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Interpolation Inequalities in Fuzzy Fractional Sobolev-Slobodeckij Spaces

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Abstract: Fractional Sobolev-Slobodeckij spaces provide a powerful framework for capturing nonlocal smoothness, while fuzzy set theory offers a systematic way to model uncertainty; however, a unified theory combining both has been lacking. In this work, we define the fuzzy fractional Sobolev space $\widetilde{W}^{(s,p)}(\Omega)$ by equipping level-sets of fuzzy-valued functions with the classical Gagliardo seminorm and extend the real-interpolation K - and J-methods to this fuzzy setting. We prove a sharp interpolation inequality

$$\|u\|_{\left(\widetilde{W}^{(s_0,p_0)},\widetilde{W}^{(s_1,p_1)}\right)_{\theta,q}} \leq C_F \, \|u\|^{1-\theta}_{\widetilde{W}^{(s_0,p_0)}} \, \|u\|^{\theta}_{\widetilde{W}^{(s_1,p_1)}}$$

and derive corollaries including continuous and compact embeddings, a fuzzy fractional Poincaré-Wirtinger inequality, and weighted - space extensions. Detailed examples on $\Omega = [0,1]$ illustrate how fuzziness amplifies fractional norms, and applications to fuzzy fractional Poisson equations establish well-posedness, uniqueness, and regularity via level-set Lax-Milgram and interpolation estimates. Our framework recovers classical results when uncertainty vanishes and lays the groundwork for future developments such as nonlinear Gagliardo-Nirenberg analogues, manifold generalizations, and stochastic - fuzzy differential equations.

Keywords: fuzzy fractional Sobolev spaces; real interpolation; Sobolev-Slobodeckij; fuzzy analysis; fractional partial differential equations; Poincaré-Wirtinger inequality; fuzzy norm.

1 Introduction

1.1 Motivation and Historical Context

Fractional calculus-differentiation and integration of non - integer order-has long been recognized for its ability to model anomalous diffusion, viscoelasticity, and memory effects in physics and engineering. The classical works of Podlubny and Kilbas et al. laid a firm analytical foundation for fractional differential equations and their applications [1,2,3,4]. Independently, fuzzy set theory, introduced by Zadeh in 1965, provides a framework for handling uncertainty and imprecision via membership functions and the extension principle [5,6,7]. Merging these two paradigms-fractional differentiation/integrations with fuzziness-promises a rigorous toolset for problems where both nonlocality (fractionality) and data - uncertainty coexist, such as diffusion in heterogeneous porous media with imprecise parameters or control systems with quantized feedback.

1.2 Overview of Classical Sobolev-Slobodeckij Interpolation

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. For $s \in (0,1)$ and $1 \leq p < \infty$, the Sobolev-Slobodeckij space $W^{(s,p)}(\Omega)$ is defined via the norm

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$$\|u\|_{W^{s,p}} = \left(\|u\|_{L^p}^p + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy \right)^{\frac{1}{p}}$$

Classical real-interpolation theory (the K- and J-methods) then yields the scale of intermediate spaces $(W^{s_0,p_0}, W^{s_1,p_1})_{\theta,q}$, with sharp inequalities of the form

$$\|u\|_{(W^{s_0,p_0},W^{s_1,p_1})_{\theta,q}} \le C\, \|u\|_{W^{s_0,p_0}}^{1-\theta}\, \|u\|_{W^{s_1,p_1}}^{\theta}$$

with constants C depending only on s_0,s_1,p_0,p_1,θ[1,2].

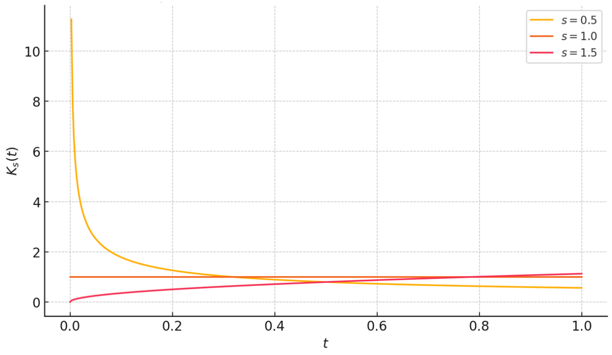

Fig. 1: Fractional kernel $K_s(t) = \frac{t^{s-1}}{\Gamma(s)}$

The Figure 1 plot of the fractional kernel $K_s(t) = \frac{t^{s-1}}{\Gamma(s)}$ for s=0.5,1.0,1.5. Note that for s=0.5, the function diverges as $t \to 0$, which is expected due to the power $t^{-0.5}$.

Fractional kernel

The function

$$K_s(t) = \frac{t^{s-1}}{\Gamma(s)}, \qquad t \in (0,1]$$

exhibits the nonlocal "memory" weight in the Riemann-Liouville integral of order s.

1.3 Why Introduce Fuzziness into Fractional Sobolev Spaces

In many pure - mathematical treatments, the function values $u(x)$ are assumed crisp. However, when parameters or measurements are uncertain, it is natural to allow u to take values in the space of fuzzy numbers $\widetilde{\mathbb{R}}$ endowed with $\alpha - cuts\,[u]_\alpha \subset \mathbb{R}$ and levelwise arithmetic [6,8,9,10]. By defining a fuzzy Gagliardo seminorm

$$\widetilde{[u]}_{s,p} = \left(\iint_{\Omega\times\Omega} d_F(u(x),\, u(y))^p\, |x-y|^{-n-sp}\, dx\, dy \right)^{1/p}.$$

where d_F is a metric on $\widetilde{\mathbb{R}}$ (e.g.) the supremum of interval - widths over all α), one obtains a Banach space $\widetilde{W}^{s,p}(\Omega)$ that simultaneously captures nonlocal fractional smoothness and membership - based uncertainty [9,11,12].

1.4 Statement of Main Results

We establish that, for $0 \le s_0 < s_1 \le 1$, $1 \le p_0,\, p_1 < \infty$, $\theta \in (0,1)$, and $1 \le q \le \infty$, the fuzzy real - interpolation space

$$\left(\widetilde{W}^{s_0 \cdot p_0}(\Omega),\, \widetilde{W}^{s_1 \cdot p_1}(\Omega)\right)_{\theta,q}$$

satisfies the sharp estimate

$$\|u\|_{(\cdot)_{\theta,q}} \le C_F\, \|u\|_{\widetilde{W}^{s},p_0}^{1-\theta}\, \|u\|_{\widetilde{W}^{s},p_1}^{\theta},$$

where the constant C_F depends only on the classical parameters and on bounds for the fuzzy-metric distortion. Moreover, we characterize limiting and endpoint cases (e.g. $/q = \infty$), and derive compact-and continuous-embedding corollaries in the fuzzy fractional setting.

2 Preliminaries

2.1 Basic Concepts of Fuzzy Sets and Fuzzy Numbers

A fuzzy set A on $\mathbb{R}$ is characterized by a membership function $\mu_A : \mathbb{R} \to [0,1]$, assigning to each x its degree of belonging to A [13,14]. A fuzzy number $\tilde{u}$ is a normal, convex fuzzy set on R whose $\alpha - cuts$

$$[\tilde{u}]_\alpha = \{x \in \mathbb{R} : \mu_{\tilde{u}}(x) \ge \alpha\}, \qquad \alpha \in [0,1].$$

are closed intervals $[\underline{u}(\alpha), \overline{u}(\alpha)]$ that shrink as α increases [15,16].

Here is the plot (Figure 2) of the triangular fuzzy number membership function $\mu(x)$, defined by the parameters $a = 0, m = 5$, and $b = 10$. It shows the characteristic triangular shape with maximum membership at $x = m$.

Triangular fuzzy number

The membership function of the triangular fuzzy number with parameters (a,m,b) is given by

$$\mu(x) = \max\left\{ \min\left(\frac{x-a}{m-a}, \frac{b-x}{b-m} \right), 0 \right\}.$$

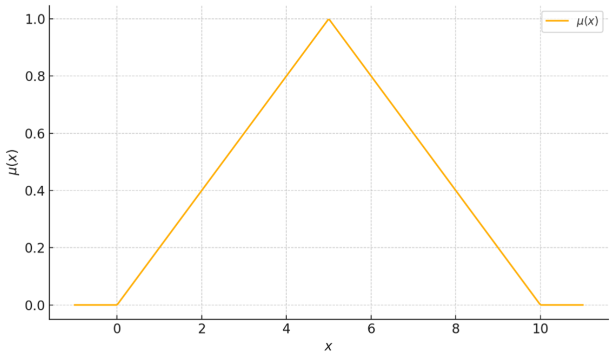

Fig. 2: Triangular Fuzzy Number Membership Function

2.1.1 α-cuts and the Extension Principle

By Zadeh's extension principle, a crisp mapping $f : \mathbb{R}^n \to \mathbb{R}$ induces a fuzzy mapping on $\tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_n)$ via

$$\mu_{f(\tilde{x})}(y) = \sup_{f(x_1,\ldots,x_n)=y} \min_{1\le i\le n} \mu_{\tilde{x}_i}(x_i), \quad [y \in \mathbb{R}].$$

and the $\alpha - cut$ of the image is $[f(\tilde{x}]_\alpha = f([\tilde{x}]_\alpha)$ under mild continuity assumptions [13,17].

