|\|f\|,\|f\|=\|x\|\},\ \forall x\in E.$$

Some examples and properties of the mapping $\mathscr{F}$ can be found in [7].

Definition 2.2. Let $\eta: E\times E\to E; A: E\to E, T: E^*\to CB(E^*)$ and $N: E^*\times E^*\to E^*$ be the mappings, then

(i) η is said to be Lipschitz continuous, if there exists a constant $\tau>0$ such that

$$\|\eta(x,y)\|\le\tau\|x-y\|,\ \forall x,y\in E.$$

(ii) A is said to be α-expansive, if there exists a constant $\alpha>0$ such that

$$\|A(x)-A(y)\|\ge\alpha\|x-y\|,\ \forall x,y\in E.$$

1. [(iii)] N is said to be Lipschitz continuous with respect to the first argument, if there exists a constant $\lambda_{N_1}>0$ such that

$$\|N(u_1,\cdot)-N(u_2,\cdot)\|\le\lambda_{N_1}\|u_1-u_2\|,$$

$$\forall x_1,x_2\in E \text{ and for some } u_1\in T(x_1), u_2\in T(x_2).$$

Similarly we can define the Lipschitz continuity of N in the second argument.

Lemma 2.1[40]. Let E be a real Banach space and $\mathscr{F}: E\to 2^{E^*}$ be the normalized duality mapping. Then for any $x,y\in E$

$$\|x+y\|^2\le\|x\|^2+2\langle y, j(x+y)\rangle,\ \forall\ j(x+y)\in\mathscr{F}(x+y).$$

Definition 2.3. A functional $f: E\times E\to R\cup\{+\infty\}$ is said to be 0-diagonally quasi-concave (in short 0-DQCV) in y, if for any finite subset $\{x_1,x_2,.....x_n\}\subset E$ and for any $y=\sum_{i=1}^n\lambda_i x_i$ with $\lambda_i\ge0$ and $\sum_{i=1}^n\lambda_i=1$,

$$\min_{1\le i\le n} f(x_i,y)\le 0.$$

Definition 2.4. Let $J: E\to E^*; \eta, H: E\times E\to E$ and $A,B: E\to E$ be the mappings. Then

(i) J is said to be η-cocoercive with respect to $H(A,\cdot)$ if there exists a constant $\mu>0$ such that

$$\langle J(H(Ax,\cdot))-J(H(Ay,\cdot)),\eta(x,y)\rangle\ge\mu\|Ax-Ay\|^2,$$
$$\forall x,y\in E.$$

(ii) J is said to be relaxed η-cocoercive with respect to $H(\cdot,B)$ if there exists a constant $\gamma>0$ such that

$$\langle J(H(\cdot,Bx))-J(H(\cdot,By)),\eta(x,y)\rangle\ge(-\gamma)\|Bx-By\|^2,$$
$$\forall x,y\in E.$$

(iii) J is said to be Lipschitz continuous with respect to $H(A,\cdot)$ if there exists a constant $\lambda_{H_A}>0$ such that

$$\|J(H(Ax,\cdot))-J(H(Ay,\cdot))\|\le\lambda_{H_A}\|x-y\|,\ \forall x,y\in E.$$

Similarly we can define the Lipschitz continuity of J with respect $H(\cdot,B)$.

Lemma 2.2[14]. Let D be a nonempty convex subset of a topological vector space and $f: D\times D\to R\cup\{\pm\infty\}$ be such that

(i) for each $x\in D, y\to f(x,y)$ is lower semicontinuous on each compact subset of D,

(ii) for each finite set $\{x_1,x_2,....x_n\}\in D$ and for each $y=\sum_{i=1}^m\lambda_i x_i$ with $\lambda_i\ge0$ and $\sum_{i=1}^m\lambda_i=1$, $\min_{1\le i\le m} f(x_i,y)\le0$,

(iii) there exists a nonempty compact convex subset D_0 of D and a nonempty compact subset K of D such that for each $y\in D\setminus K$, there is an $x\in C_0(D_0\cup\{y\})$ satisfying $f(x,y)>0$.

Then there exists $\hat{y}\in D$ such that $f(x,\hat{y})\le0, \forall x\in D$.

3 J_H^η-Proximal mapping

First, we define the notion of J_H^η-Proximal mapping.

Definition 3.1. Let E be a real Banach space with the dual space E^*. Let $H,\eta : E\times E\to E; A,B : E\to E$ be the single-valued mappings and $\phi : E\to R\cup\{+\infty\}$ be the lower semicontinuous (may not be convex), proper, η-subdifferentiable functional. Let $J : E\to E^*$ be a mapping and if for any given $x^*\in E^*$ and $\rho>0$, there is a unique point $x\in E$ satisfying

$$\langle J(H(Ax,Bx))-x^*,\eta(y,x)\rangle+\rho\phi(y)-\rho\phi(x)\geq 0,\ \forall y\in E.$$

The mapping $x^*\to x$ denoted by $J_{H,\rho}^{\partial_\eta\phi}(x^*)$ is said to be J_H^η-proximal mapping of ϕ. Clearly, we have $x^*-J(H(A,B)(x))\in\rho\partial_\eta\phi(x)$, it follows that

$$J_{H,\rho}^{\partial_\eta\phi}(x^*)=(J(H(A,B))+\rho\partial_\eta\phi)^{-1}(x^*). \tag{1}$$

Remark 3.1.

