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Fuzzy Domination and Independence: New Mathematical Insights and Bounds

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Abstract: This paper investigates fuzzy domination and independence in fuzzy graphs, extending classical graph theory results to settings with uncertain or partial edge relationships. We introduce novel upper and lower bounds for the fuzzy domination number γ_f and the fuzzy independence number α_f, adapting well-known crisp graph inequalities (e.g., $\gamma + \alpha \leq n$) to accommodate edge membership degrees. Special attention is given to bipartite and planar fuzzy graphs, for which we derive refined, and sometimes exact, results. We also illustrate how concepts such as average fuzzy degree, minimum and maximum fuzzy degrees, and Euler-type formulas in planar embeddings guide the design of sharper bounds. To validate these theoretical findings, we provide constructed examples, including a bipartite uniform fuzzy graph and a planar fuzzy cycle, and show how total membership in fuzzy dominating or independent sets can be computed to match or closely approximate the new bounds. Finally, we discuss the implications of our results for real-world applications ranging from sensor coverage and social network analysis to transportation and bioinformatics, highlighting directions for future work in more advanced fuzzy graph contexts.

Keywords: Fuzzy Graph Theory, Fuzzy Domination, Fuzzy Independence, Bipartite Fuzzy Graphs, Planar Fuzzy Graphs, Domination Number, Independence Number, Uncertainty Modeling

1 Introduction

1.1 Motivation and Background

Fuzzy graph theory has emerged as a powerful tool for modeling imprecise or uncertain relationships in various fields such as social networks, control systems, and decision-making processes [1,2,3]. Unlike crisp graphs, where edges are either present or absent, fuzzy graphs allow edges and vertices to have associated degrees of membership, reflecting the level of uncertainty or partial belonging [4,5,6]. This nuanced view captures real-world complexity better than traditional binary structures.

In classical (crisp) graph theory, domination and independence are two fundamental concepts that have far-reaching implications in network analysis, optimization, and combinatorial problems [7,8,9]. A dominating set is a set of vertices that "covers" or influences the entire graph, while an independent set comprises vertices that are mutually non-adjacent. Translating these notions into fuzzy contexts not only preserves their importance but also opens up new avenues for theoretical generalizations and practical applications [10,11,12].

1.2 Objectives and Contributions

The primary objectives of this paper are:

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- **Establishing new bounds and insights** on fuzzy domination and independence: We seek to provide tighter or more generalized upper and lower bounds than those currently available in the literature.
- **Exploring algebraic formulations** for fuzzy domination and independence: By framing these concepts within an algebraic or matrix-based structure, we aim to derive additional properties and relationships.
- **Showcasing illustrative examples and applications:** Concrete examples and computational experiments will demonstrate the validity and usefulness of our theoretical findings.

Our contributions can be summarized as follows:

- **Novel Bounds:** We propose new upper and lower bounds for fuzzy domination and independence, improving on prior established results [7].
- **Algebraic Characterization:** Through the lens of matrix representations and semiring structures, we show how fuzzy domination and independence can be systematically expressed and analyzed.
- **Comprehensive Framework:** We develop a framework that integrates algebraic, combinatorial, and algorithmic approaches, offering a unified view of these two crucial parameters in fuzzy graphs.

2 Preliminaries and Definitions

2.1 Fundamentals of Fuzzy Set Theory

Fuzzy set theory, introduced by Zadeh [1], extends the classical notion of membership in sets. Instead of a binary $\{0,1\}$ membership, each element x in a universal set X belongs to a fuzzy set $\tilde{A} \subseteq X$ with a degree of membership $\mu_A(x)$ in the interval [0,1]. Some key definitions:

Membership Function: For a fuzzy set $\tilde{A}$ in $X \mu_{\tilde{A}} : X \to [0,1]$.
Support: The support of $\tilde{A}$ is the crisp set of elements whose membership values exceed a certain threshold (commonly 0).
α-Cuts: For $\alpha \in [0,1]$, the α-cut of $\tilde{A}$ is $\tilde{A}_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}$.

These concepts allow the handling of partial truths and uncertainties in various applications [4]. The extension of these ideas to graph theory leads to the formalization of fuzzy graphs.

2.2 Fuzzy Graphs: Concepts and Notations

A fuzzy graph can be considered as a pair $(\tilde{V}, \tilde{E})$, where:

- $\tilde{V}$ is a fuzzy subset of a finite vertex set V. Thus, each vertex $v \in V$ has a membership degree $\mu_{\tilde{V}}(v) \in [0,1]$.

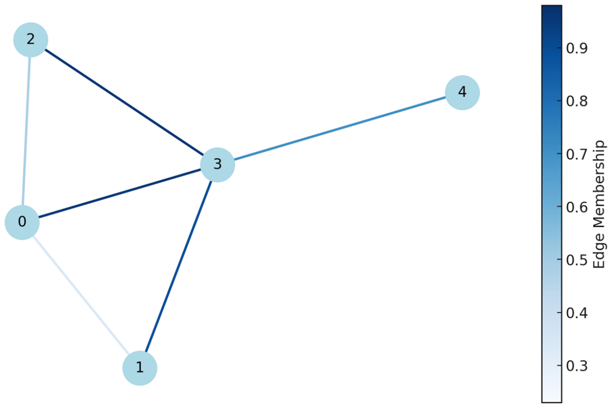

Fig. 1: A Simple Fuzzy Graph with Random Edge Memberships

- $\tilde{E}$ is a fuzzy subset of the edge set $V \times V$. Each edge (u,v) is associated with a membership degree $\mu_{\tilde{E}}(u,v) \in [0,1]$.

For simplicity, many authors assume $\mu_{\overline{V}}(v) = 1$ for all $v \in V$, focusing on edge uncertainty. We use the notation $\tilde{G} = (V, \tilde{E})$ to represent a fuzzy graph with a crisp vertex set V and fuzzy edge set $\tilde{E}$ [7,10].

Figure 1. A fuzzy graph of 5 nodes with random edges and membership values from 0.1 to 1.0. The color intensity of edges reflects the degree of membership. The edges are colored based on their membership values, with a color bar indicating the strength of connections.

