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Comprehensive Subclasses of Bi-univalent Functions Specified by Liouville–Caputo-Type Fractional Derivatives and Euler Polynomials

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Abstract: In this paper, we introduce two new inclusive subclasses, $\mathscr{F}_{\Psi}(\digamma,\delta,\rho)$ and $\mathscr{L}_{\Psi}(\varphi,\rho)$, of analytic functions defined via Euler polynomials and Liouville–Caputo-type fractional derivatives. We derive estimates for the initial Maclaurin coefficients $|b_2|$ and $|b_3|$ for functions belonging to these subclasses. Furthermore, by specializing the parameters involved in our major results, several known and new consequences are obtained as corollaries.

Keywords: Analytic, Univalent, Bi-univalent, Euler polynomials, Liouville–Caputo derivatives

1 Introduction

Euler polynomials, introduced by Leonhard Euler in the eighteenth century, play an important role in the representation of complex functions and in the study of their geometric properties. In geometric function theory, they arise in various contexts, including the research centered on Schwarz–Christoffel conformal transformations, univalent functions, and the structure of Riemann surfaces. In particular, they contribute to the characterization of conformal mappings that preserve angles locally [1]. These applications highlight the deep connections between analytic functions and geometric transformations facilitated by Euler polynomials [2]. Their fundamental properties have been widely investigated, and they have been employed in constructing subclasses of analytic functions as well as in representing solutions of certain differential equations.

Owing to their broad applicability in pure mathematics, Euler polynomials have also attracted attention in related areas of research. Recent developments in geometric function theory primarily investigates the geometric characteristics of special functions and their corresponding subclasses. For further discussions on these geometric aspects, we refer the reader to [3,4,5].

Let $\aleph$ denote the family of analytic functions $\daleth$ in the open unit disk $\eth=\{z\in\mathbb{C}:|z|<1\}$, that normalized by $\daleth(0)=\daleth'(0)-1=0$ and having the expansion:

$$\daleth(z)=z+\sum_{\ell=2}^{\infty}b_{\ell}z^{\ell},\ \ (z\in\eth). \tag{1}$$

Furthermore, We denote by $\mho$ the family of univalent functions in $\eth$.

Any function $\daleth\in\mho$ has an inverse $\daleth^{-1}$, defined by

$$\daleth^{-1}(\daleth(z))=z \text{ and } w=\daleth(\daleth^{-1}(w))\ \ (z\in\eth,\ |w|<r_0(\daleth)\geq\frac{1}{4})$$

where

$$\begin{aligned}\daleth^{-1}(w)=D(w)&=w-b_2w^2+(2b_2^2-b_3)w^3\\&-(b_4+5,b_2^3-5b_3b_2)w^4+\cdots.\end{aligned} \tag{2}$$

A function $\daleth$ is defined as bi-univalent in $\eth$ if both $\daleth$ and its inverse $\daleth^{-1}$ are univalent in $\eth$. We denote by Ψ the family of all bi-univalent functions in $\eth$ given by (1).

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Note that the function $h(z)=\frac{z}{1-z}$ is a member of Ψ but the function $h(z)=\frac{z}{1-z^2}$ is not a member of Ψ; see [6].

Miller and Mocanu [7] introduced the theory of differential subordination (see also [8] and [9]). A function $\daleth$ is defied as subordinate to an analytic function D, denoted by $\daleth \prec D$, if both $\daleth$ and D are analytic on $\eth$ and there exists a function $w \in \aleph$ analytic on $\eth$, such that

$$w(0)=0, |w(z)|<1,$$

and

$$\daleth(z)=D(w(z)) \qquad (z \in \eth).$$

Additionally, if D is univalent in $\eth$, we have

$$\daleth(z) \prec D(z) \text{ if and only if } \daleth(0)=D(0) \text{ and } \daleth(\eth) \subset D(\eth).$$

Using subordination theorems for analytic functions, many authors have studied several subordination relationships among certain subclasses of analytic functions; see [10,11,12]. Euler polynomials, a fundamental tool in mathematical analysis, have intriguing applications in geometric function theory, particularly in the study of complex analysis and conformal mappings.

