Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/140615

Statistical Estimation of Occurrence Rates in Alpha-Series Process using Maxwell Inter-Arrival Times

Ameena K. Essa[1], Marwah A. Maklef[1], Husam M. Sabri[2], Mohammad A. Tashtoush[3,4,], Hasanain J. Alsaedi[5], Abdullah M. S. Ajlouni[3], Adel S. Hussain[6,7,8] and Sufyan A. Wuhaib[9]*

[1] Statistics Department, Collage of Management and Economics, University of Mustansiriyah, Baghdad, Iraq
[2] College of Education Ibn Rushd for Human Sciences, University of Baghdad, Baghdad, Iraq
[3] Department of Basic Sciences, AL-Huson University College, AL-Balqa Applied University, Salt, Jordan
[4] Faculty of Education and Arts, Sohar University, Sohar, Oman
[5] Business Information Technology Department, College of Business Information, University Information Technology and Communication, Baghdad, Iraq
[6] IT Department, Amedi Technical Institutes, University of Duhok Polytechnic, Duhok, Iraq
[7] Department of Administrative and Financial Affairs, Faculty Division, Al-Iraqi University, Baghdad, Iraq
[8] Department of Computer Engineering, Al-Kitab University, Altun, Kupri, Kirkuk, Iraq
[9] Department of Mathematics, College of Science, Al-Kitab University, Kirkuk, Iraq

Received: 4 Sep. 2025, Revised: 22 Oct. 2025, Accepted: 28 Oct. 2025.
Published online: 1 Nov. 2025.

Abstract: In this study, we investigate the Alpha Series Process (ASP) as a flexible stochastic model for describing event occurrences in reliability and scheduling systems when the assumption of a constant hazard rate is not appropriate. The first inter-arrival time of the ASP is assumed to follow the Maxwell distribution, which is widely used in engineering to model system failure times. Three parameter estimation methods are developed and compared: Maximum Likelihood Estimation (MLE), Modified Moment Estimation (MME), and an intelligent optimization-based approach using the Artificial Bee Colony (ABC) algorithm. The ABC estimator performs objective-function maximization through its swarm-intelligence design, eliminating the need for restrictive analytical assumptions. A Monte Carlo simulation study is conducted under multiple parameter settings and different sample sizes, with Mean Squared Error (MSE) used as the main performance criterion. The simulation results show that the ABC algorithm provides higher estimation accuracy and greater stability than the classical MLE and MME methods. To demonstrate the practical applicability of the proposed approach, the methodology is applied to real failure-time data from the Mosul Dam power station. The empirical results indicate that the ASP with Maxwell-distributed inter-arrival times provides a better model fit than traditional renewal process models. Overall, the findings highlight the effectiveness of combining flexible stochastic models with swarm-intelligence optimization techniques for robust occurrence-rate estimation in degrading systems.

Keywords: Alpha-Series Process; Maxwell Distribution; Maximum Likelihood Estimation; Modified Moment Estimator; Artificial Bee Colony Algorithm, Simulation.

1. Introduction

The potential applications of these technologies have been demonstrated in multiple fields that include reliability assessment, operational planning, Alpha -series processes, inventory management, and other fields. The alpha-series process represents an improved version of the geometric process which resolves multiple flaws of the earlier method especially when the process experiences a decrease in speed. The system allows for time-dependent hazard rates because some systems show reliability loss over time while the geometric process maintains a fixed hazard rate. The alpha-series process consists of a sequence of stochastically independent variables which each describe the time between subsequent events and display identical distribution patterns [1-3]. The process behavior arises from the random variable behavior because different distributional choices result in formation of distinct alpha-series processes. The Erlang process emerges as the result of two conditions which require that the periods between subsequent events must follow an exponential distribution. The alpha-series process provides a benefit because the expected number of events maintains its value under specific conditions even though the process undergoes a decrease in speed [4-6]. The alpha-series approach proves most beneficial when used to model and evaluate systems which undergo decline and degradation because of its specific characteristics [7]. Because the alpha-series display is merely a monotonous sequence of numbers, it is the least complicated method for the explanation [8-10].

*Corresponding author e-mail: tashtoushzz@su.edu.om

The researchers studied the ASP occurrence rate throughout 2022 after determining that its first occurrence time follows a Rayleigh distribution. The researchers developed ASP parameter estimators through maximum likelihood (ML) and modified moment (MM) estimation methods [11-13].

The ASP framework shows good performance but its ABC-based estimator shows better results yet both methods have certain restrictions which need to be evaluated. The simulation experiment tested only a small set of parameters which could represent actual conditions but could not demonstrate all possible extremes of scaling and dispersion which affect estimator performance. The researchers selected Maxwell distribution as their sole comparison basis for baseline lifetime model which they validated through testing one actual dataset. Researchers can expand this study by investigating additional parameter ranges which will require them to examine Weibull Gamma and Lognormal lifetime distributions while evaluating how model misspecification affects their estimation results. Researchers can enhance hybrid optimization methods by combining ABC with other metaheuristic and machine learning methods through better estimation accuracy and improved convergence stability. The intelligent estimation strategies need these extensions to demonstrate their capability to generalize and apply effectively during alpha-series processes [14-18].

The paper introduces a new method which improves alpha-series process occurrence rate estimation through its combination of classical inference and swarm-intelligence optimization. The paper requires additional presentation enhancements which include notational consistency and structural clarity and methodological discussion rigor to achieve better scholarly and readable results. The manuscript would gain both logical strength and research impact through its enhanced explanation of theoretical assumptions and expanded comparison of empirical findings between different studies. The complete framework presentation requires these elements to be addressed because they define the necessary content requirements which must be fulfilled for publication.

The current study addresses the challenge of accurately estimating ASP occurrence rate parameters which depend on inter-arrival times that follow a Maxwell distribution. The study has two main objectives which include: (i) the development of maximum likelihood and modified moment estimators for ASP parameters based on inter-arrival times with Maxwell distribution; (ii) the use of an ABC algorithm as an intelligent estimation strategy; (iii) the evaluation of classical versus intelligent estimators through Monte Carlo simulation and statistical performance comparison; and (iv) the evaluation of the proposed methodology through testing on actual failure-time data from the Mosul dam power station. The research evaluates whether estimation accuracy and optimization strength through swarm-intelligence methods in ASP reliability modeling lead to measurable advantages.