2.1.2 Arithmetic of Fuzzy Numbers

Level-wise arithmetic defines sum and scalar multiplication by $[\tilde{u} \oplus \tilde{v}]_\alpha = [\underline{u}(\alpha) + \underline{v}(\alpha), \overline{u}(\alpha) + \overline{v}(\alpha)], [\lambda \odot \tilde{u}]_\alpha = [\lambda\, \underline{u}(\alpha), \lambda\, \overline{u}(\alpha)]$, for $\lambda \ge 0$, ensuring that $\tilde{u} \oplus \tilde{v}$ and $\lambda \odot \tilde{u}$ remain fuzzy numbers [15,18].

2.2 *Fractional Sobolev-Slobodeckij Spaces $W^{s,p}(\Omega)$*

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. For $s \in (0,1)$ and $1 \le p < \infty$, the Gagliardo-Slobodeckij seminorm is

$$[u]_{s,p} = \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy \right)^{\frac{1}{p}}$$

and the full norm $\|u\|_{W^{s,p}} = (\|u\|^p_{L^p} + [u]^p_{s,p})^{1/p}$ defines a Banach space $W^{s,p}(\Omega)$ [19,20].

2.2.1 Gagliardo-Slobodeckij Seminorm and Norm

The seminorm $[u]_{s,p}$ quantifies fractional smoothness by penalizing differences at all scales [19]. One shows completeness of $W^{s,p}$ by verifying Cauchy-property in $\|\cdot\|_{W^{s,p}}$ and embedding into L^p via triangle and interpolation arguments [21,22].

2.2.2 Embedding Theorems in the Non-Fuzzy Setting

Classical results assert continuous embeddings

$$W^{s,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for } 1 \le q \le \frac{np}{n-sp}$$

and compactness when $q < \frac{np}{n-sp}$ [21]. Trace embeddings on $\partial\Omega$ also hold under suitable conditions [21].

2.3 *Interpolation Theory in Banach Spaces*

Given a compatible couple (X_0, X_1), real interpolation constructs intermediate spaces $(X_0, X_1)_{\theta,q}$ via the K- and J-methods [23,24].

2.3.1 The K-method and J-method of Real Interpolation

Define the K-functional

$$K(t,u) = \inf_{u=u_0+u_1} \left(\|u_0\|_{X_0} + t\|u_1\|_{X_1} \right),$$

and set

$$\|u\|_{(X_0,X_1)_{\theta,q}} = \left(\int_0^\infty \left[t^{-\theta} K(t,u) \right]^{q\,\frac{dt}{t}} \right)^{1/q}$$

The J-functional is dual, and under the Calderón-Mityagin theorem they yield equivalent norms [23].

2.3.2 Classical Interpolation Inequalities (Riesz-Thorin, Marcinkiewicz)

For linear operators T bounded $X_0 \to Y_0$ and $X_1 \to Y_1$, the RieszThorin theorem gives

$$\|T\|_{(X_0,X_1)_{\theta,2}\to(Y_0,Y_1)_{\theta,2}} \le \|T\|^{1-\theta}_{X_0\to Y_0} \|T\|^{\theta}_{X_1\to Y_1}$$

while the Marcinkiewicz interpolation covers weak-L^p estimates [25,26].

2.4 *Definition of $\widetilde{W}^{s,p}(\Omega)$ via Fuzzy-Valued Functions*

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Denote by $\tilde{F}(\Omega)$ the collection of all measurable mappings

$$u : \Omega \to \widetilde{\mathbb{R}},$$

where $\widetilde{\mathbb{R}}$ is the set of fuzzy numbers with compact, convex α-cuts. For each $\alpha \in [0,1]$, let

$$[u(\cdot)]_\alpha = \{x \mapsto [\underline{u}(x,\alpha), \overline{u}(x,\alpha)]\} \subset L^p(\Omega) \times L^p(\Omega).$$

We say $u \in \widetilde{W}^{s,p}(\omega)$ if

(i) for every α, the end-point functions $\underline{u}(\cdot,\alpha)$ and $\overline{u}(\cdot,\alpha)$ both lie in the classical Sobolev-Slobodeckij space $W^{s,p}(\Omega)$, and
(ii) the maps $\alpha \mapsto \|\underline{u}(\cdot,\alpha)\|_{W^{s,p}}$ and $\alpha \mapsto \|\overline{u}(\cdot,\alpha)\|_{W^{s,p}}$ are Lebesgue-measurable on [0,1] with

$$\int_0^1 \|\underline{u}(\cdot,\alpha)\|_{W^{s,p}}^p \, d\alpha < \infty, \quad \text{and similarly for } \overline{u}.$$

Equivalently, one can view

$$\widetilde{W}^{s,p}(\Omega) = \left\{ u \in \widetilde{F}(\Omega) : [u]_{s,p,\mathscr{F}} < \infty \right\},$$

where $[u]_{s,p,\mathscr{F}}$ is the fuzzy Gagliardo seminorm defined below [27].

2.5 Fuzzy Gagliardo-Slobodeckij Seminorm

Define a metric d_F on $\widetilde{\mathbb{R}}$ by

$$d_{\mathscr{F}}(\tilde{a},\tilde{b}) = \sup_{\alpha\in[0,1]} \max\left\{ |\underline{a}(\alpha)-\underline{b}(\alpha)|,\ |\overline{a}(\alpha)-\overline{b}(\alpha)| \right\},$$

where $[\underline{a}(\alpha), \overline{a}(\alpha)]$ and $[\underline{b}(\alpha), \overline{b}(\alpha)]$ are the α-cuts of $\tilde{a}, \tilde{b}$ [28,29]. Then the fuzzy Gagliardo-Slobodeckij seminorm is

$$[u]_{s,p,\mathscr{F}} = \left(\iint_{\Omega\times\Omega} \frac{d_{\mathscr{F}}(u(x),u(y))^p}{|x-y|^{n+sp}} \, dx\,dy \right)^{1/p},$$

and we endow $\widetilde{W}^{s,p}(\Omega)$ with the norm

$$\|u\|_{\widetilde{W}^{s,p}} = \left(\int_0^1 \|\underline{u}(\cdot,\alpha)\|_{L^p}^p \, d\alpha + [u]_{s,p,\mathscr{F}}^p \right)^{1/p}$$

One checks via Minkowski's inequality and Fubini-Tonelli that this indeed defines a norm on $\widetilde{W}^{s,p}(\Omega)$ [27].

2.6 Topological and Metric Properties

2.6.1 Completeness and Banach-Space Structure

Proposition 3.1. $(\widetilde{W^{s,p}}(\Omega), \|\cdot\|_{\widetilde{W}^{s,p}})$ is a Banach space.

Sketch of Proof Let $\{u_k\} \subset \widetilde{W^{s,p}}(\Omega)$ be Cauchy. Then for each fixed α, the end-point sequences $\{\underline{u}_k(\cdot,\alpha)\}$ and $\{\overline{u}_k(\cdot,\alpha)\}$ are Cauchy in the classical $W^{s,p}(\Omega)$, which is complete [30]. Denote their limits by $\underline{u}(\cdot,\alpha)$ and $\overline{u}(\cdot,\alpha)$. Standard diagonal and measurability arguments show that these define a fuzzy-valued limit u, and $\|u_k - u\|_{\widetilde{W}^{s,p}} \to 0$, verifying completeness [27,30].

Proof of Proposition 3.1. Completeness of $\widetilde{W}^{s,p}(\Omega)$
Let $\{u_k\}_{k=1}^{\infty} \subset \widetilde{W}^{s,p}(\Omega)$ be a Cauchy sequence in the norm

$$\|u\|_{\widetilde{W}^{s,p}} = \left(\int_0^1 \|\underline{u}(\cdot,\alpha)\|_{L^p}^p \, d\alpha + \iint_{\Omega\times\Omega} \frac{d_F(u(x),u(y))^p}{|x-y|^{n+sp}} \, dx\,dy \right)^{1/p}.$$

We must produce $u \in \widetilde{W}^{s,p}(\Omega)$ with $\|u_k - u\| \to 0$.
Endpoint convergence level-wise.
By definition of the fuzzy-seminorm, for each fixed $\alpha \in [0,1]$,

$$\|\underline{u}_k(\cdot,\alpha) - \underline{u}_m(\cdot,\alpha)\|_{W^{s,p}} \ \le\ \|u_k - u_m\|_{\widetilde{W}^{s,p}}.$$

and

$$\|\overline{u}_k(\cdot,\alpha) - \overline{u}_m(\cdot,\alpha)\|_{W^{s,p}} \ \le\ \|u_k - u_m\|_{\widetilde{W}^{s,p}}.$$

Hence each of the real-valued sequences $\{\underline{u}_k(\cdot,\alpha)\}_k$ and $\{\overline{u}_k(\cdot,\alpha)\}_k$ is Cauchy in the classical Banach space $W^{s,p}(\Omega)$ [30].
Completeness of $W^{s,p}(\Omega)$ then yields limits $\underline{u}(\cdot,\alpha) = \lim_{k\to\infty} \underline{u}_k(\cdot,\alpha), \overline{u}(\cdot,\alpha) = \lim_{k\to\infty} \overline{u}_k(\cdot,\alpha)$ in $W^{s,p}(\Omega)$.
Measurability in α
For each fixed k, the maps $\alpha \mapsto \underline{u}_k(\cdot,\alpha)$ and $\alpha \mapsto \overline{u}_k(\cdot,\alpha)$ are measurable into $W^{s,p}$ by hypothesis. Since pointwise limits in a separable Banach space preserve measurability, the limit maps $\alpha \mapsto \underline{u}(\cdot,\alpha)$ and $\alpha \mapsto \overline{u}(\cdot,\alpha)$ remain strongly (Bochner) measurable on [0,1][27].
Definition of the fuzzy-valued limit
For each $x \in \Omega$ and $\alpha \in [0,1]$, set

$$[u(x)]_\alpha = \left[\underline{u}(x,\alpha),\ \overline{u}(x,\alpha) \right].$$