Euler polynomials $\Upsilon_\ell(\rho)$ are frequently defined via the generating function (see, [13]):

$$G\mathscr{B}(\rho,h)=\frac{2e^{h\rho}}{e^h+1}=\sum_{\ell=0}^{\infty}\Upsilon_\ell(\rho)\frac{h^\ell}{\ell!},\left(\frac{1}{2}<\rho\le 1,\ |h|<\Psi\right).$$

An explicit formula for $\Upsilon_\ell(\rho)$ is given by

$$\Upsilon_j(\rho)=\sum_{\ell=0}^{j}\frac{1}{2^\ell}\sum_{u=0}^{\ell}(-1)^u\binom{\ell}{u}(\rho+u)^j.$$

From the above equation, an explicit formula for $\Upsilon_\ell(\rho)$ in terms of Υ_u can be obtained as:

$$\Upsilon_\ell(\rho)=\sum_{u=0}^{\ell}\frac{\Upsilon_u}{2^u}\binom{\ell}{u}(\rho-\frac{1}{2})^{\ell-u}.$$

The initial values of Euler polynomials are:

$$\begin{aligned}\Upsilon_0(\rho)&=1;\\ \Upsilon_1(\rho)&=\frac{2\rho-1}{2};\\ \Upsilon_2(\rho)&=\rho^2-\rho;\\ \Upsilon_3(\rho)&=\frac{4\rho^3-6\rho^2+1}{4};\\ \Upsilon_4(\rho)&=\rho^4-2\rho^3+\rho.\end{aligned} \tag{3}$$

Srivastava [14] introduced the operator $R^\delta:\mho\to\mho$ as:

$$\begin{aligned}R^\delta\daleth(z)&=\Gamma(2-\delta)z^\delta\mathscr{I}_z^\delta\daleth(z)\\ &=z+\sum_{\ell=2}^{\infty}\frac{\Gamma(\ell+1)\Gamma(2-\delta)}{\Gamma(\ell+1-\delta)}b_\ell z^\ell\\ &=z+\sum_{\ell=2}^{\infty}\eth(\ell,\delta)b_\ell z^\ell,\end{aligned}$$

where $\delta\in\mathbb{R};\ \delta\ne 2,3,4,\cdots$.

Definition 1. ([15]) *Let $\daleth\in\mho$ be defined on a simply connected domain containing the origin. The fractional integral of $\daleth$ of order ξ is given by*

$$\mathscr{I}_z^{-\xi}\daleth(z)=\frac{1}{\Gamma(\xi)}\int_0^z\frac{\daleth(\chi)}{(z-\chi)^{1-\xi}}d\chi,\ \ \xi>0. \tag{4}$$

Further, the fractional derivatives of $\daleth$ of order ξ is given as

$$\mathscr{I}_z^{\xi}\daleth(z)=\frac{1}{\Gamma(1-\xi)}\frac{d}{dz}\int_0^z\frac{\daleth(\chi)}{(z-\chi)^{\xi}}d\chi,\ \ 0\le\xi<1 \tag{5}$$

the multivalued nature of $(z-\chi)^{\xi-1}$ and $(z-\chi)^{-\xi}$ is taken away by entailing $\ell og(z-\chi)$ to be real, as $z>\chi$.

Definition 2. *The fractional derivatives of $\daleth$ of the order $n+\xi$ is*

$$\mathscr{I}_z^{-n+\xi}\daleth(z)=\frac{d^n}{dz^n}\mathscr{I}_z^{\xi}\daleth(z),\quad 0\le\xi<1, n\in\mathbb{N}_0.$$

The article examines Liouville–Caputo's concept of the fractional-order derivative [16] under the assumption that

$$\mathscr{I}^{\xi}\daleth(z)=\frac{1}{\Gamma(\ell-\xi)}\int_a^z\frac{\daleth^{(\ell)}(\chi)}{(z-\chi)^{\xi+1-\ell}}d\chi \tag{6}$$

where $\ell-1<\mathrm{Re}(\xi)\le\ell,\ \ell\in\mathbb{N}$, and $\xi\in\mathbb{C}$, ξ is the starting value of $\daleth$.