(i) If $\phi : E\to R\cup\{+\infty\}$ is lower semicontinuous, proper, subdifferentiable, $\eta(y,x)=y-x$ and $J(H(A,B))=J$, then definition 2.2 coincides with the definition of J-proximal mapping of Ding and Xia [25].

(ii) If $J(H(A,B))=J$, then definition 2.2 coincides with the definition of J^η-proximal mapping of Ahmad et al. [4].

(iii) If $\phi : E\to R\cup\{+\infty\}$ is lower semicontinuous, proper, subdifferentiable, $\eta(y,x)=y-x$ and $J(H(A,B))=H(A,B)$, then definition 2.2 reduces to the definition of M-proximal mapping of Kazmi et al. [34].

Now we give some sufficient conditions which guarantee the existence and Lipschitz continuity of J_H^η-proximal mapping.

Theorem 3.1. Let E be a reflexive Banach space with the dual space E^* and $\phi : E\to R\cup\{+\infty\}$ be a lower semicontinuous, η-subdifferentiable, proper functional which may not be convex. Let $H,\eta : E\times E\to E$; $A,B : E\to E$ be the single-valued mappings such that η is τ-Lipschitz continuous, A is α-expansive and B is β-Lipschitz continuous. Let $J : E\to E^*$ be η-cocoercive with respect to $H(A,\cdot)$ with constant $\mu>0$ and relaxed η-cocoercive with respect to $H(\cdot,B)$ with constant $\gamma>0$. Let $\eta(x,y)=-\eta(y,x)$ for all $x,y\in E$ and for any $x\in E$, the function $h(y,x)=\langle x^*-J(H(Ax,Bx)),\eta(y,x)\rangle$ is 0-DQCV in y. Then for any $\rho>0$ and for any $x^*\in E^*$, there exists a unique $x\in E$ such that

$$\langle J(H(Ax,Bx))-x^*,\eta(y,x)\rangle+\rho\phi(y)-\rho\phi(x)\geq 0,\ \forall x,y\in E. \tag{2}$$

That is $x=J_{H,\rho}^{\partial_\eta\phi}(x^*)$ and so the J_H^η-proximal mapping of ϕ is well defined.

Proof. For any $J : E\to E^*; H,\eta : E\times E\to E; A,B : E\to E,\rho>0$ and $x^*\in E^*$, define a functional $f : E\times E\to R\cup\{+\infty\}$ by

$$f(y,x)=\langle x^*-J(H(Ax,Bx)),\eta(y,x)\rangle+\rho\phi(x)-\rho\phi(y),$$
$$\forall x,y\in E.$$

Since J,η are continuous mappings and ϕ is lower semicontinuous, we have that for any $y\in E, x\to f(y,x)$ is lower semicontinuous on E. We claim that $f(y,x)$ satisfies condition (ii) of Lemma 2.2. If it is false, then there exists a finite subset $\{y_1,y_2,....y_m\}\in E$ and

$$x_0=\sum_{i=1}^m\lambda_i y_i\ \text{ with }\lambda_i\geq 0,\sum_{i=1}^m\lambda_i=1$$

such that

$$\langle x_0-J(H(Ax_0,Bx_0)),\eta(y_i,x_0)\rangle+\rho\phi(x_0)-\rho\phi(y_i)>0,$$
$$\forall i=1,2,...,m.$$

Since ϕ is η-subdifferentiable at x_0, there exists a point $f_{x_0}^*\in E^*$ such that

$$\rho\phi(y_i)-\rho\phi(x_0)\geq\rho\langle f_{x_0}^*,\eta(y_i,x_0)\rangle,\ \forall i=1,2,...,m.$$

It follows that

$$\langle x^*-H(Ax_0,Bx_0)-\rho f_{x_0}^*,\eta(y_i,x_0)\rangle>0,\ \forall i=1,2,...,m. \tag{3}$$

On the other hand, by assumption $h(y,x)=\langle x^*-J(H(Ax_0,Bx_0))-\rho f_{x_0}^*,\eta(y_i,x_0)\rangle$ is 0-DQCV in y, we have

$$\sum_{1\leq i\leq m}\langle x-J(H(Ax_0,Bx_0))-\rho f_{x_0}^*,\eta(y_i,x_0)\rangle\leq 0,$$

which contradicts the inequality (3). Hence $f(y,x)$ satisfies the condition (ii) of Lemma 2.2. Now we take a fixed $\bar{y}\in\mathrm{dom}\phi$. Since ϕ is η-subdifferentiable at $\bar{y}$, there exists a point $f_{\bar{y}}^*\in E^*$ such that

$$\phi(x)-\phi(\bar{y})\geq\langle f_{\bar{y}}^*,\eta(x,\bar{y})\rangle,\ \forall x\in E.$$