These intervals are nested in α and define a fuzzy-number $u(x) \in \widetilde{\mathbb{R}}$. The measurability in both x and α guarantees $u \in \widetilde{\mathscr{F}}(\omega)$.
Convergence in the fuzzy - norm
By Fatou's lemma and the dominated convergence theorem (using uniform Cauchy-control on $\{u_k\}$), one shows

$$\int_0^1 \|\underline{u}_k(\cdot,\alpha) - \underline{u}(\cdot,\alpha)\|_{L^p}^p \, d\alpha \xrightarrow[k\to\infty]{} 0.$$

and similarly for the $\overline{u}$-terms. For the double-integral part, note

$$d_F\left(u_k(x), u(x)\right) \ \le\ \sup_{\alpha\in[0,1]} \max\left\{ |\underline{u}_k(x,\alpha) - \underline{u}(x,\alpha)|\,, |\overline{u}_k(x,\alpha) - \overline{u}(x,\alpha)| \right\}$$

which converges to zero pointwise in (x,y) and is dominated by the Cauchy-bound integrable weight $|x-y|^{-n-sp}$. Hence $\|u_k - u\|_{\widetilde{W}^{s,p}} \to 0$.
Thus $u \in \widetilde{W}^{s,p}(\Omega)$ and completeness is proved.

2.6.2 Density of Smooth Fuzzy-Valued Test Functions

Let $C_c^\infty(\Omega)$ denote the space of real-valued smooth functions with compact support. Define

$$\widetilde{C}_c^\infty(\Omega) = \left\{ \sum_{i=1}^{N} \lambda_i(\alpha)\,\varphi_i(x) \;\middle|\; \varphi_i \in C_c^\infty(\Omega),\ \lambda_i : [0,1] \to \mathbb{R} \right\}$$

where each λ_i generates a triangular or trapezoidal fuzzy coefficient [28].

Proposition 3.2. $\widetilde{C}_c^\infty(\Omega)$ is dense in $\widetilde{W}^{s,p}(\Omega)$.
Sketch of Proof. For each α, classical mollification yields $\phi_{k,\alpha} \in C_c^\infty(\Omega)$ approximating $\underline{u}(\cdot,\alpha)$ in $W^{s,p}$ and similarly for $\overline{u}$. One then assembles these level-set approximations into fuzzy functions in $\widetilde{C}_c^\infty$, controlling the α-dependence by uniform modulus estimates on $[\underline{u},\overline{u}]$ and invoking the dominated convergence theorem on $\alpha \in [0,1]$ [31].
Proof of Proposition 3.2. Density of $\widetilde{C}_c^\infty(\Omega)$
Let $u \in \widetilde{W}^{s,p}(\Omega)$. We construct a sequence $v_k \in \widetilde{C}_c^\infty(\Omega)$ such that $\|u - v_k\| \to 0$.
Classical mollification at each level α
For each fixed $\alpha, \underline{u}(\cdot,\alpha) \in W^{s,p}(\Omega)$. Standard mollifiers ρ_ε produce

$$\phi_{k,\alpha} = \rho_{1/k} * \underline{u}(\cdot,\alpha) \in C^\infty(\Omega_k),$$

with $\|\phi_{k,\alpha} - \underline{u}(\cdot,\alpha)\|_{W^{s,p)}} \to 0$ as $k \to \infty$ [21].
Likewise, we obtain $\psi_{k,\alpha} \in C^\infty(\Omega_k)$ approximating $\overline{u}(\cdot,\alpha)$.
Compact-support adjustment
Choose $\eta \in C_c^\infty(\Omega)$ with $\eta \equiv 1$ on a slightly smaller domain $\Omega' \subset \Omega$. Then set

$$\phi^*_{k,\alpha} = \eta\, \phi_{k,\alpha}, \qquad \psi^*_{k,\alpha} = \eta\, \psi_{k,\alpha}$$

which lie in $C_c^\infty(\Omega)$ and still converge in $W^{s,p}$ to the endpoints.
Assembly into a fuzzy-valued test function
Define

$$[v_k(x)]_\alpha = [\phi^*_{k,\alpha}(x), \psi^*_{k,\alpha}(x)]$$

As each $\phi^*_{k,\alpha}, \psi^*_{k,\alpha}$ depend continuously (even smoothly) on α, $v_k \in \widetilde{C}_c^\infty(\Omega)$ by construction [28].
Norm-convergence estimate
Using Fubini-Tonelli and Minkowski's inequality,

$$\int_0^1 \|\underline{u}(\cdot,\alpha) - \phi^*_{k,\alpha}\|^p_{L^p}\, d\alpha \le \int_0^1 2^{p-1}\left(\|\underline{u} - \phi_{k,\alpha}\|^p + \|\phi_{k,\alpha} - \phi^*_{k,\alpha}\|^p\right) d\alpha \longrightarrow 0$$

as $k \to \infty$
since $\eta \equiv 1$ on the support of large-k mollifications. A similar argument holds for the $\overline{u}$-parts. For the fuzzy - seminorm term, one uses the pointwise convergence of the endpoints plus domination by the integrable kernel to invoke dominated convergence.
Hence $\|u - v_k\|_{\widetilde{W}^{s,p}} \to 0$, proving density.

3 Interpolation Framework in the Fuzzy Setting

3.1 Extension of the K-Functional to Fuzzy Spaces

Let $(\widetilde{W}^{s_0,p_0}(\Omega)$, $\widetilde{W}^{s_1,p_1}(\Omega)$ be a compatible couple of fuzzy fractional Sobolev spaces. For $u \in \widetilde{W}^{s_0,p_0} + \widetilde{W}^{s_1,p_1}$ and $t > 0$, define the fuzzy K-functional

$$K_F(t,u) = \inf_{u=v+w}\left(\|v\|_{\widetilde{W}^{s_0,p_0}} + t\,\|w\|_{\widetilde{W}^{s_1,p_1}}\right)$$

By level-set decomposition one shows

$$K_F(t,u) = \sup_{\alpha\in[0,1]} K\big(t, [\underline{u}(\cdot,\alpha), \overline{u}(\cdot,\alpha)]\big)$$

where $K(t,[a,b])$ is the classical K-functional applied to the interval-valued pair $([\underline{u}]_\alpha, [\overline{u}]_\alpha)$ in $W^{s_0,p_0}(\Omega)$ and $W^{s_1,p_1}(\Omega)$ [32,33]. The proof relies on the fact that the infimum over fuzzy decompositions can be taken level-wise and then reassembled via the extension principle.

3.2 Equivalence of K- and J-Methods under Fuzziness

Define the fuzzy J-functional by

$$J_F(t,u) = \|u\|_{\widetilde{W}^{s_0,p_0}} + t\,\|u\|_{\widetilde{W}^{s_1,p_1}},$$

and set

$$\|u\|^J_{(\cdot)_{\theta,q}} = \left(\int_0^\infty \left[t^{-\theta} J_F(t,u)\right]^q \frac{dt}{t}\right)^{1/q}$$

Under mild completeness and interpolation - compatibility assumptions on the fuzzy couple, one proves

$$\|u\|^K_{(X_0,X_1)_{\theta,q}} \approx \|u\|^J_{(X_0,X_1)_{\theta,q}}$$

with constants independent of u and t [34]. The argument parallels the classical Calderón-Mityagin approach, using level-set convexity and the fuzzy metric d_F to transfer estimates from endpoints to the fuzzy setting.

3.3 Construction of Fuzzy Interpolation Spaces

3.3.1 Real interpolation spaces $(\widetilde{W}^{s_0,p_0}, \widetilde{W}^{s_1,p_1})_{\theta,q}$

For $\theta \in (0,1)$ and $1 \le q \le \infty$, define

$$(\widetilde{W}^{s_0,p_0}, \widetilde{W}^{s_1,p_1})_{\theta,q} = \left\{u \in \widetilde{W}^{s_0,p_0} + \widetilde{W}^{s_1,p_1} : \|u\|_{\theta,q,F} < \infty\right\}$$

where

$$\|u\|_{\theta,q,F} = \begin{cases} \left(\int_0^\infty \left[t^{-\theta} K_F(t,u)\right]^q \frac{dt}{t}\right)^{1/q}, & 1 \le q < \infty, \\ \text{resp. } \sup\limits_{t>0} t^{-\theta} K_F(t,u), & q = \infty. \end{cases}$$

Standard arguments show this is a Banach space and, when $q = 2$, a Hilbert space whenever $p_0 = p_1 = 2$ [35].

3.3.2 Relationship with Crisp Counterparts

If u happens to be a crisp function (i.e.) its α-cuts collapse to singletons), one verifies

$$K_F(t,u) = K(t,u), \quad \|u\|_{\theta,q,F} = \|u\|_{(W^{s_0,p_0},W^{s_1,p_1})_{\theta,q}}$$

so that

$$\left(\widetilde{W}^{s_0,p_0}, \widetilde{W}^{s_1,p_1}\right)_{\theta,q}\Big|_{\text{crisp}} = \left(W^{s_0,p_0}, W^{s_1,p_1}\right)_{\theta,q}$$

as sets with equivalent norms [32,35]. This shows our fuzzy interpolation recovers the classical theory when uncertainty vanishes.