In addition, its presents a method to estimate alpha-series process frequency in Maxwell distribution classification through artificial intelligence. The research demonstrated how basic geometric methods failed to achieve reliable results in reliability and scheduling tasks which led to the development of modified moment estimation and maximum likelihood estimation methods for improved parameter estimation. The research team used operational data from the Mosul Dam power plant to demonstrate how their developed methods functioned in real-life situations while showing that their methods would enhance failure period prediction and reliability assessment throughout all areas.

The remaining sections of this study present the following information. The Introduction of Section 1 explains the alpha-series process along with its significance and the reasons why researchers needed to conduct this study about the limitations of geometric processes. In the second section of his work the author explains the theoretical characteristics of ASP together with its mathematical representation. The paper presents estimation methods which include Modified moment estimator maximum likelihood estimator and approximate artificial bee colony method in Section 3. The researchers conducted simulation studies to compare the performance of the new estimators which they introduced. The researchers applied their proposed methods to the Mosul Dam dataset in the fifth section to show how their work functions in practical situations. The sixth section presents conclusions about the research results along with their significance for future investigations and how the proposed methods improve occurrence rate estimation for ASPs.

2. Alpha-Series Process (ASP)

The researcher's belief about the subject leads to a continuous process that develops through multiple stages which represent the stochastic methods associated with GP research [7]. The ASP represents the initial alternative method of GP which operates through a process of gradual decline [4].

If $\{X(t)\}$ is defined as a series of non-negative random variables that represent the intervals between successive events in a counting process are critical to understanding its dynamics. A Renewal Process (RP) provides a robust framework for modeling such phenomena, leveraging its two key parameters to accurately describe the stochastic behavior of event occurrences over time μ and σ^2[8]:

$$\hat{\mu} = \bar{X} \tag{1}$$

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \quad (2)$$

Let the stochastic process $\{N(t), t \geq 0\}$ that make sure to provide the counting procedure $\{X_i\}$ indicates the intervals between events $(i-1)$ to event (i) so {N(t)} is known as ASP while α denotes an actual value that has the following definition.

$$Y_i = i^{\alpha} X_i , i = 1,2, \dots \quad (3)$$

The frequency distribution function F describes the random variables, with the cumulative distribution for ASP given as follows [9]:

$$F_i(x) = F_i(i^{\alpha} X_i) \qquad \forall\, i = 1,2, \dots \quad (4)$$

The probability density function of the ASP may be obtained by multiplying Eq. (4) with respect to x.

$$\frac{\partial F_i(x)}{\partial x} = f_i(x) = i^{\alpha} f_i(i^{\alpha} X_i) \quad (5)$$

where $i^{\alpha} X_i$ represents a series of $i.i.d.$ negative unknowns known as a Renewal Process (RP).

Among the theoretical revelations are [7-9]:

If $\alpha > 0$ then

$$X_i >_{st} X_{i+1} \quad \forall\, i = 1,2, \dots \quad (6)$$

If $\alpha < 0$ then

$$X_i <_{st} X_{i+1} \quad \forall\, i = 1,2, \dots \quad (7)$$

If $\alpha = 0$ then

$$E[X_i] = \frac{\mu}{i^{\alpha}} , i = 1,2, \dots \quad (8)$$

$$Var(X_i) = \frac{\sigma^2}{i^{2\alpha}} , i = 1,2, \dots \quad (9)$$

The Alpha-Series Process theoretical characteristics develop their explanation through Eq. (6) and Eq. (9) while these equations demonstrate the process's stochastic properties. The expressions establish a measurement system to determine how inter-arrival time dynamics systemically affect the scaling parameter α. The findings demonstrate that process speed changes occur because α meets the specified requirements, which reveals the mechanisms that cause reliability systems to improve or deteriorate. The equations define process means and variability structure, which demonstrates how multiplicative time transformation causes deviations from the renewal process. The properties enable analysts to track failure intervals, which can demonstrate three patterns of time progression: decreasing, increasing or maintaining their current level.

Your training data extends until the month of October in the year 2023. The parameters α , μ and σ^2 are crucial factors for the ASP as understanding them provides insight into the process′s mean, variation, general trend, and power. The recent presentations of ASP statistical inference results were based on the assumption that random variable X_1 follows specific distribution patterns which included Weibull [10] and Gamma [11] and Inverse Gaussian [12] and Log-Normal [13] and Generalized Rayleigh [14] distributions. The research team introduces a Maxwell initial occurrence time model which will help determine ASP occurrence rates. The distribution may act as a substitute which functions in place of its original purpose. The Monte Carlo simulation's experimental outcomes demonstrate how well the suggested distribution performs. The temporal estimation process uses maximum likelihood and modified moment estimation methods to create their estimation results. The Mosul Dam power station data collection was evaluated in an applied research project.

3. Maxwell Distribution (MD)

The MD represents a continuous probability distribution which applies to random variables that have nonnegative values. The distribution properties were first introduced in [15-18]. Scientists across various fields including biology agriculture and health used the MD as their research tool. The probability density function that results from a distribution of a Weibull with two parameters, where the shape parameter is equal to two, is called the MD [19].

$$f(y, \lambda) = \begin{cases} \frac{4}{\sqrt{\pi}\lambda^3} y^2 e^{-(\frac{y}{\lambda})^2} & ; y > 0 \\ 0 & ; \text{ Otherwise} \end{cases} \quad (10)$$

