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Statistical Control Points: Approximate Methods for Solving First Order Fuzzy Initial Value Problems

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Abstract: This article proposes two approximate methods for solving first order Fuzzy Initial Value Problems (FIVPs) using new forms of Bezier curves and B-Spline control point's techniques. Fuzzy set theory principles and characteristics are utilized to update and evaluate the suggested techniques for solving both linear and nonlinear fuzzy problems. The control points on the Bezier curve are determined by minimizing the residual function using the least square method. This ensures that the curve approximates the solution to the FIVP. The Bezier curve is a mathematical curve commonly used in computer graphics and geometric modelling. Furthermore, B-Spline interpolation techniques are employed to enhance the accuracy of the solution. B-Splines are a type of mathematical spline function that allows for smooth and flexible interpolation of data points. The article presents numerical examples to illustrate the effectiveness of the proposed method. A comparison is made between the obtained results and the exact solution to demonstrate the capabilities of the method in accurately solving FIVPs. Overall, the article introduces a novel approach to solving first order fuzzy initial value problems. By utilizing Bezier curves, B-Spline interpolation techniques, and principles from fuzzy set theory, the method aims to provide accurate and efficient solutions. The numerical examples serve as evidence of the method's effectiveness in practice.

Keywords: Fuzzy Sets theory, fuzzy differential equation, Bezier curves method (BCM), residual function.

1 Introduction

The field of ambiguity and fuzziness has experienced substantial growth in the past decade, particularly within optimization, system modeling, pattern recognition, and control. This growth is reflected in mathematical literature, as highlighted by [1]. The concepts of fuzziness, has expanded its application to various domains [2]. Learning algorithms, decision-making processes, automata, linguistics, and pattern classification are among the areas that have embraced fuzzy set theory, and expanded into decision-making processes Fuzzy set theory provides a valuable framework for modelling and addressing vagueness and uncertainty within a mathematical model. By incorporating fuzzy logic and fuzzy sets, it becomes possible to represent and manipulate imprecise or uncertain information more effectively [3]. This has led to the development of real-life applications that utilize fuzzy reasoning to extend beyond traditional crisp interpretations. By embracing fuzziness, these applications are able to capture and handle the inherent uncertainty present in many real-world scenarios. Whether it is in decision-making processes, system modelling, or pattern recognition, fuzzy set theory offers a powerful tool to model and reason with imprecise and uncertain information, providing a more accurate representation of reality [4].

Ordinary Differential Equations (ODEs) are extensively utilized in applied sciences and engineering to represent physical

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processes. Researchers have extensively utilized classical ordinary differential equations to enhance the comprehensibility of numerous subjects under investigation. Data pertaining to physical phenomena sometimes encompasses uncertainties arising from variations in data collection, measurement procedures, experimental processes, and the establishment of initial values. Numerous efforts have been undertaken to quantify and elucidate the uncertainty. One approach yields a fuzzy set, resulting in fuzzy differential equations (FDEs). In a fractional integro-differential inclusion (FIDI), the unknown function is defined on a given interval, and the inclusion involves a set-valued operator that contains both fractional derivatives and integrals [37].

Historically, [5] provided the approach for solving FDEs that it emerged as a focal point for engineers and scientists. [6] established the theorem regarding the uniqueness and existence of solutions for FDEs. [7,8] examined fuzzy initial value problems (FIVPs) for fuzzy ordinary differential equations (FODEs). Currently, scholars extensively study the theory and numerical applications of fuzzy mathematics involving first order FIVP used well-known existing methods such as Homotopy Perturbation Method (HPM) and VIM [36]. Additionally, [9] investigated the Cauchy for fractional differential equations (FDEs). Consequently, the examination and resolution of FDEs succeeded investigations into the theory and applications of derivatives and integrals of uncertain functions. It is an advantageous instrument for the mathematical modelling of real-world applications characterised by uncertainties or ambiguities. In recent years, FDEs have proven to be excellent and robust instruments for the mathematical modelling of various engineering and scientific processes [10]. Note that this development took place over the course of a few years. In addition, research on FDEs and other related topics are both active and extensive around the world [11-15]. This is due to the fact that FDEs and other related topics have the property of considering all the uncertain parameters. In contrast, simple differential models typically keep these parameters constants due to the fact that FDEs and other topics in the same general area have the property of considering all of the uncertain parameters. FDEs may arise from measurement error, deficient data collection, or inaccuracies in determining initial or boundary conditions [17-19]. As a result, fuzzy set theory is a powerful tool for dealing with such problems under uncertainty [20]. Furthermore, this research aligns with the United Nations' Sustainable Development Goal 9, which emphasizes the importance of enhancing scientific research and upgrading technological capabilities in industrial sectors. By developing more accurate approximate methods for solving Fuzzy Initial Value Problems (FIVPs), this study contributes to the robust mathematical frameworks necessary for modeling complex, uncertain engineering systems. Such advancements are fundamental to designing resilient infrastructure and optimizing industrial processes, directly supporting the global objective of fostering innovation through improved computational tools."

Hence, it is best to use appropriate and useful methods to develop a mathematical construction that can handle FDEs well and solve them. The key benefit of the aforementioned methodologies is that approximate solutions can be produced even when the problem does not have a closed-form solution or an exact solution. In addition, the approximation analytical technique can provide a direct solution to nth-order FDEs without being reduced to a first-order system, minimizing the amount of computational work required [21-23]. Fortunately, there exist methods in crisp differential equations that can ensure the convergence of solutions using the concept of the BCPM and the BSM. The methods-based control point technique with the ability to control the convergence of the solution of the FODEs. Several mathematical problems were effectively approximated using the BCPM and B-Spline method BSM, which were successfully applied in the crisp environment in several mathematical problems [24-28].

These methods are beneficial in overcoming the difficulty of ensuring convergence. Other than that, the approximate solution is gained in the form of a series that rapidly converges to the exact or closed-form solution. A few approximate methods offer a gentle way to ensure the convergence of the solution series. Using similar methods to find the solution to these equations has its generous compensation but not without its obstacles. Currently, scholars extensively study the theory and numerical applications of fuzzy mathematics involving first order FIVP used well- known existing methods for solving FIVP such as Homotopy Perturbation Method (HPM) and VIM [29-32] is in term infinite series expansion and the approximate analytical solution degree is obtained using closed form or truncated form. These will lead to the possibility of truncation errors. Meanwhile for the BCPM and BSM methods, degree of the approximate solutions depends on the number of control points which is the exact degree of solution and will avoid the truncation errors as in the previous one. Therefore, there should be more investigation into approximate methods for fuzzy initial value problems (FIVPs) that have not been used yet [31-35].

The purpose of this study is to modify new forms of efficient approximate techniques. The purpose of this study is to modify new forms of efficient approximate techniques. These techniques include BCPM and BSM, which solve and analyze the approximate solution of nonlinear FIVPs, followed by fuzzy and comparison analysis. Additionally, the illustration convergence of these methods and error analysis are presented in detail. To our knowledge, no research has yet employed the Bézier control point's technique (BCPM) or the B-Spline method (BSM) for the analytical or approximation solution of first order nonlinear FIVP. We begin the discussion of defuzification of general first FIVP [26] with crisp variable t given as follows:

$$\tilde{u}'(t) = \tilde{f}\big(t, \tilde{u}(t)\big), \quad t \in [t_0, T]$$
$$\tilde{u}(t_0) = \tilde{u}_0 \qquad (1)$$

As stated in Eq. (1), the fuzzy functions [28] are represented by $\tilde{u}(t)$ and $\tilde{f}$; the fuzzy Hukuhara differentiable [28] of $\tilde{u}(t)$ is represented by $\tilde{u}'(t)$; and the fuzzy initial condition, which is identical to the fuzzy number $\tilde{u}\,\tilde{u}_0$ [29], is represented by $\tilde{u}(t_0)$. After that, the fuzzy function [30] that relates to $\tilde{u}(t)$can be represented as $[\tilde{u}]_z = [\underline{u}, \overline{u}]_z$, where $t \in [t_0, T]$ and $z \in [0,1]$. According to [29], the z – level set of $\tilde{u}(t)$ is expressed as:

$$[\tilde{u}(t)]_z = [\underline{u}(t;z), \overline{u}(t;z)], \tag{2}$$

$$[\tilde{u}(t_0)]_z = [\underline{u}(t_0;z), \overline{u}(t_0;z)]. \tag{3}$$

Also, we can represent

$$\left[\tilde{f}(t,\tilde{u})\right]_z = \left[\underline{f}(t,\tilde{u};z), \overline{f}(t,\tilde{u};z)\right], \tag{4}$$

with

$$\begin{cases} \underline{f}(t,\tilde{u};z) = \mathcal{F}[t,\underline{u},\overline{u}]_z, \\ \overline{f}(t,\tilde{u};z) \ = \mathcal{G}[t,\underline{u},\overline{u}]_z. \end{cases} \tag{5}$$

Because of $\tilde{u}'(t) = \tilde{f}\big(t,\tilde{u}(t)\big)$ and from the extension principle [31] of fuzzy function we obtain the following membership function:

$$\begin{cases} \underline{f}(t,\tilde{u}(t;z)) = min\{\tilde{f}\big(t,\widetilde{M}(z)\big) \big| \widetilde{M}(z) \in \tilde{u}(t;z)\} \\ \overline{f}(t,\tilde{u}(t;z)) = max\{\tilde{f}\big(t,\widetilde{M}(z)\big) \big| \widetilde{M}(z) \in \tilde{u}(t;z)\}, \end{cases} \tag{6}$$

where

$$\begin{cases} \underline{f}(t,\tilde{u}(t;z)) = \mathcal{F}\left(t,\underline{u}(t;z),\overline{u}(t;z)\right) = \mathcal{F}\big(t,\tilde{u}(t;z)\big) \\ \overline{f}(t,\tilde{u}(t;z)) = \mathcal{G}\left(t,\underline{u}(t;z),\overline{u}(t;z)\right) = \mathcal{G}\big(t,\tilde{u}(t;z)\big). \end{cases} \tag{7}$$

The fuzzy analysis of the proposed methods for the solution of Eq. (1) can be determined by using the above defuzzifications technique. Note: In the technique, the fuzzy function is in term of is $\tilde{f}\big(t,\tilde{u}(t)\big)$. However, our current study will also include the fuzzy numbers of coefficients as follows

$$\tilde{u}'(t) = \widetilde{W}\tilde{g}\big(t,\tilde{u}(t)\big) + \tilde{V}L(t), \tag{8}$$

where $\widetilde{W}$ and $\tilde{V}$ the the fuzzy numbers.