The generalized Salagean derivative operator [17] and the integral operator of Libera [18], was given by Owa [19].

$$\Upsilon^{\varsigma}\daleth(z)=\Gamma(2-\varsigma)z^{\varsigma}\mathscr{I}^{\xi}\daleth(z)=z+\sum_{\ell=2}^{\infty}q_\ell\, z^\ell,\ \ \varsigma\in\mathbb{R}.$$

Recently, Salah and Darus [20], introduced the operator

$$K_\xi^\varsigma\daleth(z)=\frac{\Gamma(2+\varsigma-\xi)}{\Gamma(\varsigma-\xi)}z^{\xi-\varsigma}\int_0^z\frac{\Upsilon^\varsigma\daleth(\chi)}{(z-\chi)^{\xi+1-\varsigma}}d\chi \tag{7}$$

where $\varsigma\in\mathbb{R}$ and $(\varsigma-1<\xi<\varsigma<2)$.

Primitive computations for $\daleth\in\aleph$ give

$$\begin{aligned}K_\xi^\varsigma\daleth(z)&=z+\sum_{\ell=2}^{\infty}\Theta_\ell\, b_\ell\, z^\ell\\ &=z+\sum_{\ell=2}^{\infty}\frac{\Gamma(2+\varsigma-\xi)\Gamma(2-\varsigma)(\Gamma(\ell+1))^2}{\Gamma(\ell-\varsigma+1)\Gamma(\ell+\varsigma-\xi+1)}\ b_\ell\, z^\ell.\end{aligned}$$

Furthermore, note that $K_0^0\daleth(z)=\daleth(z)$ and $K_1^1\daleth(z)=zB'(z)$.

$$\begin{aligned}K_\xi^\varsigma\daleth(z)&=z+\Theta_2b_2z^2+\Theta_3b_3z^3+\cdots\ \ z\in\mho\\ K_\xi^\varsigma D(w)&=w-\Theta_2b_2w^2+\Theta_3(2b_2^2-b_3)w^3+\cdots\ \ w\in\mho.\end{aligned}$$

The definitions of the new subclasses $\mathscr{F}_\Psi(\digamma,\delta,\rho)$ and $\mathscr{L}_\Psi(\varphi,\rho)$, which are associated with Euler polynomials and Liouville–Caputo-type fractional derivatives, are provided below.

Definition 3. *A function* $\daleth \in \Psi$ *given by (1) belongs to* $\mathscr{F}_\Psi(\digamma,\delta,\rho)$ *if it satisfies the following two subordinations:*

$$(1-\digamma)\frac{K_\xi^\varsigma \daleth(z)}{z}+\digamma\left(K_\xi^\varsigma \daleth(z)\right)'+\delta z\left(K_\xi^\varsigma \daleth(z)\right)'' \prec \tag{8}$$

$$G\mathscr{B}(\rho,z):=\sum_{\ell=0}^{\infty}\Upsilon_\ell(\rho)\frac{z^\ell}{\ell!}$$

and

$$(1-\digamma)\frac{K_\xi^\varsigma D(w)}{w}+\digamma\left(K_\xi^\varsigma D(w)\right)'+\delta z\left(K_\xi^\varsigma D(w)\right)'' \prec \tag{9}$$

$$G\mathscr{B}(\rho,w):=\sum_{\ell=0}^{\infty}\Upsilon_\ell(\rho)\frac{w^\ell}{\ell!},$$

where $\digamma \geq 1,\ \delta \geq 0,\ \frac{1}{2}<\rho\leq 1,\ z,w\ \in\ \eth$ *and* $D=\daleth^{-1}$.

Definition 4. *A function* $\daleth \in \Psi$ *given by (1) belongs to* $\mathscr{L}_\Psi(\varphi,\rho)$ *if it satisfies the following two subordinations:*

$$\left(K_\xi^\varsigma \daleth(z)\right)'+z\frac{e^{\ell\varphi}+1}{2}\left(K_\xi^\varsigma \daleth(z)\right)'' \prec G\mathscr{B}(\rho,z) \tag{10}$$

and

$$\left(K_\xi^\varsigma D(w)\right)'+w\frac{e^{\ell\varphi}+1}{2}\left(K_\xi^\varsigma D(w)\right)'' \prec G\mathscr{B}(\rho,w), \tag{11}$$

where $-\pi<\varphi\leq\pi,\ \frac{1}{2}<\rho\leq 1,\ z,w\ \in\ \eth$ *and* $D=\daleth^{-1}$.