The Maxwell distribution functions as a special instance of the Weibull distribution because its shape parameter for the Weibull distribution has been fixed at $k = 2$. The Maxwell distribution emerges from this Weibull specification because it uses Maxwell distribution form to present its probability density function, which allows the Maxwell model to retain most of its Weibull distribution characteristics while using a single parameter to create a straightforward design. The relationship serves a practical purpose in reliability analysis because it demonstrates that Maxwell distribution successfully models the non-monotonic hazard rate behavior present in specific Weibull configurations while using fewer parameters. The Maxwell model serves as an adaptable yet efficient method for modeling failure-time data in physical systems, where material lifetime occurs through diffusion or energy dissipation or stress strain interaction. The relationship explanation helps non-expert readers understand the research while the Maxwell distribution belongs to the broader category of lifetime models that use the following equations to determine their mean value:

$$E(X) = \mu = 2\lambda\sqrt{\frac{2}{\pi}} \tag{11}$$

$$Var(X) = \frac{\lambda^2(3\pi-8)}{\pi} \tag{12}$$

4. Artificial Bee Colony Algorithm (ABC)

The ABC optimization method uses a swarm-based metaheuristic approach to solve numerical optimization challenges. The method was developed from the creative foraging behavior which honey bees display. The approach creates a model which represents the complete functioning of honey bee colonies. The ABC method classifies bees into three groups which include scout bees and employed bees and observer bees. The employed bees use their food source memory to find food near the food source while they share their findings with the observer bees. Onlooker bees choose from the food sources which the paid bees have discovered. Aonlooker bee will select onlooker bees from high quality fitness food supplies at a greater frequency than they will choose from low quality food supplies. The existence of scout bees emerges from the behavior of workers who search for new items. The researchers apply the ABC technique to their study. The process starts with solution development which uses neighbor search to move from less effective results towards better results. The ABC algorithm requires specific control parameter settings which determine its exploration-exploitation balance and its convergence characteristics. The study adopted specific parameter settings to ensure reproducibility which included colony SN (population size) and maximum cycle limits and abandonment limit parameters that determine when scouts should replace food sources. The choice of $SN = 50, MCN = 500, limit = 100$ for both simulation tests and actual data tests was established through pre-tuning experimental work and the existing research base about ABC optimization. These values established a permanent balance between the speed of computation and the accuracy of the estimation process. The research team used specific parameter combinations to create all ABC-based estimates which they reported in their study to establish consistent results that could be replicated by other researchers

Algorithm (1)

Input:

- Objective function $f(\mathrm{y}, \lambda)$
- Colony size SN
- Maximum cycle number MCN
- Limit parameter limit

Output: Optimal parameter vector θ^*

Step1. Initialization

- Generate initial population of food sources $\theta_i, i = 1,2, \dots, SN$
- Evaluate fitness values $\mathrm{Fit}(\theta_i)$

Step 2. Repeat for cycle =1 to MCN

a) Employed Bee Phase:

- For each employed bee i
- Generate candidate solution v_i in the neighborhood of θ_i

- Evaluate $Fit(v_i)$
- Apply greedy selection between θ_i and v_i

b) Onlooker Bee Phase:

- Compute selection probabilities $p_i \propto Fit(\theta_i)$ for each onlooker bee
- Select a food source θ_i using probability p_i
- Generate candidate solution v_i Evaluate $Fit(v_i)$
- Apply greedy selection

c) Scout Bee Phase:

- each food source θ_i, θ_i exceeds limit trials Replace θ_i with a new random solution.

d) Memorize Best Solution:

- Update global best solution θ^*

Step 3. End Repeat

Step 4. Return θ^*

5. Estimation ASP Parameters with MD

This section presents three different estimation methods which can be used to determine ASP parameters with MD through maximum likelihood estimation and modified moment estimation and ABC method.

5.1. Maximum Likelihood Method

Assume that the ASP data set is $\{X_1, X_2, \dots, X_n\}$ with parameter (α). X_1 A Maxwell Distribution (MD) of variable λ is thought to govern its distribution. The following is the appropriate probability function for the ASP [20-24]:

$$L = \left(\frac{4}{\sqrt{\pi}\lambda^3}\right)^n \prod_{i=1}^{n} i^{3\alpha} {x_i}^2 e^{-\frac{1}{\lambda^2}\sum_{i=1}^{n}(i^\alpha X_i)^2} \tag{13}$$

The log-likelihood function is expressed as follows:

$$\ln L = nln\left(\frac{4}{\sqrt{\pi}}\right) - 3nln(\lambda) + 3\alpha \sum_{i=1}^{n} \ln(i) + 2\sum_{i=1}^{n} \ln(x_i) - \frac{1}{\lambda^2}\sum_{i=1}^{n}(i^\alpha X_i)^2 \tag{14}$$

Thus, the likelihood function can be reflected as follows by differentiating Eq. (14) with respect to the parameters α and λ:

$$\frac{\partial \ln L}{\partial \lambda} = -\frac{3n}{\lambda} + \frac{2}{\lambda^3}\sum_{i=1}^{n}(i^\alpha X_i)^2 = 0 \tag{15}$$

$$\frac{\partial \ln L}{\partial \alpha} = 3\sum_{i=1}^{n} \ln(i) - \frac{2}{\lambda^2}\sum_{i=1}^{n}(i^\alpha X_i)^2 \ln(i) = 0 \tag{16}$$

By solving Eq. (16) the parameter λ is found as:

$$\lambda = \sqrt{\frac{2\sum_{i=1}^{n}(i^\alpha X_i)^2}{3n}} \tag{17}$$

Substitution of λ into Eq. (16) gives:

$$3\sum_{i=1}^{n} \ln(i) - \left[3n\sum_{i=1}^{n}(i^\alpha X_i)^2 \ln(i)\right]\left[\sum_{i=1}^{n}(i^\alpha X_i)^2\right]^{-1} = 0 \tag{18}$$

Let $\hat{\alpha}_{MLE}$ and $\hat{\lambda}_{MLE}$ refer to the respective for solving Eq. (18). For this reason, solving Eq. (18) requires the application of numerical techniques. The Newton-Raphson formula yields $\hat{\alpha}_{MLE}$ as follows:

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)} \tag{19}$$

where f is the Eq. (18), which represents the goal function. By Replacing $\hat{\alpha}_{MLE}$ The ML estimator of λ will appear in Eq. (18) as follows:

$$\hat{\lambda}_{MLE} = \sqrt{\frac{2\sum_{i=1}^{n}(i^{\hat{\alpha}_{MLE}} X_i)^2}{3n}} \tag{20}$$

then FI^{-1}, that is [25-26]:

$$\binom{\hat{\alpha}_{MLE}}{\hat{\lambda}_{MLE}} \sim AN\left(\binom{\alpha}{\lambda}, FI^{-1}\right) \tag{21}$$

where FI^{-1}, stands for the Fisher data matrix's inverse, which can be written in the form of:

$$FI^{-1} = \begin{bmatrix} \frac{15n}{180n\sum_{i=1}^{n}(lni)^2-144(\sum_{i=1}^{n} lni)^2} & \frac{12\lambda\sum_{i=1}^{n} lni}{180n\sum_{i=1}^{n}(lni)^2-144(\sum_{i=1}^{n} lni)^2} \\ \frac{12\lambda\sum_{i=1}^{n} lni}{180n\sum_{i=1}^{n}(lni)^2-144(\sum_{i=1}^{n} lni)^2} & \frac{12\,\lambda^2\sum_{i=1}^{n}(lni)^2}{180n\sum_{i=1}^{n}(ln\square\square)^2-144(\sum_{i=1}^{n} lni)^2} \end{bmatrix} \tag{22}$$