2 Fuzzy Analysis of Bezier Curve for FIVPs

We have decided to use the polynomial solution of Eq. (2) in degree m Bezier curve, which is defined on a fuzzy domain that is presented in the form of Eq. (1), that is, in order to get and examine the approximate solution of Eq. (1) by the use of BCM based on the analysis that was conducted in [32]. $\tilde{y}(t,z) = \tilde{P}(t;z)$, where $\tilde{y}(t,r) = \Big(\underline{y}(t;z), \overline{y}(t;z)\Big)$, we have,

$$\begin{cases} \underline{y}(t;r) = \sum_{i=0}^{m}[\underline{a_i}]_r \underline{B_i^m}(t;r), \\ \overline{y}(t;r) = \sum_{i=0}^{m}[\overline{a}_i]_r \overline{B}_i^m(t;r), \end{cases} \qquad 0 \le r \le 1, t_0 \le t \le T, \tag{9}$$

where $[\underline{a_i}]_r$ and $[\overline{a}_i]_r$ are necessary to evaluate the Bezier control points. Substitute Eq. (9) into Eq. (1) and the residual purposes can be retrieved. i.e.

$$\begin{cases} \underline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{m}[\underline{a_i}]_r \underline{B_i^m}(t;r)\right) - \left(\sum_{i=0}^{m}[\underline{a_i}]_z \underline{B_i^m}(t;z)\right), \\ \overline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{m}[\overline{a}_i]_r \overline{B}_i^m(t;z)\right) - \left(\sum_{i=0}^{m}[\overline{a}_i]_z \overline{B}_i^m(t;z)\right). \end{cases} \tag{10}$$

From Eqs (9-10), the least square error of residual can be minimized by setting,

$$\begin{cases} \int_{t_0}^{T} \underline{R}(t;z)\frac{d\underline{R}(t;z)}{d\underline{a}_i}dt = 0, \\ \int_{t_0}^{T} \overline{R}(t;z)\frac{d\overline{R}(t;z)}{d\overline{a}_i}dt = 0, \end{cases} \tag{11}$$

or

$$\begin{cases}\left(\underline{R}(t;z),\frac{d\underline{R}(t;z)}{d\underline{a}_i}\right)=0,\\ \left(\overline{R}(t;z),\frac{d\overline{R}(t;z)}{d\overline{a}_i}\right)=0.\end{cases} \tag{12}$$

This will lead to a linear system, which can be solved for $\left[\underline{a}_i\right]_z$ and $[\overline{a}_i]_z$for each level sets $r\in[0,1]$. Next, substitute the coefficients in Eq. (9), approximate solution of Eq. (2) satisfying the triangular numbers [29] properties will be obtained.

3 Reliability Measures

We select the polynomial solution of Eq. (2) in degree m as the B-Spline Method (BSM) in order to get and analyze the approximate solution of Eq. (1) via BSM, as stated in [22]. The fuzzy domain is supplied in the form of (1), which is: $\tilde{y}(t,z)=\tilde{P}(t;z)$, where $\tilde{y}(t,z)=\left(\underline{y}(t,;z),\overline{y}(t;z)\right)$, we have,

$$\begin{cases}\underline{y}(t;z)=\sum_{i=0}^{m}\left[\underline{a}_i\right]_r\underline{N}_{i,j}(t;z),\\ \overline{y}(t;z)=\sum_{i=0}^{m}[\overline{a}_i]_r\overline{N}_{i,j}(t;z),\end{cases}\quad 0\le z\le 1, t_0\le t\le T\,, \tag{13}$$

where $\left[\underline{a}_i\right]_z$ and $[\overline{a}_i]_z$ are necessary to evaluate the BSM. Substitute Eq. (10) into Eq. (5) and the residual purposes can be retrieved. i.e.

$$\begin{cases}\underline{R}(t;z)=\frac{d}{dt}\left(\sum_{i=0}^{m}\left[\underline{a}_i\right]_r\underline{N}_{i,j}(t;z)\right)-\left(\left[\underline{a}_i\right]_z\underline{N}_{i,j}(t;r)\right),\\ \overline{R}(t;z)=\frac{d}{dt}\left(\sum_{i=0}^{m}[\overline{a}_i]_z\overline{N}_{i,j}(t;z)\right)-\left(\sum_{i=0}^{m}[\overline{a}_i]_z\overline{N}_{i,j}(t;r)\right).\end{cases} \tag{14}$$

Thus, the least square error of residual can be minimized by setting,

$$\begin{cases}\left(\underline{R}(t;z),\frac{d\underline{R}(t;z)}{d\underline{a}_i}\right)=0,\\ \left(\overline{R}(t;z),\frac{d\overline{R}(t;z)}{d\overline{a}_i}\right)=0.\end{cases} \tag{15}$$

These will lead to a linear system which can be solved for $\left[\underline{a}_i\right]_z$ and $[\overline{a}_i]_z$ for each value of the level set $z\in[0,1]$. Substitute the coefficients in Eq. (9), approximate solution of Eq. (2) satisfying triangular fuzzy number properties will be obtained.

4 Fuzzy Convergence Analysis

Next proceed to demonstrate that the square of the L_2– norm of a polynomial over can be approximated by taking the average of the squares of the Bézier coefficients of the polynomial, provided that the degree of the polynomial is elevated. This is in addition to the convergence of control points approaches.

Accordance with [32], we provide a definition for the following theorem. When the control point-based approaches are applied to the general fuzzy differential equation shown below, we investigate how they converge such that:

$$\begin{aligned}\mathcal{L}[\tilde{u}(t;z)]+\mathcal{N}[\tilde{u}(t;z)\tilde{\varphi}_i(t;z)]&=\tilde{f}(t;z), t\in\Gamma,\\ \mathcal{B}\left(\tilde{u},\frac{\partial\tilde{u}}{\partial n}\right)&=0, t\in\Gamma,\end{aligned} \tag{16}$$

where for $z\in[0,1]$ we have:

$\tilde{u}(t;z)=\left[\underline{u}(t;z),\overline{u}(t;z)\right]$,

$\tilde{f}(t;z)=\left[\underline{f}(t;z),\overline{f}(t;z)\right]$, and

$\tilde{\varphi}_i(t;z)=\left[\underline{\varphi}_i(t;z),\overline{\varphi}_i(t;z)\right]$ for $i=1,2,\dots,n$.

In Eq. (8), the boundary operator is denoted by $\mathcal{B}$, and the domain Ω includes the boundary Γ, $\mathcal{L}$ is a general differential operator, $\mathcal{N}$ in the nonlinear operator and $\tilde{f}(t;z)$, $\tilde{\varphi}_i(t;z)$ are the known fuzzy analytical polynomials in crisp variable t. For any type of ordinary fuzzy equations, this analysis might be carried out such that

$\mathcal{N}[\tilde{u}(t;z)\tilde{\varphi}_i(t;z)]=\tilde{u}^{(n-1)}(t;z)\tilde{\varphi}_n(t;z)+\tilde{u}^{(n-2)}(t;z)\tilde{\varphi}_{n-1}(t;z)+\tilde{u}^{(n-3)}(t;z)\tilde{\varphi}_{n-2}(t;z)+\cdots+$

$\tilde{u}''(t;z)\tilde{\varphi}_3(t;z) + \tilde{u}'(t;z)\tilde{\varphi}_2(t;z) + \tilde{u}(t;z)\tilde{\varphi}_1(t;z) + \tilde{u}(t;z)\tilde{\varphi}_0(t;z),$ (17)

where $\tilde{u}(t;z)\tilde{\varphi}_0(t;z)$be the nonlinear terms of Eq. (17). Initially, we demonstrate that using degree elevation, the square of the L_2-norm of a polynomial over Γ may be approximated by the average of the squares of the Bézier coefficients. As per illustrated in [32], we establish the subsequent theorem.

Theorem 1: If Eq. (8) has a unique, $C^{\mathcal{L}}$ continuous solution $\tilde{u}^*(t;z)$, then for every fuzzy level set $z \in [0,1]$the approximate solution obtained by the control point method converges to the exact solution $\tilde{u}^*(t;z)$ as the degree of the approximate solution tends to infinity.

