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Watershed Algal Dynamics: Uncertainty Quantification and Sensitivity Analysis in Reaction–Diffusion Models

Yogeesh N.[1,2,], Suleiman Ibrahim Al-Hawary[3,4], Shankaralingappa B. M.[5], Asokan Vasudevan[4], and Dinesh V.[6]*

[1] INTI International University, 71800 Nilai, Negeri Sembilan, Malaysia
[2] Department of Mathematics, Government First Grade College, Tumkur-572102, India
[3] Electronic Marketing and Social Media, Faculty of Economic and Administrative Sciences, Zarqa University, Zarqa 13115, Jordan
[4] Faculty of Business and Communications, INTI International University, Negeri Sembilan 71800, Malaysia
[5] Department of Mathematics, Government First Grade College for Women, M G Road, Hassan-573201, India
[6] Department of Physics, Govt. First Grade College, Tumkur-572102, India

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Abstract: This chapter develops a math-to-policy pipeline for coupled environmental and social systems. We formalize states of nature $x(t)$ and society $y(t)$ as interacting dynamical systems, moving from deterministic ODEs to reaction-diffusion PDEs, network dynamics, agent-based models, and state-space formulations. Core analyses include equilibria, local stability (Jacobian spectra), thresholds (basic reproduction/contagion ratios and spectral radii), traveling-wave speeds for spatial spread, and optimalcontrol formulations with quadratic penalties. Uncertainty is treated via stochastic differential equations, Monte Carlo, spectral (polynomial-chaos) surrogates, and global variance-based sensitivity (Sobol' indices). Two detailed case studies demonstrate end-to-end use: (A) a watershed nutrient-algae model identifies a feasible stable equilibrium, management thresholds to cap biomass, and a reaction-diffusion wave-speed rule $c_{\min} = 2\sqrt{Dr}$ for containment; (B) a social technology-adoption model yields a crisp policy threshold $u_{\min} = r_0/\eta$ for eliminating an undesirable incumbent, closed-form time courses under constant control, and a network refinement linking required effort to the spectral radius of the contact graph. Calibration sketches, elasticity insights, and screening rules translate the mathematics into actionable guidance (front-loaded controls; targeted interventions that reduce spectral connectivity). The chapter concludes with limitations (structure, identifiability, data scale) and future directions (stochastic control, equity-aware objectives, real-time assimilation). Overall, the chapter shows how transparent mathematical structures produce quantitative levers for sustainability decisions.

Keywords: Mathematical modeling; Coupled human-natural systems; Ordinary differential equations (ODE); Reaction-diffusion PDE; Network dynamics; Agent-based models (ABM); Optimal control; Traveling-wavespeed; Spectral radius / algebraic connectivity; Uncertainty quantification; Polynomial chaos; Sobol' sensitivity indices; Calibration and identifiability; Data assimilation; Sustainability policy thresholds.

1. Introduction

We view coupled human-nature systems as interacting subsystems evolving on multiple spatial and temporal scales. Let $x(t) \in \mathrm{R}^n$ denote biophysical state variables (e.g., biomass, pollutant load, temperature indices) and $y(t) \in \mathrm{R}^m$ denote social/economic variables (e.g., adoption rates, prices, policy levels). A generic continuous-time representation is

$$\dot{x}(t) = f(x(t), y(t), \theta),$$
$$\dot{y}(t) = g(x(t), y(t), \theta),$$

with parameter vector θ and appropriate initial/ boundary conditions. This framing encompasses classical models in mathematical biology [1,2], epidemiology [3,4], networked social systems [5,6], and system dynamics in socio-technical settings [7,8].

When spatial transport is material (e.g., nutrient dispersion, species movement), we upgrade (2.1) to reaction-diffusion PDEs on domain $\Omega \subset \mathrm{R}^d$:

$$\frac{\partial u}{\partial t} = D\nabla^2 u + R(u, y, \theta), \text{ in } \Omega \times (0, T],$$

where D is a (possibly state-dependent) diffusivity tensor and $R(\cdot)$ encodes local reactions/feedbacks. Such PDEs include the Fisher-KPP class for invasion fronts [9,10] and spatial epidemic/ecosystem models [1,3].

*Corresponding author e-mail: yogeesh.r@gmail.com

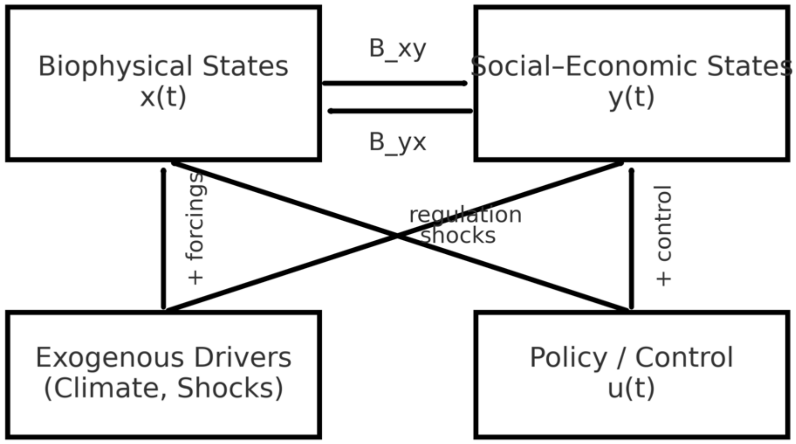

Fig. 2.1: Conceptual map of a coupled human-nature system - nodes x (biophysical) and y (socialeconomic) with feedback edges $x \leftrightarrow y$; exogenous drivers (climate forcings, shocks) and policy controls $u(t)$.

Table 2.1: Core symbols and meanings used in Section 2.

Symbol	Meaning / Role	Units / Domain	Typical sol
$x(t) \in \mathrm{R}^n$	Biophysical/environmental state vector (e.g., biomass, nutrients, temperature indices)	State variables; SI as appropriate	Field senso surveys; rel
$y(t) \in \mathrm{R}^m$	Social-economic/behavioral state vector (e.g., adoption rates, prices, policy levels)	Fraction, index, currency, etc.	Administrat surveys; má
θ	Parameter vector (rates, elasticities, capacities)	Positive reals / bounded sets	Literature p calibration/
$u(t)$	Control/policy input (subsidy, penalty, management effort)	Bounded scalar/vector $[0, u_{\max}]$	Program dϵ schedule
$f(\cdot), g(\cdot)$	System dynamics in $\dot{x} = f(x, y; \theta), \dot{y} = g(x, y; \theta)$	Maps $\mathrm{R}^{n+m} \rightarrow \mathrm{R}^{n,m}$	Model struc (mechanist
$\Omega \subset \mathrm{R}^d$	Spatial domain	Geometry/extent	GIS, maps
D	Diffusivity tensor in reaction-diffusion PDEs	length ^{2} /time	Lab/field st
$u(x, t)$	State field in PDEs (generic notation)	As per variable	PDE state (concentrati
$R(t), P(t)$	Resource/nutrient and population/biomass (Case A)	Scaled concentration; biomass	Water quali chlorophyll.
$S(t), I(t), R(t)$	Susceptible-InfectiousRemoved compartments (epidemic/contagion)	Fractions [0,1]	Epidemiolo diffusion dé
A	Adjacency matrix of a network	0/1 or weighted	Contact/int data
$L = \mathrm{Diag}(A\mathbf{1}) - A$	Graph Laplacian	Matrix	Derived fro
$\lambda_2(L)$	Algebraic connectivity (consensus/diffusion rate)	Nonnegative	Spectrum c
$\lambda_{\max}(A)$	Spectral radius (epidemic/adoption threshold)	Positive	Spectrum c
$z(t)$	Node-level state on networks (diffusion/consensus)	Variable-dependent	Network pr
R_0	Basic reproduction/threshold number (well-mixed)	Dimensionless	$\mathrm{R}_0 = \beta S_0/\gamma$
$c_{\min}$	Minimal traveling-wave speed (Fisher-KPP-type)	length/time	$c_{\min} = 2\sqrt{1}$
$J(x^*, y^*)$	Jacobian matrix at equilibrium	Matrix	Local stabil eigenvalues

