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On Characterizations of Erlang Truncated Exponential Distribution through Generalized Order Statistics with Applications to Statistical Prediction Problem

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Abstract: Erlang Truncated Exponential Distributions are characterized by distributional properties of generalized order statistics. These characterizations include known results for ordinary order statistics based on two non-adjacent m-generalized order statistics coming from two independent Erlang truncated exponential distributions. Using this method and compared with an efficient recent method given by [33], three examples of real lifetime data-sets are analyzed by that deals with non-random samples. Such type of examples predicts the accumulative new cases per million foe infection of the new COVID-19. Corollaries for Pareto and power function distributions are also derived.

Keywords: Generalized order statistics, characterization of distributions, reliability characteristics, Erlang truncated exponential, random translation.

1 Introduction

Various characterizations of Erlang truncated exponential distributions based on distributional properties of order statistics are found in the literature. Let $X_{1,n}, X_{2,n}, \ldots X_{n,n}$ denote the order statistics of a identically independent distributed *(i.i.d)* random variables $X_1, X_2, \ldots X_n, n \geq 2$ each with distribution function $F_X(x)$. The concept of generalized order statistics was introduced by [16] as a unified approach to order statistics and record values. Furthermore, a variety of other models of ordered random variables are contained in this concept. For a detailed discussion of several of these models, such as sequential order statistics, k^{th} record values, and Pfeifer's record model, we refer to ([16], Ch. I).

In this paper we present characterizations of Erlang truncated exponential distributions DF $exp(\beta\alpha_\lambda)$ with mean $\frac{1}{(\beta\alpha_\lambda)}, \beta > 0, \alpha > 0, \lambda > 0$ via distributional properties of generalized order statistics including the known results for ordinary order statistics.

An n-component system that fails if and only if at least k of the n-components fail is called a k-out-of-n:F system. The lifetime of such a system could be represented as $X_{k:n}$. The k-out-of-n:F system structure is a very popular type of redundancy in fault tolerant systems with wide applications in industrial and military systems. For two different systems say a k-out-of-n:F system and (k+1)-out-of-n:F system, the engineer may be interested in the additional lifetime $X_{k+1:n} - X_{k:n}$ for the system design and the cost purpose. Due to the complicated distribution form, one may provide a sharp bound on the survival function. Therefore, the researchers have to rely on a characterization of the assumed distribution and check if the corresponding conditions are satisfied. Classical results in characterizations can be found in [7,34,35]. Different results of characterization and its applications in terms of generalized order statistics (GOSs) and dual generalized order statistics (DGOSs) are derived by many authors. Among these authors are [1,2,3,4,5,6,8,9,10,11,12,13,14,15,17,18,19,20,21,22,23,25,26,27,28,29].

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2 Generalized order statistics

Let $X(1,n,m,k), X(2,n,m,k), ..., X(n,n,m,k)$ be generalized order statistics based on the absolutely continuous distribution function $F_X(x)$ with density function $f_X(x)$, which means that the marginal density function of the r^{th} generalized order statistic and dual generalized order statistic are given by

$$f^{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[1-F_X(x)]^{\Psi_r^{(n)}-1} f_X(x) g_m^{r-1}(F(x)) \tag{1}$$

and

$$f^{X^*(r,n,\tilde{m},k)}(x) = \frac{C_{r-1}}{(r-1)!} F_X{}^{\Psi_r^{(n)}-1} f_X(x) g_m^{r-1}(\overline{F}(x)) \tag{2}$$

(see [16], p.64, and the density function of the spacing)

$$W_{(r-1,r,n)} = X_{(r,n,m,k)} - X_{(r-1,n,m,k)}, \quad 2 \le r \le n$$

has the following representation:

$$f^{X(r,n,m,k)-X(r-1,n,m,k)}(y) = \frac{C_{r-1}}{(r-2)!}\int_{-\infty}^{\infty}[1-F_X(x)]^m f_X(x) g_m^{r-2}(F(x))[1-F_X(x+y)]^{\Psi_r^{(n)}-1} f_X(x+y)dx \tag{3}$$

with

$$C_{r-1} = \prod_{j=1}^{r} \psi_j, \quad 1 \le r \le n \tag{4}$$

$$h_m(x) = \int (1-x)^m dx = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \ne -1 \\ \log\frac{1}{1-x}, & m = -1 \end{cases} \tag{5}$$

$g_m(x) = h_m(x) - h_m(0), x \in [0,1)$ (see [16], p. 69).
Generalizing [31] result it is shown in ([16], p.81) that the normalized spacings

$$D(1,n,m,k) = \psi_1 X(1,n,m,k) \tag{6}$$

$$D(r,n,m,k) = \psi_r[X(r,n,m,k) - X(r-1,n,m,k)], \quad 2 \le r \le n \tag{7}$$

based on an exponential distribution with parameters $\beta > 0$, $\alpha > 0$, $\lambda > 0$ are independent and identically distributed according to Erlang truncated Exp$((\beta\alpha_\lambda))$.