Proof: The proof is divided into several steps:

Given an arbitrary small positive number ε > 0 then we quickly find a polynomial $\tilde{S}_N(t;z)$ of degree-N such that $\left\|\frac{d^i\tilde{S}_N(t;z)}{dt^i} - \frac{d^i\tilde{u}^*(t;z)}{dt^i}\right\|_\infty < \frac{\varepsilon}{2},\ \ i = 0,1,2,$ where $\|.\|_\infty$ stand for L_∞-norm over Γ. Generally, $\tilde{S}_N(t;z)$ does not satisfy the boundary conditions. After a small perturbation with a linear polynomial $\alpha t + \beta$ we can obtain another polynomial $\tilde{P}_N(t;z) = \tilde{S}_N(t;z) + (\alpha t + \beta)$, such hat $\tilde{P}_N(t;z)$ satisfies the boundary conditions Γ and $\left\|\frac{d^i\tilde{P}_N(t;z)}{dt^i} - \frac{d^i\tilde{u}^*(t;z)}{dt^i}\right\|_\infty < \varepsilon,\ \ i = 0,1,2,\dots,n.$

Thus, we have estimation on the residual form:

$$\begin{aligned}\left\|\mathcal{L}\tilde{P}_N(t;z) - \tilde{f}(t;z)\right\|_\infty &= \left\|\mathcal{L}\left(\tilde{P}_N(t;z) - \tilde{u}^*(t;z)\right)\right\|_\infty \\ &\le \|\tilde{\varphi}_n(t;z)\|_\infty \left\|\tilde{P}_N{}^{(n-1)}(t;z) - \tilde{u}^{*(n-1)}(t;z)\right\|_\infty + \|\tilde{\varphi}_{n-1}(t;z)\|_\infty \left\|\tilde{P}_N{}^{(n-2)}(t;z) - \tilde{u}^{*(n-2)}(t;z)\right\|_\infty \\ &+ \cdots + \|\tilde{\varphi}_0(t;z)\|_\infty \left\|\tilde{P}_N(t;z) - \tilde{u}^*(t;z)\right\|_\infty \le C_1\varepsilon,\end{aligned}$$

where $C_1\|\tilde{\varphi}_n(t;z)\|_\infty + \|\tilde{\varphi}_{n-1}(t;z)\|_\infty + \cdots + \|\tilde{\varphi}_0(t;z)\|_\infty$ is a constant.

Represent the Residual in Bézier Form

$$\mathcal{R}\left(\tilde{P}_N(t;z)\right) = \mathcal{L}\tilde{P}_N(t;z) - \tilde{f}(t;z) = \sum_{i=0}^{m}\left[\tilde{d}_{i,m}\right]_r \tilde{B}_i^m(t;z).$$

Then from Lemma1 in [32], there exists an integer $M(\ge N)$ such that when $m > M$, we have $\left|\frac{1}{m+1}\sum_{i=0}^{m}\left[\tilde{d}_{i,m}\right]_z - \int_t^\Gamma \left(\mathcal{R}\left(\tilde{P}_N(t;z)\right)\right)^2\right| < \varepsilon$, which gives

$$\frac{1}{1+m}\sum_{i=0}^{m}\left[\tilde{d}_{i,m}\right]_z < \varepsilon + \int_t^\Gamma \left(\mathcal{R}\left(\tilde{P}_N(t;z)\right)\right)^2 \le \varepsilon + C_1^2\varepsilon.$$

Now, let $\tilde{v}(t;z)$ is the approximate solution Eq. (8) with the degree $m(\ge M)$, obtained by the control point method. Let

$$\mathcal{R}\left(\tilde{v}(t;z)\right) = \mathcal{L}\tilde{v}(t;z) - \tilde{f}(t;z) = \sum_{i=0}^{k}\left[\tilde{d}_{i,k}\right]_z \tilde{B}_i^k(t;z),$$

where the degree $k \ge m \ge M$. Now, the difference of the approximate solution $\tilde{v}(t;z)$ and the exact solution $\tilde{u}^*(t;z)$ for $\Gamma = [t_0, T]$ we have:

$$\begin{aligned}\|\tilde{v}(t;z) - \tilde{u}^*(t;z)\|_{H^2\Gamma} &:= \int_t^\Gamma \sum_{j=0}^{n}\left|\frac{d^j\tilde{v}(t;z)}{dt^j} - \frac{d^j\tilde{u}^*(t;z)}{dt^j}\right|^2 dt \\ &\le C\left(|\tilde{v}(t_0;z) - \tilde{u}^*(t_0;z)| + |\tilde{v}(T;z) - \tilde{u}^*(T;z)| + \left\|R\left(\tilde{v}(t;z) - \tilde{u}^*(t;z)\right)\right\|_2\right) \\ &= C\int_{t_0}^{T}\left(\sum_{i=0}^{k}\left[\tilde{C}_{i,k}\right]_z \tilde{B}_i^k(t;z)\right)^2 dt \le \frac{C}{k+1}\sum_{i=0}^{k}\left[\tilde{C}_{i,k}^2\right]_z\end{aligned}$$

where C is a constant. Since the control point method minimizes the average of squares of the residual's Bézier coefficients, the average of $\tilde{v}(t;z)$ is smaller than that of $\tilde{P}_m(t;z)$. Therefore

$\|\tilde{v}(t;z)-\tilde{u}^*(t;z)\|_{H^2[t_0,T]} \le \frac{C}{k+1}\sum_{i=0}^{k}\left[\tilde{C}_{i,k}^2\right]_z \le \frac{C}{k+1}\sum_{i=0}^{k}\left[\tilde{d}_{i,k}^2\right]_z \le \frac{C}{m+1}\sum_{i=0}^{k}\left[\tilde{d}_{i,m}^2\right]_z \le C(\varepsilon + C_1^2\varepsilon^2)$.

This completes the proof.

The above theory fits the BSM since this method involves control points.

5 Numerical Examples

Three numerical examples are introduced to demonstrate the ability of BCPM representation with BSM and applied first-order nonlinear FIVPs. For the remainder of this work, $\tilde{E}_z$ will denote the absolute error, defined as formals:

$$\tilde{E}_z(t,r) = \left|\tilde{U}(t;z)-\tilde{u}(t;z)\right| = \begin{Bmatrix} \underline{E}(t;z) = \left|\underline{U}(t;z)-\underline{u}(t;z)\right| \\ \overline{E}(t;z) = \left|\overline{U}(t;z)-\overline{u}(t;z)\right| \end{Bmatrix} \tag{18}$$

Where $\widetilde{U}(t;z)$ refer to the exact solution and $\tilde{u}(t;z)$ is the approximate solution of the given equation

Example 5.1 [33]: For the purpose of this example, both BCPM and BSN will be tested and compared with the exact solution of the nonlinear fuzzy problem, this will be done with varying degrees of approximate solution. Consider the following first-order nonlinear FIVP

$$\begin{cases} \tilde{u}'(t) = \tilde{u}^2(t)+1, \ t\in[0,1], \\ \tilde{u}(0) = (0.1z-0.1, 0.1-0.1z), \qquad 0\le z\le 1. \end{cases} \tag{19}$$

According to [33], the exact solution of Eq. (19) is:

$$\begin{cases} \underline{U}(t,z) = tan\left(t - tan^{-1}\left(\frac{1-z}{10}\right)\right), \\ \overline{U}(t,z) = tan\left(t + tan^{-1}\left(\frac{1-z}{10}\right)\right), \qquad 0\le z\le 1. \end{cases} \tag{20}$$

According to section 3, Eq. (19) can be written in defuzzification form

$$\begin{cases} \underline{u}'(t;z) = \underline{u}^2(t;z)+1, \ \underline{u}(0;z) = 0.1z-0.1, \\ \overline{u}'(t;z) = \overline{u^2}(t;z)+1, \ \overline{u}(0;z) = 0.1-0.1z. \end{cases} \qquad 0\le z\le 1. \tag{21}$$

For numerical implementation, we consider the approximate solutions using BCPM in Eq. (19) of degrees 6 ($m=6$) and 8 ($m=8$), respectively. To obtain the residual function, the detailed results are as follows:

BCPM Solution

5.1 Degree-6 BCPM

Let

$$\begin{cases} \underline{u}(t;z) = \sum_{i=0}^{6}\underline{a_i}\,\underline{B_i^6}(t;z), \\ \overline{u}(t;z) = \sum_{i=0}^{6}\overline{a_i}\,\overline{B_i^6}(t;z), \end{cases} \tag{22}$$

where $0\le z\le 1$ and $\underline{a_i}$ and $\overline{a_i}, i=0,\dots,6$, are the Bézier control points that need to be determined, substitute Eq. (4.33) into Eq. (4.32), and the residual functions can be obtained, i.e.

$$\begin{cases} \underline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{6}\underline{a_i}\,\underline{B_i^6}(t;z)\right) - \left(\sum_{i=0}^{6}\underline{a_i}\,\underline{B_i^6}(t;z)\right)^2 - 1, \\ \overline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{6}\overline{a_i}\,\overline{B_i^6}(t;z)\right) - \left(\sum_{i=0}^{6}\overline{a_i}\,\overline{B_i^6}(t;z)\right)^2 - 1. \end{cases} \tag{23}$$

The right side of Eq. (22) constitutes a polynomial of degree six; hence, the residual function is expressed in Eq. (23) as follows.

$$\begin{cases} \underline{R}(t;z) = \sum_{i=0}^{12}\underline{b_i}\,\underline{B_i^{12}}(t;z), \\ \overline{R}(t;z) = \sum_{i=0}^{12}\overline{b_i}\,\overline{B_i^{12}}(t;z). \end{cases} \tag{24}$$

From the fuzzy initial condition in Eq. (4.32) we get $\underline{a_0}(z) = 0.1z-0.1$ and remaining control points $\underline{a_1}(z)$, $\underline{a_2}(z)$, $\underline{a_3}(z)$,

$\underline{a_4}(z)$, $\underline{a_5}(z)$ and $\underline{a_6}(z)$ are determined. Substitute $\underline{a_0}(z)$ in Eq. (24), we obtain the following system of nonlinear equations:

$0.01z^2 + 0.58z + 0.41 = 6.0\underline{a_1}$,

$1.1\underline{a_1} + 2.5a3 = z(0.1\underline{a_1} + 0.35) + 0.65$,

$2.318\underline{a_2} + 0.909\underline{a_3} = 1.273\underline{a_1} + 0.546\underline{a_1}^2 + z\left(4.453 - 41\underline{a_1} + 0.0455\underline{a_2} + 0.191\right) + 0.809$,

$1.909\underline{a_1} + 0.8182\underline{a_1 a_2} + z\left(1.336 - 40\underline{a_1} + 0.01818\underline{a_3} + 0.0955\right) + 0.905 = 3.339 - 42z^2 + 0.9545\underline{a_2} + 1.655\underline{a_3} + 0.2727\underline{a_4}$,

$1.336 - 41z^2 + 1.697\underline{a_3} + 0.7939\underline{a_4} + 0.06061\underline{a_5} = 0.4545\underline{a_2}^2 + 0.4242\underline{a_2} + 1.697\underline{a_1} + 0.4848\underline{a_1}a4 + z(3.621 - 40\underline{a_1} + 0.006061\underline{a_4} + 0.04242) + 0.9576$,

$3.525 - 41z^2 + 1.061\underline{a_3} + 1.326\underline{a_4} + 0.2591\underline{a_5} + 0.007576\underline{a_6} = 1.167\underline{a_1} + 1.326\underline{a_2} + 0.2273\underline{a_1 a_4} + 0.7576\underline{a_2 a_3} + z\left(8.015 - 40\underline{a_1} + 0.00152\underline{a_5} + 0.01591\right) + 0.9841$,

$0.4329\underline{a_3}^2 + 0.6364\underline{a_1} + 1.591\underline{a_2} + 0.07792\underline{a_1 a_5} + 0.487\underline{a_2 a_4} + z\left(1.38 - 39\underline{a_1} + 3.18e - 42\underline{a_5} + 0.000217\underline{a_6} + 0.004545\right) + 0.996 = 7.792 - 41z^2 + 1.591\underline{a_4} + 0.6364\underline{a_5} + 0.04567\underline{a_6}$,

$0.258\underline{a_1} + 1.326\underline{a_2} + 1.061\underline{a_3} + 0.0152\underline{a_1 a_6} + 0.2273\underline{a_2 a_5} + 0.758\underline{a_3 a_4} + z\left(1.766 - 39\underline{a_1} + 2.226 - 41\underline{a_5} + 0.000758\right) + 0.9992 = 1.577 - 40z^2 + 1.326\underline{a_4} + 1.167\underline{a_5} + 0.1591\underline{a_6}$,

$0.455\underline{a_4}^2 + 0.06061\underline{a_1} + 0.788\underline{a_2} + 1.697\underline{a_3} + 0.06061\underline{a_2 a_6} + 0.485\underline{a_3 a_5} + z\left(1.437e - 39\underline{a_1} + 8.905 - 41\underline{a_5} + 2.28 - 38\right) + 1.0 = 3.057 - 40z^2 + 0.4242\underline{a_4} + 1.697\underline{a_5} + 0.4242\underline{a_6}$,

$3.556 - 37\underline{a_1} + 0.2727\underline{a_2} + 1.636\underline{a_3} + 0.9545\underline{a_4} + 0.1818\underline{a_3 a_6} + 0.8182\underline{a_4 a_5} + z\left(1.069 - 40\underline{a_1} + 2.672 - 40\underline{a_5} + 4.189 - 38\right) + 1.0 = 5.844 - 40z^2 + 1.909\underline{a_5} + 0.9545\underline{a_6}$,

$1.124 - 39z^2 + 1.558 - 39\underline{a_1}z + 1.273\underline{a_5} + 1.909a7 = 9.575 - 37\underline{a_1} + 0.9091\underline{a_3} + 2.273\underline{a_4} + 0.4545\underline{a_4 a_6} + z\left(6.679 - 40\underline{a_5} + 9.404 - 38\right) + 0.5455\underline{a_5}^2 + 1.0$,

$2.204 - 39z^2 + 1.714 - 39\underline{a_1}z + 3.5\underline{a_6} = 2.132 - 36\underline{a_1} + 2.5\underline{a_4} + \underline{a_5} + \underline{a_5 a_6} + z\left(1.224 - 39\underline{a_5} + 2.508 - 37\right) + 1.0$,

$\underline{a_6}^2 + 4.326 - 36\underline{a_1} + 6.0\underline{a_5} + 7.112 - 37z + 1.0 = 4.408 - 39z^2 + 6.0\underline{a_6}$.

By using Isqnonlinear command in Matlab 2022b, the unknown lower control points are obtained

$\underline{a_1} = 0.097607637890536322333723262545391z + 0.067823354446407843609812005070125$,

$\underline{a_2} = 0.10581962363674354832276947036007z + 0.23242398762555951940989018567052$,

$\underline{a_3} = 0.10689050745940348319251711473044z + 0.39541297159946303674260548177699$,

$\underline{a_4} = 0.145663054037326933354279390187z + 0.60875290673610693570338980862289$,

$\underline{a_5} = 0.15754940261261851652818677393952z + 0.83010809585562828072369256915408$,

$\underline{a_6} = 0.29379991182216946832284065749263z + 1.2605511565741553958730492013274$.

From the fuzzy initial condition in Eq. (19) we get $\overline{a_0}(z) = 0.1 - 0.1z$ and remaining control points $\overline{a_1}(z)$, $\overline{a_2}(z)$, $\overline{a_3}(z)$, $\overline{a_4}(z)$, $\overline{a_5}(z)$ and $\overline{a_6}(z)$ are determined. Substitute $\overline{a_0}(z)$ in Eq. (22), we obtain the following system of nonlinear equations:

$0.01z^2 + 1.61 = 6.0\overline{a_1} + 0.62z$,

$0.9\overline{a_1} + 2.5\overline{a_2} + z(0.1\overline{a_1} + 0.35) = 1.35$,

$2.23\overline{a_2} + 0.91\overline{a_3} + z(4.45 - 41\overline{a_1} + 0.046\overline{a_2} + 0.191) = 0.545\overline{a_1}^2 + 1.27\overline{a_1} + 1.19,$

$1.91\overline{a_1} + 0.818\overline{a_1}\overline{a_2} + 1.1 = 3.34 - 42z^2 + (1.34 - 40\overline{a_1} + 0.0182\overline{a_3} + 0.0955)z + 0.955\overline{a_2} + 1.62\overline{a_3} + 0.273\overline{a_4},$

$1.34 - 41z^2 + (3.62 - 40\overline{a_1} + 0.00606\overline{a_4} + 0.0424)z + 1.7\overline{a_3} + 0.782\overline{a_4} + 0.0606\overline{a_5} = 0.455\overline{a_2}^2 + 0.424\overline{a_2} + 1.7\overline{a_1} + 0.485\overline{a_1}\overline{a_3} + 1.04,$

$3.52 - 41z^2 + (8.01 - 40\overline{a_1} + 0.00152\overline{a_5} + 0.0159)z + 1.06\overline{a_3} + 1.33\overline{a_4} + 0.256\overline{a_5} + 0.00758\overline{a_6} = 1.17\overline{a_1} + 1.33\overline{a_2} + 0.227\overline{a_1}\overline{a_4} + 0.758\overline{a_2 a_3} + 1.02,$

$0.433\overline{a_3}^2 + 0.636\overline{a_1} + 1.59\overline{a_2} + 0.0779\overline{a_1}a6 + 0.487\overline{a_2 a_4} + 1.0 = 7.79e - 41z^2 + (1.38 - 39\overline{a_1} + 3.18 - 42\overline{a_5} + 2.16 - 4\overline{a_6} + 0.00455)z + 1.59\overline{a_4} + 0.636\overline{a_5} + 0.0452\overline{a_6},$

$0.258\overline{a_1} + 1.33\overline{a_2} + 1.06\overline{a_3} + 0.0152\overline{a_1}\overline{a_6} + 0.227\overline{a_2}\,\overline{a_5} + 0.758\overline{a_3}\,\overline{a_4} + 1.0 = 1.58 - 40z^2 + (1.77 - 39\overline{a_1} + 2.23 - 41\overline{a_5} + 7.58 - 4)z + 1.33\overline{a_4} + 1.17\overline{a_5} + 0.159\overline{a_6},$

$0.455\overline{a_4}^2 + 0.0606\overline{a_1} + 0.788\overline{a_2} + 1.7\overline{a_3} + 0.0606\overline{a_2}\,\overline{a_6} + 0.485\overline{a_3}\,\overline{a_5} + 1.0 = 3.06 - 40z^2 + (1.44 - 39\overline{a_1} + 8.91 - 41\overline{a_5} + 2.24 - 38)z + 0.424\overline{a_4} + 1.7\overline{a_5} + 0.424\overline{a_6},$

$0.273\overline{a_2} + 1.64\overline{a_3} + 0.955\overline{a_4} + 0.182\overline{a_3}\,\overline{a_6} + 0.818\overline{a_4}\,\overline{a_5} + 1.0 = 5.84 - 40z^2 + (1.07 - 40\overline{a_1} + 2.67 - 40\overline{a_5} + 4.45 - 38)z + 3.56 - 37\overline{a_1} + 1.91\overline{a_5} + 0.955\overline{a_6},$

$1.12 - 39z^2 + (6.68 - 40\overline{a_5} + 9.69 - 38)z + 9.57 - 37\overline{a_1} + 1.27\overline{a_5} + 1.91\overline{a_6} = 0.545\overline{a_5}^2 + 0.909\overline{a_3} + 2.27\overline{a_4} + 0.455\overline{a_4}\,\overline{a_6} + 1.56 - 39\overline{a_1}z + 1.0,$