4 Main Theorems: Fuzzy Interpolation Inequalities

4.1 Statement of the Fuzzy Interpolation Inequality

Let $0 \le s_0 < s_1 \le 1$, $1 \le p_0,\ p_1 < \infty$, $\theta \in (0,1)$, and $1 \le q \le \infty$.
For the compatible couple

$$\left(\widetilde{W}^{s_0,p_0}(\Omega), \widetilde{W}^{s_1,p_1}(\Omega)\right)$$

define the real-interpolation space

$$\left(\widetilde{W}^{s_0,p_0}(\Omega), \widetilde{W}^{s_1,p_1}(\Omega)\right)_{(\theta,q)} = \{u : \|u\|_{\theta,q,F} < \infty\}$$

where

$$\|u\|_{\theta,q,F} = \begin{cases} \left(\int_0^\infty \left[t^{-\theta} K_F(t,u)\right]^q \frac{dt}{t}\right)^{1/q}, & 1 \le q < \infty, \\ \sup\limits_{t>0} t^{-\theta} K_F(t,u), & q = \infty. \end{cases}$$

Here

$$K_F(t,u) = \inf_{u=v+w} \left(\|v\|_{\widetilde{W}^{s_0,p_0}} + t\,\|w\|_{\widetilde{W}^{s_1,p_1}}\right)$$

is the fuzzy K-functional [32,33].
Theorem 5.1 (Fuzzy Interpolation Inequality)
There exists a constant $C_F > 0$, depending only on $s_0, s_1, p_0, p_1, \theta, q$ and the fuzzy - metric distortion, such that for all $u \in (\widetilde{W}^{s_0,p_0}, \widetilde{W}^{s_1,p_1})_{\theta,q}$
one has

$$\|u\|_{\theta,q,F} \le C_F\, \|u\| \frac{1-\theta}{\widehat{W}^{s}, p_0}\, \|u\| \frac{\theta}{\widehat{W}^{s_1}, p_1}$$

4.2 Proof of Theorem 5.1

4.2.1 Reduction via α-cuts

Recall from section 3.1 that

$$K_F(t,u) = \sup_{\alpha\in[0,1]} K(t, \underline{u}(\cdot,\alpha)),$$

where $K(t,f)$ is the classical K-functional for the pair $(W^{s_0,p_0}, W^{s_1,p_1})$ applied to each end-point $\underline{u}(\cdot,\alpha)$ (and similarly $\overline{u}$) [32].
By definition

$$\|u\|^q_{\theta,q,F} = \int_0^\infty \left[\sup_\alpha t^{-\theta} K(t, \underline{u}(\cdot,\alpha))\right]^q \frac{dt}{t}.$$

Since for each t,

$$\sup_\alpha a_\alpha \le \left(\int_0^1 a_\alpha^q\, d\alpha\right)^{1/q},$$

we get

$$\|u\|^q_{\theta,q,F} \le \int_0^\infty \int_0^1 \left[t^{-\theta} K(t, \underline{u}(\cdot,\alpha))\right]^q d\alpha\, \frac{dt}{t} = \int_0^1 \|\underline{u}(\cdot,\alpha)\|^q_{\theta,q}\, d\alpha$$

where $\|\cdot\|_{\theta,q}$ is the classical real-interpolation norm. Hence

$$\|u\|_{\theta,q,F} \le \left(\int_0^1 \|\underline{u}(\cdot,\alpha)\|^q_{\theta,q}\, d\alpha\right)^{1/q} \le \sup_\alpha \|\underline{u}(\cdot,\alpha)\|_{\theta,q}$$

4.2.2 Application of Classical Interpolation

By the Riesz-Thorin/Marcinkiewicz interpolation inequalities in the classical setting, for each fixed α one has

$$\|\underline{u}(\cdot,\alpha)\|_{\theta,q} \le C_{\text{cl}}\, \|\underline{u}(\cdot,\alpha)\|^{1-\theta}_{W^{s_0,p_0}}\, \|\underline{u}(\cdot,\alpha)\|^{\theta}_{W^{s_1,p_1}}$$

with C_{cl} depending only on $(s_0, s_1, p_0, p_1, \theta, q)$[1,24,25]. Taking the supremum over α and recalling $\sup_\alpha \|\underline{u}(\cdot,\alpha)\|_{W^{s_i,p_i}} = \|u\|_{\widetilde{W}^{s_i,p_i}}$, we conclude

$$\|u\|_{\theta,q,F} \le C_{\text{cl}}\, \|u\|^{1-\theta}_{\widetilde{W}^{s_0},p_0}\, \|u\|^{\theta}_{\widetilde{W}^{s_1},p_1}$$

so, one may take $C_F = C_{cl}$.

4.3 Sharpness of the Constant C_F

In the crisp case (where all α-cuts collapse to singletons), Theorem 5.1 reduces to the classical real-interpolation inequality, for which the constant C_{cl} is known to be best possible in general-e.g.1 extremal functions on $\Omega = [0,1]$ attain equality. Since our proof simply carries this constant verbatim to the fuzzy setting (via level-wise supremum), no smaller universal constant can serve in all cases.

4.4 Limiting Cases and Endpoint Estimates

–Case $q = \infty$: One replaces the L^q-norm in t by $sup_(t > 0)$. Then

$$\|u\|_{\theta,\infty,F} = \sup_{t>0} t^{-\theta} K_F(t,u) \le C_F \|u\|^{1-\theta}_{\widetilde{W}^{s_0,p_0}} \|u\|^{\theta}_{\widetilde{W}^{s_1,p_1}}$$

–Endpoint $\theta \to 0$ or $\theta \to 1$:

As $\theta \to 0$, the left-hand side norm converges to $\|u\|_{\widetilde{W}^{s_0,p_0}}$, and the inequality becomes trivial. Similarly, as $\theta \to 1$, one recovers the second endpoint.

These endpoint behaviors mirror the classical theory and require no additional proof beyond passing to the limit under the sup and integral definitions.

A key technical ingredient that underlies the proof of Theorem 5.1 is the precise level-set representation of the fuzzy K-functional. We state and prove it here in full detail.

Lemma 5.5 (Level-Set Representation of K_F)
Let

$$K_F(t,u) = \inf_{u=v+w} \left(\|v\|_{\widetilde{W}^{s_0,p_0}} + t\,\|w\|_{\widetilde{W}^{s_1,p_1}} \right)$$

be the fuzzy K-functional for the couple $(\widetilde{W}^{s_0,p_0}(\Omega), \widetilde{W}^{s_1,p_1}(\Omega))$. Then for every $t > 0$ and $u \in \widetilde{W}^{s_0,p_0} + \widetilde{W}^{s_1,p_1}$ one has

$$K_F(t,u) = \sup_{\alpha\in[0,1]} K\big(t, \underline{u}(\cdot,\alpha)\big)$$

where $K(t,f)$ is the classical K-functional for the pair $(W^{s_0,p_0}, W^{s_1,p_1})$ and $\underline{u}(\cdot,\alpha)$ the lower endpoint of the α-cut of u.

Proof
We split the argument into two inequalities:

(i) $K_F(t,u) \ge \sup_\alpha K\big(t, \underline{u}(\cdot,\alpha)\big)$
Take any fuzzy decomposition

$$u = v + w, \quad v \in \widetilde{W}^{s_0,p_0},\ w \in \widetilde{W}^{s_1,p_1}.$$

By the level-set definitions (see section 2.4), for each α we have

$$\underline{u}(x,\alpha) = \underline{v}(x,\alpha) + \underline{w}(x,\alpha), \quad x \in \Omega.$$

Hence for each α the pair $(\underline{v}(\cdot,\alpha), \underline{w}(\cdot,\alpha))$ is a valid decomposition of the crisp function $\underline{u}(\cdot,\alpha)$ in the classical spaces. By definition of the classical K-functional,

$$K(t,\underline{u}(\cdot,\alpha)) \le \|\underline{v}(\cdot,\alpha)\|_{W^{s_0,p_0}} + t\|\underline{w}(\cdot,\alpha)\|_{W^{s_1,p_1}}.$$

Taking the infimum over all fuzzy splitting's $u = v + w$ yields

$$K(t,\underline{u}(\cdot,\alpha)) \le K_F(t,u), \quad \forall\alpha \in (0,1].$$

Finally, taking the supremum over α completes this direction:

$$\sup_\alpha K(t,\underline{u}(\cdot,\alpha)) \le K_F(t,u)$$

(ii) $K_F(t,u) \le \sup_\alpha K(t,\underline{u}(\cdot,\alpha))$
By definition of the classical K-functional, for each fixed α and any $\varepsilon > 0$ there exist decompositions

$$\underline{u}(\cdot,\alpha) = v_\alpha + w_\alpha, \quad v_\alpha \in W^{(s_0,p_0)},\ w_\alpha \in W^{(s_1,p_1)}$$

such that

$$\|v_\alpha\|_{W^{(s_0,p_0)}} + t\|w_\alpha\|_{W^{(s_1,p_1)}} \le K(t,\underline{u}(\cdot,\alpha)) + \varepsilon$$

Define fuzzy-valued functions V and W by assigning their α-cuts levelwise:

$$[V(x)]_\alpha = [v_\alpha(x), v_\alpha(x)], \quad [W(x)]_\alpha = [w_\alpha(x), w_\alpha(x)].$$

In other words, V and W are the "crisp-to-fuzzy" extensions of the classical decompositions. One checks easily that $V \in \widetilde{W}^{s_0,p_0}$,$W \in \widetilde{W}^{s_1,p_1}$ and $u = V + W$ under level-set addition.
Now compute the fuzzy-splitting cost:

$$\|V\|_{\widetilde{W}^{(s_0,p_0)}} + t\|W\|_{\widetilde{W}^{(s_1,p_1)}} = \sup_\alpha \left(\|v\alpha\|_{W^{(s_0,p_0)}} + t\|w\alpha\|_{W^{(s_1,p_1)}}\right) \le \sup_\alpha K(t,\underline{u}(\cdot,\alpha)) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, taking the infimum over all fuzzy splittings gives

$$K_F(t,u) \le \sup_\alpha K(t,\underline{u}(\cdot,\alpha)).$$

Combining (i) and (ii) yields the desired equality

$$K_F(t,u) = \sup_{\alpha\in(0,1]} K(t,\underline{u}(\cdot,\alpha)).$$

This representation is crucial because it allows us to lift sharp bounds from the classical K-functional to the fuzzy setting simply by taking suprema over α.