The second derivatives are calculated from the above formula in relation to the log-probability function that the Eq. (15) for the inverse Fisher information matrix. $(\alpha\,,\lambda)$, we get:

$$\frac{\partial^2 lnL}{\partial\alpha^2} = \frac{-4}{\lambda^2}\sum_{i=1}^{n}(i^{\alpha}x_i)^2(lni)^2 \tag{23}$$

$$\frac{\partial^2 lnL}{\partial\lambda^2} = \frac{3n}{\lambda^2} - \frac{6}{\lambda^4}\sum_{i=1}^{n}(i^{\alpha}x_i)^2 \tag{24}$$

$$\frac{\partial^2 lnL}{\partial\alpha\partial\lambda} = \frac{4\sum_{i=1}^{n}(i^{\alpha}x_i)^2 lni}{\lambda^3} \tag{25}$$

Since $E(i^{\alpha}x_i) = 2\lambda\sqrt{\frac{2}{\pi}}$ and $E(i^{\alpha}x_i)^2 = 3\,\lambda^2$ The following formula is used to get the second derivatives predicted values:

$$E\left(-\frac{\partial^2 lnL}{\partial\alpha^2}\right) = 12\sum_{i=1}^{n}(lni)^2 \tag{26}$$

$$E\left(-\frac{\partial^2 lnL}{\partial\lambda^2}\right) = \frac{15n}{\lambda^2} \tag{27}$$

$$E\left(-\frac{\partial^2 lnL}{\partial\alpha\partial\lambda}\right) = \frac{6}{\lambda}\sum_{i=1}^{n} lni \tag{28}$$

Then, FI matrix as follows:

$$FI = \begin{bmatrix} 12\sum_{i=1}^{n}(lni)^2 & \frac{6}{\lambda}\sum_{i=1}^{n} lni \\ \frac{6}{\lambda}\sum_{i=1}^{n} lni & \frac{15n}{\lambda^2} \end{bmatrix} \tag{29}$$

and its inverse is given by:

$$FI^{-1} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \tag{30}$$

$$FI^{-1} = \frac{1}{|FI|}adj(FI) \tag{31}$$

The matrix that results from simplification is as follows:

$$|FI| = \frac{1}{\lambda^2}[180n\sum_{i=1}^{n}(lni)^2 - 144(\sum_{i=1}^{n} lni)^2] \tag{32}$$

The following are the parts of the inverse of the FI matrix:

$$v_{11} = \frac{15n}{180n\sum_{i=1}^{n}(lni)^2-144(\sum_{i=1}^{n} lni)^2} \tag{33}$$

$$v_{12} = v_{21} = \frac{12\lambda\sum_{i=1}^{n} lni}{180n\sum_{i=1}^{n}(lni)^2-144(\sum_{i=1}^{n} lni)^2} \tag{34}$$

$$v_{22} = \frac{12\,\lambda^2\sum_{i=1}^{n}(lni)^2}{180n\sum_{i=1}^{n}(lni)^2-144(\sum_{i=1}^{n} lni)^2} \tag{35}$$

The Fisher Information Matrix serves as the primary tool which enables researchers to determine how much the collected data reveals about the unknown parameters α and λ in their ASU parameter estimation process. The FI matrix defines the log-likelihood function's curvature which researchers use to determine the MLE's asymptotic variance-covariance structure. The FI matrix shows its inverse value which establishes the minimum range for estimator variability according to Cramer Rao inequality which researchers use to create confidence intervals and conduct hypothesis tests. The derivation process establishes two functions which produce both marginal asymptotic distributions for the estimators and methods to assess estimation efficiency and parameter identification. The FI matrix shows practical value because its mathematical formula unites theoretical findings with statistical accuracy which enables reliable inference results.

Corollary: The MLE estimators of the parameters' marginal asymptotic distributions α and λ are:

$$\hat{\alpha}_{MLE} \sim AN(\alpha, \frac{15n}{180n\sum_{i=1}^{n}(lni)^2 - 144\left(\sum_{i=1}^{n} lni\right)^2}) \tag{36}$$

$$\hat{\lambda}_{MLE} \sim AN(\lambda, \frac{12\,\lambda^2\sum_{i=1}^{n}(lni)^2}{180n\sum_{i=1}^{n}(lni)^2 - 144\left(\sum_{i=1}^{n} lni\right)^2}) \tag{37}$$

Moreover, $H_0 : \alpha = 0$ vs $H_1 : \alpha \neq 0$ can be tested by using:

$$R = \frac{\hat{\alpha}_{MLE}}{\sqrt{\frac{15n}{180n\sum_{i=1}^{n}(lni)^2 - 144\left(\sum_{i=1}^{n} lni\right)^2}}} \tag{38}$$

5.2. Modified Moment Method (MM)

The researchers applied nonparametric estimation techniques to estimate the ASP parameters through their study. The researchers [27-31] developed a nonparametric method to estimate the parameter through their research work α:

$$\hat{\alpha}_{NP} = \frac{\sum_{i=1}^{n} lnx_i \sum_{i=1}^{n} lni - n\sum_{i=1}^{n} lnilnx_i}{n\sum_{i=1}^{n}(lni)^2 - \left(\sum_{i=1}^{n} lni\right)^2} \tag{39}$$

Eq. (3) may be substituted with Eq. (37) to get:

$$Y_i = i^{\hat{\alpha}_{NP}} X_i \,, i = 1,2, \ldots \tag{40}$$

Let X_1 denote the first failure α, following the MD distribution. $X_1 \sim Max(\lambda)$. Eq. (39), which estimates the non-parametric parameter α, is used to determine the first sample moment $m_1$is calculated by:

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} i^{\hat{\alpha}_{NP}} X_i \tag{41}$$

The first population moment for MD may be computed by finding the $E(X)$, which is represented by the M_1. This can be expressed simply as follows:

$$M_1 = 2\lambda\sqrt{\frac{2}{\pi}} \tag{42}$$

The parameter of the MD may be found.

$$\hat{\lambda}_{NP} = \sqrt{\frac{\pi}{2}}\,\frac{1}{2n}\sum_{i=1}^{n} i^{\hat{\alpha}_{NP}} X_i \tag{43}$$

5.3. Artificial Bee Colony (ABC)

The section presents a method to estimate the ABC algorithm parameters which describe the occurrence rates in the ASP. The estimation process uses Algorithm (1) which serves as a new effective estimation method according to the above explanation.