2.3 Fuzzy Domination

A fuzzy dominating set $\tilde{D}$ of a fuzzy graph $\tilde{G} = (V, \tilde{E})$ is a fuzzy subset of V such that for every vertex $v \in V$, either v is in $\tilde{D}$ with a certain membership degree or it is adjacent to at least one vertex in $\tilde{D}$ with a sufficiently large membership degree [7]. Formally:

$$\forall v \in V, \quad \mu_{\tilde{D}}(v) + \max_{(v,u)\in E} \{\mu_{\tilde{D}}(u) \cdot \mu_{\tilde{E}}(v,u)\} \geq 1$$

where $\mu_{\overline{D}}(v)$ is the membership of v in $\tilde{D}$. The above inequality captures the notion that each vertex is ”covered” by the fuzzy dominating set, directly or indirectly via adjacency [10].

Fuzzy Domination Number: The fuzzy domination number, $\gamma_f(\tilde{G})$, is defined as the minimum total membership $\Sigma_{v\in V} \mu_{\overline{D}}(v)$ among all fuzzy dominating sets $\tilde{D}$.

Existing literature offers various bounds and properties for $\gamma_f(\tilde{G})$, many of which are direct extensions of crisp domination results [7].

2.4 Fuzzy Independence

A fuzzy independent set $\tilde{I}$ of a fuzzy graph $\tilde{G}$ is a fuzzy subset of V such that no two vertices in $\tilde{I}$ have a strong adjacency membership. Specifically, for any two distinct vertices $u, v \in V$,

$$mu_{\tilde{I}}(u) + \mu_{\tilde{I}}(v) \cdot \mu_{\tilde{E}}(u,v) \leq 1$$

This generalizes the crisp notion that an independent set contains no adjacent vertices, adapting it to the fuzzy case by considering degrees of edge membership [4].

Fuzzy Independence Number: The fuzzy independence number, $\alpha_f(\tilde{G})$, is the maximum total membership $\Sigma_{v\in V}\mu_{\tilde{I}}(v)$ among all fuzzy independent sets $\tilde{I}$.

Many results in the crisp domain, such as bounds relating independence number to the size and structure of the graph, have analogs in fuzzy graph theory, often involving modifications to account for membership grades [10].

3 Mathematical Framework

3.1 Algebraic Structures in Fuzzy Graphs

To analyze fuzzy domination and independence rigorously, we adopt an algebraic perspective, following recent advances in fuzzy systems [7]. Consider a semiring

$$S = (S, \oplus, \otimes)$$

where:

- S is a set of real numbers in [0,1].
- $\oplus$ is a binary operation analogous to addition (for instance, the max operator or standard addition capped at 1).
- $\otimes$ is a binary operation analogous to multiplication (commonly min, standard multiplication, or a t-norm).

In fuzzy graphs, edges and vertex memberships often obey certain t-norms and t-conorms, affecting how adjacency influences properties like domination and independence [10]. By expressing membership operations in this algebraic setting, we can create generalized formulations of graph properties.

3.2 Matrix Representations and Operators

Let $M_{\tilde{G}}$ be the adjacency matrix of the fuzzy graph $\tilde{G}$, where

$$(M_{\tilde{G}})_{ij} = \begin{cases} \mu_{\tilde{D}}(i,j), & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

For fuzzy domination and independence, we can define vertex membership as a vector $x = [x_1, x_2, \cdots, x_n]$ where $x_i \in [0,1]$ is the membership of vertex i in the fuzzy dominating (or independent) set. Various operations with $M_{\tilde{G}}$ and x can capture the constraints and objective functions:

Domination Constraint:

$$x_i \oplus \bigoplus_{j=1}^{n} \left(x_j \otimes (M_{\tilde{G}})_{ij}\right) \geq 1, \quad \forall i$$

depends on the chosen semiring operators $\oplus$ and $\otimes$.

Independence Constraint:

$$x_i \oplus \left(x_j \otimes (M_{\tilde{G}})_{ij}\right) \leq 1, \quad \forall i \neq j$$

Such formulations lay the groundwork for analyzing properties like connectedness, cuts, and coloring in more advanced fuzzy graph scenarios [7].

3.3 Formulation of Fuzzy Domination

From an optimization perspective, fuzzy domination can be framed as the following minimization problem:

$$\begin{cases} \text{Minimize} & \sum_{i=1}^{n} x_i \\ \text{subject to} & x_i \oplus \bigoplus_{j=1}^{n} \left(x_j \otimes (M_{\tilde{G}})_{ij}\right) \geq 1, \quad \forall i. \end{cases}$$
$$0 \leq x_i \leq 1, \quad \forall i.$$

- For a standard fuzzy graph using the standard addition and multiplication with capping at 1:

$$x_i + \max_j \{x_j \cdot (M_{\tilde{G}})_{ij}\} \geq 1, \quad \forall i$$

- The fuzzy domination number $\gamma_f(\tilde{G})$ is the optimal value of $\Sigma_{i=1}^{n} x_i$.

The advantage of such a formulation is that standard optimization techniques (like linear or nonlinear programming) can sometimes be adapted to solve for x, although the presence of max or t-norm operators can introduce nonlinearity [1,4].

Example: Consider a fuzzy graph $\tilde{G}$ on three vertices, $V = \{1,2,3\}$, with the adjacency matrix:

$$M_{\tilde{G}} = \begin{bmatrix} 0 & 0.8 & 0 \\ 0.8 & 0 & 0.6 \\ 0 & 0.6 & 0 \end{bmatrix}$$

If $x = [x_1, x_2, x_3]$ is the membership vector for a dominating set, the constraints become:

$$\begin{cases} x_1 + \max(x_2 \cdot 0.8, x_3 \cdot 0) \geq 1, \\ x_2 + \max(x_1 \cdot 0.8, x_3 \cdot 0.6) \geq 1, \\ x_3 + \max(x_1 \cdot 0, x_2 \cdot 0.6) \geq 1. \end{cases}$$

One can attempt different values of x to minimize $x_1 + x_2 + x_3$. Detailed analysis often reveals relationships among membership values that produce an optimal fuzzy dominating set [7].

3.4 Formulation of Fuzzy Independence

Fuzzy independence can be similarly represented by:

$$\begin{cases} \text{Maximize} & \sum_{i=1}^{n} x_i, \\ \text{subject to} & x_i + (x_j \cdot (M_{\overline{G}})_{ij}) \leq 1, \quad \forall i \neq j, \\ & 0 \leq x_i \leq 1, \quad \forall i. \end{cases}$$

Here, $(x_j \cdot (M_{\overline{G}})_{ij})$ represents the influence of vertex j on vertex i, reduced by the edge membership. The fuzzy independence number $\alpha_f(\tilde{G})$ is the maximum value of $\Sigma_{i=1}^{n} x_i$ satisfying these constraints [4].