Many subclasses are obtained by choosing certain choices for the parameters $\digamma,\delta$, and ρ in Definition 3, and for φ and ρ in Definition 4.

Lemma 1. ([21]) *If* $\daleth\in\Omega$, *then* $|m_\ell|\leq 2$ *for each* ℓ, *where* Ω *is the family of all analytic functions in* $\eth$ *for which*

$$Re\left(\daleth(z)\right)>0,\ \daleth(z)=1+m_1z+m_2^2z+\cdots\quad(z\in\eth).$$

In recent years, the geometric function theory including coefficient estimates has been the subject of numerous investigations. Non-sharp estimates on the coefficients $|b_2|$ and $|b_3|$ in the Taylor-Maclaurin series summation were introduced, along with a number of subclasses of the class Ψ. For example, El-Ityan et al. [22] defined the class $\mathscr{T}_\Sigma(\theta,sin)$ using the sine function. Amourah et al. [23] defined the class $\mathscr{G}_\Sigma(\rho,\daleth,\varsigma,x)$ using the Jacobi polynomials. Illafe et al. [24] defined the class $\mathscr{B}_\Sigma^\alpha(x,\tau,\eth,\mu)$ using the Gegenbauer polynomials. Al-Hawary et al. [25] investigated the classes $\mathscr{Y}_\Gamma(\kappa,\eta,\sigma),\mathscr{W}_\Gamma(\alpha,\varphi)$ and $\mathscr{K}_\Gamma(\alpha,\varphi)$ using Gregory numbers. Yousef et al. [26] introduced some subclasses by Frasin differentia operator.

In this work, we employ the Euler polynomials and Liouville–Caputo-type fractional derivatives to develop two new comprehensive subfamilies of bi-univalent functions, namely $\mathscr{F}_\Psi(\digamma,\delta,\rho)$ and $\mathscr{L}_\Psi(\varphi,\rho)$. We then derive upper bounds for the initial coefficients $|b_2|$ and $|b_3|$. Furthermore, several new results are obtained as corollaries.

In addition to the existing literature, the present study aims to further enrich the theory of bi-univalent functions by combining Euler polynomials with Liouville–Caputo-type fractional derivatives within a unified framework. The motivation behind this approach lies in the growing interest in fractional operators and special functions in geometric function theory. By integrating these tools, we obtain new subclasses that generalize several previously studied families and provide broader insight into coefficient problems associated with bi-univalent functions.

2 Main Results for the Subclasses $\mathscr{F}_\Psi(\digamma,\delta,\rho)$ and $\mathscr{L}_\Psi(\varphi,\rho)$

In this section, we establish the main coefficient estimates for the newly introduced subclasses $\mathscr{F}_\Psi(\digamma,\delta,\rho)$ and $\mathscr{L}_\Psi(\varphi,\rho)$. The results are obtained by applying the principle of subordination together with the properties of Euler polynomials and suitable coefficient comparison techniques. These estimates extend and complement several known results in the literature.

Theorem 1. *Let* $\daleth\in\Psi$ *be defined by (1) and suppose it belongs to the family* $\mathscr{F}_\Psi(\digamma,\delta,\rho)$ *where* $\digamma\geq 1,\ \delta\geq 0,\ \frac{1}{2}<\rho\leq 1,\ z,w\ \in\ \eth$ *and* $D=\daleth^{-1}$. *Then*

$$|b_2|\leq\sqrt{\frac{(2\rho-1)^3}{2\Theta_2^2\left|\begin{array}{l}(6\delta+2\digamma+1)(2\rho-1)^2\\ \quad-2(2\delta+\digamma+1)^2(\rho^2-3\rho+1)\end{array}\right|}},$$

and

$$|b_3|\leq\frac{(2\rho-1)^2}{4\Theta_3\,(2\delta+\digamma+1)^2}+\frac{2\rho-1}{2\Theta_3\,(6\delta+2\digamma+1)}.$$