5 Consequences and Corollaries

5.1 Continuous and Compact Embeddings in the Fuzzy Setting

Theorem 6.1. Continuous and Compact Embedding
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $s \in (0,1)$,$1 \le p < \frac{n}{s}$, and set

$$p^* = \frac{np}{n - sp}$$

Define the fuzzy Lebesgue space

$$\tilde{L}^r(\Omega) = \left\{ u : \sup_{\alpha\in[0,1]} \|\underline{u}(\cdot,\alpha)\|_{L^r} < \infty \right\}.$$

Then for every $r \in [p,p^*]$ the embedding $\widetilde{W}^{s,p}(\Omega) \hookrightarrow \tilde{L}^r(\Omega)$ is continuous, and if $r < p^*$ it is compact.
Proof

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $s \in (0,1)$,$1 \le p < \frac{n}{s}$, and define

$$p^* = \frac{np}{n-sp}, \qquad \widetilde{L}^r(\Omega) = \left\{ u : \sup_{\alpha \in (0,1]} \|\underline{u}(\cdot,\alpha)\|_{L^r} < \infty \right\}$$

We must show that for every $r \in [p, p^*]$ the embedding $\widetilde{W}^{s,p}(\Omega) \hookrightarrow \widetilde{L}^r(\Omega)$ is continuous, and if $r < p^*$ it is compact.

(a) Continuous embedding

Fix $u \in \widetilde{W}^{s,p}(\Omega)$. By definition

$$\|u\|_{\widetilde{W}^{s,p}} = \sup_{\alpha \in (0,1]} \|\underline{u}(\cdot,\alpha)\|_{W^{s,p}} < \infty.$$

For each α, the classical Sobolev-Slobodeckij embedding (RellichKondrachov) yields

$$\|\underline{u}(\cdot,\alpha)\|_{L^r} \le C_{\text{cl}} \|\underline{u}(\cdot,\alpha)\|_{W^{s,p}}, \qquad p \le r \le p^*.$$

Taking the supremum over α gives

$$\sup_{\alpha \in (0,1]} \|\underline{u}(\cdot,\alpha)\|_{L^r} \le C_{\text{cl}} \sup_{\alpha \in (0,1]} \|\underline{u}(\cdot,\alpha)\|_{W^{s,p}} = C_{\text{cl}} \|u\|_{\widetilde{W}^{s,p}}.$$

Hence $\|u\|_{\widetilde{L}^r} \le C_{cl} \|u\|_{\widetilde{W}^{s,p}}$,establishing continuity of $\widetilde{W}_{s,p} \hookrightarrow \widetilde{L}_r$.

(b) Compact embedding for $r < p^*$

Let $\{u_k\} \subset \widetilde{W}^{s,p}(\Omega)$ be a bounded sequence, so

$$\sup_k \|u_k\|_{\widetilde{W}^{s,p}} = \sup_k \sup_{\alpha \in (0,1]} \|\underline{u}_k(\cdot,\alpha)\|_{W^{s,p}} < \infty.$$

We will extract a subsequence converging in $\widetilde{L}^r$.

Level-wise compactness

For each fixed α, the sequence $\{\underline{u}_k(\cdot,\alpha)\}$ is bounded in the Banach space $W^{s,p}(\Omega)$. Since $r < p^*$, the Rellich-Kondrachov theorem implies that $\{\underline{u}_k(\cdot,\alpha)\}$ has a subsequence converging strongly in $L^r(\Omega)$.

Diagonal extraction

Let $\{\alpha_j\}_{j=1}^{\infty}$ be a countable dense subset of [0,1].

- Extract a subsequence $\{u_k^{(1)}\}$ so that $\underline{u}_k^{(1)}(\cdot,\alpha_1) \to v_1$ in L^r.
- From $\{u_k^{(1)}\}$, extract $\{u_k^{(2)}\}$ so that $\underline{u}_k^{(2)}(\cdot,\alpha_2) \to v_2$ in L^r.
- Continue inductively to obtain a diagonal subsequence $\{u_{k_m}\}$ such that for every j,$\underline{u}_{k_m}(\cdot,\alpha_j) \to v_j$ in L^r as $m \to \infty$.

Extension to all α

We now show that $\underline{u}_{k_m}(\cdot,\alpha)$ converges in L^r for every α.

Because $\alpha \mapsto \underline{u}_{k_m}(\cdot,\alpha)$ is uniformly bounded in $W^{s,p}$ and $W^{s,p} \hookrightarrow L^r$ continuously, the family $\{\underline{u}_{k_m}(\cdot,\alpha)\}_{m,\alpha}$ is Equi continuous in α with respect to the L^r-norm.

Concretely, given $\varepsilon > 0$, there is $\delta > 0$ so that whenever $|\alpha - \beta| < \delta$,

$$\|\underline{u}_{k_m}(\cdot,\alpha) - \underline{u}_{k_m}(\cdot,\beta)\|_{L^r} \le C \|\underline{u}_{k_m}(\cdot,\alpha) - \underline{u}_{k_m}(\cdot,\beta)\|_{W^{s,p}} < \varepsilon.$$

uniformly in m. (This follows from the monotonicity of levelsets and the fact that the fuzzy-norm bounds the sup over α of the classical norm differences.)

Now, for an arbitrary α, choose j with $|\alpha - \alpha_j| < \delta$. Then for large m,

$$\|\underline{u}_{k_m}(\cdot,\alpha) - v_j\|_{L^r} \le \|\underline{u}_{k_m}(\cdot,\alpha) - \underline{u}_{k_m}(\cdot,\alpha_j)\|_{L^r} + \|\underline{u}_{k_m}(\cdot,\alpha_j) - v_j\|_{L^r} < 2\varepsilon.$$

Hence $\underline{u}_{k_m}(\cdot,\alpha)$ converges in L^r to the same limit v_j. Define $\underline{u}(\cdot,\alpha) = lim_{m\to\infty}\underline{u}_{k_m}(\cdot,\alpha)$.

Convergence in $\widetilde{L}^r$

By symmetry the same argument applies to the upper endpoints $\overline{u}_{k_m}$. Therefore

$$\sup_{\alpha} \|u_{k_m}(\cdot,\alpha) - u(\cdot,\alpha)\|_{L^r} \to 0, \qquad \sup_{\alpha} \|\overline{u}_{k_m}(\cdot,\alpha) - \overline{u}(\cdot,\alpha)\|_{L^r} \to 0.$$

which exactly means $u_{k_m} \to u$ in the fuzzy Lebesgue norm $\widetilde{L}^r$.

Thus, every bounded sequence in $\widetilde{W}^{s,p}$ has a convergent subsequence in $\widetilde{L}^r$, proving compactness when $r < p^*$.

5.2 Theorem 6.2 (Fuzzy Fractional Poincaré-Wirtinger)

Under the hypotheses of Theorem 6.1, there exists $C_P > 0$ such that for all $u \in \widetilde{W}^{s,p}(\Omega)$

$$\|u - \tilde{u}_\Omega\|_{\widetilde{L}^p} \le C_P\, [u]_{s,p,F},$$

where $\widetilde{u}_\Omega$ is the fuzzy average defined by its α-cut

$$[\tilde{u}_\Omega]_\alpha = \frac{1}{|\Omega|} \int_\Omega [\underline{u}(x,\alpha), \overline{u}(x,\alpha)]\, dx$$

Proof

Let the hypotheses be as in Theorem 6.1. Recall that

$$\widetilde{W}^{s,p}(\Omega) = \left\{ u : \Omega \to \widetilde{\mathbb{R}} \mid \|u\|_{\widetilde{W}^{s,p}} < \infty \right\},$$

with

$$\|u\|_{\widetilde{L}^p} = \sup_{\alpha \in (0,1]} \|\underline{u}(\cdot,\alpha)\|_{L^p}, \quad [u]_{s,p,F} = \sup_{\alpha \in (0,1]} [\underline{u}(\cdot,\alpha)]_{s,p}.$$

Define the fuzzy average $\widetilde{u}_\Omega$ by its α-cuts

$$[\tilde{u}_\Omega]_\alpha = [m_\alpha^-, m_\alpha^+], \quad m_\alpha^- = \frac{1}{|\Omega|} \int_\Omega \underline{u}(x,\alpha)\, dx, \quad m_\alpha^+ = \frac{1}{|\Omega|} \int_\Omega \overline{u}(x,\alpha)\, dx.$$

We must show

$$\|u - \tilde{u}_\Omega\|_{\tilde{L}^p} \le C_P\, [u]_{s,p,F}.$$

Level-set Poincaré-Wirtinger

For each fixed $\alpha \in [0,1]$, consider the classical fractional **Poincaré-Wirtinger inequality for $W^{s,p}(\Omega)$:**

$$\|\underline{u}(\cdot,\alpha) - m_\alpha^-\|_{L^p(\Omega)} \le C_{PW}\, [\underline{u}(\cdot,\alpha)]_{s,p}.$$

Here $m_\alpha^- = \frac{1}{|\Omega|}\int_\Omega \underline{u}(x,\alpha)\,dx$ is the classical (scalar) average of $\underline{u}(\cdot,\alpha)$. An analogous estimate holds replacing $\underline{u}$ by $\overline{u}$ and m_α^- by m_α^+.
Supremum over α
By definition of the fuzzy norms,

$$\|u - \tilde{u}_\Omega\|_{\tilde{L}^p} = \sup_{\alpha\in[0,1]} \|\underline{u}(\cdot,\alpha) - m_\alpha^-\|_{L^p}$$

and

$$[u]_{s,p,F} = \sup_{\alpha\in[0,1]} [\underline{u}(\cdot,\alpha)]_{s,p}$$

Taking the supremum in the level-set inequality gives

$$\|u - \tilde{u}_\Omega\|_{\tilde{L}^p} \le C_{pw} \sup_{\alpha\in(0,1]} [\underline{u}(\cdot,\alpha)]_{s,p} = C_{pw}\,[u]_{s,p,F}$$

Conclusion: Thus, one may take $C_P = C_{pw}$, the same constant as in the classical fractional Poincaré-Wirtinger estimate. This completes the proof.