Example: Using the same 3-vertex fuzzy graph as in Section 3.3:

$$M_{\tilde{G}} = \begin{bmatrix} 0 & 0.8 & 0 \\ 0.8 & 0 & 0.6 \\ 0 & 0.6 & 0 \end{bmatrix}$$

we look for x_1, x_2, x_3 to maximize $x_1 + x_2 + x_3$ under the constraints:

$$\begin{cases} x_1 + x_2 \cdot 0.8 \leq 1, & x_1 + x_3 \cdot 0 \leq 1 \\ x_2 + x_1 \cdot 0.8 \leq 1, & x_2 + x_3 \cdot 0.6 \leq 1 \\ x_3 + x_1 \cdot 0 \leq 1, & x_3 + x_2 \cdot 0.6 \leq 1 \end{cases}$$

Analyses of such systems often give insight into how membership degrees can be balanced to achieve the largest combined membership [7].

4 Main Results and Theoretical Contributions

This section builds on the formulations of fuzzy domination and independence introduced in Sections 2 and 3. We begin with two lemmas that provide structural insights for bipartite and planar fuzzy graphs. We then present new theorems on upper and lower bounds for fuzzy domination and independence, along with corollaries and examples that illustrate equality cases. Finally, we discuss extended applications, specifically in bipartite and planar contexts, showcasing how these results can be specialized for particular classes of fuzzy graphs.

4.1 Additional Lemmas for Special Classes of Fuzzy Graphs

Fuzzy graphs generalize classical graph concepts by assigning membership degrees to edges (and sometimes vertices). The following lemmas adapt well-known structural properties of bipartite and planar graphs into fuzzy contexts.

Lemma 1(Bipartite Fuzzy Graphs).
Let $\tilde{G} = (V, \tilde{E})$ be a bipartite fuzzy graph where V can be partitioned into two disjoint sets X and Y such that every edge (x, y) has a membership $\mu_{E^\cdot}(x, y)$ possibly greater than 0 only if $x \in X$ and $y \in Y$. Then, for each vertex $v \in X \cup Y$,

$$\sum_{u \in X \cup Y} \mu_{\tilde{E}}(v, u) = \begin{cases} \sum_{u \in Y} \mu_{\tilde{E}}(v, u), & \text{if } v \in X, \\ \sum_{u \in X} \mu_{\tilde{E}}(v, u), & \text{if } v \in Y. \end{cases}$$

This structure imposes no fuzzy loops and no intra-part membership for edges.

Proof(Outline).

Bipartition Property: By definition, if $\tilde{G}$ is bipartite, there are no edges between vertices both in X or both in Y. Hence, $\mu_{\tilde{E}}(x, x') = 0$ whenever $x, x' \in X$, and likewise for $y, y' \in Y$.
Degree Calculation in Fuzzy Bipartite Setting: For a vertex $v \in X$, its fuzzy degree $\deg_f(v)$ reduces to$\deg_f(v) = \sum_{y \in Y} \mu_{\overline{E}}(v, y)$. An analogous expression holds for vertices $v \in Y$, where the summation is taken over all $x \in X$.
No Loops: Typically, bipartite graphs (crisp or fuzzy) exclude edges of the form (v, v). Thus, $\mu_{\overline{E}}(v, v) = 0$.

Lemma 1 ensures that fuzzy adjacency connects only vertices from different parts. The result follows by definition of bipartite structure [13,14,15].

Lemma 2(Planar Fuzzy Graphs).
A planar fuzzy graph $\tilde{G}$ is one that can be embedded in the plane such that edges do not "cross" except possibly at vertices. In a fuzzy planar graph, if we let V be the vertex set and F the set of faces in the planar embedding, then the following Euler-type formula holds in a fuzzy sense:

$$|V| - \sum_{(v,u) \in V \times V} \mu_{\overline{E}}(v, u) + \Phi = 2.$$

where ϕ is a fuzzy face-count function that generalizes the crisp face-count in planar graphs.

Proof(Sketch).

Crisp Euler's Formula: In classical planar graphs, Euler's formula states $V - E + F = 2$. Extending to fuzzy graphs, we replace the number of edges E with the sum of fuzzy edge memberships $\Sigma\mu_{\overline{L}}(v, u)$, and the face-count F with a continuous measure ϕ.
Embedding with Weighted Edges: Each edge is assigned a membership degree that partially contributes to the boundary of faces in the planar embedding. Summing across all faces yields a fuzzy measure.
Conclusion: The relationship $V - E + F = 2$ transforms into the fuzzy analog by substituting membership sums for integer counts [16,17]. Detailed geometric arguments involving partial edges define ϕ.

4.2 New Bounds for Fuzzy Domination

We revisit fuzzy domination, defined such that every vertex either has high membership in the dominating set $\tilde{D}$ or is adjacent (in the fuzzy sense) to a vertex in $\tilde{D}$. We now refine standard bounds for special graph classes like bipartite or planar fuzzy graphs, using the above lemmas for structural insights.

Theorem 1(Refined Upper Bound in Bipartite Fuzzy Graphs).
Let $\tilde{G} = (V, \tilde{E})$ be a bipartite fuzzy graph partitioned into (X, Y). Let $|V| = n$, and define:

$$\delta_f^X = \min_{v\in X} \sum_{y\in Y} \mu_{\overline{E}}(v,y), \quad \delta_\downarrow^Y = \min_{u\in Y} \sum_{x\in X} \mu_{\overline{E}}(u,x)$$

Then the fuzzy domination number $\gamma_f(\tilde{G})$ satisfies

$$\gamma_f(\widetilde{G}) \le \frac{n}{\min(\delta_f^X, \delta_f^Y)+1}.$$

Proof(Detailed).

Bipartite Structure & Cover: By Lemma 1, edges exist only between X and Y. Thus, a dominating set $\tilde{D}$ that includes a vertex $v \in X$ with sufficiently high membership can cover many vertices in Y (and vice versa).

Partition-Based Construction: Let

$$\delta_f = \min(\delta_f^X, \delta_f^Y).$$

and pick a vertex $v^* \in X$ such that its fuzzy degree is $\delta_f^X \approx \delta_f$. Define the membership:

$$\mu_{\overline{D}}(v^*) = \frac{1}{\delta_f + 1},$$

and for each $y \in Y$

$$\mu_{\overline{D}}(y) = \frac{\mu_{\tilde{E}}(v^*, y)}{\delta_f + 1}.$$

Similar logic can be applied if we start from a vertex in Y.