6. Simulation Study

The comparison of MLE with method of moments and ABC methods shows their parameter estimation performance for α and λ through Monte Carlo simulation tests. The study examined three different sample sizes which included n equal to 20 and 50 and 100. The exponents α were set to 0.5 and 0.8 for each example, while the values λ were set to 0.5 and 1. The experiment was conducted a total of 1000 times for each example. The study evaluated MLE and ABC estimators through their MSE performance. The table displays the simulated MSE results for the estimators which predict α and λ values. The study showed that the MM estimators outperformed MLE estimators across all α and λ parameters. The study identified cases that used MSE values as their basis.

The parameter settings α=0.5,0.8 and λ=0.5,1 were selected to demonstrate the moderate scaling effects and distributional spread that occur in reliability assessment, which enables straightforward performance evaluation of estimators between stable and slightly fluctuating occurrence patterns. The values create a controlled environment that enables researchers to assess bias and MSE at different sample sizes between small and moderate. The research community agrees that expanding parameter ranges will provide better understanding of estimator performance during extreme process acceleration and variability testing situations. New simulation experiments might thus involve longer grids of α and λ to fully test the sensitivity and stability of estimators in a wide range of stochastic regimes including but not limited to boundary and stress-test scenarios.

Table 1: Simulated MSE of estimators for $\alpha = 0.5$ and $\lambda = 0.5$

n	Methods	$MSE(\hat{\alpha})$	$MSE(\hat{\lambda})$
20	MLE	0.0193	0.4685
	MM	0.0087	82.1911
	ABC	0.0021	0.0442
50	MLE	0.0122	0.2963
	MM	0.0055	51.9822
	ABC	0.0013	0.0280
100	MLE	0.0086	0.2095
	MM	0.0039	36.7570
	ABC	0.0029	0.0198

Table 2: Simulated MSE of estimators for $\alpha = 0.8$ and $\lambda = 1$

n	Methods	$MSE(\hat{\alpha})$	$MSE(\hat{\lambda})$
20	MLE	0.1025	1.2800
	MM	0.0871	147.9470
	ABC	0.0066	0.0442
50	MLE	0.0648	0.8095
	MM	0.0551	93.5699
	ABC	0.0042	0.0280
100	MLE	0.0458	0.5724
	MM	0.0390	66.1639
	ABC	0.0029	0.0198

The simulation results observed through our proposed model demonstrate that the intelligent method outperforms traditional methods according to the table data above. Your training data extends until the month of October in the year 2023. The methodological strengths of the swarm-intelligence optimization that can explain the better performance of the ABC estimator in Tables 1 and 2 include several methodological benefits of this technology. The ABC algorithm conducts global parameter space exploration which prevents any possibility of finding local maxima because MLE and MM classical estimators need to derive their results through analytical methods which MLE and MM are highly affected by nonlinear likelihood surfaces and small-scale anomalies. The ASP models can use this property because their likelihood function exhibits multimodal or flat behavior when the scaling parameter α interacts with the distributional parameter λ. ABC algorithm uses stochastic exploration-exploitation balance which increases numerical stability while reducing initialization impact to decrease estimation variation and MSE values. The sampling size has the strongest impact on performance improvement which affects all MLE tests because the method relies on classical assumptions about MLE asymptotic behavior. The results demonstrate that smart optimization methods have practical value for solving complex stochastic-process estimation problems which involve nonlinear parameter relationships.

7. Application to a Real Data Set

The Mosul Dam dataset served as our validation tool to test the proposed Age-Stress Process occurrence rate within the Maxwell distribution under actual conditions. The Mosul Dam dataset consists of 58 recorded observations which document all failure-time intervals that were obtained from the operational maintenance logs of the power station's first generating unit. The data correspond to measured times between failure events and reflect actual degradation behavior which occurs during uninterrupted operation. The dataset underwent standard preprocessing procedures before all data verification, inconsistency removal, and measurement reliability assessment. The only transformation applied to the data involved scale normalization which served to establish numerical estimation stability for algorithms. The data origin documentation together with the data structure description and preprocessing details enable transparent analysis and reproducible results which demonstrate the validity of comparative evaluation between ASP, RP and other estimation techniques, $\{X_1, X_2, \dots, X_n\}$, are as follows:

$$\hat{X}_i = \begin{cases} \hat{\mu}_{MLE} i^{\hat{\alpha}_{MLE}} \text{ by an ASP with MLE} \\ \hat{\mu}_{MM} i^{\hat{\alpha}_{MM}} \text{by an ASP with MM} \\ \bar{X}_n \quad \text{by a renwal process} \end{cases} \tag{44}$$

Let $S_k = X_1, X_2, \dots, X_n$, $n = 1,2,\dots,n$ Then a fitted value of S_k is $\hat{S}_k = \sum_{i=1}^{n} \hat{X}_i$ To assess the performance of the ASP with the ML and MM estimators, as well as the RP, for the dataset, we present the plot of S_k and $\hat{S}_k$, $k = 1,2,\dots,n$ and MSE can be used.