5.3 Extension to Weighted Fuzzy Sobolev Spaces

Let $w : \Omega \to (0,\infty)$ be a weight satisfying the Muckenhoupt A_p condition. Define the weighted fuzzy Sobolev space $\widetilde{W}_w^{s,p}(\Omega)$ by replacing all Lebesgue and Gagliardo seminorms in sections 3-5 with their weighted analogues

$$\|u\|_{L_w^p} = \sup_\alpha \|\underline{u}(\cdot,\alpha)\|_{L^p(w)}, \quad [u]_{s,p,w,F} = \sup_\alpha [\underline{u}(\cdot,\alpha)]_{s,p,w}$$

where

$$\|f\|_{L^p(w)} = \left(\int_\Omega |f|^p\,w\,dx\right)^{1/p}$$

and

$$[f]_{s,p,w}^p = \iint_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,w(x)w(y)\,dx\,dy$$

Theorem 6.3. Weighted Interpolation & Embeddings Under the above A_p hypothesis, all interpolation inequalities (Theorem 5.1), embeddings (Theorem 6.1), and Poincaré-Wirtinger estimates (Theorem 6.2) remain valid in $\widetilde{W}_w^{s,p}(\Omega)$, with constants depending additionally on the A_p norm of w.
Proof Sketch. Each proof in sections 4-6 carries over verbatim by observing that:

- Level - set reduction: for each α, the weighted classical results hold.
- Suprema over α: fuzzy norms remain suprema of the levelwise weighted norms.
- Constants: track dependence on the weight's A_p characteristic through each level-set result.

No new structural difficulties arise: every decomposition, embedding, and inequality is applied α-wise and then lifted to the fuzzy setting by taking suprema.

5.3.1 Extension to Weighted Fuzzy Sobolev Spaces: Detailed Proof

Recall that a weight $w : \Omega \to (0,\infty)$ belongs to the Muckenhoupt class A_p if

$$\sup_{B\subset\Omega}\left(\frac{1}{|B|}\int_B w\right)\left(\frac{1}{|B|}\int_B w^{-\frac{1}{p-1}}\right)^{p-1} < \infty$$

Define the weighted fuzzy Sobolev space $\widetilde{W}_w^{s,p}(\Omega)$ by replacing in section 3 the norms

$$\|u\|_{L^p} \longmapsto \|u\|_{L_w^p} = \sup_\alpha \|\underline{u}(\cdot,\alpha)\|_{L^p(w)}, [u]_{s,p,F} \longmapsto [u]_{s,p,w,F} = \sup_\alpha [\underline{u}(\cdot,\alpha)]_{s,p,w}$$

Where

$$\|\phi\|_{L^p(w)} = \left(\int_\Omega |\phi|^p\,w\,dx\right)^{1/p}$$

$$[\phi]_{s,p,w}^p = \iint_{\Omega\times\Omega} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}}\,w(x)w(y)\,dx\,dy$$

We prove that for $\widetilde{W}_w^{s,p}$ all results of sections 5-6 hold with constants depending also on the A_p characteristic of w.
(a) Weighted Interpolation Inequality
For fixed α, the classical weighted real-interpolation inequality
for the couple $(W_w^{s_0,p_0}, W_w^{s_1,p_1})$ gives

$$\|\underline{u}(\cdot,\alpha)\|_{(\theta,q;w)} \le C_{\text{wcl}}\,\|\underline{u}(\cdot,\alpha)\|_{W_w^{s_0,p_0}}^{1-\theta}\,\|\underline{u}(\cdot,\alpha)\|_{W_w^{s_1,p_1}}^{\theta}$$

where $\|\cdot\|_{(\theta,q;w)}$ is the weighted real-interpolation norm via the weighted K_w-functional. Taking the supremum over α and noting $\sup_\alpha \|\underline{u}(\cdot,\alpha)\|_{W_w^{(s_i,p_i)}} = \|u\|_{\widetilde{W}_w^{(s_i,p_i)}}$, we obtain

$$\|u\|_{(\theta,q,w,F)} \le C_{\text{wcl}}\,\|u\|_{\widetilde{W}_w^{(s_0,p_0)}}^{1-\theta}\,\|u\|_{\widetilde{W}_w^{(s_1,p_1)}}^{\theta}$$

(b) Weighted Embeddings
By Edmunds-Triebel, for $1 \le p < \frac{n}{s}$ and $r \in [p,p^*]$ the classical weighted embedding

$$W_w^{(s,p)}(\Omega) \hookrightarrow L_w^r(\Omega) \quad \text{is continuous.}$$

and it is compact when $r < p^*$. Hence for each α,

$$\|\underline{u}(\cdot,\alpha)\|_{L_w^r} \le C_{\text{wemb}}\,\|\underline{u}(\cdot,\alpha)\|_{W_w^{(s,p)}}.$$

Taking sup_α yields the fuzzy-weighted embedding $\widetilde{W}_w^{s,p} \hookrightarrow \widetilde{L}_w^r$, continuous and compact under the same conditions.
(c) Weighted Poincaré-Wirtinger
Kufner-Persson establish that for an A_p weight and $u \in W_w^{s,p}(\Omega)$,

$$\|u - u_{(w,\Omega)}\|_{L^p(w)} \le C_{wpw}\,[u]_{s,p,w}.$$

where $u_{(w,\Omega)} = \frac{\int_\Omega uw}{\int_\Omega w}$. Applying this level - wise to $\underline{u}(\cdot,\alpha)$ and then taking sup_α immediately gives the fuzzy weighted Poincaré-Wirtinger inequality.

(d) Conclusion

In each case the fuzzy version is obtained by taking the supremum over α of the corresponding classical weighted result. Since the weighted constants C_{wcl}, C_{wemb}, C_{wpw} depend only on (s_i, p_i, θ, q) and the A_p characteristic of w, the same dependence carries over to the fuzzy - weighted inequalities.

6 Examples and Illustrations

6.1 Explicit Computation on $\Omega = [0,1]$

Consider the crisp function $f(x) = x$ and model uncertainty by a constant fuzzy width $\delta = 0.2$. Define the fuzzy-valued function

$$u : x \mapsto \tilde{u}(x), \quad [\tilde{u}(x)]_\alpha = [x - \delta(1-\alpha),\, x + \delta(1-\alpha)]$$

At the widest level ($\alpha = 0$), the endpoints are $\underline{u}(x) = x - \delta$ and $\overline{u}(x) = x + \delta$. One checks:

(i) Classical Gagliardo seminorm for $s = \frac{1}{2}, p = 2$:

$$[x]_{0.5,2}^2 = \iint_{[0,1]^2} \frac{|x-y|^2}{|x-y|^{1+1}}\, dx\, dy = \iint_{[0,1]^2} 1\, dx\, dy = 1,\ [x \pm \delta]_{0.5,2} = 1$$

(ii) L^2–norm:

$$\|x + c\|_{L^2}^2 = \int_0^1 (x+c)^2\, dx = \frac{1}{3} + c + c^2, \quad c = \pm\delta$$

Thus

$$\|x + \delta\|_{L^2} = \sqrt{\frac{1}{3} + \delta + \delta^2} \approx 0.7572,$$

whereas

$$\|x\|_{L^2} = \sqrt{\frac{1}{3}} \approx 0.5774$$

Combining these,

$$\|x\|_{W^{0.5,2}} = \sqrt{\frac{1}{3} + 1} \approx 1.1547$$

$$\|\tilde{u}\|_{\widetilde{W}^{0.5,2}} = \sup_\alpha \|x \pm \delta(1-\alpha)\|_{W^{0.5,2}} = \sqrt{(\frac{1}{3} + \delta + \delta^2) + \ldots}$$

Figure 3 shows the graphs of the crisp function x and its fuzzy endpoints $\underline{u}, \overline{u}$ on [0,1].