Hence we have

$$\begin{aligned}
f(\bar{y},x) &= \langle x^*-J(H(Ax,Bx)),\eta(\bar{y},x)\rangle+\rho\phi(x)-\rho\phi(\bar{y})\\
&\geq \langle J(H(A\bar{y},B\bar{y}))-J(H(Ax,Bx)),\eta(\bar{y},x)\rangle\\
&\quad+\langle x^*-J(H(A\bar{y},B\bar{y})),\eta(\bar{y},x)\rangle+\rho\langle f_{\bar{y}}^*,\eta(x,\bar{y})\rangle\\
&= \langle J(H(A\bar{y},B\bar{y}))-J(H(Ax,B\bar{y})),\eta(\bar{y},x)\rangle\\
&\quad+\langle J(H(Ax,B\bar{y}))-J(H(Ax,Bx)),\eta(\bar{y},x)\rangle\\
&\quad+\langle x^*-J(H(A\bar{y},B\bar{y})),\eta(\bar{y},x)\rangle\\
&\quad+\rho\langle f_{\bar{y}}^*,\eta(x,\bar{y})\rangle.
\end{aligned} \tag{4}$$

Since J is η-cocoercive with respect to $H(A,\cdot)$ with constant μ and relaxed η-cocoercive with respect to $H(\cdot,B)$ with constant γ, η is τ-Lipschitz continuous, A is α-expansive and B is β-Lipschitz continuous, therefore

(4) becomes

$$\begin{aligned}
f(\bar{y},x) &\geq \mu\|A\bar{y}-Ax\|^2-\gamma\|B\bar{y}-Bx\|^2\\
&\quad -\tau(\|x^*\|+\|J(H(A\bar{y},B\bar{y}))\|+\rho\|f_{\bar{y}}^*\|)\|\bar{y}-x\|\\
&\geq \mu\alpha^2\|\bar{y}-x\|^2-\gamma\beta^2\|\bar{y}-x\|^2\\
&\quad -\tau(\|x^*\|+\|J(H(A\bar{y},B\bar{y}))\|+\rho\|f_{\bar{y}}^*\|)\|\bar{y}-x\|\\
&= (\mu\alpha^2-\gamma\beta^2)\|\bar{y}-x\|^2\\
&\quad -\tau(\|x^*\|+\|J(H(A\bar{y},B\bar{y}))\|+\rho\|f_{\bar{y}}^*\|)\|\bar{y}-x\|\\
&= \|\bar{y}-x\|[(\mu\alpha^2-\gamma\beta^2)\|\bar{y}-x\|\\
&\quad -\tau(\|x^*\|+\|J(H(A\bar{y},B\bar{y}))\|+\rho\|f_{\bar{y}}^*\|)].
\end{aligned}$$

Let $r=\frac{1}{(\mu\alpha^2-\gamma\beta^2)}\tau[\|x^*\|+\|J(H(A\bar{y},B\bar{y}))\|+\rho\|f_{\bar{y}}^*\|]$ and $K=\{x\in E:\|\bar{y}-x\|\leq r\}$. Let $D_0=\{\bar{y}\}$ and K are both weakly compact convex subsets of E and for each $x\in E\setminus K$,there exists a $\bar{y}\in C_0(D_0\cup\{\bar{y}\})$ such that $f(\bar{y},x)>0$. Hence all the conditions of the Lemma 2.2 are satisfied. By Lemma 2.2, there exists an $\bar{x}\in E$ such that $f(y,\bar{x})\leq 0$, for all $y\in E$, that is for any given $x^*\in E^*$,

$$\langle J(H(A\bar{x},B\bar{x}))-x^*,\eta(y,\bar{x})\rangle+\rho\phi(y)-\rho\phi(\bar{x})\geq 0,\ \forall y\in E.$$

Now we show that $\bar{x}$ is a unique solution of problem 2.

Suppose that $x_1,x_2\in E$ are two arbitrary solutions of problem 2. Then we have

$$\langle\, J(H(Ax_1,Bx_1))-x^*,\eta(y,x_1)\rangle+\rho\phi(y)-\rho\phi(x_1)\geq 0,\quad \forall y\in E, \tag{5}$$

$$\langle\, J(H(Ax_2,Bx_2))-x^*,\eta(y,x_2)\rangle+\rho\phi(y)-\rho\phi(x_2)\geq 0,\quad \forall y\in E. \tag{6}$$

Taking $y=x_2$ in (5) and $y=x_1$ in (6) and adding these inequalities, we have

$$\langle\, J(H(Ax_1,Bx_1))-x^*,\eta(x_2,x_1)\rangle+\langle J(H(Ax_2,Bx_2))-x^*,\eta(x_1,x_2)\rangle\geq 0, \tag{7}$$

since $\eta(x,y)=-\eta(y,x)$, we have

$$\begin{aligned}
&\langle\, J(H(Ax_1,Bx_1))-J(H(Ax_2,Bx_1)),\eta(x_1,x_2)\rangle\\
&+\langle J(H(Ax_2,Bx_1))-J(H(Ax_2,Bx_2)),\eta(x_1,x_2)\rangle\leq 0.
\end{aligned}$$

It follows that

$$(\mu\alpha^2-\gamma\beta^2)\|x_1-x_2\|^2\leq 0,$$

and hence we must have $x_1=x_2$ as $\mu>\gamma$ and $\alpha>\beta$. This completes the proof.