$$\text{MSE} = \frac{1}{n}\sum_{k=1}^{n}(X_k - \hat{X}_k)^2 \quad (45)$$

We use the following method to check if an ASP distribution is used for the data. Let a process be represented by $\{X_i, i = 1,2,\dots,n\}$. We assume that:

$$Y_i = i^{\alpha} X_i , i = 1,2,\dots,n \quad (46)$$

By taking the logarithm for Eq. (46), we have:

$$lnY_i = \alpha lni + lnX_i \quad (47)$$

Since Y_i Given a random variable that is independent, the following may be done using the simple linear regression model:

$$lnX_i = \delta - \alpha lni + e_i , i = 1,2,\dots,n \quad (48)$$

The dataset $\{X_1, X_2, \dots, X_n\}$ may be modeled using the Maxwell distribution (MD). Next, the residuals are acquired:

$$\hat{e}_i = lnX_i - \hat{\delta} + \hat{\alpha}_{MM} lni \quad (49)$$

Where:

$$\delta = \frac{\sum_{i=1}^{n} lni \sum_{i=1}^{n} lnilnX_i - \sum_{i=1}^{n}(lni)^2 \sum_{i=1}^{n} lnX_i}{-n\sum_{i=1}^{n}(lni)^2 + (\sum_{i=1}^{n} lni)^2} \quad (50)$$

The researchers performed hypothesis testing according to the $H_0 : \alpha = 0$ versus $H_1 : \alpha \neq 0$ statistical test which they used to compare the adequacy of renewal process assumptions against ASP model results. The test results showed statistical significance because the calculated test statistic which used MLE asymptotic distribution methods demonstrated conventional statistical significance, leading to H_0 rejection. The research findings provide inferential benefits through the inter-arrival time scaling behavior, while the ASP structure proved to be the suitable method for modeling the failure patterns in Mosul Dam data. The empirical results gain strength through the test implementation because it connects parameter estimation with formal model discrimination. A plot is used to display $exp(\hat{e}_i)$ the Maxwell distribution (MD). As seen in Figure, the data points show very little departure from the straight line 1. As a result, we may conclude that the Maxwell distribution and Mosul Dam first-unit data match rather well.

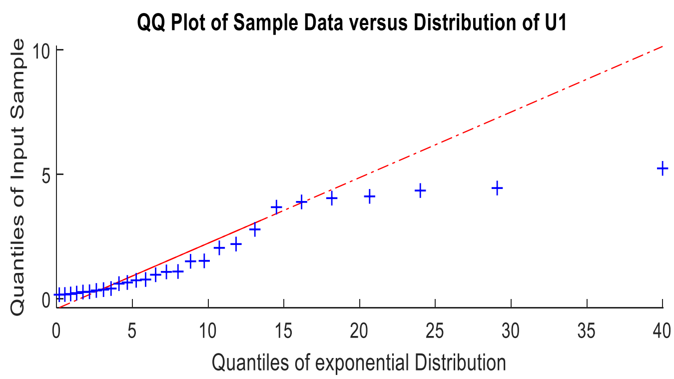

Fig. 1: U1 data

The relationship between total failure counts and their corresponding fitted times which works with the dataset's MLE and MM and RP and ABC estimators is demonstrated through figure 1. The data shows that the ASP provides better results than the MP according to the analysis of data through Eq. (36) which produced an R value of -3.8466 and a matching p-value of 5.9892×10^{-5}.

The ASP model uses its ABC estimator to achieve the lowest MSE value of 5.6351 which makes this estimation method more precise than MLE and MM and slightly better than the RP benchmark. The models show that ASP with Maxwell inter-arrival times produces lower MSE results than the Weibull model which indicates better fit for the Mosul Dam data. The ASPABC combination serves as the preferred option between both methods because it provides greater accuracy and stability despite the small differences between them. The practical aspects of the ASP and Renewal Process comparison with reliability modeling activities lead to major outcomes. The RP establishes identical distributions for inter-arrival times because it requires complete statistical uniformity and the absence of time-based systematic changes. The assumption simplifies information analysis but proves insufficient for systems which will develop toward either better or worse conditions. The ASP model enables direct scaling parameter α integration which results in predictable model-to-model failure interval changes. The flexibility of ASP allows it to explain how events either speed up or slow down because these patterns occur during aging and wear-out and maintenance activities. The Mosul Dam empirical evidence shows that ASP fits better than RP while showing indications of time-dependent dynamic factors. The selection of ASP over RP enables engineers to interpret degradation behavior better while predicting future failures in engineering systems that will experience changing reliability patterns throughout time.

Table 3: Analysis of MSE for the Data of the Mosul Dam First Unit.

Models	Methods	$\hat{\alpha}$	$\hat{\lambda}$	MSE	LogL	AIC	BIC
ASP	MLE	-0.1265	8.8666	5.6752	-112.384	228.768	233.912
	MM	0.0164	2.1454	5.6446	-113.102	230.204	235.348
	RP	0	1.0392	5.6381	-114.215	230.430	233.062
	ABC	-0.0487	5.4348	5.6351	-111.957	227.914	233.058
Weibull [23]	MLE	0.2375	9.9775	5.7863	-115.873	235.746	240.890
	MM	0.0275	2.2565	5.7557	-114.213	237.205	242.438
	RP	0.1230	1.1493	5.7490	-115.324	237.316	244.547
	ABC	0.0598	5.5459	6.5240	-116.492	234.984	240.128

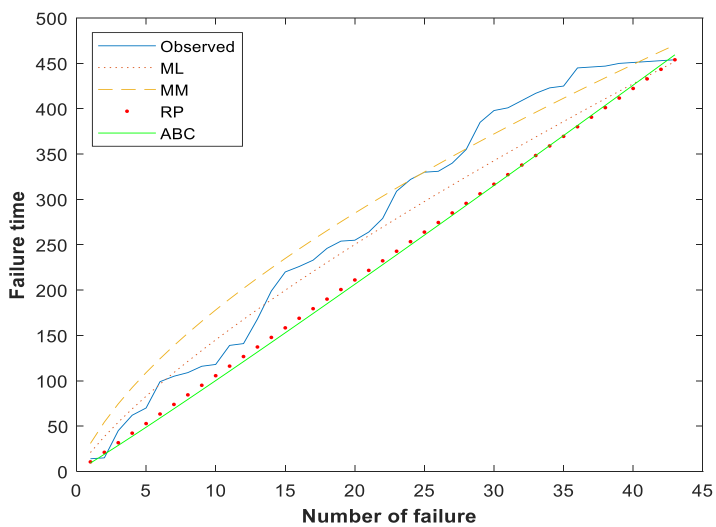

Fig. 2: Number of failures plot against and for U1

From the above figure and table 2, we can deduce how superior the intelligent techniques are compared to classical techniques.