Proof. Since $\daleth(z)=z+\sum_{\ell=2}^{\infty}b_\ell z^\ell\in\mathscr{F}_\Psi(\digamma,\delta,\rho)$, so from Definition 3, we have

$$(1-\digamma)\frac{K_\xi^\varsigma \daleth(z)}{z}+\digamma\left(K_\xi^\varsigma \daleth(z)\right)'+\delta z\left(K_\xi^\varsigma \daleth(z)\right)'' \prec G\mathscr{B}(\rho,z) \tag{12}$$

and

$$(1-\digamma)\frac{K_\xi^\varsigma D(w)}{w}+\digamma\left(K_\xi^\varsigma D(w)\right)'+\delta z\left(K_\xi^\varsigma D(w)\right)'' \prec G\mathscr{B}(\rho,w). \tag{13}$$

One can find two functions $r,s:\eth\to\eth$, with $r(0)=s(0)=0$ and $|r(z)|<1,\ |s(w)|<1$ for all $z,w\in\eth$. Thus, we can define $\gamma,\ \eth\in\Omega$ as follows:

$$\gamma(z)=\frac{r(z)+1}{1-r(z)}=1+\gamma_1z+\gamma_2z^2+\gamma_3z^3+\cdots,\ |\gamma_\ell|\leq 2,\ \ell\in\mathbb{N}.$$

$$\Rightarrow r(z)=\frac{\gamma(z)-1}{\gamma(z)+1}=\frac{\gamma_1}{2}z+\left(\frac{\gamma_2}{2}-\frac{\gamma_1^2}{4}\right)z^2 +\frac{1}{2}\left(\gamma_3-\gamma_1\gamma_2+\frac{\gamma_1^3}{4}\right)z^3+\cdots \tag{14}$$

and

$$\eth(w)=\frac{s(w)+1}{1-s(w)}=1+\eth_1 w+\eth_2 w^2+\eth_3 w^3+\cdots,\ |\eth_\ell|\le 2,\ \ell\in\mathbb{N}.$$

$$\Rightarrow s(w)=\frac{\eth(w)-1}{\eth(w)+1}=\frac{\eth_1}{2}w+\left(\frac{\eth_2}{2}-\frac{\eth_1^2}{4}\right)w^2 +\frac{1}{2}\left(\eth_3-\eth_1\eth_2+\frac{\eth_1^3}{4}\right)w^3+\cdots. \tag{15}$$

Using (14) and (15), we get

$$\begin{aligned} G\mathscr{B}(\rho,r(z))&=\Upsilon_0(\rho)+\frac{\Upsilon_1(\rho)}{2}\gamma_1 z\\ &+\left(\frac{\Upsilon_1(\rho)}{2}\left(\gamma_2-\frac{\gamma_1^2}{2}\right)+\frac{\Upsilon_2(\rho)}{8}\gamma_1^2\right)z^2\\ &+\left(\begin{array}{c}\frac{\Upsilon_1(\rho)}{2}\left(\gamma_3-\gamma_1\gamma_2+\frac{\gamma_1^3}{4}\right)\\ +\frac{\Upsilon_2(\rho)}{4}\left(\gamma_1\gamma_2-\frac{\gamma_1^3}{2}\right)+\frac{\Upsilon_3(\rho)}{48}\gamma_1^3\end{array}\right)z^3+\cdots \end{aligned} \tag{16}$$

and

$$\begin{aligned} G\mathscr{B}(\rho,s(w))&=\Upsilon_0(\rho)+\frac{\Upsilon_1(\rho)}{2}\eth_1 w\\ &+\left(\frac{\Upsilon_1(\rho)}{2}\left(\eth_2-\frac{\eth_1^2}{2}\right)+\frac{\Upsilon_2(\rho)}{8}\eth_1^2\right)w^2\\ &+\left(\begin{array}{c}\frac{\Upsilon_1(\rho)}{2}\left(\eth_3-\eth_1\eth_2+\frac{\eth_1^3}{4}\right)\\ +\frac{\Upsilon_2(\rho)}{4}\left(\eth_1\eth_2-\frac{\eth_1^3}{2}\right)+\frac{\Upsilon_3(\rho)}{48}\eth_1^3\end{array}\right)w^3+\cdots \end{aligned} \tag{17}$$