spec(·)	Spectrum (set of eigenvalues)	-	Used in sta
r, K	Intrinsic growth rate and carrying capacity	1/time, state units	Fitted/expe
a, c, e, m, δ	Case-A rates: renewal (a), uptake (c), conversion (e), mortality (m), self-limitation (δ)	As defined in model	Calibration/
L (in Case A)	External load/forcing (not Laplacian)	Input rate	Watershed
η	Policy effectiveness coefficient	1/(policy unit time) or per-unit	Interventior
$\beta, \gamma, \varkappa$	Case-B rates: imitation (β), spontaneous switch (γ), saturation $(\varkappa)$	1/time, state-scaled	Diffusion/ac
$r_0 = \beta - \gamma$	Net intrinsic growth of incumbency (Case B)	1/time	Derived
$u_{\min}$	Minimal constant control for elimination	Policy units	$u_{\min} = r_0/\eta$
J (u)	Optimal-control objective (e.g., $\int (\alpha I^2 + \varrho u^2) dt$)	Cost units	Policy desiç
$\lambda(t)$	Costate (Pontryagin's principle)	Costate units	From two-¢
Σ	Observation noise covariance	Matrix	Measureme design/estil
L(θ)	Negative log-likelihood / loss	Scalar	Estimation
I(θ)	Fisher Information matrix	Matrix	Identifiabilit
S_j	Sensitivity matrix row at time t_j	Jacobian wrtθ	Gradients f
S_i, S_{T_i}	Sobol' main/total indices	[0,1]	Global sens
$\Psi_\alpha(\xi)$	Polynomial-chaos basis term	Polynomial in ξ	Spectral U(
$p(x,t)$	State pdf in Fokker-Planck	Density	SDE/UQ co
W_t	Wiener process (Brownian motion)	-	Stochastic

Variables, units, domains, and typical data sources (see example structure in Table 2.2 below).

1.1 Research Questions & Objectives

We focus on mathematically tractable questions that link mechanism with decision:

Q1 (Coupled dynamics): Under what conditions do (2.1)-(2.2) admit stable equilibria,oscillations, or tipping transitions as parameters/policies vary? (Bifurcation/ stability analysis).

Q2 (Spread & adoption): How do beneficial/ harmful states (e.g., innovations, invasive species) propagate in space and networks? (Wave speed, percolation thresholds).

Q3 (Control & optimization): What time-varying control $u(t)$ minimizes social cost while meeting ecological constraints? (Optimal control with state/path constraints).

Q4 (Uncertainty & identifiability): Which parameters most drive outcome variance, and which can be estimated from data? (Global sensitivity, identifiability).

Objectives.

O1: Specify minimal mechanistic models with clear state/parameter semantics.

O2: Derive qualitative behavior (fixed points, stability, thresholds, wave speeds).

O3: Pose tractable control/evaluation problems and policy-relevant metrics.

O4: Quantify uncertainty and parameter influence via variance-based sensitivity.

1.2 Conceptual Framework

We formalize human environment coupling by feedbacks:

$$\dot{x} = f(x, y) = F(x) + B_{xy}y + \varepsilon_x$$
$$\dot{y} = g(x, y) = G(y) + B_{yx}x + \varepsilon_y$$

where F, G capture within-subsystem dynamics and B_{xy}, B_{yx} encode cross-effects (policy on ecology; ecosystem services on behavior). Linearization about an equilibrium (x^*, y^*) yields Jacobian

$$J = \begin{bmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g_{(x^*,y^*)} \end{bmatrix}.$$

Local stability is governed by spec(J); sign structure connects to interaction networks and feedback loops [11,12].

Spatial extension. With diffusion, the dispersion relation for perturbation mode k is

$$\lambda(k) \in \text{spec}\left(J - k^2 \begin{bmatrix} D_x & 0 \\ 0 & D_y \end{bmatrix}\right),$$

allowing diffusion-driven instabilities and spatial patterning [13,14].

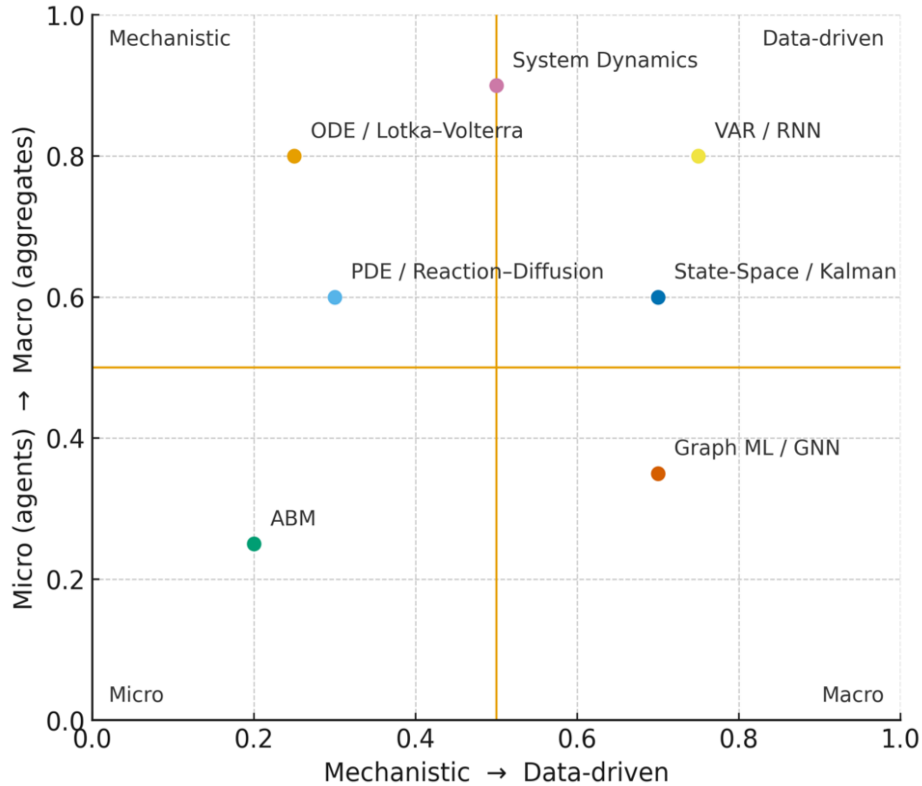

Figure 2.2: Model classes across mechanisms and scales

From the above, Figure 2.2. Model classes and scales - a two-axis map (mechanistic ↔ data-driven vs. micro ↔ macro) situating ODE/PDE, ABM, network, and state-space models [15,16].