The cumulative distribution function (CDF) $F_X(x)$ and probability density function (PDF) $f_X(x)$ of the Extended Erlang-Truncated Exponential (EETE) distribution are given by:

$$F_X(x) = \left[1 - e^{-\beta(\alpha_\lambda)x}\right]^{\alpha}, \quad 0 \le x < \infty; \quad \alpha, \beta, \lambda > 0 \tag{8}$$

and

$$f_X(x) = \alpha\beta(\alpha_\lambda)e^{-\beta(\alpha_\lambda)x}\left[1 - e^{-\beta(\alpha_\lambda)x}\right]^{\alpha-1}, \quad 0 \le x < \infty, \quad \alpha, \beta, \lambda > 0 \tag{9}$$

where α and β are the shape parameters and λ is the scale parameter. The Extended Erlang-Truncated Exponential (EETE) distribution reduces to Erlang-Truncated Exponential (ETE) when $\alpha = 1$.

In Section 4, we reveal some characterization properties for the Erlang truncated exponential distribution based on two nonadjacent generalized order statistic ($\tilde{m}-$GOSs) (consequently $\tilde{m}$ dual generalized order statistic ($\tilde{m}-$DGOSs) from two independent Erlang truncated exponential distributions. The cumulative distribution function $F_X(x)$ and probability density function $f_X(x)$ of the Erlang truncated exponential distribution are given by:

$$F_X(x) = \left[1 - e^{-\beta(\alpha_\lambda)x}\right], \quad 0 \le x < \infty, \quad \beta > 0, \alpha > 0, \lambda > 0 \tag{10}$$

where $\alpha_\lambda = (1 - e^{-\lambda})$ and

$$f_X(x) = \beta\alpha_\lambda e^{-\beta(\alpha_\lambda)x}, \quad 0 \le x < \infty, \quad \beta > 0, \alpha > 0, \lambda > 0 \tag{11}$$

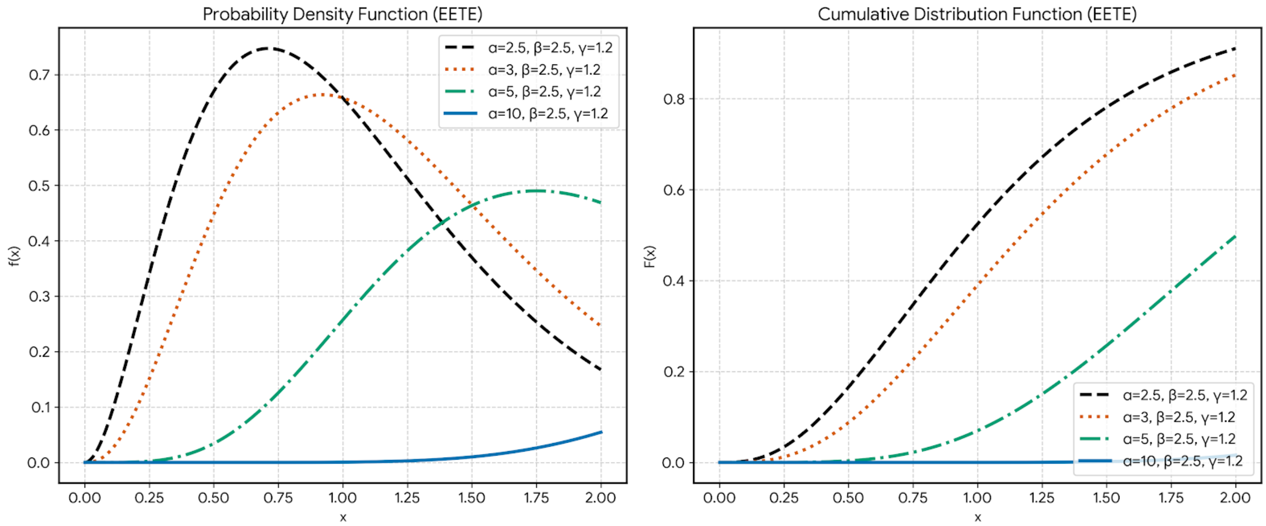

Fig. 1: Possible shapes of the probability density function $f(x)$ (left) and cumulative distribution function $F(x)$ (right) of the Extended Erlang-Truncated Exponential (EETE) distribution for fixed parameter values of β and λ

The cumulative distribution function $F_X(x)$ of the Pareto distribution is

$$F_X(x) = \left[1 - x^{-\beta(\alpha_\lambda)}\right], \quad 1 < x < \infty, \quad \beta > 0, \alpha > 0, \lambda > 0 \tag{12}$$

and the probability density function $f_X(x)$ of the Pareto distribution is

$$f_X(x) = -\beta\alpha_\lambda x^{-(\beta(\alpha_\lambda)+1)}, \quad 0 \le x < \infty, \quad \beta > 0, \alpha > 0, \lambda > 0 \tag{13}$$

The cumulative distribution function $F_X(x)$ of the Pareto distribution is

$$F_X(x) = x^{\beta(\alpha_\lambda)}, \quad 0 < x < 1, \quad \beta > 0, \alpha > 0, \lambda > 0 \tag{14}$$

and the probability density function $f_X(x)$ of the Power distribution is

$$f_X(x) = \beta\alpha_\lambda x^{(\beta(\alpha_\lambda)-1)}, \quad 0 \le x < \infty, \quad \beta > 0, \alpha > 0, \lambda > 0 \tag{15}$$