From (12), (13) and (16), (17), we have

$$(2\delta+\digamma+1)\,\Theta_2 b_2=\frac{\Upsilon_1(\rho)}{2}\gamma_1, \tag{18}$$

$$(6\delta+2\digamma+1)\,\Theta_3 b_3=\frac{\Upsilon_1(\rho)}{2}\left(\gamma_2-\frac{\gamma_1^2}{2}\right)+\frac{\Upsilon_2(\rho)}{8}\gamma_1^2, \tag{19}$$

$$-(2\delta+\digamma+1)\,\Theta_2 b_2=\frac{\Upsilon_1(\rho)}{2}\eth_1, \tag{20}$$

and

$$(6\delta+2\digamma+1)\left(2\Theta_2^2 b_2^2-\Theta_3 b_3\right)=\frac{\Upsilon_1(\rho)}{2}\left(\eth_2-\frac{\eth_1^2}{2}\right)+\frac{\Upsilon_2(\rho)}{8}\eth_1^2. \tag{21}$$

Adding equation (18) to (20) and some simplification, we have

$$\gamma_1=-\eth_1 \text{ and } \gamma_1^2=\eth_1^2 \tag{22}$$

and

$$8\,(2\delta+\digamma+1)^2\,\Theta_2^2 b_2^2=\Upsilon_1^2(\rho)(\gamma_1^2+\eth_1^2). \tag{23}$$

$$\Rightarrow b_2^2=\frac{\Upsilon_1^2(\rho)(\gamma_1^2+\eth_1^2)}{8\Theta_2^2\,(2\delta+\digamma+1)^2} \tag{24}$$

Adding (19) to (22) gives

$$\begin{aligned} &8\,(6\delta+2\digamma+1)\,\Theta_2^2 b_2^2\\ &=2\Upsilon_1(\rho)(\gamma_2+\eth_2)+(\gamma_1^2+\eth_1^2)\left(\frac{1}{2}\Upsilon_2(\rho)-\Upsilon_1(\rho)\right). \end{aligned}$$

By (22), we have

$$\begin{aligned} &8\,(6\delta+2\digamma+1)\,\Theta_2^2 b_2^2\\ &=2\Upsilon_1(\rho)(\gamma_2+\eth_2)+\gamma_1^2\,(\Upsilon_2(\rho)-2\Upsilon_1(\rho)) \end{aligned} \tag{25}$$

Also, applying (22) in (23)

$$\gamma_1^2=\frac{4\,(2\delta+\digamma+1)^2\,\Theta_2^2 b_2^2}{\Upsilon_1^2(\rho)}. \tag{26}$$

Replacing γ_1^2 in (25)

$$b_2^2=\frac{\Upsilon_1^3(\rho)\,(\gamma_2+\eth_2)}{2\Theta_2^2\left[\begin{array}{c}2\,(6\delta+2\digamma+1)\,\Upsilon_1^2(\rho)\\ -(2\delta+\digamma+1)^2\,(\Upsilon_2(\rho)-2\Upsilon_1(\rho))\end{array}\right]} \tag{27}$$

$$\Rightarrow |b_2|^2=\frac{\Upsilon_1^3(\rho)\,(|\gamma_2|+|\eth_2|)}{2\Theta_2^2\left|\begin{array}{c}2(6\delta+2\digamma+1)\,\Upsilon_1^2(\rho)\\ -(2\delta+\digamma+1)^2\,(\Upsilon_2(\rho)-2\Upsilon_1(\rho))\end{array}\right|}.$$

Applying Lemma 1 and (3), we get:

$$|b_2|\le\sqrt{\frac{(2\rho-1)^3}{2\Theta_2^2\left|\begin{array}{c}(6\delta+2\digamma+1)(2\rho-1)^2\\ -2(2\delta+\digamma+1)^2(\rho^2-3\rho+1)\end{array}\right|}}.$$

Subtracting (22) from (19), and performing some computations in view of (22), we get

$$\Theta_3 b_3=\Theta_2^2 b_2^2+\frac{\Upsilon_1(\rho)\,(\gamma_2-\eth_2)}{4\,(6\delta+2\digamma+1)}. \tag{28}$$

By (24) and (22)

$$b_3=\frac{\Upsilon_1^2(\rho)\Theta_2^2\gamma_1^2}{4\Theta_3\,(2\delta+\digamma+1)^2}+\frac{\Upsilon_1(\rho)\,(\gamma_2-\eth_2)}{4\Theta_3\,(6\delta+2\digamma+1)}. \tag{29}$$

Applying Lemma 1 and (3), we have:

$$|b_3|\le\frac{(2\rho-1)^2}{4\Theta_3\,(2\delta+\digamma+1)^2}+\frac{2\rho-1}{2\Theta_3\,(6\delta+2\digamma+1)}.$$

This concludes the proof of Theorem 1.