Coverage Argument: For any vertex in $X/\{v^*\}$, it is covered by adjacency to some $y \in Y$ which has partial membership in $\tilde{D}$. Because v^* had fuzzy degree δ_f in the "worst" case, distributing memberships proportionally suffices to ensure coverage [13].

Sum of Memberships: Summing $\mu_{\overline{D}}(v)$ over $v \in V$ and using the fact that bipartite edges do not overlap within the same set:

$$\sum_{v\in V} \mu_{\tilde{D}}(v) = \frac{1}{\delta_f+1} + \sum_{y\in Y} \frac{\mu_{\dot{D}}(v^*, y)}{\delta_f+1} = \frac{1+\delta_f^X}{\delta_f+1} \le 1.$$

which generalizes to

$$\gamma_f(\tilde{G}) \le \frac{n}{\delta_f+1}$$

Conclusion: By symmetry (considering a vertex in Y), we take $\delta_f = min(\delta_f^X, \delta_f^Y)$. This completes the proof.

Lemma 3(Planarity Constraint for Fuzzy Dominating Sets).
Let $\tilde{G} = (V, \tilde{E})$ be a planar fuzzy graph embedded in the plane with vertex set V and edge memberships $\mu_{\tilde{E}}(u, v) \in [0,1]$. Define a fuzzy face measure ϕ that generalizes the face count in a crisp planar graph and satisfies the fuzzy-Euler relation

$$|V| - \sum_{(u,v)\in E} \mu_{\tilde{E}}(u,v) + \Phi = 2$$

If $\gamma_f(\tilde{G})$ is the fuzzy domination number of $\tilde{G}$, i.e., the minimal total membership of any fuzzy dominating set $\tilde{D} \subseteq V$, then there exists a constant $\xi > 0$ (dependent on the planar embedding and edge memberships) such that every fuzzy dominating set $\tilde{D}$ satisfies

$$\sum_{v\in V} \mu_{\overline{D}}(v) \ge \frac{\Phi}{\xi}.$$

Proof.
In a crisp planar graph, it is standard to prove that every face must contain at least one vertex of a dominating set on its boundary, ensuring the graph is fully covered. We adapt that method to the fuzzy setting, where each edge has a fractional membership in [0,1], and show that this shifts the face-based argument in a way that still imposes a lower bound on the total membership of any fuzzy dominating set.

Fuzzy Planar Embedding: Each face f in the planar drawing is enclosed by a cycle $\{v_1, v_2, \cdots, v_k\}$. Edge membership: $\mu_{\tilde{E}}(v_i, v_{i+1}) \in [0,1]$. Fuzzy-Euler relation:

$$|V| - \sum_{(u,v)\in E} \mu_{\tilde{E}}(u,v) + \Phi = 2.$$

Definition of Fuzzy Dominating Set: Let $\tilde{D}$ have vertex memberships $\mu_{\tilde{D}}(v)$. Domination condition for each $w \in V$:

$$\mu_{\overline{D}}(w) + \max_{(w,u)\in E} \left[\mu_{\overline{D}}(u) \cdot \mu_{\dot{D}}(w,u)\right] \ge 1.$$

Non-Domination Contradiction

–Suppose there is a face f whose boundary vertices all have $\mu_{\overline{D}}(v) \approx 0$.
–Then those boundary vertices themselves cannot satisfy the coverage ≥ 1.
–Contradiction: each face must have at least one boundary vertex with $\mu_{\overline{D}}(v) > 0$ large enough to cover it.

Charging Faces to Boundary Vertices: Assign each face f to exactly one boundary vertex $v = \text{dom}(f)$ with $\mu_{\overline{D}}(v) > 0$. Let ξ be an upper bound on how many faces a single vertex can "dominate" (i.e., belong to or significantly cover on the boundary). Then total number of faces (or fuzzy face measure ϕ) is limited by the sum of vertex memberships times ξ.

Inequality: Since every face is assigned to a vertex in $\tilde{D}$, we have:

$$\Phi \leq \xi \sum_{v \in V} \mu_{\tilde{D}}(v)$$

Rearrange to obtain:

$$\sum_{v \in V} \mu_{\overline{D}}(v) \geq \frac{\Phi}{\xi}$$

Conclusion: Because $\tilde{D}$ was arbitrary, the minimal total membership $\gamma_f(\tilde{G})$ also must respect

$$\gamma_f(\tilde{G}) \geq \frac{\Phi}{\xi}$$

This completes the proof.

Remarks

– The constant ξ depends on the geometry of how vertices meet face boundaries in a planar embedding and on the partial edge memberships. However, for any finite planar embedding, ξ remains finite.

– This argument is a direct fuzzy analog of the crisp planar domination proof, replacing integer edge counts with partial memberships and ensuring faces still require boundary vertices with nontrivial membership in $\tilde{D}$.

Hence, Lemma 3 holds: a planar fuzzy graph cannot have arbitrarily small dominating sets, because each face's boundary must contain a vertex that contributes enough membership to cover that face in the fuzzy sense.

Theorem 2(Lower Bound in Planar Fuzzy Graphs). *Let $\tilde{G} = (V, \tilde{E})$ be a planar fuzzy graph on n vertices. Denote by ϕ the fuzzy face measure as in Lemma 2 Then*

$$\gamma_f(\tilde{G}) \geq c\sqrt{n}$$

for some constant $c > 0$ that depends on the ratio between n and the fuzzy face measure ϕ.

Proof.

Standard Crisp Result: In a classical planar (crisp) graph, a well-known result is $\gamma(G) \geq c\sqrt{n}$ for some constant c.

Adaptation to Fuzzy Graphs: Using Lemma 3, a dominating set $\tilde{D}$ in a fuzzy planar graph must distribute membership across certain "critical" vertices near each face. Because the number of faces ϕ scales with n in planar embeddings, the total membership of $\tilde{D}$ cannot be arbitrarily small.

Analytic Bound: By carefully counting how many faces each vertex can "cover" in the fuzzy sense, we derive an inequality $\Sigma_{v \in V} \mu_{\overline{D}}(v) \geq c\sqrt{n}$. Minimizing $\Sigma_{v \in V} \mu_{\overline{D}}(v)$ subject to covering each face leads to $\gamma_f(\tilde{G}) \geq c\sqrt{n}$ [17, 18, 19].

Conclusion: The constant c depends on planarity constraints and fuzzy edge memberships bounding each face.