3 Reliability Characteristics

Ensuring the reliability of systems, objects, and processes is one of the main goals in their creation and further operation. Redundancy serves this aim, and a k-out-of-n:F model is a very popular configuration for it. This is a model of a system that consists of n components in parallel that fails when at least k of them fail. Hereinafter, we will use this notation omitting the symbol “F”. Due to the wide range of practical applications of k-out-of-n systems, many papers have been devoted to their study. The reliability function $R(x)$ is an important tool for characterizing life phenomenon. $R(x)$ is analytically expressed as $R(x) = 1 - F(x)$. Under certain predefined conditions, the reliability function $R(x)$ gives the probability that a system will operate without failure until a specified time x. The reliability function of the Extended Erlang-Truncated Exponential distribution is given by

$$R(x) = 1 - \left(1 - e^{-\beta(\alpha_\lambda)x}\right)^{\alpha}, \quad 0 \le x < \infty, \quad \alpha, \beta, \lambda > 0 \tag{16}$$

Another important reliability characteristic is the failure rate function. The failure rate function gives the probability of failure for a system that has survived up to time x. The failure rate function $h(x)$ is mathematically expressed as

$h(x) = f(x)/R(x)$. The failure rate function of the Extended Erlang-Truncated Exponential (EETE) distribution is given by:

$$h(x) = \frac{\alpha\beta(\alpha_\lambda)e^{-\beta(\alpha_\lambda)x}\left[1-e^{-\beta(\alpha_\lambda)x}\right]^{\alpha-1}}{1-\left[1-e^{-\beta(\alpha_\lambda)x}\right]^{\alpha}}, \quad 0 \le x < \infty, \quad \alpha,\beta,\lambda > 0 \tag{17}$$

Hence, in the following characterization results it remains to show that the respective properties determine Erlang truncated exponential distributions uniquely.

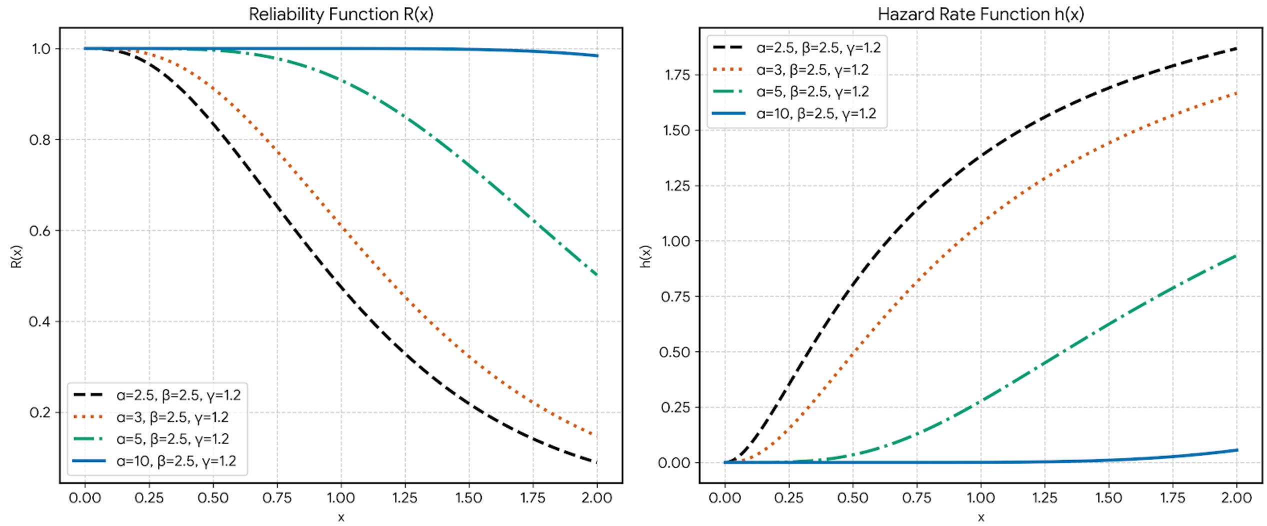

Fig. 2: Possible shapes of the reliability function $R(x)$ (left) and failure rate function $h(x)$ (right) of the Extended Erlang-Truncated Exponential (EETE) distribution for fixed parameter values of β and λ.

4 A characterization of the Erlang-Truncated Exponential distribution

In this section we consider a relation characterizing the Erlang-Truncated Exponential distribution based on generalized order statistics. This generalizes some previous characterization results and uses upper as well as lower generalized order statistics. It has been assumed here throughout that the df is differentiable w.r.t. its argument.

Theorem 4.1. A random variable $Y(r,n;\tilde{m},k)$ be a sequence of i.i.d. non-negative random variables with an absolutely continuous distribution having be the r^{th} m-generalized order statistics (m-GOS) from a sample of size n drawn from a continuous DF $F_Y(y)$ with PDF $f_y(y)$. Furthermore, given the following statements:

$$Y_{(R-N_0+r,R;\tilde{m},k)} \stackrel{d}{=} Y_{(r,N_0;\tilde{m},k)} + \tilde{Z}, \tag{18}$$

be satisfied for all $1 \le r < N_0 < R$, then $\tilde{Z} \stackrel{d}{=} Y_{(R-N_0,R;\tilde{m},k)}$ and $Z \sim$ Erlang truncated exponential $(\beta\alpha_\lambda)$ if and only if $Y \sim$ Erlang truncated exponential $(\beta\alpha_\lambda)$, $\beta > 0$, $\alpha > 0$, $\lambda > 0$.