Theorem 2. *Let $\daleth \in \Psi$ be defined by (1) and suppose it belongs to the family $\mathscr{L}_{\Psi}(\varphi,\rho)$ where $-\pi < \varphi \leq \pi$, $\frac{1}{2} < \rho \leq 1$, $z,w \in \eth$ and $D = \daleth^{-1}$. Then*

$$|b_2| \leq \sqrt{\frac{(2\rho-1)^3}{2\Theta_2^2 \left| \begin{array}{l} 3\left(e^{\ell\varphi}+2\right)(2\rho-1)^2 \\ \quad -2\left(e^{\ell\varphi}+3\right)^2(\rho^2-3\rho+1) \end{array} \right|}}$$

and

$$|b_3| \leq \frac{2\rho-1}{6\Theta_3\left(e^{\ell\varphi}+2\right)} + \frac{(2\rho-1)^2}{4\Theta_3\left(e^{\ell\varphi}+3\right)^2}.$$

Proof. Since $\daleth(z) = z + \sum_{\ell=2}^{\infty} b_\ell z^\ell \in \mathscr{L}_{\Psi}(\varphi,\rho)$, so from Definition 4 and the equations (16) and (17), we can write

$$\left(K_{\xi}^{\varsigma}\daleth(z)\right)' + z\frac{e^{\ell\varphi}+1}{2}\left(K_{\xi}^{\varsigma}\daleth(z)\right)'' \prec G\mathscr{B}(\rho,z) \tag{30}$$

and

$$\left(K_{\xi}^{\varsigma}D(w)\right)' + w\frac{e^{\ell\varphi}+1}{2}\left(K_{\xi}^{\varsigma}D(w)\right)'' \prec G\mathscr{B}(\rho,w). \tag{31}$$

By comparing the coefficients in (30) and (31), where $G\mathscr{B}(\rho,z)$ and $G\mathscr{B}(\rho,w)$, respectively given by (16) and (17), we get

$$\left(e^{\ell\varphi}+3\right)\Theta_2 b_2 = \frac{\Upsilon_1(\rho)}{2}\gamma_1, \tag{32}$$

$$3\left(e^{\ell\varphi}+2\right)\Theta_3 b_3 = \frac{\Upsilon_1(\rho)}{2}\left(\gamma_2 - \frac{\gamma_1^2}{2}\right) + \frac{\Upsilon_2(\rho)}{8}\gamma_1^2, \tag{33}$$

$$-\left(e^{\ell\varphi}+3\right)\Theta_2 b_2 = \frac{\Upsilon_1(\rho)}{2}\eth_1, \tag{34}$$

and

$$3\left(e^{\ell\varphi}+2\right)\left(2\Theta_2^2 b_2^2 - \Theta_3 b_3\right) = \frac{\Upsilon_1(\rho)}{2}\left(\eth_2 - \frac{\eth_1^2}{2}\right) + \frac{\Upsilon_2(\rho)}{8}\eth_1^2. \tag{35}$$

Using the same technique as in the proof of Theorem 1, we get the estimates that are asserted by Theorem 2.

3 Some Corollaries

In this section, we illustrate how various previously known subclasses can be recovered as special cases of our main results. By assigning particular values to the involved parameters, we derive several corollaries that demonstrate the flexibility and generality of the proposed classes.

By particularizing the parameters $\digamma$ and δ in Theorem 1, and φ in Theorem 2, we get several subresults related to the classes $\mathscr{F}_{\Psi}(\digamma,\delta,\rho)$ and $\mathscr{L}_{\Psi}(\varphi,\rho)$.

If we set $\digamma = 1$ in Theorems 1, we get the following result.