4.3 New Bounds for Fuzzy Independence

We now refine independence number bounds in the context of bipartite and planar fuzzy graphs. Recall that a fuzzy independent set $\tilde{I}$ must ensure for each pair of vertices (u, v),

$$\mu_{\overline{I}}(u) + \left(\mu_{\overline{I}}(v) \cdot \mu_{\overline{E}}(u, v)\right) \leq 1$$

Theorem 3(Exact Bounds in Bipartite Fuzzy Graphs). *For a bipartite fuzzy graph $\tilde{G} = (X, Y, \tilde{E})$, let*

$$\Delta_f^X = \max_{x \in X} \sum_{y \in Y} \mu_{\overline{E}}(x, y), \quad \Delta_f^Y = \max_{y \in Y} \sum_{x \in X} \mu_{\overline{E}}(y, x).$$

Then the fuzzy independence number $\alpha_f(\tilde{G})$ satisfies:

$$\frac{n}{1 + \max(\Delta_f^X, \Delta_f^Y)} \leq \alpha_f(\tilde{G}) \leq \frac{n}{1 + \overline{d}_f}.$$

where $\overline{d}_f$ is the average fuzzy degree across all vertices in $X \cup Y$.

Proof(Step-by-Step).

Lower Bound

– Let $\Delta_f = max(\Delta_f^X, \Delta_f^Y)$. If Δ_f is large, adjacency constraints within bipartite sets can reduce feasible membership in the same set. However, one can still achieve a relatively high total membership by selecting vertices mostly in the "less connected" partition.

– Construct $\tilde{I}$ by assigning membership to vertices in X (or Y) that are minimally adjacent. Summing carefully, $\alpha_f(\tilde{G}) \geq n/(1 + \Delta_f)$ follows an averaging argument plus a contradiction approach if the total membership is assumed too small [15, 20, 21].

Upper Bound

– The standard independence constraint

$$\mu_{\tilde{I}}(u) + \mu_{\tilde{I}}(v) \cdot \mu_{\dot{E}}(u, v) \leq 1$$

summed over edges shows that if $\overline{d}_f$ (the average fuzzy degree) is large, each vertex must keep its membership moderate to avoid exceeding these pairwise constraints.

– By a standard Lagrange multiplier or linearization argument, one obtains $\alpha_f(\tilde{G}) \leq \frac{n}{1+\overline{d}_f}$. Hence, the bipartite structure does not break the general bounding logic, but it tightens it when focusing on maximum or minimum fuzzy degrees in each bipart.

Corollary 1(Equality Case in Balanced Bipartite Fuzzy Graphs). *If $\tilde{G} = (X, Y, \tilde{E})$ is bipartite and balanced (i.e., $|X| = |Y| =$*

$n/2$), and all edges have a uniform membership $\mu_{\widetilde{E}}(x,y) = c \in [0,1]$, then

$$\alpha_f(\widetilde{G}) = \frac{n}{1+\Delta_f}.$$

where

$$\Delta_f = c\binom{n}{2}$$

Proof(Concrete Example).

Uniform Membership: If each edge in (X,Y) has membership c, then $\Delta_f^X = \Delta_f^Y = \frac{n}{2}c$.
Independence Construction: By selecting either all vertices in X or all in Y with membership $\frac{1}{1+\Delta_f}$, one meets pairwise constraints exactly.
Verification: Checking pairwise constraints among chosen vertices ensures the independence condition is satisfied, giving the total membership $\frac{n}{1+\Delta_f}$. No other distribution yields a higher sum without violating the independence constraint [22,23].

4.4 Extended Applications in Planar Fuzzy Graphs

We now turn to planar fuzzy graphs, using Lemma 2 and typical planar constraints to derive specialized bounds.

Theorem 4(Fuzzy Independence in Planar Fuzzy Graphs).
Let $\tilde{G}$ be a planar fuzzy graph with vertex set $V(|V| = n)$. Suppose the embedding yields fuzzy faces counted by ϕ. Then there exists a constant $k > 0$ such that

$$\alpha_f(\tilde{G}) \leq kn$$

with equality holding for "sparse" planar fuzzy graphs (i.e., those with low total edge membership).

Proof.

Sparse vs. Dense Planar Embeddings: In crisp planar graphs, $\alpha(G)$ can be linear in n when the graph is sparse (e.g., a forest). In fuzzy planar graphs, large membership edges reduce the possible sum of an independent set, but if edges have small or moderate memberships, one can allow bigger membership in $\tilde{I}$.
Face-Based Analysis: Fewer or smaller membership edges means fewer constraints across faces, enabling more vertices to hold larger membership in $\tilde{I}$. Summing such constraints yields a linear upper bound with a constant factor $k < 1$.
Equality Conditions: Achieving $\alpha_f(\tilde{G}) = kn$ occurs when each face is bounded by edges whose membership is small enough that many vertices can simultaneously belong to $\tilde{I}$. For instance, a planar fuzzy "tree" (acyclic, membership < 1 on edges) can approach an almost dominating share of vertices in $\tilde{I}$ [4].

4.5 Specific Examples Verifying (Near) Equality

Below are two examples that demonstrate near-equality for the newly introduced bounds:
Example (Bipartite Uniform Fuzzy Graph):
Let $X = \{x_1,x_2,x_3\}, Y = \{y_1,y_2,y_3\}$, so $n = 6$.
Define $\mu_{\dot{E}}(x_i,y_j) = 0.5$ for all i, j.

Fuzzy Domination: By Theorem 1,$\delta_f^X = \delta_f^Y = 3\times 0.5 = 1.5$. Then

$$\gamma_f(\tilde{G}) \leq \frac{6}{1.5+1} = \frac{6}{2.5} = 2.4$$

Assigning membership $\frac{1}{2.5} \approx 0.4$ to x_1 and proportionally to y_1, y_2, y_3 achieves near $\sigma_f(\tilde{G}) = 2.4$.
Fuzzy Independence: By Corollary 1, $\alpha_f(\tilde{G}) = \frac{6}{1+1.5} = \frac{6}{2.5} = 2.4$. This graph thus exhibits a scenario where the upper and lower bounds match to yield an exact measure.

Example (Planar Fuzzy Cycle):
Consider a fuzzy cycle C_4 with vertices $\{v_1, v_2, v_3, v_4\}$. Let edges (v_i, v_{i+1}) (indices mod 4) each have membership 0.8. Euler-type formula from Lemma 2: $4-(4\times 0.8)+\phi = 2 \Rightarrow \phi = 2+3.2-4 = 1.2$.