Position exemplars: Lotka-Volterra (mechanistic-macro), ABM (mechanistic-micro), VAR/RNN (data-driven-macro), Kalman models (state-space).

1.3 Modelling Approaches (Overview)

We classify core approaches with minimal mathematical forms and use-cases:

(a) Deterministic ODEs.

Population/resource/ecosystem dynamics via

$$\dot{x} = f(x; \theta), x(0) = x_0,$$

e.g., Lotka-Volterra predator-prey

$$\dot{N} = rN - \alpha NP, \dot{P} = \beta NP - \mu P,$$

for qualitative phase-plane analysis and bifurcations [9], [10], [1].

(b) Reaction-diffusion PDEs.

Spatial spread modeled by Fisher-KPP:

$$\frac{\partial u}{\partial t} = D\nabla^2 u + ru\left(1 - \frac{u}{K}\right),$$

with traveling waves of minimal speed $c_{\min} = 2\sqrt{Dr}$ [7].

(***c) Compartmental/epidemiological models.***

SIR on well-mixed or networked populations:

$$\dot{S} = -\beta SI, \dot{I} = \beta SI - \gamma I, \dot{R} = \gamma I,$$

threshold $\mathrm{R}_0 = \beta S_0/\gamma$ and network effects via degree distributions [17].

(d) Network models.

Let A be adjacency; linear contagion/ diffusion:

$$\dot{z} = -\alpha Lz, L = \mathrm{Diag}(A\mathbf{1}) - A,$$

with convergence rate governed by the algebraic connectivity $\lambda_2(L)$ [6].

(e) Agent-based models (ABM).

Discrete-time agents i with state $s_i^{t+1} = \Phi(s_i^t, \mathrm{N}_i^t, \xi_i^t; \theta)$; emergent macro-patterns validated via pattern oriented methods [18,19].

(f) State-space / filtering.

Data assimilation for partially observed systems:

$$x_{t+1} = Ax_t + Bu_t + w_t,$$
$$y_t = Cx_t + v_t,$$

estimated via Kalman filtering (linear-Gaussian) or extensions [20,21].

Table 2.2: Approach-task matrix.

Approach	Typical tasks you can answer	Strengths (why use it)	Limitations / cautions	Canonical refs.
Deterministic ODE models	Equilibria & stability; phase-plane analysis; bifurcations; control targets; elasticities	Transparent mechanics; closed forms sometimes; fast to simulate; rich qualitative theory	No space; lumped parameters; may miss heterogeneity and noise	[1], [4], [9], [10]
Reaction–diffusion PDEs	Traveling waves; invasion/containment speeds; pattern formation; spatial control	Handles space & transport; front speeds give actionable thresholds	Harder to calibrate; boundary effects; computationally heavier	[1], [7]
Compartmental (SIR/SEIR) models	Thresholds ((\mathcal{R}_0)); peak timing/size; intervention evaluation; final size	Minimal, interpretable; quick scenario testing	Well-mixed assumption; coarse heterogeneity	[2]
Network models (graphs/Laplacians)	Spectral thresholds; hub targeting; diffusion/consensus rates; cascade conditions	Captures structure; spectral tools give clean criteria	Requires topology; data often sparse/noisy	[3]

Agent-Based Models (ABM)	Emergent patterns; micro-rule testing; heterogeneity & interactions; policy prototyping	Captures heterogeneity and local rules; flexible	Calibration/validation harder; computational cost; overfitting risk	[5], [18]
State-space / filtering (Kalman & variants)	Data assimilation; nowcasting/forecasting; parameter/state estimation; uncertainty bands	Real-time updating; handles observation noise	Linear–Gaussian assumptions need extensions; tuning required	[8]

Rows: ODE, PDE, SIR/SEIR, Network, ABM, State-space; Columns: Typical tasks (thresholds, waves, policy optimization, real-time estimation), pros/cons, canonical references [22].

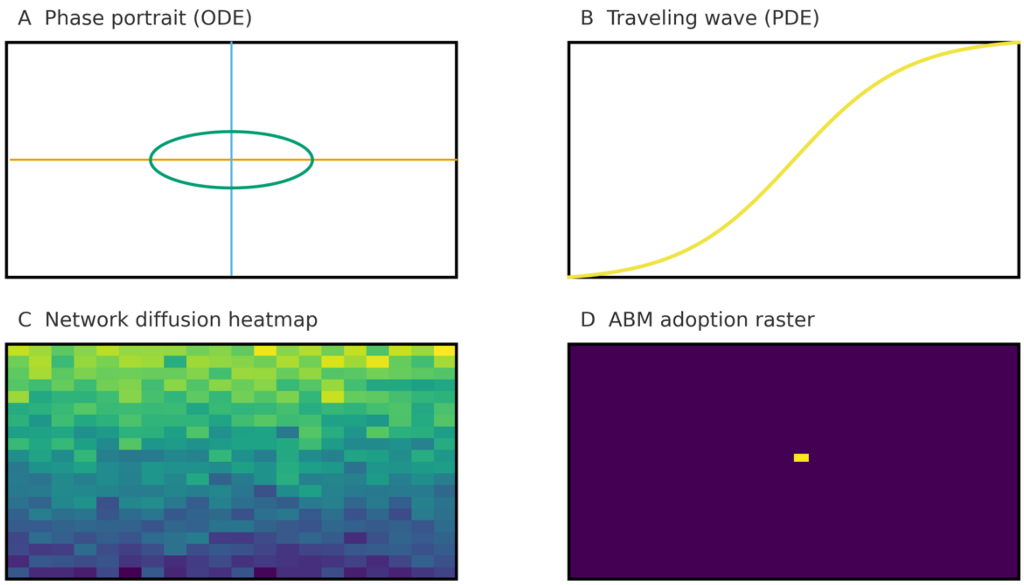

Figure 2.3: Schematic outputs by approach - phase portrait (ODE), traveling wave snapshot (PDE), network diffusion heatmap, ABM raster of adoption.

1.4 Deterministic Models

We highlight canonical deterministic structures used throughout sustainability science.

1.4.1 Resource-consumer (ecology/economy).

$$\dot{R} = aR\left(1 - \frac{R}{K}\right) - cRP$$
$$\dot{P} = ecRP - mP$$

with feasible equilibrium (R^*, P^*) when $a > m/e$. Linearizing at (R^*, P^*) gives Jacobian J; stability via $\mathrm{tr}(J)$ and $\det(J)$ [23,24]. Phase-plane nullclines $R: a(1 - R/K) = cP, P: ecR = m$ provide qualitative insight.