Proof: By using the easy-to-prove relation. Let the MGF of $Y_{(r,N_0,\tilde{m},k)}$ be $M_{Y_{(r,N_0,\tilde{m},k)}}(t)$. Then, (18) implies that

$$M_{Y_{(R-N_0+r,R,\tilde{m},k)}}(t) = M_{Y_{(r,N_0,\tilde{m},k)}}(t) \cdot M_{\tilde{Z}}(t) \tag{19}$$

Let us now derive the moment generating function of the r^{th} $\tilde{m}$-generalized order statistics $Y_{(r,n,\tilde{m},k)}$ based on Erlang truncated $\exp(\beta\alpha_\lambda)$. From (1), we have

$$M_{Y_{(r,N_0,\tilde{m},k)}}(t)=\frac{\beta(\alpha_\lambda)c_{r-1}^{(N_0)}}{(r-1)!(\tilde{m}+1)^{r-1}}\int_0^\infty e^{-y(\beta\alpha_\lambda\Psi_r^{(N_0)}-t)}[1-e^{-\beta(\alpha_\lambda)(\tilde{m}+1)y})]^{r-1}dy$$

which by using the transformation $u=e^{-\beta\alpha_\lambda(m+1)y}$ takes the form

$$\begin{aligned}
&=\frac{C_{r-1}^{(N_0)}}{(r-1)!(\widetilde{m}+1)^r}\int_0^1 u^{\left(\frac{\Psi_r^{(N_0)}}{\widetilde{m}+1}-\frac{t}{\beta(\alpha_\lambda)(m+1)}\right)^{-1}}(1-u)^{r-1}du\\
&=\frac{C_{r-1}^{(N_0)}}{(\widetilde{m}+1)^r}\frac{\Gamma\left(\frac{\Psi_r^{(N_0)}}{(\widetilde{m}+1)}-\frac{t}{\beta\alpha_\lambda(\widetilde{m}+1)}\right)}{\Gamma\left(\frac{\Psi_r^{(N_0)}}{(\widetilde{m}+1)}-\frac{t}{\beta\alpha_\lambda(\widetilde{m}+1)}+r\right)}\\
&=\prod_{k=1}^{r}\left(\frac{\frac{\Psi_r^{(N_0)}}{(\widetilde{m}+1)}}{\frac{\Psi_r^{(N_0)}}{(\widetilde{m}+1)}-\frac{1}{\beta(\alpha_\lambda)(\widetilde{m}+1)}+r-k}\right)\\
&=\prod_{k=1}^{r}\left(1-\frac{t}{\beta(\alpha_\lambda)\Psi_k^{(N_0)}}\right)^{-1}
\end{aligned}\tag{20}$$

On the other hand, in view of (19) and by using the relations $C_{R-N_0+r-1}^{(R)}=C_{r-1}^{(N_0)}C_{R-N_0-1}^{(R)}$ and $\Psi_{R-N_0+r}^{(R)}=\Psi_r^{(N_0)}$, we get

$$\begin{aligned}
M_{\tilde{Z}}(t)=\frac{M_{Y_{(R-N_0+r,R,\widetilde{m},k)}}(t)}{M_{Y_{(r,N_0,\widetilde{m},k)}}(t)}&=\frac{C_{R-N_0-1}^{(R)}}{(\widetilde{m}+1)^{R-N_0}}\frac{\Gamma\left(\frac{\Psi_r^{(N_0)}}{(\widetilde{m}+1)}-\frac{t}{\beta(\alpha_\lambda)(\widetilde{m}+1)}+r\right)}{\Gamma\left(\frac{\Psi_r^{(N_0)}}{(\widetilde{m}+1)}-\frac{t}{\beta(\alpha_\lambda)(\widetilde{m}+1)}+R-N_0+r\right)}\\
&=\frac{C_{R-N_0-1}^{(R)}}{(\widetilde{m}+1)^{R-N_0}}\frac{\Gamma\left(\frac{\Psi_{R-N_0}^{(R)}}{(\widetilde{m}+1)}-\frac{t}{\beta(\alpha_\lambda)(\widetilde{m}+1)}\right)}{\Gamma\left(\frac{\Psi_{R-N_0}^{(R)}}{(\widetilde{m}+1)}-\frac{t}{\beta(\alpha_\lambda)(\widetilde{m}+1)}+R-N_0\right)}\\
&=\prod_{k=1}^{R-N_0}\left(\frac{\frac{\Psi_i^{N_0}}{(\widetilde{m}+1)}}{\frac{\Psi_{R-N_0}^{(N_0)}}{(\widetilde{m}+1)}-\frac{t}{\beta(\alpha_\lambda)(\widetilde{m}+1)}+R-N_0-k}\right)\\
&=\prod_{k=1}^{R-N_0}\left(1-\frac{t}{\beta(\alpha_\lambda)\Psi_k^{(R)}}\right)^{-1}
\end{aligned}\tag{21}$$

Since $\frac{\Psi_r^{(N_0)}}{(\widetilde{m}+1)}+r=\frac{\Psi_{R-N_0}^{(R)}}{(\widetilde{m}+1)}$. On comparing (21) with (20), we deduce that $M_{\tilde{Z}}(t)$ is the moment generating function of $Z_{(R-N_0,R,\tilde{m},k)}$, i.e., the $(R-N_0)^{th}$ m-generalized order statistics from a sample of size R drawn from the distribution function (df) Erlang truncated exponential $(\beta(\alpha_\lambda))$. Hence the proved necessity part.