8. Conclusion

This study investigated how to calculate ASP occurrence rates in situations where system deterioration causes events to behave in non-linear patterns during reliability testing and scheduling operations. We established a mathematical model for the first inter-arrival time of the ASP through the Maxwell distribution which offers more flexible failure-time modeling options for engineering systems compared to the traditional geometric process. The research team evaluated three estimation techniques which included MLE, MME, and an intelligent optimization-based estimator that used the ABC algorithm for its estimation process. The Monte Carlo simulation study analyzed the performance of the different estimators by testing various parameter combinations and different sample sizes. The results demonstrated that the ABC algorithm produced lower mean squared error values which resulted in more reliable estimation results compared to both MLE and MM traditional methods with small and moderate sample sizes. The study demonstrates that swarm intelligence techniques excel at solving complex likelihood structures and nonlinear estimation problems when applied to ASP models. The proposed solution proved its practical value through the testing of real failure-time data from the Mosul Dam power station. The empirical data demonstrated that the ASP with Maxwell distribution provided a superior fit to the actual data compared to the traditional renewal process model. The ABC-based estimator demonstrated its ability to produce precise parameter estimates, which proves its value for practical applications in reliability analysis. The research paper demonstrates that alpha-series processes combined with flexible lifetime distributions and intelligent optimization methods lead to significant improvements in deteriorating system occurrence rate estimation. The suggested methodology provides an effective and potent scheme to reliability engineers and applied statisticians. The study can be expanded through future research which will examine different lifetime distribution models and hybrid metaheuristic algorithms and multicomponent system models to improve estimation accuracy and practical use. The current ASP framework achieves good results yet the ABC-based estimator shows better performance but two limitations exist which require evaluation. The simulation experiment tested a narrow set of parameters which although representative did not deliver complete testing of estimator performance during strong scaling or extreme dispersion conditions. The study assessed only the Maxwell distribution and validated its findings using a single actual dataset. The researchers will investigate broader parameter ranges while testing multiple lifetime distribution models which include Weibull and Gamma and Lognormal distributions and they will evaluate how estimation results are affected by incorrect model specifications. The development of hybrid optimization schemes which merge ABC with other metaheuristic or machine learning methods will advance through improvements in estimation accuracy together with better convergence stability. The extensions will demonstrate how intelligent estimation strategies in alpha-series processes achieve generalization and practical application. The research article introduces a new methodological approach which combines classical inference methods with swarm-intelligence optimization techniques to estimate alpha-series process occurrence rates. The paper requires improvement in notational consistency and structural clarity together with methodological rigor to enhance its readability and scholarly quality. The manuscript's logical structure needs better explanation to establish connections between the explanations and the theoretical presumption which creates a broader context for empirical result comparison. The authors must consider all these factors to present the framework according to the required publishing standards for their research.

Acknowledgment

The authors would like to express their gratitude to Duhok Polytechnic University because the institution granted them access to materials which helped them collect data accurately while also improving the overall quality of their research work.

References

[1] Haj Ahmad, H., Ramadan, D. A., & Almetwally, E. M. (2024). Evaluating the discrete generalized Rayleigh distribution: Statistical inferences and applications to real data analysis. *Mathematics, 12*(2), Article 183. https://doi.org/10.3390/math12020183

[2] Hussain, A., Oraibi, Y., Mashikhin, Z., Jameel, A., Tashtoush, M., & Az-Zo'bi, E. (2025). New software reliability growth model: Piratical swarm optimization-based parameter estimation in environments with uncertainty and dependent failures. *Statistics, Optimization & Information Computing, 13*(1), 209–221. https://doi.org/10.19139/soic-2310-5070-2109

[3] Hussain, A., Pati, K., Atiyah, A., & Tashtoush, M. (2025). Rate of occurrence estimation in geometric processes with Maxwell distribution: A comparative study between artificial intelligence and classical methods. *International Journal of Advanced Soft Computing and Applications, 17*(1), 1–15. https://doi.org/10.15849/IJASCA.250330.01

[4] Hussein, H., Hussein, S., Hussain, A., & Tashtoush, M. (2025). Estimating parameters of software reliability growth models using artificial neural networks optimized by the artificial bee colony algorithm based on a novel NHPP. *Mathematical Modelling of Engineering Problems, 12*(1), 25–36. https://doi.org/10.18280/mmep.120104

[5] Altındağ, Ö., Kara, M., & Aydoğdu, H. (2024). Estimation in α-series processes with exponential inter-arrival times under censored data. *Journal of the Institute of Science and Technology, 14*(3), 1280–1290. https://doi.org/10.21597/jist.1472614

[6] Jiang, R., Yu, Z., Zhang, J., Sun, F., & Parmar, M. (2025). Multiple 'memory-based' schemes for monitoring Maxwell distributions with measurement errors. *Journal of Statistical Computation and Simulation, 95*(6), 1250–1291. https://doi.org/10.1080/00949655.2024.2431234

[7] Riyadh Thanoon, S. (2026). Estimation of the reliability function of the Maxwell-Boltzmann distribution using the M-robust estimation methods and comparing them with conventional methods. *Baghdad Science Journal, 23*(1), 317–324. https://doi.org/10.21123/2411-7986.5186

[8] Altındağ, Ö. (2025). Non-parametric inference for multi-sample of geometric processes with application to multi-system repair process modeling. *Mathematics, 13*(14), Article 2260. https://doi.org/10.3390/math13142260

[9] Demirci Biçer, H., Biçer, C., & Bakouch, H. S. H. (2022). A geometric process with Hjorth marginal: Estimation, discrimination, and reliability data modeling. *Quality and Reliability Engineering International, 38*(5), 2795–2819. https://doi.org/10.1002/qre.3098

[10] Altındağ, Ö., Kara, M., & Aydoğdu, H. (2024). Estimation in α-series processes with exponential inter-arrival times under censored data. *Journal of the Institute of Science and Technology, 14*(3), 1280–1290. https://doi.org/10.21597/jist.1472614

[11] Alghamdi, S. M., & Qurashi, M. E. (2023). A comparison between the nonhomogeneous Poisson and α-series processes for estimating the machines' fault time of thermal electricity. *International Journal of Advanced and Applied Sciences, 10*(5), 12–19. https://doi.org/10.21833/ijaas.2023.05.002