Corollary 1. *Let $\daleth \in \Psi$ be defined by (1) and suppose it belongs to the family $\mathscr{F}_{\Psi}(1,\delta,\rho)$ where $\delta \geq 0$, $\frac{1}{2} < \rho \leq 1$, $z,w \in \eth$ and $D = \daleth^{-1}$. Then*

$$|b_2| \leq \sqrt{\frac{(2\rho-1)^3}{2\Theta_2^2 \left| \begin{array}{l} (6\delta+3)(2\rho-1)^2 \\ \quad -8(\delta+1)^2(\rho^2-3\rho+1) \end{array} \right|}},$$

and

$$|b_3| \leq \frac{(2\rho-1)^2}{8\Theta_3(\delta+1)^2} + \frac{2\rho-1}{6\Theta_3(2\delta+1)}.$$

If we set $\delta = 0$ in Theorems 1, we get the following result.

Corollary 2. *Let $\daleth \in \Psi$ be defined by (1) and suppose it belongs to the family $\mathscr{F}_{\Psi}(\digamma,0,\rho)$ where $\digamma \geq 1$, $\frac{1}{2} < \rho \leq 1$, $z,w \in \eth$ and $D = \daleth^{-1}$. Then*

$$|b_2| \leq \sqrt{\frac{(2\rho-1)^3}{2\Theta_2^2 \left| \begin{array}{l} (2\digamma+1)(2\rho-1)^2 \\ \quad -2(\digamma+1)^2(\rho^2-3\rho+1) \end{array} \right|}},$$

and

$$|b_3| \leq \frac{(2\rho-1)^2}{4\Theta_3(\digamma+1)^2} + \frac{2\rho-1}{2\Theta_3(2\digamma+1)}.$$

For $\delta = 0$ in Corollary 1 or $\digamma = 1$ in Corollary 2 or $\varphi = \pi$ in Theorems 2, we get the following result.

Corollary 3. *Let $\daleth \in \Psi$ be defined by (1) and suppose it belongs to the family $\mathscr{F}_{\Psi}(1,0,\rho) \equiv \mathscr{L}_{\Psi}(\Psi,\rho)$ where $\frac{1}{2} < \rho \leq 1$, $z,w \in \eth$ and $D = \daleth^{-1}$. Then*

$$|b_2| \leq \sqrt{\frac{(2\rho-1)^3}{2\Theta_2^2\left|3(2\rho-1)^2 - 8(\rho^2-3\rho+1)\right|}},$$

and

$$|b_3| \leq \frac{(2\rho-1)^2}{8\Theta_3} + \frac{2\rho-1}{6\Theta_3}.$$

If we set $\varphi = 0$ in Theorems 2, we get the following result.

Corollary 4. *Let $\daleth \in \Psi$ be defined by (1) and suppose it belongs to the family $\mathscr{L}_{\Psi}(0,\rho)$ where $\frac{1}{2} < \rho \leq 1$, $z,w \in \eth$ and $D = \daleth^{-1}$. Then*

$$|b_2| \leq \sqrt{\frac{(2\rho-1)^3}{2\Theta_2^2\left|9(2\rho-1)^2 - 32(\rho^2-3\rho+1)\right|}}$$

and

$$|b_3| \leq \frac{2\rho-1}{18\Theta_3} + \frac{(2\rho-1)^2}{64\Theta_3}.$$

4 Conclusions

In this paper, we introduced two new subclasses of analytic and bi-univalent functions, namely $\mathscr{F}_{\Psi}(\digamma,\delta,\rho)$ and $\mathscr{L}_{\Psi}(\varphi,\rho)$, defined via Euler polynomials and Liouville–Caputo-type fractional derivatives. Upper bound estimates for the initial coefficients were obtained for functions belonging to these classes. However, determining the sharp bounds for $|b_2|$ and $|b_3|$ remains an interesting problem, while establishing sharp estimates for $|b_\ell|$, $\ell \geq 3$, continues to be an open problem.

The structure of the proposed subclasses and the techniques developed herein provide a foundation for further investigation. In particular, future research may explore sharp coefficient problems, alternative special functions, or related subfamilies of analytic and bi-univalent functions of complex order. Exploring the coupling of Euler polynomials and Liouville–Caputo-type fractional derivatives may also reveal additional developments in geometric function theory and its applications.

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