To prove the sufficiency part, in view of (18) being satisfied with $\tilde{Z}\overset{d}{=}Z_{(R-N_0,R,\tilde{m},k)}$ and $Z\sim\exp(\beta(\alpha_\lambda))$. Furthermore, let $Y_{(R-N_0+r,R,\tilde{m},k)}$ and $Y_{(r,N_0,\tilde{m},k)}$ in (18) be $\tilde{m}$-generalized order statistics ($\tilde{m}-$GOSs), which are based on an unknown DF $F_Y(y)$ and they are independent of $Z_{(R-N_0,R,\tilde{m},k)}$. Therefore, the convolution relation (18) implies that

$$f_{Y(R-N_0+r,\,R,\,\widetilde{m},\,k)}(y)=\int_0^\infty f_{Y(r,N_0,\widetilde{m},k)}(z)f_{Z((R-N_0,\,R,\,\widetilde{m},\,k))}(y-z)dz$$

$$=\frac{\beta(\alpha_\lambda)c_{(R-N_0-1)}^{(R)}}{(R-N_0-1)!(\widetilde{m}+1)^{(R-N_0-1)}}\int_0^\infty\left[e^{-\beta(\alpha_\lambda)(y-z)}\right]^{\psi_{(R-N_0)}^{(n_1)}}\left[1-(e^{-\beta(\alpha_\lambda)(y-z)})^{\widetilde{m}+1}\right]^{R-N_0-1}f_{Y(r,N_0,\widetilde{m},k)}(z)dz\tag{22}$$

Differentiating equation (22) on both sides with respect to y, we get

$$\begin{aligned}\frac{d}{dy}f_{Y(R-N_0+r,\,R,\,\widetilde{m},\,k)}(y) = {} & \frac{(\beta(\alpha_\lambda))^2 C^{(R)}_{R-N_0-1}}{(R-N_0-2)!(\widetilde{m}+1)^{R-N_0-1}}\int_0^y \left[e^{-\beta(\alpha_\lambda)(y-z)}\right]^{\psi^{(R)}_{R-N_0}+(\widetilde{m}+1)} \\ & \left[1-(e^{-\beta(\alpha_\lambda)(y-z)})^{\widetilde{m}+1}\right]^{R-N_0-2} f_{Y(r,N_0,\widetilde{m},k)}(z)dz \\ & -\frac{(\beta(\alpha_\lambda))^2 \psi^{(R)}_{R-N_0} C^{(R)}_{R-N_0-1}}{(R-N_0-1)!(\widetilde{m}+1)^{R-N_0-1}}\int_0^y \left[e^{-\beta(\alpha_\lambda)(y-z)}\right]^{\psi^{(R)}_{R-N_0}} \\ & \left[1-(e^{-\beta(\alpha_\lambda)(y-z)})^{\widetilde{m}+1}\right]^{R-N_0-1} f_{Y(r,N_0,\widetilde{m},k)}(z)dz\end{aligned} \tag{23}$$

by using the relation

$$e^{-\beta(\alpha_\lambda)\psi^{(R)}_{R-N_0}v}\left(1-\left(e^{-\beta(\alpha_\lambda)v}\right)^{\tilde{m}+1}\right)^{R-N_0-1} = e^{-\beta(\alpha_\lambda)\psi^{(R)}_{R-N_0}v}\left(1-e^{-\beta(\alpha_\lambda)(\tilde{m}+1)v}\right)^{R-N_0-2} - e^{-\beta(\alpha_\lambda)\left(\psi^{(R)}_{R-N_0}+(\tilde{m}+1)v\right)}\left(1-e^{-\beta(\alpha_\lambda)(\tilde{m}+1)v}\right)^{R-N_0-} \tag{24}$$

and by using the relation (22), we get

$$\begin{aligned}& \frac{\beta(\alpha_\lambda) C^{(R)}_{R-N_0-1}}{(R-N_0-2)!(\widetilde{m}+1)^{R-N_0-2}}\int_0^y \left[e^{-\beta(\alpha_\lambda)(y-z)}\right]^{\psi^{(R)}_{R-N_0}+(\widetilde{m}+1)} \\ & \left[1-(e^{-\beta(\alpha_\lambda)(y-z)})^{\widetilde{m}+1}\right]^{R-N_0-2} f_{Y(r,N_0,\widetilde{m},k)}(z)dz \\ & = \frac{C^{(R)}_{R-N_0-1}}{C^{(R-1)}_{R-N_0-2}} f_{Y(R-N_0-1,R-1,\widetilde{m},k)}(y) - (\widetilde{m}+1)(R-N_0-1) f_{Y(R-N_0,R,\widetilde{m},k)}(y)\end{aligned} \tag{25}$$

Thus, by using the relation

$$\psi^{(R)}_{R-N_0} + (\tilde{m}+1)(R-N_0-1) = \psi^{(R)}_1 and \frac{C^{(R)}_{R-N_0-1}}{C^{(R-1)}_{R-N_0-2}} = \frac{\prod_{i=1}^{R-N_0}\psi^{(R)}_i}{\prod_{i=2}^{R-N_0}\psi^{(R)}_i} = \psi^{(R)}_1$$

and by combining (31) and (23), we get

$$\frac{d}{dy}f_{Y_{(R-N_0+r,R,\tilde{m},k)}}(y) = \beta(\alpha_\lambda)\psi^{(R)}_1\left[f_{Y_{(R-N_0+r-1,R-1,\tilde{m},k)}}(y) - f_{Y_{(R-N_0+r,R,\tilde{m},k)}}(y)\right]$$

or equivalently, by integrating from 0 to y:

$$f_{Y(R-N_0+r,R,\widetilde{m},k)}(y) = \beta(\alpha_\lambda)\,\psi^{(R)}_1\left[F_{Y(R-N_0+r-1,R-1,\widetilde{m},k)}(y) - F_{Y(R-N_0+r,R,\widetilde{m},k)}(y)\right] \tag{26}$$