$2.2 - 39z^2 + (1.22 - 39\overline{a_5} + 2.35 - 37)z + 2.13 - 36\overline{a_1} + 3.5\overline{a_6} = 2.5\overline{a_4} + \overline{a_5} + \overline{a_5}\,\overline{a_6} + 1.71 - 39\overline{a_1}z + 1.0,$

$\overline{a_6}^2 + 6.0\overline{a_5} + 1.0 = 4.41 - 39z^2 + 6.17 - 37z + 4.33 - 36\overline{a_1} + 6.0\overline{a_6}.$

By using Isqnonlinear command in Matlab 2012b, the unknown upper control points are obtained

$\overline{a_1} = 0.26438943466198472354022896979586 - 0.0989584423250427225315917212356z,$

$\overline{a_2} = 0.45212834246020250361652870196849 - 0.11388473119790543108820202178322z,$

$\overline{a_3} = 0.6072455656400680412687620446377 - 0.10494208658120418586889854850597z,$

$\overline{a_4} = 0.92743592438549837808636766567361 - 0.17301996361207994112874075653963z,$

$\overline{a_5} = 1.1528502304025891600502973233233 - 0.16519273193435179969412729406031z,$

$\overline{a_6} = 1.9381270839607176448282643832499 - 0.38377601556442053620799015334342z.$

From the above control points the approximate solutions function of BCPM obtained via Matlab 2012b for all z ∈ [0,1] is given below:

$\underline{u}(t,z) = 0.1z - 1.0t^5(1.0321z + 0.8679) - 1.0t(0.014354z - 1.0069) + t^4(0.93882z + 0.75527) + t^2(0.15907z - 0.048341) + t^6(0.49729z + 0.48236) - 1.0t^3(0.35491z - 0.032221) - 0.1.$

$\overline{u}(t,z) = t^5(2.3672z - 4.2673) - 1.0t^4(2.1109z - 3.805) - 1.0t^6(1.1036z - 2.0832) - 0.1z + t^3(0.79674z - 1.1194) - 1.0t^2(0.23952z - 0.35024) + t(0.0062493z + 0.98634) + 0.1.$

The outcomes and precision of the sixth-degree approximate solutions of Eq. (19) at $t = 1$, obtained by BCPM for various fuzzy level sets ranging from 0 to 1, are presented in Tables 1 and 2, and highlighted in Figures 1 and 2 as follows:

Table 1: Approximate and exact values for lower solutions of degree-6 BCPM for Eq. (19) at $t = 1$.

z	**BCPM**	**Exact Solution**	$\underline{E}(t;z)$ **of BCPM**
0	**1.260551156574155**	**1.261016102725077**	**4.649461509211861e-04**
0.2	**1.313063190339934**	**1.313727034166937**	**6.638438270034630e-04**
0.4	**1.368483404850189**	**1.369441041285457**	**9.576364352676947e-04**
0.6	**1.427026537759410**	**1.428422288413148**	**1.395750653737427e-03**
0.8	**1.488911657454789**	**1.490966858599166**	**2.055201144377739e-03**
1	**1.554351068396325**	**1.557407724654902**	**3.056656258577428e-03**

Table 2: Approximate and exact values for upper solutions of degree-6 BCPM for Eq. (19) at $t = 1$.

z	BCPM	Exact solution	$\overline{E}(t;z)$ of BCPM
0	1.938127083960715	1.963150263095175	2.502317913445995e-02
0.2	1.854175714601331	1.870452269733388	1.627655513205761e-02
0.4	1.773514520768108	1.784124258798744	1.060973803063625e-02
0.6	1.696578967128923	1.703531445981465	6.952478852542354e-03
0.8	1.623530901720750	1.628120679100946	4.589777380195725e-03
1	1.554351068396325	1.557407724654902	3.056656258577428e-03

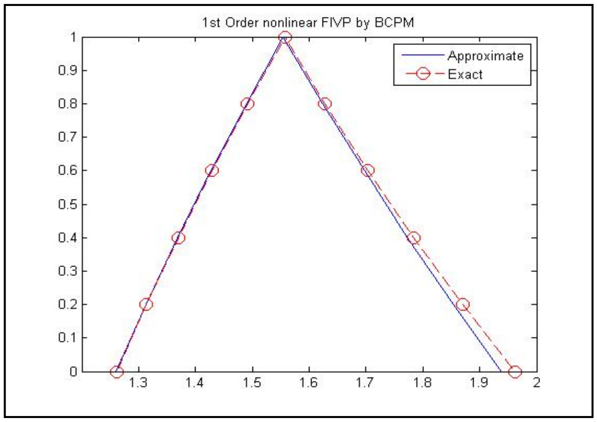

Fig. 1: Approximate and exact solutions of Eq. (19) at $t = 1$ and degree-6 BCPM for all $z \in [0,1]$.

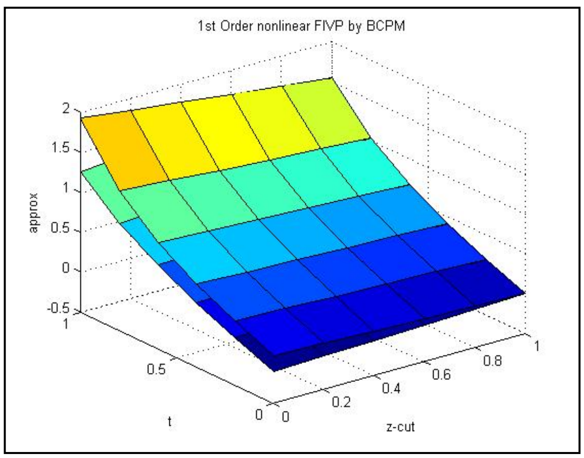

Fig. 2: Approximate solutions of Eq. (19) at $t \in [0,1]$ and degree-6 BCPM for all $z \in [0,1]$.

5.2 Degree-8 BCPM

Let

$$\begin{cases} \underline{u}(t;z) = \sum_{i=0}^{8} \underline{a_i}\, \underline{B_i^8}(t;z), \\ \overline{u}(t;z) = \sum_{i=0}^{8} \overline{a_i}\, \overline{B_i^8}(t;z), \end{cases} \tag{25}$$

where $0 \leq z \leq 1$ and $\underline{a_i}$ and $\overline{a_i}, i = 0, \dots, 8.$ are the Bézier control points that need to be determined, substitute Eq. (25) into Eq. (19) and the residual functions can be obtained, i.e.

$$\begin{cases} \underline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{8} \underline{a_i}\, \underline{B_i^8}(t;z)\right) - \left(\sum_{i=0}^{8} \underline{a_i}\, \underline{B_i^8}(t;z)\right)^2 - 1, \\ \overline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{8} \overline{a_i}\, \overline{B_i^8}(t;z)\right) - \left(\sum_{i=0}^{8} \overline{a_i}\, \overline{B_i^8}(t;z)\right)^2 - 1. \end{cases} \tag{26}$$

The right-hand side of Eq. (26) is a polynomial of degree-8, and therefore, the residual function can be represented in Eq. (19) as follows:

$$\begin{cases} \underline{R}(t;z) = \sum_{i=0}^{16} \underline{b_i}\ \underline{B_i^{16}}(t;z)\,, \\ \overline{\mathrm{R}}(t;z) = \sum_{i=0}^{16} \overline{b_i}\ \overline{B_i^{16}}(t;z). \end{cases} \tag{27}$$

From the fuzzy initial condition in Eq. (19) we get $\underline{\mathrm{a}_0}(z) = 0.1\mathrm{z} - 0.1$ and remaining control points $\underline{\mathrm{a}_1}(z)$, $\underline{\mathrm{a}_2}(z)$, $\underline{\mathrm{a}_3}(z)$, $\underline{\mathrm{a}_4}(z)$, $\underline{\mathrm{a}_5}(z)$, $\underline{\mathrm{a}_6}(z)$, $\underline{\mathrm{a}_7}(z)$ and $\underline{\mathrm{a}_8}(z)$ are determined. Substitute $\underline{\mathrm{a}_0}(z)$ in Eq. (26), we obtain the following system of nonlinear equations:

$$0.01z^2 + 0.78z + 0.21 = 8.0\underline{\mathrm{a}_1},$$

$$1.1\underline{\mathrm{a}_1} + 3.5\underline{\mathrm{a}_2} = z\left(0.1\underline{\mathrm{a}_1} + 0.45\right) + 0.55,$$

$$0.533\underline{\mathrm{a}_1}^2 + 1.8\underline{\mathrm{a}_1} + 3.06 - 42z^2 + \left(0.0467\underline{\mathrm{a}_2} + 0.24\right)z + 0.76 = 2.85\underline{\mathrm{a}_2} + 1.4\underline{\mathrm{a}_3},$$

$$0.9\underline{\mathrm{a}_2} + 2.22\underline{\mathrm{a}_3} + 0.5\underline{\mathrm{a}_4} = 9.18 - 42z^2 + \left(0.02\underline{\mathrm{a}_3} + 0.12\right)z + 2.4\underline{\mathrm{a}_1} + 0.8\underline{\mathrm{a}_1\mathrm{a}_2} + 0.88,$$

$$1.94\underline{\mathrm{a}_3} + 1.24\underline{\mathrm{a}_4} + 0.154\underline{\mathrm{a}_5} + 2.58 - 41\underline{\mathrm{a}_3}z = 0.431\underline{\mathrm{a}_2}^2 + 0.738\underline{\mathrm{a}_2} + 2.16 - 41z^2 + \left(0.00769\underline{\mathrm{a}_4} + 0.0554\right)z + 2.03\underline{\mathrm{a}_1} + 0.492\underline{\mathrm{a}_1}\,\underline{\mathrm{a}_3} + 0.945,$$