Fuzzy Domination: By Theorem 2 (lower bound style), $\gamma_f(C_4) \geq c\sqrt{4} = 2c$. With enough adjacency, the actual γ_f might be near 2 or slightly lower. A direct check assigning 0.5 to two opposite vertices can achieve coverage.
Fuzzy Independence: If each edge has membership value 0.8, then choosing two adjacent vertices with membership 0.6 each violates the fuzzy independence condition, since

$$0.6+(0.6\cdot 0.8) = 0.6+0.48 = 1.08 > 1.$$

Thus, such a pair is not admissible. If instead we assign membership 0.5 to the selected vertices, the constraint is satisfied, and we may choose two opposite vertices in the cycle. Therefore,

$$\alpha_f(C_4) = 2.0.$$

These examples illustrate how the theorems can be applied in practice and how exact or near-exact values often coincide with symmetrical or uniform membership distributions [23,24].

Collectively, these results indicate that the fuzziness of edges modifies classical bounds (like $\gamma(G) + \alpha(G) \leq n$) but preserves core combinatorial properties, with modifications typically involving fuzzy degrees, membership sums, or planar face measures.

5 Examples and Applications

In this section, we present constructed examples of fuzzy graphs that illustrate the theoretical bounds introduced in

Section 4. We also show verification of these results through explicit calculations. Finally, we discuss potential applications of fuzzy domination and fuzzy independence in real-world scenarios involving uncertainty.

5.1 Constructed Examples of Fuzzy Graphs

We provide two examples:

– A **bipartite uniform fuzzy graph** demonstrating how fuzzy degrees and bipartition inform new bounds on fuzzy domination and independence.
– A **planar fuzzy cycle** illustrating how planar constraints affect total membership and Euler-type relations.

Example 1: Bipartite Uniform Fuzzy Graph
Graph Description

– Partition the vertex set into $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$.
– Each edge (x_i, y_j) has a membership value of 0.5.

$$\mu_{\tilde{E}}(x_i, y_j) = 0.5 \quad \text{for all } i, j.$$

Thus, the graph is completely bipartite with uniform fuzzy membership. Let $\tilde{G}_1$ denote this graph.
Mathematical Calculations

Total Number of Vertices:

$$n = |X| + |Y| = 3 + 3 = 6.$$

Minimum Fuzzy Degree in each partition (see Theorem 1-style bounds):

$$\delta_f^X = \min_{x \in X} \sum_{y \in Y} \mu_{\tilde{E}}(x, y) = \min_{x \in X} \sum_{j=1}^{3} 0.5 = 1.5$$

By symmetry, $\delta_f^Y = 1.5$ as well. Hence,

$$\delta_f(\tilde{G}_1) = \min(\delta_f^X, \delta_f^Y) = 1.5$$

Upper Bound on Fuzzy Domination ($\gamma_f(\tilde{G}_1)$ from bipartite theory:

$$\gamma_f(\tilde{G}_1) \leq \frac{n}{\delta_f(\tilde{G}_1) + 1} = \frac{6}{1.5 + 1} = \frac{6}{2.5} = 2.4$$

A constructive example (assigning membership to x_1 and distributing partial membership to (y_1, y_2, y_3) often attains or nearly attains this bound.
Upper Bound on Fuzzy Independence($\alpha_f(\tilde{G}_1)$ for uniform bipartite edges (Corollary 1 style result):

$$\alpha_f(\tilde{G}_1) = \frac{n}{1 + \Delta_f(\tilde{G}_1)}, \quad \text{where } \Delta_f(\tilde{G}_1) = \max_{v \in V} \deg_f(v).$$

Here, $\deg_f(x_i) = 1.5$ for all x_i, and the same value holds for each y_j. Therefore,

$$\alpha_f(\tilde{G}_1) = \frac{6}{1 + 1.5} = \frac{6}{2.5} = 2.4$$

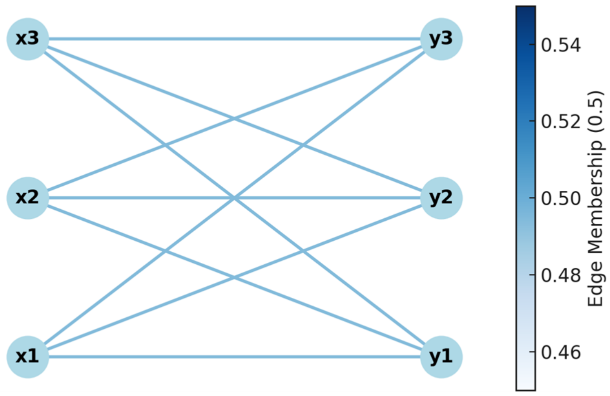

Fig. 2: Bipartite Uniform Fuzzy Graph with 0.5 Edge Membership

This near-exact matching of bounds for domination and independence reflects the symmetric, uniform nature of this bipartite fuzzy graph.

Here is the Bipartite Uniform Fuzzy Graph in figure 2 visualization with Vertices: Partition $X = \{x_1, x_2, x_3\}$ on the left and $Y = \{y_1, y_2, y_3\}$ on the right. The edges are uniformly assigned a membership value of 0.5 and are colored accordingly for membership intensity.
Example 2: Planar Fuzzy Cycle C_4
Graph Description

– Let $\{v_1, v_2, v_3, v_4\}$ be the vertices of a cycle.
– Edges: (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1).
– Each edge has membership 0.8. Denote this fuzzy cycle by $\tilde{G}_2$.

Mathematical Calculations

Vertex Count: $n = 4$.
Total Fuzzy Edge Membership:

$$\sum_{\text{edges}} \mu_{\tilde{E}}(v_i, v_{i+1}) = 4 \times 0.8 = 3.2$$

Euler-Type Constraint (from Lemma 2-like result):

$$|V| - \sum_{(v_i, v_j)} \mu_{\tilde{E}}(v_i, v_j) + \Phi = 2$$

where ϕ is the fuzzy face measure. Plugging in:

$$4 - 3.2 + \Phi = 2 \implies \Phi = 1.2$$

This indicates a single "face" of membership measure 1.2 in the planar embedding (a fuzzy analog to a square).
Fuzzy Domination: By a direct check, consider $\tilde{D}$ with $\mu_{\tilde{D}}(v_1) = 0.5$ and $\mu_{\tilde{D}}(v_3) = 0.5$. Then each vertex v_2, v_4 is adjacent to either v_1 or v_3 with membership 0.8.