(see Kamps 1995 [16], p. 75, and by using the relation (II)), we get

$$\begin{aligned}& F_{Y(R-N_0+r-1,R-1,\widetilde{m},k)}(y) - F_{Y(R-N_0+r,R,\widetilde{m},k)}(y) \\ & = \frac{C^{(R-1)}_{R-N_0+r-2}}{(R-N_0+r-1)!(\widetilde{m}+1)^{R-N_0+r-1}}\left[\overline{F}_Y(y)\right]^{\psi^{(R)}_{R-N_0+r}-1}\left[1-(\overline{F}_Y(y))^{\widetilde{m}+1}\right]^{R-N_0+r-1}\end{aligned} \tag{27}$$

Therefore, in view of (22), (25) and (26), we have

$$\frac{f_Y(y)}{\bar{F}_Y(y)} = \beta(\alpha_\lambda)$$

Hence, the complete sufficient part,

$$F_Y(y) = \left[1 - e^{-\beta(\alpha_\lambda)y}\right], \quad y > 0,\ \beta > 0,\ \alpha > 0,\ \lambda > 0.$$

Corollary 4.1. A random variables (RVs) Y and Z are independent, as we assumed in Theorem 4.1. By replacing the additive relation (18) by the multiplication relation

$$Y\left(R - N_0 + r, R, \widetilde{m}, k\right) \stackrel{d}{=} Y\left(r, N_0, \widetilde{m}, k\right) \cdot \widetilde{Z} \tag{28}$$

Then, $\tilde{Z} \stackrel{d}{=} Z_{(R-N_0,R,\tilde{m},k)}$ and $Z \sim \text{Pareto}(\beta(\alpha_\lambda))$ (i.e., $F(y) = [1 - z^{-\beta(\alpha_\lambda)}], z > 1$) if and only if $Y \sim \text{Pareto}(\beta(\alpha_\lambda))$, $\beta > 0$, $\alpha > 0$, $\lambda > 0$.
Proof: By noting that if $Y \sim \text{Pareto}(\beta(\alpha_\lambda))$, then $\log Y \sim \exp(\beta(\alpha_\lambda))$ and

$$\log Y_{(R-N_0+r,R,\tilde{m},k)} \stackrel{d}{=} \log Y_{(r,N_0,\tilde{m},k)} + \log \tilde{Z}$$

which implies

$$Y_{(R-N_0+r,R,\tilde{m},k)} \stackrel{d}{=} Y_{(r,N_0,\tilde{m},k)} \cdot \tilde{Z}$$

Corollary 4.2. The random variables (RVs) Y and Z are independent. Let $Y_{(r,n,\tilde{m},k)}$ and $Z_{(r,n,\tilde{m},k)}$ be the r^{th} m-dual generalized order statistics (m-DGOS) based on a sample of size n drawn from $F_Y(y)$ and $F_Z(z)$, respectively. By replacing the additive relation (18) by the multiplicative relation

$$Y^*\left(R - N_0 + r, R, \widetilde{m}, k\right) \stackrel{d}{=} Y^*\left(r, N_0, \widetilde{m}, k\right) \cdot Z^* \tag{29}$$

Then, $\tilde{Z} \stackrel{d}{=} Z_{(R-N_0,R,\tilde{m},k)}$ and $\tilde{Z} \sim \text{Power}(\beta(\alpha_\lambda))$, $\beta > 0$, $\alpha > 0$, $\lambda > 0$ (i.e., $F_Z(z) = z^{\beta(\alpha_\lambda)}$ for $0 < z < 1$) if and only if $Y \sim \text{Power}(\beta(\alpha_\lambda))$.
Proof: By noting that if $Y \sim \text{Power}(\beta(\alpha_\lambda))$, then $-\log Y \sim \exp(\beta(\alpha_\lambda))$ and

$$-\log Y_{(R-N_0+r,R,\tilde{m},k)} \stackrel{d}{=} -\log Y_{(r,N_0,\tilde{m},k)} - \log \tilde{Z}$$

implies

$$Y_{(R-N_0+r,R,\tilde{m},k)} \stackrel{d}{=} Y_{(R-N_0,R,\tilde{m},k)} \cdot \tilde{Z}$$

Theorem 4.2. Let $Y_{(r,n;\tilde{m},k)}$ be a sequence of i.i.d. non-negative random variables with an absolutely continuous distribution, and let it be the r^{th} m-generalized order statistics (m-GOS) from a sample of size n drawn from a continuous distribution function (df) $F_Y(y)$ with probability density function (PDF) $f_Y(y)$. Given the following statements:

$$Y\left(R - N_0 + r,\ R, \widetilde{m}, k\right) \stackrel{d}{=} Y\left(r, N_0, \widetilde{m}, k\right) + \widetilde{Z} \tag{30}$$