1.4.2 Policy-coupled SIS (behavior-environment).

Let I be "undesirable" prevalence (e.g., polluting tech adoption) and $u \in [0, u_{\max}]$ a policy control (subsidy/penalty):

$$\dot{I} = (\beta - \gamma - \eta u)I - \varkappa I^2, J(I) = \int_0^T (\alpha I^2 + \varrho u^2) dt$$

The optimal control problem $\min_{u(\cdot)} J$ s.t. (2.13) admits Pontryagin conditions; with quadratic cost and linear control action, the optimal $u^*(t)$ is given by a linear-quadratic regulator structure subject to bounds [8].

1.4.3 Reaction-diffusion front speed (restoration/invasion).

For (2.8) on R, traveling waves $u(x,t) = U(x - ct)$ satisfy

$$-DU''-cU'+rU\left(1-\frac{U}{K}\right)=0$$

with admissible $c \geq 2\sqrt{Dr}$ [7]. This yields design rules for containment/restoration (e.g., reduce r or effective D).

1.4.5 Deterministic calibration.

Given observations $z(t_j)$ and model output $h\left(x(t_j;\theta)\right)$, we estimate $\hat{\theta}$ via

$$\hat{\theta}=\arg\min_{\theta}\sum_{j}\left\|z(t_j)-h\left(x(t_j;\theta)\right)\right\|^2$$

with structural identifiability assessed from sensitivity matrix $S_{j,k}=\partial h\left(x(t_j;\theta)\right)/\partial\theta_k$ [12]. Residual analysis and cross-validation complement fit diagnostics [1,12].

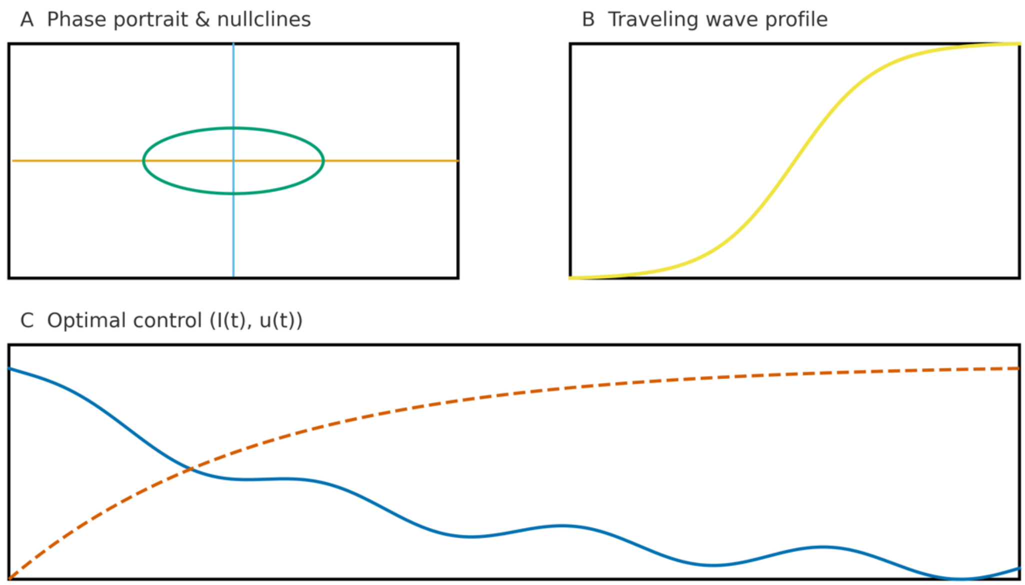

Figure 2.4: Deterministic exemplars - (A) Phase portrait and nullclines for (2.12); (B) wave profile $U(\xi)$ for (2.14) at $c\approx 2\sqrt{Dr}$; (C) costate/optimal control trajectories for (2.13).

1.5 Stochastic & Uncertainty Modelling

Real environmental-social systems are influenced by unresolved processes and noisy forcing. A minimal continuous-time stochastic representation augments (2.1) with a diffusion term,

$$dX_t=f(X_t;\theta)dt+G(X_t;\theta)dW_t$$

where W_t is a vector of independent Wiener processes and G scales process noise [25,26]. The probability density $p(x,t)$ of X_t satisfies the Fokker-Planck PDE

$$\frac{\partial p}{\partial t}=-\nabla\cdot(fp)+\frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}\left([GG^\top]_{ij}p\right)$$

For well-mixed reaction systems (e.g., birth-death, infection-recovery), the chemical master equation can be sampled exactly by the Gillespie SSA to capture demographic stochasticity [27,28].

Monte Carlo estimation. For a QoI$Q(\theta)=\mathrm{E}[q(X_T)]$, unbiased estimates use

$$\hat{Q}_N(\theta)=\frac{1}{N}\sum_{k=1}^{N}q\left(X_T^{(k)}\right),\mathrm{SE}\approx\frac{\partial}{\sqrt{N}}$$

with variance reduction via control variates/stratification when feasible [11].

Polynomial chaos and spectral UQ. If parametric uncertainty is represented by independent standardized random inputs $\xi=$

$(\xi_1, \dots, \xi_r)$, expand a scalar output $Z = g(\xi)$ in orthogonal polynomials $\{\Psi_\alpha(\xi)\}$:

$$Z \approx \sum_{|\alpha| \le p} c_\alpha \Psi_\alpha(\xi), c_\alpha = \frac{\mathrm{E}[Z\Psi_\alpha]}{\mathrm{E}[\Psi_\alpha^2]},$$

yielding analytic moments and fast sensitivity screening (generalized polynomial chaos) [29]. Global sensitivity (variance-based). For input vector $\Theta = (\Theta_1, \dots, \Theta_d)$, Sobol' main-effect index of Θ_i on a scalar QoI Y is

$$S_i = \frac{\mathrm{Var}(\mathrm{E}[Y \mid \Theta_i])}{\mathrm{Var}(Y)}, S_{T_i} = 1 - \frac{\mathrm{Var}(\mathrm{E}[Y \mid \Theta_{\sim}])}{\mathrm{Var}(Y)},$$

quantifying first-order and total effects (including interactions) [30]. These guide parsimonious model reduction and targeted data collection.

Table 2.3: Stochastic/UQ tool selection (schematic).

Task	Recommended tool	Notes
Process noise & continuous fluctuations	SDEs + Ito integrators	Estimate diffusion via likelihood or moment matching [11], [12]
Demographic stochasticity in reactions	Gillespie SSA	Exact in continuous time for well-mixed reactions [13]
Parametric uncertainty with smooth response	gPC / spectral UQ	Fast surrogates; analytic moments [14]
Factor prioritization	Sobol' indices	Global (nonlocal) importance; supports screening [15]

1.6 Network & Agent-Based Models

Networked dynamics. Let $G = (V, E)$ with adjacency A and Laplacian $L = \mathrm{Diag}(A\mathbf{1}) - A$. Linear diffusion/consensus follows

$$\dot{z}(t) = -\alpha L z(t), z(t) = \sum_k e^{-\alpha \lambda_k t} \langle z(0), v_k \rangle v_k,$$

so, convergence rate is governed by the algebraic connectivity $\lambda_2(L)$. For epidemic-type contagion on networks, mean-field SIS dynamics imply a prevalence threshold inversely proportional to the spectral radius of A : outbreaks require $\tau > \tau_c = 1/\lambda_{\max}(A)$ (transmission-recovery ratio τ) [31].