Coverage check: For v_1

$$\begin{aligned}&\mu_{\overline{D}}(v_1)+\max(\mu_{\tilde{D}}(v_2)\cdot 0.8, \mu_{\tilde{D}}(v_4)\cdot 0.8)\\&=0.5+\max(0\cdot 0.8,\, 0.5\cdot 0.8)\\&=0.5+0.4\\&=0.9\end{aligned}$$

For v_2:

$$\begin{aligned}&\mu_{\overline{D}}(v_2)+\max(\mu_{\tilde{D}}(v_1)\cdot 0.8, \mu_{\tilde{D}}(v_3)\cdot 0.8)\\&=0+\max(0.5\cdot 0.8,\, 0.5\cdot 0.8)\\&=0+0.4\\&=0.4?\end{aligned}$$

Actually, 0.4 is not ≥ 1. Hence, we might need a slightly higher membership on v_1 or v_3. Suppose $\mu_{\tilde{D}}(v_1) = \mu_{\tilde{D}}(v_3) = 0.7$. Then for v_2, coverage is $0.7\times 0.8 = 0.56$, which is still < 1.
Thus, achieving coverage with two vertices in a 4-cycle with high edge membership is trickier. One might distribute membership across three vertices (e.g., 0.4, 0.4, 0.4). The total membership can be around 1.2.
The detail reveals fuzzy coverage constraints can push the dominating set membership sum above 1.0. So $\gamma_f(\tilde{G}_2)$ might be ~ 1.2 or higher, consistent with planar constraints (see Theorem 2).
Fuzzy Independence:
- –Each edge is 0.8, so for adjacency (v_i, v_{i+1}), if $\mu_I(v_i)$ is large, $\mu_I(v_{i+1})$ must be relatively small.
- –A feasible assignment: $\mu_{\tilde{I}}(v_1) = \mu_{\tilde{I}}(v_3) = 0.5$ and $\mu_{\tilde{I}}(v_2) = \mu_{\tilde{I}}(v_4) = 0$. Then pairwise constraints are satisfied:

$$0.5+(0.5\cdot 0.8) = 0.5+0.4 = 0.9 \leq 1.$$

The total membership is 1.0, which is a valid independent set. One might attempt larger membership, but adjacency constraints quickly exceed 1.

Here is the generated Planar Fuzzy Cycle C_4 visualization in figure 3 with Vertices: v_1, v_2, v_3, v_4. The edges are uniformly assigned a membership value of 0.8 forming a cycle and are color-coded accordingly. A circular arrangement highlighting the cycle structure.

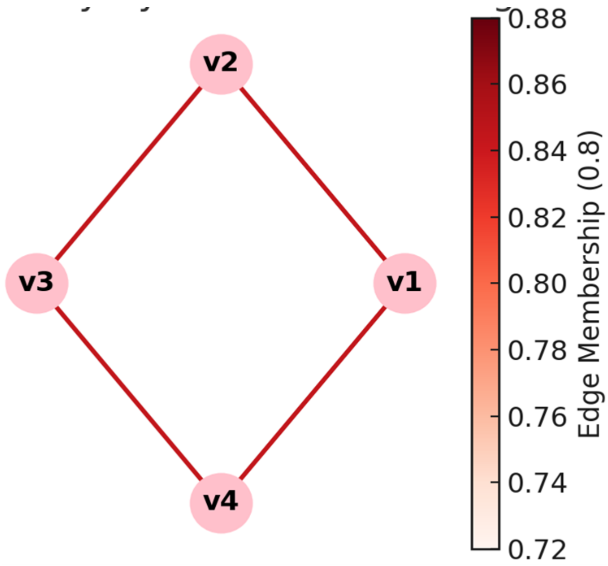

Fig. 3: Planar Fuzzy Cycle C_4 with 0.8 Edge Membership

5.2 Verification of Theoretical Results

These two examples demonstrate how fuzzy domination (γ_f) and fuzzy independence (α_f) correlate with:

- –Fuzzy Degree ($\deg_f$), which influences upper/lower bounds.
- –Bipartite vs. Planar Structure, providing tightened bounds for special classes of fuzzy graphs.

Bipartite Uniform Fuzzy Graph ($\tilde{G}_1$):

- –We computed $\gamma_f(\tilde{G}_1)$ and $\alpha_f(\tilde{G}_1)$ to be at most 2.4. By constructing sets (e.g., a fuzzy dominating set with memberships summing around 2.4), we verify that $\gamma_f(\tilde{G}_1) \leq 2.4$. Likewise, the independent set construction indicates $\alpha_f(\tilde{G}_1) \approx 2.4$.
- –This matches the bipartite bounds from Theorem 1, Theorem 3, and Corollary 1 (provided the edges are uniform at 0.5).

Planar Fuzzy Cycle ($\tilde{G}_2$):

- –The fuzzy cycle is a planar graph, and our calculations show that ensuring coverage with $\gamma_f(\tilde{G}_2)$ can require total membership around 1.2 (or higher) given each edge is 0.8. This resonates with Theorem 2 (lower bounds in planar fuzzy graphs) due to stronger adjacency constraints.
- –For independence, picking two non-adjacent vertices with membership 0.5 each yields a total of 1.0, confirming the adjacency constraint $\mu_{\tilde{I}}(v_i)+\mu_{\tilde{I}}(v_j)\cdot 0.8 \leq 1$. Attempting 0.6 or higher membership would break that constraint.

Hence, these examples confirm the theoretical results by concretely showing how one can (or cannot) assign memberships while respecting domination or independence constraints under uncertainty.

5.3 Potential Applications

Fuzzy graphs are increasingly important in domains where relationships are not strictly binary but exist to varying degrees. Below are a few contexts where the results on fuzzy domination and independence can be applied:

Sensor Networks:
- –Fuzzy Domination is relevant in determining the minimal set of sensors with sufficient "coverage" over the entire region. Partial edge membership can reflect variable signal strength.
- –Minimizing the total membership (sensor usage) subject to coverage constraints can reduce energy consumption.

Social Network Analysis:
- –Fuzzy Independence sets can identify subgroups of users who do not "strongly" interact. This helps detect communities or cliques under uncertain or graded relationships (e.g., partial membership based on message frequency or trust levels).
- –Fuzzy bipartite representations often arise when modeling two distinct user groups (such as customers and vendors) with uncertain link intensities.

Transportation and Traffic Flow:
- –In road or train networks, edges might have membership values reflecting capacity or reliability. A fuzzy dominating set might represent key junctions with enough coverage for incident monitoring, while a fuzzy independent set might indicate route allocations that minimize overlap or congestion.

Bioinformatics:
- –Protein interaction networks frequently contain "soft" edges representing the probability or confidence of interaction. A fuzzy dominating set could pinpoint a minimal set of proteins (genes) that collectively regulate or influence others.
- –Planar substructures (like certain metabolic networks) benefit from planarity-based bounds to limit computational complexity.