be satisfied for all $1 \le r < N_0 < R$. Then, $\tilde{Z} \stackrel{d}{=} Y_{(r,N_0,\tilde{m},k)}$ and $Z \sim \exp(\beta\alpha_\lambda)$ if and only if $Y \sim \exp(\beta\alpha_\lambda)$, $\beta > 0$, $\alpha > 0$, $\lambda > 0$.
Proof: The necessary part can be proved easily using mgf. Namely, let in view of (29) be satisfied with $\tilde{Z} \stackrel{d}{=} Y_{(r,N_0,\tilde{m},k)}$ and $Z \sim \exp(\beta\alpha_\lambda)$. Furthermore, let $Y_{(R-N_0+r,R,\tilde{m},k)}$ and $Y_{(R-N_0,R,\tilde{m},k)}$ in (29) be m-generalized order statistics (m-GOSs), which are based on an unknown distribution function (df) $F_Y(y)$ and they are independent of $Z_{(r,N_0,\tilde{m},k)}$. Therefore, by the convolution method, (29) implies

$$\begin{aligned} f_{Y(R-N_0+r,\ R,\widetilde{m},k)}(y) &= \int_0^y f_{Y(R-N_0,\ R,\widetilde{m},k)}(z) f_{Z(r,\ N_0,\widetilde{m},k)}(y-z)dz \\ &= \frac{\beta(\alpha_\lambda)C_{r-1}^{(N_0)}}{(r-1)!(\widetilde{m}+1)^{r-1}} \int_0^y e^{-\beta(\alpha_\lambda)\psi_r^{(N_0)}(y-z)} \\ &\quad \times \left[1 - (e^{-\beta(\alpha_\lambda)(y-z)})^{\widetilde{m}+1}\right]^{r-1} f_{Y(R-N_0,\ R,\widetilde{m},k)}(z)dz \end{aligned} \tag{31}$$

Differentiating both sides of (30) with respect to y, we get

$$\begin{aligned}\frac{d}{dy}f_{Y_{(R-N_0+r,R,\tilde{m},k)}}(y) &= \frac{(\beta(\alpha_\lambda))^2 C_{r-1}^{(N_0)}}{(r-2)!(\tilde{m}+1)^{r-2}}\int_0^y \left[e^{-\beta(\alpha_\lambda)\left(\psi_r^{(N_0)}+(\tilde{m}+1)\right)(y-z)}\right] \\ &\times\left[1-\left(e^{-\beta(\alpha_\lambda)(y-z)}\right)^{\tilde{m}+1}\right]^{r-2} f_{Y_{(R-N_0,R,\tilde{m},k)}}(z)\,dz \\ &-\frac{(\beta(\alpha_\lambda))^2\psi_r^{(N_0)}C_{r-1}^{(N_0)}}{(r-1)!(\tilde{m}+1)^{r-1}}\int_0^y e^{-\beta(\alpha_\lambda)\left(\psi_r^{(N_0)}(y-z)\right)} \\ &\times\left[1-\left(e^{-\beta(\alpha_\lambda)(y-z)}\right)^{\tilde{m}+1}\right]^{r-1} f_{Y_{(R-N_0,R,\tilde{m},k)}}(z)\,dz \\ &=\beta(\alpha_\lambda)\psi_r^{(n_2)}\left[f_{Y_{(R-N_0+r-1,R,\tilde{m},k)}}(y)-f_{Y_{(R-N_0+r,N_0,\tilde{m},k)}}(y)\right]\end{aligned}$$

Or equivalently, by integrating from 0 to y,

$$f_{Y(R-N_0+r,\ R,\tilde{m},k)}(y)=\beta\left(\alpha_\lambda\right)\psi_r^{(n_2)}\left[F_{Y(R-N_0+r-1,\ R,\tilde{m},k)}(y)-F_{Y(R-N_0+r,\ R,\tilde{m},k)}(y)\right] \tag{32}$$

(see [[16], p. 75] and by using the relation (II)), we get

$$\begin{aligned}&F_{Y(R-N_0+r-1,R,\tilde{m},k)}(y)-F_{Y(R-N_0+r,R,\tilde{m},k)}(y) \\ &=\frac{C_{R-N_0+r-2}^{(R)}}{(R-N_0+r-1)!(\widetilde{m}+1)^{R-N_0+r-1}}\left[\overline{F}_Y(y)\right]^{\psi_{R-N_0+r}^{(R)}-1}\left[1-(\overline{F}_Y(y))^{\widetilde{m}+1}\right]^{R-N_0+r-1}\end{aligned} \tag{33}$$

Therefore, in view of (30), (31) and (32), we get

$$\frac{f_Y(y)}{\bar{F}_Y(y)}=\beta(\alpha_\lambda)$$

which implies that

$$F_Y(y)=\left[1-e^{-\beta(\alpha_\lambda)y}\right],\quad \beta>0,\ \alpha>0,\ \lambda>0,\ y>0$$

This completes the proof of Theorem 4.2.

Corollary 4.3. The random variables (RVs) Y and Z are independent, as we assumed in Theorem 4.2. By replacing the additive relation (29) by the multiplicative relation

$$Y_{(R-N_0+r,R,\tilde{m},k)}\overset{d}{=}Y_{(R-N_0,R,\tilde{m},k)}\cdot\tilde{Z}, \tag{34}$$

then $\tilde{Z}\overset{d}{=}Z_{(r,n_2,\tilde{m},k)}$ and $Z\sim$ Pareto$(\beta(\alpha_\lambda))$ if and only if $Y\sim$ Pareto$(\beta(\alpha_\lambda))$.
Proof: The proof of Corollary 4.3 follows from Corollary 4.2.