$$0.923\underline{\mathrm{a}_3} + 1.73\underline{\mathrm{a}_4} + 0.541\underline{\mathrm{a}_5} + 0.0385\underline{\mathrm{a}_6} + 1.29 - 40a4z = 4.41 - 41z^2 + \left(0.00256\underline{\mathrm{a}_5} + 0.0231\right)z + 1.38\underline{\mathrm{a}_1} + 1.62\underline{\mathrm{a}_2} + 0.256\underline{\mathrm{a}_1}\,\underline{\mathrm{a}_4} + 0.718\underline{\mathrm{a}_2}\,\underline{\mathrm{a}_3} + 0.977,$$

$$1.68\underline{\mathrm{a}_4} + 1.07\underline{\mathrm{a}_5} + 0.183\underline{\mathrm{a}_6} + 0.00699\underline{\mathrm{a}_7} + 3.88 - 40a4z = 0.392\underline{\mathrm{a}_3}^2 + 0.294\underline{\mathrm{a}_3} + 8.11 - 41z^2 + \left(1.88 - 40\underline{\mathrm{a}_1} + 6.99 - 4\underline{\mathrm{a}_6} + 0.00839\right)z + 0.797\underline{\mathrm{a}_1} + 1.76\underline{\mathrm{a}_2} + 0.112\underline{\mathrm{a}_1\mathrm{a}_5} + 0.49\underline{\mathrm{a}_2}\,\underline{\mathrm{a}_4} + 0.992,$$

$$1.03\underline{\mathrm{a}_4} + 1.53\underline{\mathrm{a}_5} + 0.485\underline{\mathrm{a}_6} + 0.0435\underline{\mathrm{a}_7} + 6.99 - 4\underline{\mathrm{a}_8} + 9.04 - 40\underline{\mathrm{a}_3}z = 1.39 - 40z^2 + \left(1.32 - 39\underline{\mathrm{a}_1} + 1.4 - 4\underline{\mathrm{a}_7} + 0.00252\right)z + 0.386\underline{\mathrm{a}_1} + 1.44\underline{\mathrm{a}_2} + 1.23\underline{\mathrm{a}_3} + 0.0392\underline{\mathrm{a}_1}\,\underline{\mathrm{a}_6} + 0.274a3\underline{\mathrm{a}_5} + 0.685\underline{\mathrm{a}_3}\,\underline{\mathrm{a}_4} + 0.997,$$

$$0.381\underline{\mathrm{a}_4}^2 + 2.27 - 40z^2 + \left(5.26 - 39\underline{\mathrm{a}_1} + 4.68 - 40\underline{\mathrm{a}_2} + 2.28 - 43\underline{\mathrm{a}_7} + 1.55 - 5\underline{\mathrm{a}_8} + 5.59 - 4\right)z + 0.151\underline{\mathrm{a}_1} + 0.94\underline{\mathrm{a}_2} + 1.64\underline{\mathrm{a}_3} + 0.00995\underline{\mathrm{a}_1}\,\underline{\mathrm{a}_7} + 0.122\underline{\mathrm{a}_2}\,\underline{\mathrm{a}_6} + 0.487\underline{\mathrm{a}_3\mathrm{a}_5} + 0.999 = 1.64\underline{\mathrm{a}_5} + 0.94\underline{\mathrm{a}_6} + 0.151\underline{\mathrm{a}_7} + 0.00561\underline{\mathrm{a}_8} + 2.04 - 39a4z,$$

$$3.49 - 40z^2 + \left(1.58 - 38\underline{\mathrm{a}_1} + 4.21 - 39\underline{\mathrm{a}_2} + 1.03 - 42\underline{\mathrm{a}_7} + 6.99 - 5\right)z + 0.0434\underline{\mathrm{a}_1} + 0.485\underline{\mathrm{a}_2} + 1.53\underline{\mathrm{a}_3} + 1.03\underline{\mathrm{a}_4} + 0.0014\underline{\mathrm{a}_1}\,\underline{\mathrm{a}_8} + 0.0392\underline{\mathrm{a}_2}\,\underline{\mathrm{a}_7} + 0.274\underline{\mathrm{a}_3}\,\underline{\mathrm{a}_6} + 0.685\underline{\mathrm{a}_4}\,\underline{\mathrm{a}_5} + 1.0 = 1.23\underline{\mathrm{a}_5} + 1.44\underline{\mathrm{a}_6} + 0.386\underline{\mathrm{a}_7} + 0.0252\underline{\mathrm{a}_8} + 5.36 - 39\underline{\mathrm{a}_3}z,$$

$$0.392\underline{\mathrm{a}_5}^2 + 4.9 - 40z^2 + \left(3.95 - 38\underline{\mathrm{a}_1} + 2.18 - 38\underline{\mathrm{a}_2} + 5.87 - 42\underline{\mathrm{a}_7} + 4.21 - 37\right)z + 0.00699\underline{\mathrm{a}_1} + 0.182\underline{\mathrm{a}_2} + 1.07\underline{\mathrm{a}_3} + 1.68\underline{\mathrm{a}_4} + 0.00699\underline{\mathrm{a}_2}\,\underline{\mathrm{a}_8} + 0.112\underline{\mathrm{a}_3}\,\underline{\mathrm{a}_7} + 0.49\underline{\mathrm{a}_4\mathrm{a}_6} + 1.0 = 0.294\underline{\mathrm{a}_5} + 1.76\underline{\mathrm{a}_6} + 0.797\underline{\mathrm{a}_7} + 0.0839\underline{\mathrm{a}_8} + 1.59 - 38\underline{\mathrm{a}_3}z,$$

$$5.85 - 40z^2 + \left(8.65 - 38\underline{\mathrm{a}_1} + 8.54 - 38\underline{\mathrm{a}_2} + 4.57 - 41\underline{\mathrm{a}_7} + 8.32 - 37\right)z + 7.94 - 37\underline{\mathrm{a}_1} + 0.0385\underline{\mathrm{a}_2} + 0.538\underline{\mathrm{a}_3} + 1.73\underline{\mathrm{a}_4} + 0.923\underline{\mathrm{a}_5} + 0.0256\underline{\mathrm{a}_3}\,\underline{\mathrm{a}_8} + 0.256\underline{\mathrm{a}_4}\,\underline{\mathrm{a}_7} + 0.718\underline{\mathrm{a}_5}\,\underline{\mathrm{a}_6} + 1.0 = 1.62\underline{\mathrm{a}_6} + 1.38\underline{\mathrm{a}_7} + 0.231\underline{\mathrm{a}_8} + 4.64 - 38\underline{\mathrm{a}_3}z,$$

$$0.431\underline{\mathrm{a}_6}^2 + 4.67 - 40z^2 + \left(1.69 - 37\underline{\mathrm{a}_1} + 2.81 - 37\underline{\mathrm{a}_2} + 2.74 - 40\underline{\mathrm{a}_7} + 1.59 - 36\right)z + 5.5 - 36\underline{\mathrm{a}_1} + 0.154\underline{\mathrm{a}_3} + 1.23\underline{\mathrm{a}_4} + 1.94\underline{\mathrm{a}_5} + 0.0769\underline{\mathrm{a}_4}\,\underline{\mathrm{a}_8} + 0.492\underline{\mathrm{a}_5}\,\underline{\mathrm{a}_7} + 1.0 = 0.738\underline{\mathrm{a}_6} + 2.03\underline{\mathrm{a}_7} + 0.554\underline{\mathrm{a}_8} + 1.2e - 37\underline{\mathrm{a}_3}z,$$

$$1.91 - 35\underline{\mathrm{a}_1} + 0.5\underline{\mathrm{a}_4} + 2.2\underline{\mathrm{a}_5} + 0.9\underline{\mathrm{a}_6} + 0.2\underline{\mathrm{a}_5}\,\underline{\mathrm{a}_8} + 0.8\underline{\mathrm{a}_6}\,\underline{\mathrm{a}_7} + z\left(2.96 - 37\underline{\mathrm{a}_1} + 8.17 - 37\underline{\mathrm{a}_2} + 1.15 - 39\underline{\mathrm{a}_7} + 3.1 - 36\right) + 1.0 = 2.02 - 40z^2 + 2.66 - 37\underline{\mathrm{a}_3}z + 3.75 - 35\underline{\mathrm{a}_3} + 2.4\underline{\mathrm{a}_7} + 1.2\underline{\mathrm{a}_8},$$

$$1.97 - 39z^2 + 4.77 - 37\underline{\mathrm{a}_3}z + 7.14 - 35\underline{\mathrm{a}_3} + 1.8\underline{\mathrm{a}_7} + 2.4\underline{\mathrm{a}_8} = 5.14 - 35\underline{\mathrm{a}_1} + 1.4\underline{\mathrm{a}_5} + 2.8\underline{\mathrm{a}_6} + 0.467\underline{\mathrm{a}_6\mathrm{a}_8} + z\left(4.39 - 37\underline{\mathrm{a}_1} + 2.16 - 36\underline{\mathrm{a}_1} + 3.43 - 39\underline{\mathrm{a}_7} + 6.58 - 36\right) + 0.533\underline{\mathrm{a}_7}^2 + 1.0,$$

$1.18-34\underline{a}_1+3.5\underline{a}_6+\underline{a}_7+\underline{a}_7\,\underline{a}_8+z\left(4.7-37\underline{a}_1+5.27-36\underline{a}_2+6.43-39\underline{a}_7+1.53-35\right)+1.0=\ 5.51-39z^2+5.97-37\underline{a}_3z+1.4-34\underline{a}_3+4.5\underline{a}_8,$