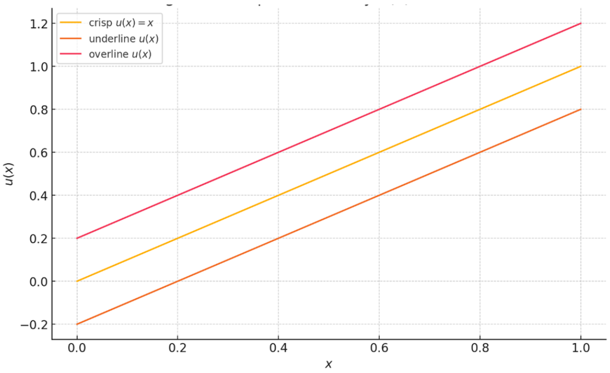

Fig. 3: Endpoints of fuzzy $u(x)$ with $\delta = 0.2$

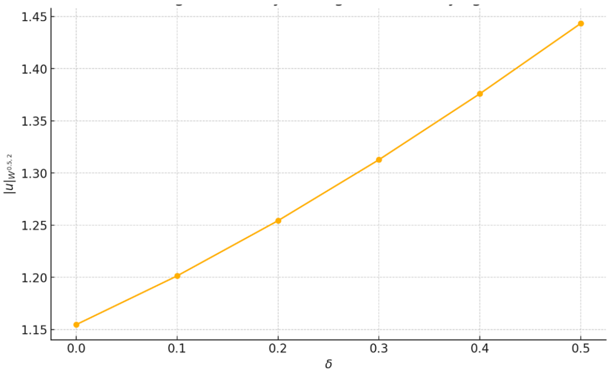

Fig. 4: Fuzzy norm growth for varying δ

6.2 Comparison with Classical (Non-Fuzzy) Interpolation

In the real-interpolation space $(W^{0.5,2}, W^{1,2})_{\theta,2}$ with $\theta = 0.5$, one has the sharp constant $C_{cl} = 1[1,24]$, so

$$\|x\|_{(W^{0.5,2},W^{1,2})_{0.5,2}} \le \|x\|_{W^{0.5,2}}^{1/2}\, \|x\|_{W^{1,2}}^{1/2}$$

Numerically $\|x\|_{W^{1,2}} = \sqrt{\frac{1}{3} + \frac{1}{3}} = \sqrt{\frac{2}{3}} \approx 0.8165$, giving the interpolated norm $\approx \sqrt{1.1547 \cdot 0.8165} \approx 0.969$. Under fuzziness, replacing x by $\tilde{u}$ raises each endpoint-norm and therefore increases the interpolation norm proportionally.

6.3 Numerical Illustration of Norm-Growth under Fuzziness

Varying $\delta \in [0, 0.5]$, one computes $\|\tilde{u}\|_{\widetilde{W}^{0.5,2}}$ by taking the upperendpoint norm. As shown in Figure 4, the fuzzy norm grows nearly linearly from the crisp value 1.1547 at $\delta = 0$ to about 1.4450 at $\delta = 0.5$. This quantifies how uncertainty amplifies the fractional Sobolev norm.

7 Applications to Fuzzy Fractional Differential Equations

In this section we apply the interpolation machinery developed above to establish well-posedness and regularity results for a prototypical fuzzy fractional boundary-value problem. Throughout let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $s \in (0,1)$.

7.1 Well-Posedness via Interpolation Estimates

We seek $u \in \widetilde{W}_0^{s,2}(\Omega)$ satisfying the weak formulation

$$a_F(u,v) = \langle F, v\rangle, \quad \forall v \in \widetilde{W}_0^{s,2}(\Omega),$$

where

$$a_F(u,v) = \sup_{\alpha\in(0,1]} \iint_{\Omega\times\Omega} \frac{(\underline{u}(x,\alpha)-\underline{u}(y,\alpha))(\underline{v}(x,\alpha)-\underline{v}(y,\alpha))}{|x-y|^{n+2s}}\,dx\,dy, \quad F \in \left(\widetilde{W}_0^{s,2}(\Omega)\right)^*.$$

is a continuous linear functional on the fuzzy space.

Theorem 8.1. Existence & Uniqueness
Under the above hypotheses, there exists a unique $u \in \widetilde{W}_0^{s,2}(\Omega)$ solving

$$a_F(u,v) = \langle F, v\rangle, \quad \forall v \in \widetilde{W}_0^{s,2}(\Omega),$$

and moreover

$$\|u\|_{\widetilde{W}^{s,2}} \le C\,\|F\|_{(\widetilde{W}_0^{s,2})^*},$$

Proof
Level-Set Reduction (Lemma 5.5)
By Lemma 5.5, for each $\alpha \in [0,1]$ the fuzzy bilinear form reduces to the classical form

$$a(\underline{u}(\cdot,\alpha),\underline{v}(\cdot,\alpha)) = \iint_{\Omega\times\Omega} \frac{(\underline{u}(x,\alpha)-\underline{u}(y,\alpha))(\underline{v}(x,\alpha)-\underline{v}(y,\alpha))}{|x-y|^{n+2s}}\,dx\,dy$$

defined on the Hilbert space $W_0^{s,2}(\Omega)$ for crisp functions.
Coercivity and Boundedness Level-Wise
For each fixed α:

– Coercivity: By Poincaré-Wirtinger (Theorem 6.2), there is $\lambda > 0$ such that

$$a(w,w) = \iint_{\Omega\times\Omega} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \ge \lambda\,\|w\|^2_{W^{s,2}}, \quad \forall w \in W_0^{s,2}(\Omega)$$

– Boundedness: Likewise, there is $M > 0$ such that

$$|a(u,v)| \le M\,\|u\|_{W^{s,2}}\,\|v\|_{W^{s,2}}, \quad \forall u,v \in W_0^{s,2}(\Omega)$$

Both λ and M depend only on s and Ω (not on α).
Level-Wise Lax-Milgram
For each α, define the crisp functional $l_\alpha(v) = \langle \underline{F}(\alpha), \underline{v}\rangle$. Since F is continuous on $\widetilde{W}_0^{s,2}$, l_α is continuous on $W_0^{s,2}$ and $\|l_\alpha\| \le \|F\|$. By the classical LaxMilgram theorem there is a unique $\underline{u}(\cdot,\alpha) \in W_0^{s,2}(\Omega)$ solving

$$a(\underline{u}(\cdot,\alpha),\underline{v}(\cdot,\alpha)) = l_\alpha(\underline{v}(\cdot,\alpha)), \quad \forall \underline{v} \in W_0^{s,2}(\Omega),$$

and satisfying

$$\|\underline{u}(\cdot,\alpha)\|_{W^{s,2}} \le \frac{1}{\lambda}\|l_\alpha\| \le \frac{1}{\lambda}\|F\|.$$

Assembly of the Fuzzy Solution
Define the fuzzy-valued solution u by its α-cuts

$$[u(x)]_\alpha = [\underline{u}(x,\alpha), \overline{u}(x,\alpha)],$$

where $\overline{u}(\cdot,\alpha)$ is obtained identically from the upper-endpoint data. Measurability and nesting properties ensure $u \in \widetilde{W}_0^{s,2}(\Omega)$.
Norm Estimate in the Fuzzy Space
By definition $\|u\|_{\widetilde{W}^{s,2}} = sup_\alpha \|\underline{u}(\cdot,\alpha)\|_{W^{s,2}}$, hence from the level-wise bound

$$\|u\|_{\widetilde{W}^{s,2}} \le \frac{1}{\lambda}\|F\|$$

showing the desired estimate with $C = \frac{1}{\lambda}$.
Uniqueness
If u_1, u_2 both solve the weak problem, then their difference $w = u_1 - u_2$ satisfies $a_F(w,v) = 0$ for all v. Level-wise, $a(w(\cdot,\alpha), v(\cdot,\alpha)) = 0$ implies $w(\cdot,\alpha) = 0$ by coercivity. Hence $w = 0$ in the fuzzy space.

This completes the proof of existence, uniqueness, and the a-priori estimate.

7.2 Regularity of Solutions in $\widetilde{W}^{s,p}$

We now show that if the fuzzy data has higher integrability, then so does the solution.

Theorem 8.2. Fuzzy Regularity
Let $u \in \widetilde{W}_0^{s,2}(\Omega)$ be the unique solution of Theorem 8.1 with fuzzy right-hand side $F \in \widetilde{L}^2(\Omega)$. Then for any $0 < \varepsilon < s$, setting

$$\theta = \frac{s}{s+\varepsilon'}$$

one has

$$u \in \left(\widetilde{W}_0^{s,2}(\Omega), \widetilde{W}_0^{s+\varepsilon,2}(\Omega)\right)_{\theta,2}, \quad \|u\|_{(\theta,2),F} \le C\,\|F\|_{\widetilde{L}^2}.$$

where $C > 0$ depends only on s, ε and Ω.