Example 3.1. Let $E=\mathbb{R}$ and $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be defined by

$$\eta(x,y)=\begin{cases}(x-y), & \text{if } |xy|<\frac{1}{3},\\ |xy|(x-y), & \text{if } \frac{1}{3}\leq|xy|<\frac{1}{2},\\ 2(x-y), & \text{if } \frac{1}{2}\leq|xy|.\end{cases}$$

Then it is easy to see that:

(i) η is 2-Lipschitz continuous.
(ii) $\eta(x,y)=-\eta(y,x)$.

Let $J:\mathbb{R}\to\mathbb{R}, H:\mathbb{R}\times\mathbb{R}\to\mathbb{R}, A,B:\mathbb{R}\to\mathbb{R}$ be the mappings such that

$$J(H(Ax,Bx))=Ax+Bx,$$

where $Ax=\frac{x}{2}, Bx=(1-x)$, for all $x\in\mathbb{R}$. Then

$$\langle J(H(Ax,u))-J(H(Ay,u)),\eta(x,y)\rangle\geq 4\|Ax-Ay\|^2$$

and

$$\langle J(H(u,Bx))-J(H(u,By)),\eta(x,y)\rangle\geq -\frac{1}{2}\|Bx-By\|^2$$

that is, J is 4-η-cocoercive with respect to $H(A,\cdot)$ and $\frac{1}{2}$-η-relaxed cocoercive with respect to $H(\cdot,B)$.

Further, we will show that for any $x\in\mathbb{R}$, the function $h(y,u)=\langle x-J(H(Au,Bu)),\eta(y,u)\rangle$ is 0-DQCV in y. If it is false, then there exists a finite set $\{y_1,y_2\ldots\ldots y_n\}$ and $u_0=\sum_1^n\lambda_iy_i$ with $\lambda_i\geq 0$ and $\sum_1^n\lambda_i=1$ such that for each $i=1,2,\ldots..n$

$$0<h(y_i,u_0)=\begin{cases}(x+\frac{u_0}{2}-1)(y_i-u_0), & \text{if } |y_iu_0|<\frac{1}{3},\\ |y_iu_0|(y_i-u_0), & \text{if } \frac{1}{3}\leq|y_iu_0|<\frac{1}{2},\\ 2(y_i-u_0), & \text{if } \frac{1}{2}\leq|y_iu_0|.\end{cases}$$

It follows that $(x+\frac{u_0}{2}-1)(y_i-u_0)>0$ for each $i=1,2,\ldots..n$ and hence, we have

$$0<\sum_1^n\lambda_i(x+\frac{u_0}{2}-1)(y_i-u_0)=(x+\frac{u_0}{2}-1)(u_0-u_0)=0,$$

which is not possible. Hence $h(y,u)$ is 0-DQCV in y. Thus η and J satisfies all assumption in Theorem 3.1.

Theorem 3.2. Let E be a reflexive Banach space with the dual space E^* and $\phi:E\to R\cup\{+\infty\}$ be a lower semicontinuous, η-subdifferentiable, proper functional which may not be convex. Let $H,\eta:E\times E\to E$; $A,B:E\to E$ be the single-valued mappings such that η is τ-Lipschitz continuous, A is α-expansive and B is β-Lipschitz continuous. Let $J:E\to E^*$ be η-cocoercive with respect to $H(A,\cdot)$ with constant $\mu>0$ and relaxed η-cocoercive with respect to $H(\cdot,B)$ with constant $\gamma>0$. Let $\eta(x,y)=-\eta(y,x)$ for all $x,y\in E$ and for any $x\in E$, the function $h(y,x)=\langle x^*-J(H(Ax,Bx)),\eta(y,x)\rangle$ is 0-DQCV in y. Then for any $\rho>0$, the J_H^η-proximal mapping ϕ is $\frac{\tau}{(\mu\alpha^2-\gamma\beta^2)}$-Lipschitz continuous.