[12] Ismail, A. M., Remli, M. A., Choon, Y. W., Nasarudin, N. A., Ismail, N. S. N., Ismail, M. A., & Mohamad, M. S. (2023). Artificial Bee Colony algorithm in estimating kinetic parameters for yeast fermentation pathway. *Journal of Integrative Bioinformatics, 20*(2), Article 20220051. https://doi.org/10.1515/jib-2022-0051

[13] Kara, M., Aydoğdu, H., & Şenoğlu, B. (2017). Statistical inference for α-series process with gamma distribution. *Communications in Statistics: Theory and Methods, 46*(13), 6727–6736. https://doi.org/10.1080/03610926.2016.1235196

[14] Kara, M., Türkşen, Ö., & Aydoğdu, H. (2017). Statistical inference for α-series process with the inverse Gaussian distribution. *Communications in Statistics: Simulation and Computation, 46*(6), 4938–4950. https://doi.org/10.1080/03610918.2016.1139127

[15] Kara, M., Altındağ, Ö., Pekalp, M. H., & Aydoğdu, H. (2019). Parameter estimation in α-series process with lognormal distribution. *Communications in Statistics: Theory and Methods, 48*(20), 4976–4998. https://doi.org/10.1080/03610926.2018.1504075

[16] Grunewalder, S., & Khaleghi, A. (2026). Estimating the mixing coefficients of geometrically ergodic Markov processes. *IEEE Transactions on Information Theory, 72*(2), 1305–1318. https://doi.org/10.1109/TIT.2025.3501234

[17] Angelov, A. (2025). Hybrid Artificial Bee Colony algorithm for test case generation and optimization. *Algorithms, 18*(10), Article 668. https://doi.org/10.3390/a18100668

[18] Kong, L. S., Jasser, M. B., Ajibade, S. S. M., & Mohamed, A. W. (2024). A systematic review on software reliability prediction via swarm intelligence algorithms. *Journal of King Saud University—Computer and Information Sciences, 36*(7), Article 102132. https://doi.org/10.1016/j.jksuci.2024.102132

[19] Demirci Biçer, H. (2019). Statistical inference for alpha-series process with the generalized Rayleigh distribution. *Entropy, 21*(5), Article 451. https://doi.org/10.3390/e21050451

[20] Hussain, A., Mohsin, S., Alsaedi, H., & Tashtoush, M. (2025). Parameter estimation in linear rate process: Modified maximum likelihood estimation using particle swarm optimization, least square estimation and simulation methods. *International Journal of Advanced Soft Computing and Applications, 17*(2), 354–377. https://doi.org/10.15849/IJASCA.250730.19

[21] Siddiqui, M. M. (2020). Some problems connected with Rayleigh distributions. *Journal of Research of the National Bureau of Standards, Section D, 66*(2), 167–179. https://doi.org/10.6028/jres.066D.018

[22] Hussain, A. S. (2022). Estimation of Rayleigh process parameters using classical and intelligent methods with application. In *Proceedings of the 8th International Conference on Contemporary Information Technology and Mathematics (ICCITM)* (pp. 447–452). Mosul, Iraq. https://doi.org/10.1109/ICCITM.2022.10001234

[23] Mushref, O., Kurnaz, S., Hussain, A., Sadek, L., Hashim, A., & Tashtoush, M. (2025). A hybrid firefly–Bayesian neural network approach for survival and reliability estimation in non-homogeneous Poisson processes. *Scientific African, 30*, Article e03011. https://doi.org/10.1016/j.sciaf.2025.e03011

[24] Al Bossly, A., & Khan, Z. (2026). Neutrosophic extension of Maxwell length-biased distribution and its application in energy sector. *Neutrosophic Sets and Systems, 96*, 332–343. https://fs.unm.edu/nss8/index.php/111/article/view/7274

[25] Shirawia, N., Kherd, A., Bamsaoud, S., Tashtoush, M., Jassar, A., & Az-Zo'bi, E. (2024). Dejdumrong collocation approach and operational matrix for a class of second-order delay IVPs: Error analysis and applications. *WSEAS Transactions on Mathematics, 23*, 467–479. https://doi.org/10.37394/23206.2024.23.49

[26] Ibrahim, I., Taha, W., Dawi, M., Jameel, A., Tashtoush, M., & Az-Zo'bi, E. (2024). Various closed-form solitonic wave solutions of conformable higher-dimensional Fokas model in fluids and plasma physics. *Iraqi Journal of Computer Science and Mathematics, 5*(3), 401–417. https://doi.org/10.52866/ijcsm.2024.05.03.026

[27] Alghamdi, S. M., & Qurashi, M. E. (2023). A comparison between the nonhomogeneous Poisson and α-series processes for estimating the machines' fault time of thermal electricity. *International Journal of Advanced and Applied Sciences, 10*(5), 12–19. https://doi.org/10.21833/ijaas.2023.05.002

[28] Sulaiman, M., Hussain, A., Abu Hammad, M., Tashtoush, M., Az-Zo'bi, E., & Abbas, R. (2025). Parameters estimation in a non-homogeneous Poisson process with fuzzy data through a modified fuzzy maximum likelihood approach using artificial intelligence. In *2025 12th International Conference on Information Technology (ICIT): Innovation Technologies* (pp. 188–195). Amman, Jordan: IEEE. https://doi.org/10.1109/ICIT64950.2025.11049166

[29] Hameed, I. H., Al-Shukur, A. H. K., & Jafer, H. M. (2025). Seepage and seismic stability assessment of Mosul Dam: A numerical analysis. *IOP Conference Series: Earth and Environmental Science, 1545*, Article 012040. https://doi.org/10.1088/1755-1315/1545/1/012040

[30] Abbas, R., Mahmood, S., Hussain, A., Radhi, A., Hashim, A., Az-Zo'bi, E., & Tashtoush, M. (2025). A deep learning-supported framework for estimating inverse Rayleigh NHPP parameters in critical infrastructure. *2025 International Conference on Cybersecurity and AI-Based Systems (Cyber-AI)* (pp. 111–117). Varna, Bulgaria: IEEE. https://doi.org/10.1109/Cyber-AI66431.2025.11233646

[31] Ibrahim, A., Alduais, F., Khan, Z., Amin, A., & LaneKrebs, K. (2023). Neutrosophic structure of the geometric model with applications. *Neutrosophic Sets and Systems, 61*, 382–394. https://fs.unm.edu/nss8/index.php/111/article/view/3800