$1.1-38z^2+3.37-34\underline{a}_3+8.0\underline{a}_8=\underline{a}_8{}^2+2.48-34\underline{a}_1+8.0\underline{a}_7+z\left(1.2-35\underline{a}_2+3.76-35\right)+1.0.$

By using Isqnonlinear command in Matlab 2020b, the unknown lower control points are obtained

$\underline{a}_1=0.09870381182382463591817867154532z+0.02622344005504652700300205481198 6,$

$\underline{a}_2=0.10141635628342737796003802941414z+\ 0.14908730284508536767518194210425,$

$\underline{a}_3=0.10574684622438690340118228050414z+\ 0.27344912602762072584283714604 68,$

$\underline{a}_4=0.11885780035011206390294091761461z+\ 0.40856364585422355606425526275416,$

$\underline{a}_5=0.12884827648293994961647968011675z+\ 0.55263254784317961121331563845 16,$

$\underline{a}_6=0.16054178680497699360785190947354z+0.72968165375516169657288401140249,$

$\underline{a}_7=0.19204818651070010293580025972915z+0.93729643912732985011615483017522,$

$\underline{a}_8=0.29632991484854032115947575221071z\ +1.26100923126521657913201579503 95.$

From the fuzzy initial condition in Eq. (19) we get $\overline{a_0}(z)=0.1-0.1\text{z}$ and remaining control points $\overline{a_1}(z)$, $\overline{a_2}(z)$, $\overline{a_3}(z)$, $\overline{a_4}(z)$, $\overline{a_5}(z)$, $\overline{a_6}(z)$, $\overline{a_7}(z)$ and $\overline{a_8}(z)$ are determined. Substitute $\overline{a_0}(z)$ in Eq. (26), we obtain the following system of nonlinear equations.

$0.01z^2+1.81=\ 8.0\overline{a_1}+\ 0.82z,$

$0.9\overline{a_1}+3.5\overline{a}_2+z(0.1\overline{a}_1+0.45)=1.45,$

$0.533\overline{a_1}^2+1.8\overline{a_1}+3.06-42z^2+1.24=2.75\overline{a_2}+1.4\overline{a_3}+z(0.0467\overline{a_2}+0.24),$

$0.9\overline{a}_2+2.18\overline{a}_3+0.5\overline{a}_4+z(0.02\overline{a}_3+0.12)=9.18-42z^2+2.4\overline{a}_1+0.8\overline{a}_1\overline{a}_2+1.12,$

$1.94\overline{a}_3+1.22\overline{a}_4+0.154\overline{a}_5+z(0.00769\overline{a}_4+0.0554)=0.431\overline{a}_2{}^2+0.738\overline{a}_2+2.16-41z^2+2.58-41\overline{a}_3z\ +2.03\overline{a}_1+0.492\overline{a}_1\overline{a}_3+1.06,$

$0.923\overline{a}_3+1.73\overline{a}_4+0.536\overline{a}_5+0.0385\overline{a_6}+z(0.00256\overline{a}_5+0.0231)=4.41-41z^2+1.29-40\overline{a}_3z+1.38\overline{a}_1+1.62\overline{a}_2+0.256\overline{a}_1\overline{a}_4+0.718\overline{a}_2\,\overline{a}_3+1.02,$

$1.68\overline{a}_4+1.07\overline{a}_5+0.181\overline{a_6}+0.00699\overline{a_7}+z(1.88-40\overline{a}_1+6.99e-4\overline{a_6}+0.00839)=0.392\overline{a}_3{}^2+3.88-40\overline{a}_3z+0.294\overline{a}_3+8.11-41z^2+0.797\overline{a}_1+1.76\overline{a}_2+0.112\overline{a}_1\overline{a}_5+0.49\overline{a}_2\,\overline{a}_4+1.01,$

$1.03\overline{a}_4+1.53\overline{a}_5+0.485\overline{a_6}+0.0432\overline{a_7}+6.99-4\overline{a_8}+z(1.32-39\overline{a}_1\ +1.4-4\overline{a_7}+0.00252)=1.39-40z^2+9.04-40\overline{a}_3z+0.386\overline{a}_1+1.44\overline{a}_2+1.23\overline{a}_3+\ 0.0392\overline{a}_1\overline{a_6}+0.274\overline{a}_2\,\overline{a}_5+0.685\overline{a}_3\,\overline{a}_4+1.0,$

$0.381\overline{a}_4{}^2+2.27-40z^2+2.04-39\overline{a}_3z+0.151\overline{a}_1+0.94\overline{a}_2+1.64\overline{a}_3+0.00995\overline{a}_1\overline{a_7}+0.122\overline{a}_2\,\overline{a_6}+0.487\overline{a}_3\ \ \overline{a}_5+1.0=1.64\overline{a}_5+0.94\overline{a_6}+0.151\overline{a_7}+0.00558\overline{a_8}+z(5.26-39\overline{a}_1+\ 4.68-40\overline{a}_2+2.28-43\overline{a_7}+1.55-5\overline{a_8}+5.59-4),$

$3.49-40z^2+5.36-39\overline{a}_3z+0.0434a2+0.485\overline{a}_2+1.53\overline{a}_3+1.03\overline{a}_4+0.0014\overline{a}_1\overline{a_8}+0.0392\overline{a}_2\,\overline{a_7}+0.274\overline{a}_3\,\overline{a_6}+0.685\overline{a}_4\,\overline{a}_5+1.0=1.23\overline{a}_5+1.44\overline{a_6}+0.386\overline{a_7}+0.0252\overline{a_8}+z(1.58-38\overline{a}_1\ +4.21-39\overline{a}_2+1.03-42\overline{a_7}+6.99-5),$

$0.392\overline{a}_5{}^2+4.9-40z^2+1.59-38\overline{a}_3z+0.00699\overline{a}_1+0.182\overline{a}_2+1.07\overline{a}_3+1.68\overline{a}_4+0.00699\overline{a}_2\,\overline{a_8}+0.112\overline{a}_3\,\overline{a_7}+0.49\overline{a}_4\,\overline{a_6}\ +1.0=0.294\overline{a}_5+1.76\overline{a_6}+0.797\overline{a_7}+0.0839\overline{a_8}+z(3.95-38\overline{a}_1+2.18-38\overline{a}_2+5.87-42\overline{a_7}+1.56-37),$

$5.85-40z^2+4.64-38\overline{a}_3z+0.0385\overline{a}_2+0.538\overline{a}_3+1.73\overline{a}_4+0.923\overline{a}_5+0.0256\overline{a}_3\,\overline{a_8}+0.256\overline{a}_4\,\overline{a_7}+0.718\overline{a}_5\,\overline{a_6}+1.0=7.94-37\overline{a}_1+1.62\overline{a_6}+1.38\overline{a_7}+0.231\overline{a_8}+z(8.65-38\overline{a}_1+8.54-38\overline{a}_2+4.57-41\overline{a_7}+2.58-37),$

$0.431\overline{a_6}^2+4.67-40z^2+1.2-37\overline{a}_3z+0.154\overline{a}_3+1.23\overline{a}_4+1.94\overline{a}_5+0.0769\overline{a}_4\,\overline{a_8}+0.492\overline{a}_5\,\overline{a_7}+1.0=5.5-36\overline{a}_1+0.738\overline{a_6}+2.03\overline{a_7}+0.554\overline{a_8}+z(1.69-37\overline{a}_1+2.81-37\overline{a}_2+2.74-40\overline{a_7}+4.47-37),$

$1.69-35\overline{a}_3+0.5\overline{a}_4+\ 2.2\overline{a}_5+0.9\overline{a_6}+0.2\overline{a}_5\,\overline{a_8}+0.8\overline{a_6}\,\overline{a_7}+2.66-37\overline{a}_3z+1.0=2.02e-40z^2+(2.96e-37\overline{a}_1+8.17-37\overline{a}_2+\ 1.15-39\overline{a_7}+9.62-37)z+1.91-35\overline{a}_1+2.4\overline{a_7}+1.2\overline{a}_8,$

$1.97-39z^2+(4.39-37\overline{a}_1+2.16-36\overline{a}_2+3.43-39\overline{a_7}+2.72-36)z+5.14-35\overline{a}_1+1.8\overline{a_7}+2.4\overline{a_8}=$

$0.533\overline{a_7}^2 + 3.29 - 35\overline{a_3} + 1.4\overline{a_5} + 2.8\overline{a_6} + 0.467\overline{a_6}\,\overline{a_8} + 4.77 - 37\overline{a_3}z + 1.0,$

$9.18 - 35\overline{a_3} + 3.5\overline{a_6} + \overline{a_7} + \overline{a_7}\,\overline{a_8} + 5.97 - 37\overline{a_3}z + 1.0 = 5.51 - 39z^2 + (4.7 - 37\overline{a_1} + 5.27 - 36\overline{a_2} + 6.43 - 39\overline{a_7} + 8.64 - 36)z + 1.18 - 34\overline{a_1} + 4.5\overline{a_8},$

$\overline{a_8}^2 + 3.37 - 34\overline{a_3} + 8.0\overline{a_7} + 1.0 = 1.1 - 38z^2 + (1.2 - 35\overline{a_2} + 2.66 - 35)z + 2.48 - 34\overline{a_1} + 8.0\overline{a_8}.$

By using Isqnonlinear command in Matlab 2022b, the unknown upper control points are obtained.