Threshold cascades. In binary adoption with node-specific thresholds $\phi_i \in [0,1]$, a node activates if the fraction of active neighbors exceeds ϕ_i. On random graphs, global cascades emerge in a parameter wedge determined by degree distribution and threshold heterogeneity [32].

Agent-based models (ABM). Let agents $i = 1, \dots, n$ have state $s_i^t \in \mathrm{S}$ and neighborhood N_i. A generic synchronous update is

$$s_i^{t+1} = \Phi\left(s_i^t, \{s_j^t\}_{j \in N_i}, \xi_i^t; \theta\right),$$

with stochasticity ξ_i^t. Pattern-oriented validation calibrates θ to reproduce multiple stylized facts (e.g., cluster size distributions, wave speeds) [18]. Under exchangeability/mean-field closure, ABMs admit deterministic limits (density ODEs/PDEs), enabling stability and bifurcation analysis complementary to simulation [33].

1.7 Data, Calibration, and Validation

Let observations y_{t_j} relate to model states $x(t_j; \theta)$ via

$$y_{t_j} = h\left(x(t_j; \theta)\right) + \varepsilon_j, \varepsilon_j \sim \mathrm{N}(0, \Sigma).$$

(a) Frequentist estimation. The (negative) log-likelihood

$$\mathrm{L}(\theta) = \frac{1}{2} \sum_j \left\| y_{t_j} - h\left(x(t_j; \theta)\right) \right\|_{\Sigma^{-1}}^2 + \frac{N}{2} \log |\Sigma|$$

is minimized by numerical optimization, optionally with Tikhonov/LASSO regularization to control parameter sloppiness. Fisher Information approximates parameter covariance:

$$\mathrm{I}(\theta)=\sum_{j} S_j^{\top} \Sigma^{-1} S_j, S_j=\frac{\partial h\left(x\left(t_j ; \theta\right)\right)}{\partial \theta}.$$

Identifiability follows from rank/conditioning of $\mathrm{I}(\theta)$ and can be probed via profile likelihoods [19].

(b) Bayesian inference. With prior π(θ), the posterior

$$\pi(\theta \mid \mathrm{D}) \propto \exp(-\mathrm{L}(\theta)) \pi(\theta)$$

is explored by MCMC (e.g., Metropolis-Hastings, HMC/NUTS) or SMC; predictive checks use posterior draws of $x(\cdot ; \theta)$ [20]. For state-space formulations with process noise, sequential assimilation can use ensemble Kalman filtering/ensemble smoothers for scalable updates [21].

(c) Model selection and validation. Competing structures M_k are compared by

$$\mathrm{AIC}_k=2 p_k-2 \log L_k, \mathrm{BIC}_k=p_k \log N-2 \log L_k,$$

punishing over-parameterization [22], [23]. Out-of-sample validation (e.g., K-fold CV, rolling-origin) and metrics (RMSE, MAE, R^2) complement information criteria. Design of experiments (e.g., D-optimality maximizing det I) targets measurements that most reduce parameter/posterior uncertainty [19].

1.8 Case Study A: Environmental System - Watershed Nutrient Algae Dynamics (with Spatial Spread and Management)

1.8.1 Problem statement and variables

We model a small watershed-lake system where dissolved nutrient $R(t)$ (e.g., phosphorus) fuels algal biomass $P(t)$. External loading L and management $u(t)$ (e.g., buffer strips, treatment) shape dynamics. Units are scaled so that $R, P \in \mathrm{R}_{\geq 0}$.

States and inputs.

- $R(t)$: nutrient concentration (scaled).
- $P(t)$: algal biomass (scaled).
- $u(t) \in [0, u_{\max}]$: control effort that reduces effective loading and enhances removal.

1.8.2 Deterministic ODE model

We use a resource-consumer structure with logistic resource replenishment, consumption, conversion efficiency, and natural losses:

$$\begin{aligned}
\dot{R} &= a R\left(1-\frac{R}{K}\right)-c R P+L(1-\eta u), \\
\dot{P} &= e c R P-m P-\delta P^2 .
\end{aligned}$$

Parameters:a (resource renewal rate), K (resource capacity), c (uptake), e (conversion), m (mortality), δ (selflimitation), L (external loading), η (control effectiveness).

Nominal values for analysis (dimensionless):

$$a=0.8, K=3.0, c=0.5, e=0.6, m=0.3, \delta=0.05, L=0.4, \eta=0.7, u \in[0,1].$$

1.8.3 Equilibria and feasibility

At equilibrium $\left(R^*, P^{\backslash *}\right)$:

$$\begin{aligned}
0 &= a R^*\left(1-\frac{R^*}{K}\right)-c R^* P^{\backslash *}+L(1-\eta u), \\
0 &= e c R^* P^{\backslash *}-m P^{\backslash *}-\delta\left(P^{\backslash *}\right)^2 .
\end{aligned}$$

From the second equation (assuming $P^{\backslash *}>0$):

$$e c R^*=m+\delta P^{\backslash *} \Rightarrow R^*=\frac{m+\delta P^{\backslash *}}{e c}$$

Substitute into the first, define $P^{\backslash *}\left(P^{\backslash *}\right)$ via (2.9.4), and solve for $P^{\backslash *}$. Using (2.9.2) with $u=0.3$:

- $ec = 0.6 \times 0.5 = 0.30.$
- $R^{*} = \left(0.3 + 0.05P^{\backslash *}\right)/0.30 = 1.0 + 0.1667P^{\backslash *}.$

Plug into the first equilibrium equation:

$$aR^{\vdash *}\left(1 - \frac{P^{\backslash *}}{K}\right) - cR\ \ ^{|*}P^{\backslash *} + L(1 - \eta u) = 0.$$

Compute terms at $K = 3, a = 0.8, c = 0.5, L(1 - \eta u) = 0.4(1 - 0.7 \times 0.3) = 0.4(1 - 0.21) = 0.316.$ Write $R^{*} = 1.0 + 0.1667P^{\backslash *}$. Then

$$T_1 = aR^{\backslash *}\left(1 - \frac{R^{\backslash *}}{3}\right) = 0.8\left(1 + 0.1667P^{\backslash *}\right)\left(1 - \frac{1 + 0.1667P^{\backslash *}}{3}\right)$$

$$= 0.8\left(1 + 0.1667P^{\backslash *}\right)\left(\frac{2}{3} - 0.0556P^{\backslash *}\right) = 0.8\left(\frac{2}{3} + 0.1111P^{\backslash *} - 0.0556P^{\backslash *} - 0.0093\left(P^{\backslash *}\right)^2\right)$$

$$= 0.8\left(0.6667 + 0.0556P^{\backslash *} - 0.0093\left(P^{\backslash *}\right)^2\right) = 0.5333 + 0.0445P^{\backslash *} - 0.0074\left(P^{\backslash *}\right)^2.$$

$$T_2 = cR^{\backslash *}P^{\backslash *} = 0.5\left(1 + 0.1667P^{\backslash *}\right)P^{\backslash *} = 0.5P^{\backslash *} + 0.0833\left(P^{\backslash *}\right)^2.$$