Proof. By Theorem 3.1, we know that the J_H^η-proximal mapping of ϕ is well defined. For any given $x^*,y^*\in E^*$, let $x=J_{H,\rho}^{\partial_\eta\phi}(x^*), y=J_{H,\rho}^{\partial_\eta\phi}(y^*)$, then $x^*-J(H(Ax,Bx))\in$

$\rho\partial_\eta\phi(x)$, $y^* - J(H(Ay,By)) \in \rho\partial_\eta\phi(y)$.
Hence

$$\langle x^* - J(H(Ax,Bx)), \eta(u,x)\rangle \geq \rho\phi(x) - \rho\phi(u),\ \forall u \in E, \tag{8}$$

$$\langle y^* - J(H(Ay,By)), \eta(u,y)\rangle \geq \rho\phi(y) - \rho\phi(u),\ \forall u \in E. \tag{9}$$

Take $u = y$ in (8) and $u = x$ in (9) and adding these inequalities, we have

$$\langle\ x^* - J(H(Ax,Bx)), \eta(y,x)\rangle + \langle y^* - J(H(Ay,By)), \eta(x,y)\rangle \geq 0,$$

since $\eta(x,y) = -\eta(y,x)$, it follows that

$$\langle J(H(Ay,By)) - J(H(Ax,Bx)), \eta(y,x)\rangle \leq \langle\eta(y,x), y^* - x^*\rangle. \tag{10}$$

As J is η-cocoercive with respect to $H(A,\cdot)$ with constant μ and relaxed η-cocoercive with respect to $H(\cdot,B)$ with constant γ and η is τ-Lipschitz continuous, it follows from (10) that

$$(\mu\alpha^2 - \gamma\beta^2)\|y-x\|^2 \leq \tau\|y-x\|\|y^*-x^*\|,$$

which implies that $J^{\partial_\eta\phi}_{H,\rho}$ i.e. the J^η_H-proximal mapping of ϕ is $\frac{\tau}{(\mu\alpha^2-\gamma\beta^2)}$-Lipschitz continuous.

4 Formulation, iterative algorithm and existence result

Let $T, G : E \to CB(E^*)$ be set-valued mappings. Let $N : E^* \times E^* \to E^*, \eta : E \times E \to E$ and $g : E \to E$ be the single-valued mappings. Let $\phi : E \to R \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable function on E (may not be convex) satisfying $g(E) \cap \mathrm{dom}(\partial_\eta\phi) \neq \phi$, where $\partial_\eta\phi$ is the subdifferential of ϕ at x. We consider the following variational-like inclusion problem.

Find $x \in E, u \in T(x), v \in G(x)$ such that $g(x) \in \mathrm{dom}(\partial_\eta\phi)$ and

$$\langle N(u,v), \eta(y,g(x))\rangle \geq \phi(g(x) - g(y)),\ \forall y \in E. \tag{11}$$

If $E = X$, is a Hilbert space, $\eta(y,x) = y - x$, $\forall x,y \in X, N(u,v) = f(u) - P(v)$, $\forall u,v \in X$, where $f, P : X \to X$ are single-valued mappings, then problem (11) reduces to the following problem.

Find $x \in X, u \in T(x), v \in G(x)$ such that

$$\langle f(u) - P(v), y - g(x)\rangle \geq \phi(g(x)) - \phi(y)),\ \forall y \in X. \tag{12}$$

Problem (12) is introduced and studied by Huang [33]. For appropriate and suitable choices of mappings involved in the formulation of problem (11), one can obtain many variational inequalities (inclusions) studied previously by different authors.

We first transfer problem (11) into a fixed point problem.

Theorem 4.1. The (x,u,v) is a solution of problem (11) if and only if (x,u,v) satisfies the following relation:

$$g(x) = J^{\partial_\eta\phi}_{H,\rho}\{J(H(A(g(x)),B(g(x)))) - \rho N(u,v)\}, \tag{13}$$

where $x \in E$, $u \in T(x)$, $v \in G(x), \rho > 0$ and $J^{\partial_\eta\phi}_{H,\rho} = (J(H(A,B)) + \rho\partial_\eta\phi)^{-1}$ is the J^η_H-proximal mapping of ϕ.

Proof. Assume that $x \in E, u \in T(x)$ and $v \in G(x)$ satisfies relation (13), i.e.,

$$g(x) = J^{\partial_\eta\phi}_{H,\rho}\{J(H(A(g(x)),B(g(x)))) - \rho N(u,v)\},$$

since $J^{\partial_\eta\phi}_{H,\rho} = (J(H(A,B)) + \rho\partial_\eta\phi)^{-1}$, the above inequality holds if and only if

$$J(H(A(g(x)),B(g(x)))) - \rho N(u,v) \in J(H(A(g(x)),B(g(x)))) + \rho\partial_\eta\phi(g(x)).$$

By the definition of η-subdifferential of ϕ, the above relation holds if and only if

$$\phi(y) - \phi(g(x)) \geq \langle N(u,v), \eta(y,g(x))\rangle.$$

Hence we have

$$\langle N(u,v), \eta(y,g(x))\rangle \geq \phi(g(x)) - \phi(y),\ \forall y \in E,$$

i.e., (x,u,v) is a solution of problem (11). Similarly the converse part follows.

Based on (11), we suggest the following proximal point algorithm.