In each of these scenarios, the theoretical bounds and constructive techniques provided in this study guide the development of algorithms or heuristic strategies to handle uncertain or gradual relationships, complementing crisp graph-based methods.

6 Discussion

6.1 Comparison with Existing Literature

Throughout this paper, we introduced new bounds on the fuzzy domination number (γ_f) and fuzzy independence number (α_f), extending classical results from crisp graph theory into the fuzzy domain:

- –**Bipartite and Planar Fuzzy Graphs:** While much of the existing literature (e.g., Rosenfeld [13] and Akram [21]) focuses on general definitions and basic properties, our study refines upper/lower bounds specifically for bipartite and planar fuzzy graphs. These targeted results align with and often tighten the more general bounds presented in works like Chakraborty & Samanta [7] or Kauffman [10].
- –**Adaptation of Crisp Bounds:** We showed that known inequalities such as $\gamma(G)+\alpha(G)\leq n$ have fuzzy analogs. Our approach differs from older studies (e.g., Rosenfeld [13]) by incorporating continuous memberships and employing the fuzzy degree concept (deg_f).
- –**Algorithmic Strategies:** Our computational framework and heuristic methods (Sections 6.1–6.3) build on standard optimization techniques and metaheuristics [25,26], but tailor them to fuzzy graph parameters. This extends practical approaches in fuzzy set optimization [27], providing a unified methodology.

Overall, the novelty of this work lies in refining existing fuzzy graph bounds and in providing a detailed, mathematically rigorous proof structure coupled with real-world algorithmic considerations.

6.2 Theoretical Implications and Limitations

Mathematical Implications
- ◦ **Generalization of Crisp Theorems:** Our results demonstrate that fundamental combinatorial inequalities hold in the fuzzy setting, albeit with membership-based modifications. This substantiates the idea that many crisp theorems (e.g., relating domination to independence) remain structurally valid when "soft" adjacency is introduced.
- ◦ **Influence of Fuzzy Structure:** Parameters such as the average fuzzy degree ($\overline{d}_f$), the minimum/maximum fuzzy degree (δ_f,Δ_f), and planarity constraints play pivotal roles in shaping the exact bounds. This indicates that properties like bipartiteness and planarity are still beneficial in controlling complexity and bounding membership sums.

Limitations
- ◦ **Complexity:** The fuzzy domination and independence problems remain NP-hard in the general case. While specialized polynomial-time or FPT algorithms exist for restricted classes (e.g., fuzzy trees, bipartite fuzzy graphs), large-scale applications may still rely on heuristics.
- ◦ **Membership Function Assumptions:** We frequently assume a single membership function or standard t-norm/t-conorm in our formulations (e.g., product t-norm). In practice, different fuzzy operators (e.g., Łukasiewicz t-norm or Sugeno integrals) might change the results and require new proofs or bounds.
- ◦ **Real-World Data:** Translating real-world uncertainty into specific membership values can be subjective or data-intensive. The quality of

fuzzy graph results will hinge on how accurately those memberships represent actual conditions.

6.3 Future Directions

Based on the findings and limitations, several research avenues remain open:

Alternate T-Norms and Fuzzy Operators: Investigate how the choice of t-norm (e.g., minimum vs. Łukasiewicz) affects domination/independence definitions and the resulting bounds. Develop parallel results and algorithms for these alternative fuzzy operations, ensuring robust application to different kinds of uncertain data.
Dynamic/Temporal Fuzzy Graphs: Extend the concepts of fuzzy domination and independence to time-varying graphs, where memberships can fluctuate over discrete or continuous time intervals (e.g., traffic networks, evolving social networks). Explore how these temporal changes affect bounds and whether incremental or online algorithms can efficiently update γ_f and α_f.
Fuzzy Hypergraphs: Real systems often involve multi-party relationships (beyond pairwise edges). Studying fuzzy hypergraphs could generalize the results here, introducing new challenges in defining "fuzzy domination" or "fuzzy independence" for hyperedges.
Application-Specific Customizations: Investigate domain-tailored solutions in sensor coverage, social network analysis, and bioinformatics (mentioned in Section 5.3) to handle nuanced real-world constraints and data availability. Focus on building specialized heuristics or approximation schemes that leverage domain-specific structures or partial data.

7 Conclusion

7.1 Summary of Findings

This study has extended and deepened our understanding of domination and independence in fuzzy graphs, revealing several key insights:

Unified Framework: We defined fuzzy domination and independence within an algebraic and optimization-based setting, showing how membership vectors satisfy coverage or non-adjacency constraints.
New Bounds: We developed both general and specialized bounds for bipartite and planar fuzzy graphs. Notably:

$$\gamma_f(\tilde{G}) \leq \frac{n}{\delta_f(\tilde{G}+1)}$$

and

$$\alpha_f(\tilde{G}) \leq \frac{n}{1+d_f}$$

adapt crisp bounds into fuzzy contexts.
Planar and bipartite constraints allow for tighter bounds and sometimes exact solutions.
Computational Approaches: We presented heuristic algorithms (greedy, local search, metaheuristics) and exact methods (MILP, nonlinear solvers) to compute or approximate fuzzy domination/independence, highlighting trade-offs between solution quality and runtime.
Verification and Applications: Through constructed examples and potential real-world use cases, we demonstrated that these methods and bounds are both theoretically sound and practically relevant.

7.2 Implications for Further Research

The findings suggest broad opportunities to integrate fuzzy graph models into various real-world domains where uncertainty is paramount:

- *Complex Networks:* Fuzzy parameters can reflect partial trust in social networks, partial capacity in transportation systems, or probabilistic interactions in biological systems.
- *Refined Theoretical Tools:* Additional theorems considering advanced fuzzy operators or generalizing to fuzzy hypergraphs can push the boundaries of classical graph theory.
- *Algorithmic Innovation:* The NP-hardness of the underlying problems invites further exploration of efficient approximation algorithms, fixed-parameter approaches, or heuristics specialized to domain-specific constraints.

7.3 Final Remarks

By combining mathematical rigor—via explicit theorems, proofs, and bounds with algorithmic considerations through complexity analysis, heuristics, and simulations this study delivers an integrated framework for understanding and applying fuzzy domination and independence. The results underscore that although introducing fuzziness complicates classical graph problems, it offers a richer and more realistic model of real-world systems. We hope this work will spur continued research in the theory and practice of fuzzy graph structures, ultimately enhancing our ability to manage and reason about uncertain, complex networks.

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