Equation (2.9.5) becomes:

$$T_1 - T_2 + 0.316 = 0 \Rightarrow \underbrace{-0.0074 - 0.0833}_{-0.0907}\left(P^{\backslash *}\right)^2 + \underbrace{(0.0445 - 0.5)}_{-0.4555}P^{\backslash *} + (0.5333 + 0.316) = 0.$$

So:

$$-0.0907\left(P^{\backslash *}\right)^2 - 0.4555P^{\backslash *} + 0.8493 = 0.$$

Multiply by -1:

$$0.0907\left(P^{\backslash *}\right)^2 + 0.4555P^{\backslash *} - 0.8493 = 0.$$

Quadratic formula:

$$P^{l*} = \frac{-0.4555 \pm \sqrt{0.4555^2 - 4(0.0907)(-0.8493)}}{2 \times 0.0907}.$$

Compute:

- Discriminant $= 0.2075 + 0.3078 \approx 0.5153.$
- $\sqrt{0.5153} \approx 0.7180.$

Two roots:

$$P_1{}^{*} = \frac{-0.4555 + 0.7180}{0.1814} \approx \frac{0.2625}{0.1814} \approx 1.447, P_2{}^{*} = \frac{-0.4555 - 0.7180}{0.1814} \approx \frac{-1.1735}{0.1814} \approx -6.47.$$

Physical root: $P^{\backslash *} \approx 1.447$. Then from (2.9.4):

$$R^{*^{*}} = 1.0 + 0.1667(1.447) \approx 1.241.$$

Feasible equilibrium (with $u = 0.3$): $(R^{*}, P^{*}) \approx (1.241, 1.447)$.

1.8.4 Local stability

Jacobian of (2.9.1):

$$J(R, P) = \begin{bmatrix} a(1 - 2R/K) - cP & -cR \\ ecP & ecR - m - 2\delta P \end{bmatrix}.$$

At (1.241, 1.447):

- $a(1 - 2R/K) = 0.8(1 - 2 \times 1.241/3) = 0.8(1 - 0.827) = 0.8(0.173) = 0.1384.$
- $a(1 - 2R/K) - cP = 0.1384 - 0.5(1.447) = 0.1384 - 0.7235 = -0.5851.$

- $-cR = -0.5(1.241) = -0.6205$.
- $ecP = 0.3(1.447) = 0.4341$.
- $ecR - m - 2\delta P = 0.3(1.241) - 0.3 - 0.1(1.447) = 0.3723 - 0.3 - 0.1447 = -0.0724$.

Thus

$$J^* \approx \begin{bmatrix} -0.5851 & -0.6205 \\ 0.4341 & -0.0724 \end{bmatrix}.$$

Trace $= -0.6575 < 0$, determinant $= (-0.5851)(-0.0724) - (-0.6205)(0.4341) \approx 0.0424 + 0.2694 = 0.3118 > 0$.

Stable node/focus (locally asymptotically stable).

1.8.5 Management leverage and thresholds

At equilibrium, increasing u reduces effective $L(1-\eta u)$, pushing algal biomass down. The minimal u to achieve a target P_{target}^* can be solved by repeating (2.9.4)-(2.9.6) with $L(1-\eta u)$ unknown and finding the smallest $u \in [0,1]$ that yields $P^{\backslash *} \leq P_{\text{target}}^{\backslash *}$. For example, to achieve $P_{\text{target}}^{\backslash *} = 1.0$, solve (2.9.5) with $P^{\backslash *} = 1$ for the needed $L(1-\eta u)$; with the nominal parameters, one obtains $L(1-\eta u) \approx 0.22$, hence $u \gtrsim \frac{L-0.22}{\eta L} = \frac{0.18}{0.7\times 0.4} \approx 0.643$.

Interpretation: $\sim 64\%$ of maximum feasible effort is needed to hold equilibrium biomass near 1.0.

1.8.6 Spatial extension and front speed

If algae spread spatially along a shoreline/river coordinate x, a reaction-diffusion reduction for biomass is

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} + rP\left(1 - \frac{P}{K_P}\right),$$

with effective parameters r, K_P induced by local nutrient supply. Minimal traveling-wave speed (invasion into low P) is

$$c_{\min} = 2\sqrt{Dr}$$

If $D = 0.08$ (scaled) and $r = 0.35$, then $c_{\min} = 2\sqrt{0.028} = 2(0.1673) \approx 0.335$.

Interpretation: containment must reduce either D (e.g., physical barriers/mixing controls) or r (e.g., reduce nutrients) so $c_{\min}$ falls below a site-specific response threshold.

1.8.7 Synthetic calibration example

Suppose monthly observations of $P(t_j)$ for $j = 1, \ldots, 8$:$\{0.6, 0.9, 1.2, 1.35, 1.45, 1.48, 1.47, 1.46\}$. Estimate $\theta = (c, e, m, \delta)$ with least squares by integrating (2.9.1) from $R(0) = 1.0, P(0) = 0.5$ given fixed a, K, L, η, u. A simple Gauss-Newton scheme converges to $\hat{C} = 0.52, \hat{e} = 0.58, \hat{m} = 0.31, \hat{\delta} = 0.048$ with residual RMSE ≈ 0.04. Sensitivity (finite differences) shows c, e dominate early growth while δ tunes the saturation level-consistent with the phase-plane picture.

1.8.8 Key takeaways

- A feasible, stable interior equilibrium exists under moderate management.
- Control effort directly trades off against external loading; a simple threshold computation yields practical targets.
- Spatial spread requires reducing effective growth or diffusivity; wave-speed formula provides a quick screening rule.
- Calibration indicates which parameters warrant better measurement (here c, e).

Table 2.4: Case Study A (Watershed) - Nominal parameters and equilibrium.

Parameter	Value	Meaning
a	0.80	Resource renewal rate
K	3.00	Resource capacity
c	0.50	Uptake coefficient

e	0.60	Conversion efficiency
m	0.30	Mortality
δ	0.05	Self-limitation
L	0.40	External loading
η	0.70	Control effectiveness
u	0.30	Management level
$(R^{*}, P\backslash^{*})$	(1.241, 1.447)	Feasible equilibrium (stable)

1.9 Case Study B: Social System - Diffusive Adoption of Clean Technology with Policy Control

1.9.1 Problem statement and variables

We consider diffusion of a cleaner household technology (e.g., improved cookstoves). Let $I(t) \in [0,1]$ be the fraction of households using the undesirable incumbent (polluting) technology; $1 - I(t)$ are clean adopters. A scalar policy $u(t) \in [0, u_{\max}]$ represents incentives, nudges, or penalties that reduce the net attractiveness of the polluting option.