Remark(2). For OOSs model the relation (33) takes the form

$$Y_{(R-N_0+r,R;0,1)}\overset{d}{=}Y_{(R-N_0,R;0,1)}\cdot Z_{(r,N_0;0,1)}$$

which implies the relation $Y_{(s,R;0,1)}\overset{d}{=}Y_{(r,R;0,1)}\cdot Z_{(s-r,R-r;0,1)}$ that belongs to Castaño-Martínez et al. (2012).

Corollary 4.4. The random variables (RVs) Y and Z are independent. Also let $Y_{(r,n,\tilde{m},k)}$ and $Z_{(r,n,\tilde{m},k)}$ be the r^{th} m-dual generalized order statistics (m-DGOS) based on a sample of size n drawn from $F_Y(y)$ and $F_Z(z)$, respectively. By replacing the additive relation (29) by the multiplicative relation

$$Y_{(R-N_0+r,R,\tilde{m},k)}\overset{d}{=}Y_{(R-N_0,R,\tilde{m},k)}\cdot\tilde{Z}, \tag{34}$$

then $\tilde{Z}\overset{d}{=}Z_{(r,N_0,\tilde{m},k)}$ and $\tilde{Z}\sim$ Power$(\beta(\alpha_\lambda))$ if and only if $Y\sim$ Power$(\beta(\alpha_\lambda))$, $\beta>0$, $\alpha>0$, $\lambda>0$.
Proof: The proof of Corollary 4.4 follows from Corollary 4.3.

5 Applications to the Prediction Problem

Many authors have considered prediction problems based on samples of random sizes. The importance of the order statistics in the reliability theory is attributed to the fact that the r^{th} order statistic $(n-r+1)$ out-of-n system made up of n identical components with independent life lengths. On the other hand, in dealing with censored samples, where the life-test is terminated after observing the r^{th} failure (Type II censoring), or the termination of the test occurs after a given time lapse (Type I censoring), the complete survival times cannot usually be observed (due to time or cost). In many biological and agriculture problems, we often come across a situation where the sample size is not deterministic because either some observations get lost for various reasons, or the size of the target population and its representative sample cannot be determined well. For example, assume that the inhabitants of a populous town are exposed to a dose of radiation resulting from an atomic accident, or exposed to an infection of an unknown epidemic. Furthermore, assume that our interest focuses on the time at which r persons would die among a big random sample of size n that is drawn from the residents of this town. Since the number of infected people in this town is unknown and changes randomly with time, the drawn sample contains a random number of infected and non-infected people. Accordingly, the sample size of the sub-sample of the infected people will be a non-negative integer valued RV, e.g. N, and it will be described by a sequence of independent and identically distributed RVs $X_1, X_2, \cdots, X_N$. Therefore, the r^{th} smallest order statistic will be denoted by $X_{(r:N)}$, which represents the time at which r persons will die.

Lemma 5.1. Let us assume that there are two independent lifetime experiments. The first one contains n_1 items, which follow $Y \sim \exp(\beta\alpha_\lambda)$. Furthermore, let us assume that s items were observed until they failed. The second experiment contains $n_1 = n_2 - s$ items, which follow $Z \sim \exp(\beta\alpha_\lambda)$. Furthermore, in the second experiment let us assume that r failure times were observed. Theorem 2.2 enables us to predict

$$Y_{(s+1,R,\tilde{m},k)}, Y_{(s+2,R,\tilde{m},k)}, \cdots, \text{ and } Y_{(s+r,R,\tilde{m},k)} \tag{35}$$

by $Y_{(s,R,\tilde{m},k)} + Z_{(1,R,\tilde{m},k)}, Y_{(s,R,\tilde{m},k)} + Z_{(2,R,\tilde{m},k)}, \cdots,$ and $Y_{(s,R,\tilde{m},k)} + Z_{(r,R,\tilde{m},k)}$, respectively.

Lemma 5.2. Let us assume that there are two independent lifetime experiments. The first one contains n_2 items, which follow $Y \sim \exp(\beta\alpha_\lambda)$. Furthermore, let us assume that r items were observed until they failed. The second experiment contains n_1 items, which follow $Z \sim \exp(\beta\alpha_\lambda)$, where $n_1 > n_2$. Furthermore, in the second experiment, let us assume that $s = n_1 > n_2$ failure times were observed. If we decided to enlarge the number of installed items in the first experiment to n_1, Theorem 1 would enable us to predict

$$Y_{(s+1,R,\tilde{m},k)}, Y_{(s+2,R,\tilde{m},k)}, \cdots, \text{ and } Y_{(s+r,R,\tilde{m},k)} \tag{36}$$

by $Y_{(1,R,\tilde{m},k)} + Z_{(s,R,\tilde{m},k)}, Y_{(2,N_0,\tilde{m},k)} + Z_{(s,R,\tilde{m},k)}, \cdots,$ and $Y_{(r,N_0,\tilde{m},k)} + Z_{(s,R,\tilde{m},k)}$, respectively.

6 Conclusions

In this paper, we consider the equality by distribution of the form $Y \stackrel{d}{=} XV$, where X and V are two independent random variables. It should be noted that the random contraction–dilation schemes have important applications in many areas, such as economic modelling and reliability. The characterization results given in Section 4 can be used in developing goodness-of-fit tests for the corresponding probability distributions. This paper deals with the generalized order statistics and dual generalized order statistics within a class of Erlang-Truncated Exponential distribution.. Two theorems for characterizing the general form of distribution based on generalized order statistics and dual generalized order statistics are given. Special cases are also deduced.

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