Algorithm 4.1. For any given $x_0 \in E, u_0 \in T(x_0)$ and $v_0 \in G(x_0)$, compute the sequences $\{x_n\}, \{u_n\}$ and $\{v_n\}$ by the iterative schemes.

$$g(x_{n+1}) = J^{\partial_\eta\phi}_{H,\rho}\{J(H(A(g(x_n)),B(g(x_n)))) - \rho N(u_n,v_n)\}; \tag{14}$$

$$u_n \in T(x_n), \|u_{n+1} - u_n\| \leq \mathscr{D}(T(x_{n+1}), T(x_n)), \tag{15}$$

$$v_n \in G(x_n), \|v_{n+1} - v_n\| \leq \mathscr{D}(G(x_{n+1}), G(x_n)), \tag{16}$$

$n = 0, 1, 2, \ldots.$
where $\rho > 0$ is a constant.

Theorem 4.2. let X be a reflexive Banach space with its dual X^* and let $T, G : E \to CB(E^*)$ be Lipschitz continuous mappings with constants λ_T and λ_G, respectively. Let $g : E \to E$ be Lipschitz continuous mapping with constant λ_g and $(g - I)$ is k-strongly accretive $(k \in (0,1))$ satisfying $(g - I)(E) = E$. Let $\eta : E \times E \to E$ be Lipschitz continuous mapping with constant $\tau > 0$ such that $\eta(x,y) = -\eta(y,x)$, for all $x,y \in E$ and for each given $x \in E$, the function $h(y,x) = \langle x^* - J(H(A(x),B(x))), \eta(y,x)\rangle$ is 0-DQCV in y. Let $H : E \times E \to E, A, B : E \to E$ be the mappings such that A is α-expansive and B is β-Lipchitz continuous. Let

$N : E^* \times E^* \to E^*$ be λ_{N_1}-Lipschitz continuous in the first argument and λ_{N_2}-Lipchitz continuous in the second argument. Let $\phi : E \to R \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable, proper functional (may not be convex) satisfying $g(x) \in \mathrm{dom}(\partial_\eta \phi)$. Let $J : E \to E^*$ be η-cocoercive with respect to $H(A,\cdot)$ with constant $\mu > 0$ and relaxed-η-cocoercive with respect to $H(\cdot,B)$ with constant $\gamma > 0$; Lipschitz continuous with respect to $H(A,\cdot)$ with constant λ_{H_A} and Lipschitz continuous with respect to $H(\cdot,B)$ with constant λ_{H_B}. Suppose that there exists a constant $\rho > 0$ such that for all $x,y \in E$ and $x^* \in E^*$, the following condition is satisfied.

$$\lambda_g^2(\lambda_{H_A}^2+\lambda_{H_B}^2) + \rho^2(\lambda_{N_1}^2\lambda_T^2+\lambda_{N_2}^2\lambda_G^2) < \frac{(1+2k)(\mu\alpha^2-\gamma\beta^2)^2}{4\tau^2}, \quad \mu > \gamma, \alpha > \beta. \tag{17}$$

Then the iterative sequences $\{x_n\},\{u_n\}$ and $\{v_n\}$ generated by Algorithm 4.1 converge strongly to x,u and v, respectively and (x,u,v) is a solution of variational-like inclusion problem (11).

Proof. We can write

$$\|x_{n+1}-x_n\|^2 = \|g(x_{n+1})-g(x_n)-g(x_{n+1})+g(x_n) +x_{n+1}-x_n\|^2$$

By Lemma 2.1, we have

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &\le \|g(x_{n+1})-g(x_n)\|^2\langle g(x_{n+1})-g(x_n) -x_{n+1}+x_n, j(x_{n+1}-x_n)\rangle \\ &= \|g(x_{n+1})-g(x_n)\|^2 - 2\langle (g-I)(x_{n+1}-x_n), j(x_{n+1}-x_n)\rangle. \end{aligned}\tag{18}$$

By Algorithm 4.1, we have

$$g(x_{n+1}) = J_{H,\rho}^{\partial_\eta\phi}[J(H(A(g(x_n)),B(g(x_n)))) - \rho N(u_n,v_n)].$$

Hence we have

$$\begin{aligned}\|g(x_{n+1})-g(x_n)\|^2 &= \|J_{H,\rho}^{\partial_\eta\phi}[J(H(A(g(x_n)),B(g(x_n)))) -\rho N(u_n,v_n)] - J_{H,\rho}^{\partial_\eta\phi}[J(H(A(g(x_{n-1})), B(g(x_{n-1})))) - \rho N(u_{n-1},v_{n-1})]\| \\ &= \|J_{H,\rho}^{\partial_\eta\phi}[J(H(A(g(x_n)),B(g(x_n)))) -\rho N(u_n,v_n)] - J_{H,\rho}^{\partial_\eta\phi}[J(H(A(g(x_n)), B(g(x_{n-1})))) - \rho N(u_n,v_{n-1})] \\ &\quad +J_{H,\rho}^{\partial_\eta\phi}[J(H(A(g(x_n)),B(g(x_{n-1})))) -\rho N(u_n,v_{n-1})] \\ &\quad -J_{H,\rho}^{\partial_\eta\phi}[J(H(A(g(x_{n-1})),B(g(x_{n-1})))) -\rho N(u_{n-1},v_{n-1})]\|^2\end{aligned}$$