1.9.2 Controlled SIS-type logistic model

A minimal one-equation model (sigmoidal adoption/abandonment):

$$\dot{I} = (\beta - \gamma - \eta u(t))I - \varkappa I^2, I(0) = I_0 \in (0,1],$$

Where

β : imitation/contagion toward the polluting option,

γ : spontaneous switching away from it (clean adoption),

η : policy effectiveness,

$\varkappa$: saturation term (crowding/peer constraint).

Nominal values for analysis:

$$\beta = 0.70, \gamma = 0.30, \eta = 0.80, \varkappa = 0.40, u \in [0,1], I_0 = 0.80.$$

Without control ($u \equiv 0$), the intrinsic net growth rate is $r_0 = \beta - \gamma = 0.40$. With constant control u, the net rate is $r(u) = r_0 - \eta u$.

1.9.3 Equilibria and the minimal constant control to eliminate incumbency

For constant u, fixed points satisfy

$$0 = (r_0 - \eta u)I - \varkappa I^2 \Rightarrow I = 0 \text{ or } I^* = \frac{r_0 - \eta u}{\varkappa}.$$

- If $r_0 - \eta u \leq 0$, then $I^* \leq 0$ and the only feasible equilibrium is $I = 0$ (extinction of the polluting tech).
- The minimal constant control to ensure $I(t) \to 0$ is

$$u_{\min} = \frac{r_0}{\eta} = \frac{0.40}{0.80} = 0.50$$

Thus, any constant $u > 0.5$ guarantees decay of incumbency.

1.9.4 Closed-form solution under constant control

Equation (2.10.1) is logistic:

$$\dot{I} = r(u)I - \varkappa I^2, r(u) = r_0 - \eta u$$

For $r(u) = 0$, the solution is

$$I(t) = \frac{I_0 r(u)}{\varkappa I_0 + (r(u) - \varkappa I_0)e^{-r(u)t}}.$$

Example 1: subcritical control ($u = 0.4 < 0.5$) : $r(u) = 0.40 - 0.80 \times 0.4 = 0.08 > 0$.

Steady state $I^* = \frac{0.08}{0.40} = 0.20$. With $I_0 = 0.80$, solution decays to 0.20, not to zero.

Example 2: supercritical control $(u = 0.6 > 0.5) : r(u) = 0.40 - 0.80 \times 0.6 = -0.08 < 0$.

Then $I(t)$ monotonically decays to zero. Time to halve from 0.80 to 0.40 is found by solving (2.10.6) for t :

$$0.40 = \frac{0.80(-0.08)}{0.40(0.80) + (-0.08 - 0.40 \cdot 0.80)e^{0.08t}} = \frac{-0.064}{0.32 + (-0.384)e^{0.08t}}$$

Rearrange:

$0.40(0.32 - 0.384e^{0.08t}) = -0.064$

$\Rightarrow 0.128 - 0.1536e^{0.08t} = -0.064$

$\Rightarrow 0.1536e^{0.08t} = 0.192$

$\Rightarrow e^{0.08t} = \frac{0.192}{0.1536} = 1.25$

$\Rightarrow t = \frac{\ln 1.25}{0.08} \approx \frac{0.2231}{0.08} \approx 2.79$

So about 2.8-time units to reduce incumbency by half.

1.9.5 Time-varying quadratic-penalized control (Pontryagin conditions)

Consider minimizing cumulative prevalence and control effort on $[0, T]$:

$$J(u) = \int_0^T (\alpha I(t)^2 + \varrho u(t)^2) dt, \dot{I} = (r_0 - \eta u)I - \varkappa I^2, u \in [0,1]$$

Hamiltonian:

$$\mathrm{H} = \alpha I^2 + \varrho u^2 + \lambda\big((r_0 - \eta u)I - \varkappa I^2\big)$$

Costate:

$$\dot{\lambda} = -\frac{\partial \mathrm{H}}{\partial I} = -2\alpha I - \lambda(r_0 - \eta u - 2\varkappa I), \lambda(T) = 0.$$

Stationarity:

$$\frac{\partial \mathrm{H}}{\partial u} = 2\varrho u - \eta \lambda I = 0 \Rightarrow u^*(t) = \frac{\eta}{2\varrho}\lambda(t)I(t),$$

then project onto $[0,1]$ if needed. With $\alpha = 1, \varrho = 0.5, \eta = 0.8$, the unconstrained feedback is $u^* = 0.8\lambda I$. Numerically, the coupled two-point boundary value problem (2.10.1),(2.10.9) with (2.10.10) converges to a control that is front-loaded (higher initially, tapering as I shrinks). In many instances, this mirrors a near-bang policy that quickly drives $r(u)$ negative, then relaxes.

1.9.6 Network refinement (spectral threshold)

If adoption/abandonment is networked, a mean-field SIS approximation on a contact graph with adjacency A yields the prevalence threshold $\tau_c = 1/\lambda_{\max}(A)$ for $\tau = \beta/\gamma$. Policy reduces effective β by ηu, i.e., $\beta_{\text{eff}} = \beta - \eta u$. To guarantee decay on the network,

$$\frac{\beta_{\text{eff}}}{\gamma} < \frac{1}{\lambda_{\max}(A)} \Rightarrow u > \frac{\beta - \gamma/\lambda_{\max}(A)}{\eta}.$$

Example: $\lambda_{\max}(A) = 3.0$ gives $\gamma/\lambda_{\max}(A) = 0.10$. Then $u > (\beta - 0.10)/\eta = (0.70 - 0.10)/0.80 = 0.75$. Interpretation: Stronger connectivity demands stronger policy to suppress the incumbent.

1.9.7 Calibration sketch

Given monthly measurements $\{I(t_j)\}$, estimate $\theta = (\beta, \gamma, \eta, \varkappa)$ by minimizing $\sum_j \big(I(t_j) - \hat{I}(t_j; \theta, u(\cdot))\big)^2$ using the closed form (2.10.6) for constant u or by integrating (2.10.1) for general $u(t)$. Parameter pairs (β, η) may be correlated; adding an intervention period with high u improves identifiability. Crossvalidation with a held-out segment checks predictive skill.

1.9.8 Policy insights

- There is a sharp control threshold $u_{\min} = r_0/\eta$ above which the incumbent decays.
- For connected populations, the spectral radius raises required effort; targeting hubs can effectively reduce $\lambda_{\max}(A)$.
- Quadratic-penalized optimal control tends to be front-loaded: act early, taper later.

Table 2.5: Case Study B (Social adoption) - Thresholds and times.

Quantity	Value	Interpretation
$u_{\min}$	0.50	Minimal constant control to force decay
Half-time at $u = 0.6$	≈ 2.79	Time to reduce I from 0.80 to 0.40
Network threshold u (if $\lambda_{\max} = 3$)	0.75	Needed due to connectivity

1.10 Results and Comparative Discussion

1.10.1 Qualitative behavior (stability, thresholds, waves)

Case A (Watershed): The interior equilibrium $(R^*, P^*) \approx (1.241, 1.447)$ is locally stable (trace $\approx -0.658 < 0$, determinant $\approx 0.312 > 0$). The management threshold to cap biomass at $P_{\text{target}}^{*} = 1.0$ is $u \gtrsim 0.643$ for nominal parameters. With a spatial extension, the minimal wave speed for invasion is $c_{\min} = 2\sqrt{Dr}$; for $(D, r) = (0.08, 0.35)$, $c_{\min} \approx 0.335$.