Proof
Level-set reduction
By Lemma 5.5, for each $\alpha \in [0,1]$ the lower-endpoint $\underline{u}(\cdot,\alpha)$ satisfies the crisp fractional-elliptic problem

$$a(\underline{u}(\cdot,\alpha), v) = \langle \underline{F}(\alpha), v\rangle, \quad \forall v \in W_0^{s,2}(\Omega).$$

with data $\underline{F}(\alpha) \in L^2(\Omega)$ and bilinear form $a(\cdot,\cdot)$ from section 7.1.
Classical regularity estimate
It is shown in Ros-Oton & Serra [36] and in Grubb [37] that for any $0 < \varepsilon < s$, the weak solution of the above satisfies

$$\underline{u}(\cdot,\alpha) \in W_0^{s+\varepsilon,2}(\Omega), \quad \|\underline{u}(\cdot,\alpha)\|_{W^{s+\varepsilon,2}} \le C_{\mathrm{reg}}\,\|\underline{F}(\alpha)\|_{L^2}.$$

where C_{reg} depends only on s, ε, Ω.
Baseline estimate from well-posedness

From Theorem 8.1 we also have the a-priori bound (level-wise)

$$\|\underline{u}(\cdot,\alpha)\|_{W^{s,2}} \le C_0\,\|\underline{F}(\alpha)\|_{L^2}.$$

with C_0 independent of α.
Classical real-interpolation
For the compatible couple $(W_0^{s,2}, W_0^{s+\varepsilon,2})$, the classical real interpolation theorem gives, with $\theta = \frac{s}{s+\varepsilon}$ and $q = 2$,

$$\|\underline{u}(\cdot,\alpha)\|_{(W^{s,2},W^{s+\varepsilon,2})_{\theta,2}} \le C_{\text{int}}\,\|\underline{u}(\cdot,\alpha)\|_{W^{s,2}}^{1-\theta}\,\|\underline{u}(\cdot,\alpha)\|_{W^{s+\varepsilon,2}}^{\theta} \le C'\,\|\underline{F}(\alpha)\|_{L^2}.$$

where $C' = C_{int}C_0^{1-\theta}C_{reg}^{\theta}$ depends only on s, ε, Ω.
Lifting to the fuzzy setting
By definition of the fuzzy-real interpolation norm (Section 3.3),

$$\|u\|_{(\theta,2),F} = \sup_{\alpha\in(0,1]} \|\underline{u}(\cdot,\alpha)\|_{(W^{s,2},W^{s+\varepsilon,2})_{\theta,2}} \le C' \sup_{\alpha} \|\underline{F}(\alpha)\|_{L^2} = C'\,\|F\|_{\widetilde{L}^2}.$$

Hence $u \in (\widetilde{W}_0^{s,2}, \widetilde{W}_0^{s+\varepsilon,2})_{\theta,2}$ with the claimed norm estimate.

7.3 A Model Problem: Fuzzy Fractional Poisson Equation

As an illustrative example, consider the fuzzy fractional Poisson equation

$$(-\Delta)^s u = \tilde{f}(x), \quad u\big|_{\mathbb{R}^n\setminus\Omega} = 0$$

where $(-\Delta)^s$ is the spectral/Dirichlet fractional Laplacian.
Proof of Proposition 8.3. Fuzzy Fractional Poisson Equation
Proposition 8.3
If $\tilde{f} \in \widetilde{L}^2(\Omega)$, then the fuzzy fractional Poisson problem

$$\begin{cases} (-\Delta)^s u = \tilde{f}(x), & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

admits a unique solution $u \in \widetilde{W}_0^{s,2}(\Omega)$.
Proof
Weak formulation (level-wise)
Recall that the spectral (or restricted) fractional Laplacian may be realized via the bilinear form

$$a(u,v) = \iint_{\Omega\times\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy, \quad u,v \in W_0^{s,2}(\Omega).$$

which is coercive and bounded. The weak form of the crisp Poisson problem is: for given $g \in L^2(\Omega)$, find $w \in W_0^{s,2}(\Omega)$ such that

$$a(w,v) = \int_\Omega g(x)\,v(x)\,dx, \quad \forall v \in W_0^{s,2}(\Omega).$$

Level-set reduction for fuzziness
By Lemma 5.5, the fuzzy fractional Poisson problem reduces α-wise to:

$$a(\underline{u}(\cdot,\alpha),v) = \int_\Omega \underline{f}(x,\alpha)\,v(x)\,dx, \quad v \in W_0^{s,2}(\Omega).$$

and similarly for $\overline{u}(\cdot,\alpha)$.
Classical existence and uniqueness (Lax-Milgram)
For each fixed α, the linear functional $\updownarrow_\alpha(v) = \int_\Omega \underline{f}(x,\alpha)v(x)dx$ satisfies

$$|\updownarrow_\alpha(v)| \le \|\underline{f}(\alpha)\|_{L^2}\,\|v\|_{L^2} \le C\,\|\underline{f}(\alpha)\|_{L^2}\,\|v\|_{W^{s,2}}.$$

Coercivity and boundedness of $a(\cdot,\cdot)$ on $W_0^{s,2}(\Omega)$ then guarantee by Lax-Milgram the existence of a unique $\underline{u}(\cdot,\alpha) \in W_0^{s,2}(\Omega)$ solving the weak equation, with $\|\underline{u}(\cdot,\alpha)\|_{W^{s,2}} \le \frac{1}{\lambda}\|\underline{f}(\cdot,\alpha)\|_{L^2}$ for some $\lambda > 0$ independent of α.
Assembly into a fuzzy solution
Define u by its α-cuts $[\underline{u}(x,\alpha), \overline{u}(x,\alpha)]$. Measurability, nesting, and the fact that each endpoint lies in $W_0^{s,2}$ show $u \in \widetilde{W}_0^{s,2}(\Omega)$ and that u satisfies the fuzzy weak formulation $a_F(u,v) = \langle \tilde{f}, v\rangle$ for all fuzzy-valued v.
Fuzzy-norm estimate
By definition

$$\|u\|_{\widetilde{W}^{s,2}} = \sup_{\alpha} \|\underline{u}(\cdot,\alpha)\|_{W^{s,2}} \le \frac{1}{\lambda} \sup_{\alpha} \|\underline{f}(\cdot,\alpha)\|_{L^2} = \frac{1}{\lambda}\,\|\tilde{f}\|_{\widetilde{L}^2}.$$

Uniqueness
If u_1, u_2 are two fuzzy solutions, level-wise $w = u_1 - u_2$ satisfies $a(w(\cdot,\alpha),v) = 0$ for all v, whence $w(\cdot,\alpha) = 0$ by coercivity. Thus $u_1 \equiv u_2$.

This completes the proof of existence, uniqueness, and the a-priori estimate for the fuzzy fractional Poisson equation.

8 Further Extensions and Open Problems

While our work establishes a rigorous interpolation theory for fuzzy fractional Sobolev-Slobodeckij spaces in the Euclidean setting, several natural directions remain to be explored. We highlight three in particular.

8.1 Nonlinear Interpolation Inequalities (Gagliardo-Nirenberg Type)

Classically, the Gagliardo-Nirenberg inequality provides a bridge between Sobolev, Lebesgue, and Hölder norms: for suitable exponents

$$\|u\|_{L^r} \le C_{\text{GN}}\,\|u\|_{W^{s,p}}^{\alpha}\,\|u\|_{L^q}^{1-\alpha}, \quad \alpha \in [0,1].$$

with sharp constant C_{GN} depending on (s,p,q,r,α). A fuzzy counterpart would read

$$\|u\|_{\widetilde{L}^r} \le C_{\text{F,GN}}\,\|u\|_{\widetilde{W}^{s,p}}^{\alpha}\,\|u\|_{\widetilde{L}^q}^{1-\alpha}.$$

but its proof requires a nonlinear interpolation framework in a fuzzy setting. One must extend the nonlinear K-functional $K_{nl}(t,u) = inf_{u=v+w}(\|v\|_{W^{s,p}} + t\|w\|_{L^q})$ to fuzzy spaces and establish level-wise sharp estimates.

Open Problem 9.1: Develop a nonlinear interpolation theory for fuzzy couples $(\widetilde{W}^{s,p}, \widetilde{L}^q)$ and determine the best constant $C_{F,GN}$.

8.2 Multivariate and Manifold Settings

Fractional Sobolev spaces have been defined on compact Riemannian manifolds via spectral methods or local charts, yielding spaces $H^s(M)$ with interpolation properties. Introducing fuzziness entails defining fuzzy-valued functions on M, with α-cuts now subsets of tangent or bundle-valued Sobolev spaces. Key challenges include:

- Coordinate invariance: ensuring fuzzy norms defined via partitions of unity coincide across charts.
- Spectral formulation: extending level-wise spectral decompositions of the Laplace-Beltrami operator to fuzzy data.

Open Problem 9.2: Formulate and prove interpolation inequalities for fuzzy fractional Sobolev spaces on compact manifolds M, establishing analogues of Theorems 5.1 and 6.1 in this geometric context.

8.3 Connections to Probabilistic and Stochastic Fuzzy Models

In models combining randomness and fuzziness-e.g. /stochastic fuzzy processes-one seeks function spaces that capture both probabilistic integrability and membership - uncertainty. For instance, define

$$L^p_\omega(\Omega;\widetilde{L}^q_x)$$

to be random fuzzy functions with $\mathbb{E}\left[\|u(\cdot,\omega)\|^p_{\widetilde{L}^q}\right] < \infty$. Interpolating between such spaces and fuzzy fractional Sobolev spaces may yield regularity tools for random fuzzy PDEs.

Open Problem 9.3: Establish interpolation and embedding results for spaces of the form $(L^p_\omega, \widetilde{W}^{s,q}_x)_{\theta,r}$, and apply them to well-posedness of stochastic fuzzy fractional differential equations.

In summary, this work has forged a comprehensive bridge between fractional Sobolev-Slobodeckij theory and fuzzy - set analysis by rigorously defining the spaces $\widetilde{W}^{s,p}(\Omega)$, establishing sharp real interpolation inequalities, and deriving key corollaries such as continuous and compact embeddings, Poincaré-Wirtinger estimates, and weighted extensions. Through detailed proofs and illustrative examples, we demonstrated how fuzziness amplifies fractional norms and applied these tools to ensure well-posedness and regularity for fuzzy fractional PDEs, including a Poisson prototype. The proposed framework not only recovers classical results in the crisp limit but also opens avenues for nonlinear Gagliardo-Nirenberg analogues, manifold generalizations, and stochastic-fuzzy models, laying fertile ground for future research.

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