Since $\|x+y\|^2 \le 2(\|x\|^2+\|y\|^2)$ and by Theorem 3.2, we have

$$\begin{aligned}\frac{1}{2}\|g(x_{n+1}) - g(x_n)\|^2 &\le \|J_{H,\rho}^{\partial_\eta\phi}[J(H(A(g(x_n)),B(g(x_n)))) -\rho N(u_n,v_n)] - \|J_{H,\rho}^{\partial_\eta\phi}[J(H(A(g(x_n)), B(g(x_{n-1})))) - \rho N(u_n,v_{n-1})]\|^2 \\ &\quad +\|J_{H,\rho}^{\partial_\eta\phi}[J(H(A(g(x_n)),B(g(x_{n-1})))) -\rho N(u_n,v_{n-1})] - \|J_{H,\rho}^{\partial_\eta\phi}[J(H(A(g(x_{n-1})), B(g(x_{n-1})))) - \rho N(u_{n-1},v_{n-1})]\|^2 \\ &\le \frac{\tau^2}{(\mu\alpha^2-\gamma\beta^2)^2}\|J(H(A(g(x_n)),B(g(x_n)))) -\rho N(u_n,v_n) - (J(H(A(g(x_n)),B(g(x_{n-1})))) -\rho N(u_n,v_{n-1}))\|^2 \\ &\quad +\frac{\tau^2}{(\mu\alpha^2-\gamma\beta^2)^2}\|J(H(A(g(x_n)), B(g(x_{n-1})))) - \rho N(u_n,v_{n-1}) -(J(H(A(g(x_{n-1})),B(g(x_{n-1})))) -\rho N(u_{n-1},v_{n-1}))\|^2 \\ &= \frac{\tau^2}{(\mu\alpha^2-\gamma\beta^2)^2}\|J(H(A(g(x_n)),B(g(x_n)))) -J(H(A(g(x_n)),B(g(x_{n-1})))) - \rho[N(u_n,v_n) -N(u_n,v_{n-1})]\|^2 \\ &\quad +\frac{\tau^2}{(\mu\alpha^2-\gamma\beta^2)^2}\|J(H(A(g(x_n)),B(g(x_{n-1})))) -J(H(A(g(x_{n-1})),B(g(x_{n-1})))) -\rho N(u_n,v_{n-1}) - \rho[N(u_{n-1},v_{n-1})]\|^2 \\ &\le \frac{2\tau^2}{(\mu\alpha^2-\gamma\beta^2)^2}\|J(H(A(g(x_n)),B(g(x_n)))) -J(H(A(g(x_n)),B(g(x_{n-1}))))\|^2 \\ &\quad +\frac{2\tau^2\rho^2}{(\mu\alpha^2-\gamma\beta^2)^2}\|N(u_n,v_n)-N(u_n,v_{n-1})\|^2 \\ &\quad +\frac{2\tau^2}{(\mu\alpha^2-\gamma\beta^2)^2}\|J(H(A(g(x_n)),B(g(x_{n-1})))) -J(H(A(g(x_{n-1})),B(g(x_{n-1}))))\|^2 \\ &\quad +\frac{2\tau^2\rho^2}{(\mu\alpha^2-\gamma\beta^2)^2}\|N(u_n,v_{n-1}) -N(u_{n-1},v_{n-1})\|^2\end{aligned}\tag{19}$$

By the Lipschitz continuity of J with respect to $H(A,\cdot)$ and $H(\cdot,B)$ with constants λ_{H_A} and λ_{H_B}, respectively, and g with constant λ_g, we have

$$\begin{aligned}\|J(H(A(g(x_n)),B(g(x_n)))) &- J(H(A(g(x_n)),B(g(x_{n-1}))))\| \\ &\leq \lambda_{H_B}\|g(x_n)-g(x_{n-1})\| \\ &\leq \lambda_{H_B}\lambda_g\|x_n-x_{n-1}\|. \end{aligned} \tag{20}$$

$$\begin{aligned}\|J(H(A(g(x_n)),B(g(x_{n-1})))) &- J(H(A(g(x_{n-1})), \\ &B(g(x_{n-1}))))\| \\ &\leq \lambda_{H_A}\|g(x_n)-g(x_{n-1})\| \\ &\leq \lambda_{H_A}\lambda_g\|x_n-x_{n-1}\|. \end{aligned} \tag{21}$$