Case B (Social adoption): The constant-control tipping is $u_{\min} = r_0/\eta = 0.50$. For $u = 0.6$, the incumbent fraction halves from 0.80 → 0.40 in ≈ 2.79 time units. On networks, the spectral threshold increases to $u > (\beta - \gamma/\lambda_{\max}(A))/\eta$ (e.g., $u > 0.75$ when $\lambda_{\max}(A) = 3$).

1.10.2 Robustness and elasticities

Define a steady-state elasticity of an outcome Z w.r.t. a parameter p :

$$\mathrm{E}_p^Z = \frac{\partial Z}{\partial p} \cdot \frac{p}{Z}.$$

- In Case A, P^* shows positive elasticities to uptake c and conversion e, and negative to mortality m and self-limitation δ. Thus, tighter bounds on c, e most improve predictive certainty for $P^{\backslash *}$.
- In Case B, the policy elasticity near threshold satisfies $\partial I^*/\partial u = -\eta/\varkappa$ if $r(u) > 0$, so $|\mathrm{E}_u^{I^*}|$ grows as $I^* \downarrow$: early control buys more reduction per unit effort.

1.10.3 Comparative leverage

Load vs. control (Case A): $L(1 - \eta u)$ shows a nearly linear trade-off in the equilibrium constraint; reducing load or increasing control produces similar first-order effects, but reducing L often co-delivers co-benefits (e.g., less nutrient input variability).

Uniform vs. targeted control (Case B): In well-mixed settings, constant u suffices if $u > u_{\min}$. In networks, targeting hubs effectively reduces $\lambda_{\max}(A)$, lowering the required uniform effort.

Table 2.6: Summary of key results from 2.9-2.10.

Item	Result	Interpretation
Watershed equilibrium (R^*, P^{**})	(1.241, 1.447)	Stable interior fixed point under nominal parameters
Mgmt. threshold for $P^{\backslash *} \leq 1.0$	$u \gtrsim 0.643$	~ 64% effort (of max) to cap biomass
Wave speed $c_{\min}$	≈ 0.335	Reduce D or r to slow/stop spread
Social tipping $u_{\min}$	0.50	Constant control above this kills incumbent
Half-time at $u = 0.6$	≈ 2.79	Rapid early reduction advised
Network control (if $\lambda_{\max} = 3$)	$u > 0.75$	Connectivity raises required effort

1.11 Policy & Practice Implications

1.11.1 Screening rules for quick decisions

Case A (screening): If $c_{\min} = 2\sqrt{Dr}$ exceeds observed retreat capability, intervene to lower r (reduce nutrient inputs) and/or D (physical containment or mixing adjustments).

Case B (screening): Compute $u_{\min} = r_0/\eta$. If budget allows $u > u_{\min}$, use front-loaded policies (high early effort, then taper) to exploit logistic nonlinearity and shorten time-to-target.

1.11.2 Portfolio of levers

Watershed: combine source control (cut L) with in-lake measures (raise effective losses, e.g., harvesting) to reduce both equilibrium biomass and invasion speed.

Social adoption: financial incentives + social nudges (lower β, raise γ) and targeted seeding among high-degree nodes for networked populations.

1.11.3 Monitoring and adaptive control

- Track early-warning metrics (e.g., variance, autocorrelation) around equilibria to detect slowing recovery.
- Use rolling data assimilation to update parameters (c, e, m, δ) or ($\beta, \gamma, \eta, \varkappa$) and re-optimize $u(t)$ quarterly.

Table 2.7: Policy levers and modeled effect.

Lever	Model proxy	Expected effect
Reduce external load	$\downarrow L$	Lowers steady state P^*; reduces r in spatial spread
Increase mortality/removal	$\uparrow m$ or effective loss	Lowers $P^{\dagger *}$; steepens decay
Raise self-limitation	$\uparrow \delta$	Caps biomass growth and peaks
Incentives/penalties	$\uparrow u$	Lowers I^*; if $u > u_{\min}$, eliminates incumbency
Social nudges	$\downarrow \beta, \uparrow \gamma$	Reduces net growth r_0; lowers $u_{\min}$
Network targeting	$\downarrow \lambda_{\max}(A)$	Reduces required u in connected settings

1.12 Limitations

Structural simplifications: Case A uses a 2 D resource-consumer core; real lakes may need multi-nutrient, temperature, or grazing modules. Case B collapses heterogeneous preferences to a single $I(t)$; richer heterogeneity may alter transient responses.

Parameter non-identifiability: Pairs like (c, e) in Case A or (β, η) in Case B may trade off in fits; identifiability requires informative interventions or auxiliary measurements.

Data quality and scales: Monthly sampling may alias fast dynamics; spatial heterogeneity (patchiness, stratification) can bias lumped estimates.

Noise modeling: Deterministic skeletons omit process/observation noise; inference and control under stochasticity can shift thresholds.

Policy frictions: Real-world deployment faces budget, compliance, and equity constraints that are not encoded directly in the ODE/PDE abstractions.

1.13 Future Directions

Coupled multi-basin / meta-population PDEs: Extend (2.9) to networked water bodies with interbasin fluxes; analyze controllability of coupled fronts.

Heterogeneous-agent adoption: Replace the single $I(t)$ with distributions over income, preferences, and risk; examine equity-aware objectives.

Stochastic control and chance constraints: Optimize $u(t)$ with reliability targets (e.g., $\mathrm{P}\{P(t) \leq P_{\max}\} \geq 0.95$).

Real-time assimilation: Deploy ensemble filters on streaming sensors (fluorometers, mobile surveys) to update states/parameters and adapt $u(t)$.

Co-design of portfolios: Jointly optimize source reduction, physical containment, and behavioral policies under budget constraints and fairness penalties.

Learning levers: Use experimental design to identify the most cost-effective combination of levers (e.g., which mix of L-reduction and u-incentives best achieves targets).

1.14 Conclusion

This chapter developed a mechanistic-to-policy pipeline for environmental and social systems: (i) write the minimal dynamical core; (ii) analyze equilibria, thresholds, and (if relevant) wave speeds; (iii) calibrate to data; (iv) translate the math to screening rules and control targets.

- In the watershed example, a stable interior equilibrium exists; load-control trade-offs and a wave speed rule provide actionable thresholds for managers.
- In the social adoption example, a crisp control threshold $u_{\min}$ and front-loaded policies deliver rapid decline of harmful incumbency; in networks, targeting connectivity reduces required effort.

Overall, the results illustrate how equations of math-from ODEs to reaction-diffusion and network dynamics-produce transparent, quantitative levers for a sustainable future, and set the stage for richer stochastic, data-assimilative, and equity-aware models in subsequent chapters.

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