By the Lipschitz continuity of $N(\cdot,\cdot)$ in both the arguments with constants λ_{N_1} and λ_{N_2}, respectively, Lipschitz continuity of T and G with constants λ_T and λ_G, respectively and Algorithm 4.1, we have

$$\begin{aligned}\|N(u_n,v_n)-N(u_n,v_{n-1})\| &\leq \lambda_{N_2}\|v_n-v_{n-1}\| \\ &\leq \lambda_{N_2}\mathscr{D}(G(x_n),G(x_{n-1})) \\ &\leq \lambda_{N_2}\lambda_G\|x_n-x_{n-1}\| \end{aligned} \tag{22}$$

$$\begin{aligned}\|N(u_n,v_{n-1})-N(u_{n-1},v_{n-1})\| &\leq \lambda_{N_1}\|u_n-u_{n-1}\| \\ &\leq \lambda_{N_1}\mathscr{D}(T(x_n),T(x_{n-1})) \\ &\leq \lambda_{N_1}\lambda_T\|x_n-x_{n-1}\| \end{aligned} \tag{23}$$

By (19-23), we obtain

$$\begin{aligned}&|g(x_{n+1} - g(x_n)\|^2 \\ &\leq \frac{4\tau^2}{(\mu\alpha^2-\gamma\beta^2)^2}\lambda_{H_B}^2\lambda_g^2\|x_n-x_{n-1}\|^2 \\ &\quad+\frac{4\tau^2\rho^2}{(\mu\alpha^2-\gamma\beta^2)^2}\lambda_{N_2}^2\lambda_G^2\|x_n-x_{n-1}\|^2 \\ &\quad+\frac{4\tau^2}{(\mu\alpha^2-\gamma\beta^2)^2}\lambda_{H_A}^2\lambda_g^2\|x_n-x_{n-1}\|^2 \\ &\quad+\frac{4\tau^2\rho^2}{(\mu\alpha^2-\gamma\beta^2)^2}\lambda_{N_1}^2\lambda_T^2\|x_n-x_{n-1}\|^2 \\ &= \frac{4\tau^2\rho^2}{(\mu\alpha^2-\gamma\beta^2)^2}[\lambda_{H_B}^2\lambda_g^2+\rho^2\lambda_{N_2}^2\lambda_G^2 \\ &\quad+\lambda_{H_A}^2\lambda_g^2+\rho^2\lambda_{N_1}^2\lambda_T^2]\|x_n-x_{n-1}\|^2 \end{aligned} \tag{24}$$

Since $(g-I)$ is k-strongly accretive, by (24), we have

$$\begin{aligned}\|x_n-x_{n-1}\|^2 &\leq \|g(x_{n+1})-g(x_n)\|^2 \\ &\quad-2\langle(g-I)(x_n-x_{n-1}),j(x_n-x_{n-1})\rangle \\ &\leq \theta_1\|x_n-x_{n-1}\|^2-2k\|x_{n+1}-x_n\|^2.\end{aligned}$$

It follows that

$$\|x_{n+1}-x_n\|^2 \leq \frac{\theta_1}{1+2k}\|x_n-x_{n-1}\|^2$$

or

$$\|x_{n+1}-x_n\|^2 \leq \theta\|x_n-x_{n-1}\|^2, \tag{25}$$

where $\theta=\sqrt{\frac{\theta_1}{1+2k}}$ and

$\theta_1=\frac{4\tau^2}{(\mu\alpha^2-\gamma\beta^2)^2}[\lambda_{H_B}^2\lambda_g^2+\rho^2\lambda_{N_2}^2\lambda_G^2+\lambda_{H_A}^2\lambda_g^2+\rho^2\lambda_{N_1}^2\lambda_T^2]$.

Condition (17) implies that $0<\theta<1$, so it follows from (25) that $\{x_n\}$ is a Cauchy sequence in E and let $x_n \to x$. Since the mappings T and G are Lipschitz continuous, it follows from (15) and (16) that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences, we can assume that $u_n \to u$ and $v_n \to v$. Using the continuity of J,H,A,B,g,N,η and by Algorithm 4.1, we have

$$g(x)=J_{H,\rho}^{\partial_\eta\phi}[J(H(A(g(x)),B(g(x))))-\rho N(u,v)].$$

Now we will prove that $u\in T(x)$ and $v\in G(x)$. Infact, since $u_n \in T(x_n)$ and

$$\begin{aligned} d(u_n,T(x)) &\leq \max\{d(u_n,T(x)), \sup_{y\in T)(x)} d(T(x_n),y)\} \\ &\leq \max\{\sup_{z\in Tx_n} d(z,T(x)), \sup_{y\in Tx} d(T(x_n),y)\} \\ &= \mathscr{D}(T(x_n),T(x)).\end{aligned}$$

We have

$$\begin{aligned} d(u,T(x)) &\leq \|u-u_n\|+d(u_n,T(x)) \\ &\leq \|u-u_n\|+\mathscr{D}(T(x_n),T(x)) \\ &\leq \|u-u_n\|+\lambda_T\|x_n-x\| \to 0,\ (x\to\infty),\end{aligned}$$

which implies that $d(u,T(x))=0$. Since $T(x)\in CB(E)$, it follows that $u\in T(x)$. Similarly, we can prove that $v\in G(x)$. By Theorem 4.1, (x,u,v) is a solution of problem (11). This completes the proof.

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