$\overline{a_1} = 0.22599629167616214209246550126409 - 0.10106903979729096876294391904594z,$

$\overline{a_2} = 0.35752484751657837014349183846207 - 0.10702118838806568001942309820151z,$

$\overline{a_3} = 0.49049106575472956048855621702387 - 0.11129509350272193124453679047292z,$

$\overline{a_4} = 0.66268197946502438444582594456733 - 0.13526053326068876447862976419856z,$

$\overline{a_5} = 0.82124753224939339713017716348986 - 0.13976670792327394732268430743716z,$

$\overline{a_6} = 1.0850439108865743698117967142025 - 0.19482047032643556860875833081082z,$

$\overline{a_7} = 1.3565310207283329813066075075767 - 0.22718639509030325029925734270364z,$

$\overline{a_8} = 1.9621823461672029953462015328114 - 0.40484320005344631709931491059251z.$

The approximate solution function of BCPM derived from the aforementioned control points using Matlab 2022b for all $z \in [0,1]$ is presented below:

$\underline{u}(t,z) = 0.1z + t^4(0.66873z + 0.3078) + t^6(2.3633z + 1.4652) + t^8(0.58786z + 0.39726) - 1.0t(0.01037z - 1.0098) - 1.0t^7(1.789z + 1.1318) - 1.0t^5(1.6025z + 0.86521) - 1.0t^3(0.13388z - 0.27202) + t^2(0.11224z - 0.094068) - 0.1.$

$\overline{u}(t,z) = t^5(4.9533z - 7.421) - 1.0t(0.0085523z - 1.008) - 0.1z + t^3(0.36744z - 0.2293) - 1.0t^2(0.13673z - 0.1549) + t^7(5.4417z - 8.3625) - 1.0t^8(1.7391z - 2.7243) - 1.0t^6(7.2276z - 11.056) - 1.0t^4(1.9552z - 2.9317) + 0.1.$

The results and accuracy of 8^{th}-degree approximate solutions of Eq. (19) at $t = 1$ via BCPM for different values of fuzzy level sets $0 \leq z \leq 1$ compared with the exact solutions are illustrated in Tables 3 and 4, that summarized in Figures 3 and 4 as follows:

Table 3: Approximate and exact values for lower solutions of degree-8 BCPM for Eq. (19) at *t*=1.

z	BCPM	Exact solution	$\underline{E}(t,z)$ of BCPM
0	1.261009231265222	1.261016102725077	6.871459854673745e-06
0.2	1.313716421951129	1.313727034166937	1.061221580811456e-05
0.4	1.369424446916392	1.369441041285457	1.659436906464862e-05
0.6	1.428396007327389	1.428422288413148	2.628108575875032e-05
0.8	1.490924689029005	1.490966858599166	4.216957016112133e-05
1	1.557339146113736	1.557407724654902	6.857854116648632e-05

Table 4: Approximate and exact values for upper solutions of degree-8 BCPM for Eq. (19) at *t*=1.

z	BCPM	Exact solution	$\overline{E}(t;z)$ of BCPM
0	1.962182346167196	1.963150263095175	9.679169279792621e-04
0.2	1.869899126074916	1.870452269733388	5.531436584720328e-04
0.4	1.783803242370609	1.784124258798744	3.210164281348682e-04
0.6	1.703342312806001	1.703531445981465	1.891331754644288e-04
0.8	1.628007599711873	1.628120679100946	1.130793890729009e-04
1	1.557339146113736	1.557407724654902	6.857854116648632e-05

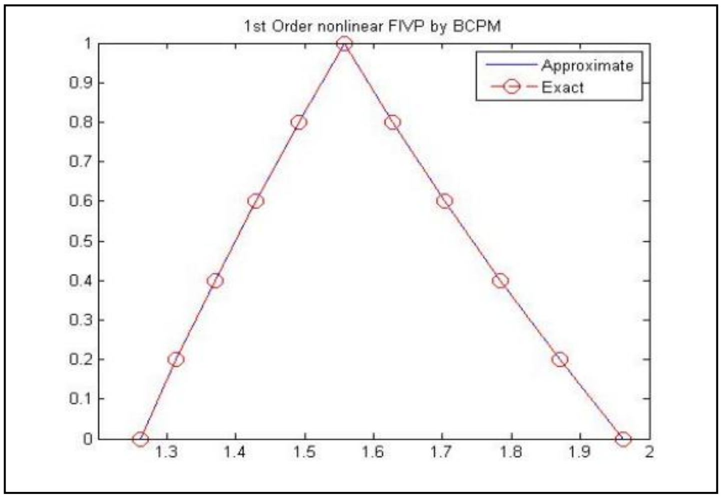

Fig. 3: Approximate and exact solutions of Eq. (19) at $t = 1$ and degree-8 BCPM for all $z \in [0,1]$.

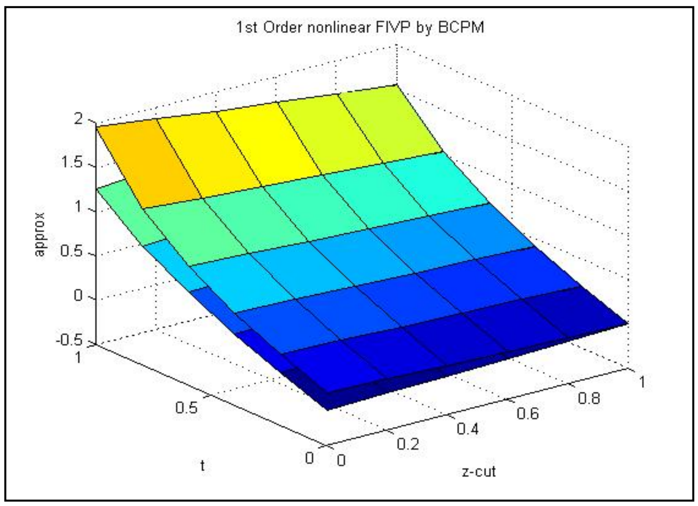

Fig. 4: Approximate solutions of Eq. (19) at $t \in [0,1]$ and degree-8 BCPM for all $z \in [0,1]$.

BSM Solution

5.3 Degree-6 BSM

Let,

$$\begin{cases} \underline{u}(t;z) = \sum_{i=0}^{6} \underline{a_i}\ \underline{N_{i,j}}(t;z), \\ \overline{u}(t;z) = \sum_{i=0}^{6} \overline{a_i}\ \overline{N_{i,j}}(t;z), \end{cases} \tag{28}$$

where $0 \le z \le 1$ and $\underline{a_i}$ and $\overline{a_i}, j = 0, \dots, 6$ are the B-Spline control points that need to be determined, substitute Eq. (28) into Eq. (19) and the residual functions can be obtained, i.e.

$$\begin{cases} \underline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{6} \underline{a_i}\ \underline{N_{i,j}}(t;z),\right) - \left(\sum_{i=0}^{6} \underline{a_i}\ \underline{N_{i,j}}(t;z)\right)^2 - 1, \\ \overline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{6} \overline{a_i}\ \overline{N_{i,j}}(t;z)\right) - \left(\sum_{i=0}^{6} \overline{a_i}\ \overline{N_{i,j}}(t;z)\right)^2 - 1. \end{cases} \tag{29}$$

The right-hand side of Eq. (29) is a polynomial of degree-6, and therefore, the residual function can be represented in the form of Eq. (19) as follows

$$\begin{cases} \underline{R}(t;z) = \sum_{i=0}^{12} \underline{b_i}\ \underline{N_{i,j}}(t;z), \\ \overline{R}(t;z) = \sum_{i=0}^{12} \overline{b_i}\ \overline{N_{i,j}}(t;z). \end{cases} \tag{30}$$

By using Isqnonlinear command in Matlab 2022b, the unknown lower control points from the above Eq. (30) are obtained:

$\underline{a_0} = 0.1162222954300204946775920689106z - 0.48444016627371411232161335647106,$

$\underline{a_1} = 0.10555675907900424759233715121809z - 0.32053266708365479864539793197764,$

$\underline{a_2} = 0.10000753636236597854392016415659z - 0.17073469383515199604417489354091,$

$\underline{a_3} = 0.098633041120675682411622631207138z - 0.027909962914643191012054046495905,$

$\underline{a_4} = 0.10121777540316707899581416540968z + 0.11377977809835400357929557912939,$

$\underline{a_5} = 0.10817483920762138716042954911245z + 0.26004117687401001601799066520471,$

$\underline{a_6} = 0.12072453051369286258776014619798z + 0.4174453749159588444861412881437.$

By using Isqnonlinear command in Matlab 2022b, the unknown upper control points from the above Eq. (30) are obtained:

$\overline{a_0} = -0.10817602235587120418358608731069z - 0.26004184848786149331090200576,$

$\overline{a_1} = -0.10122005473383267892817372057834z - 0.11375585327082222975025871392063,$

$\overline{a_2} = 0.027904085390887782558433372059881 - 0.098631242863687740296541051066015z,$

$\overline{a_3} = 0.17073105604472216256795036315452 - 0.100007977783869180834770418186963z,$

$\overline{a_4} = 0.32055923136339897761004635867721 - 0.10556167786179182499495254887734z,$

$\overline{a_5} = 0.48443658365678904376139257692557 - 0.1162205675749250488593133923132z,$

$\overline{a_6} = 0.67221449489227080853481766098412 - 0.13404458946228048343840555389761z.$

From the above control points the approximate solutions function of BSM obtained via Matlab 2022b for all $z \in [0,1]$ is given below:

$\underline{u}(t;z) = 0.1z - 1.0t(0.010004z - 1.0101) + t^6(0.045114z - 0.027832) + t^4(0.067763z - 0.071512) + t^2(0.10102z - 0.10079) - 1.0t^3(0.013248z - 0.34487) - 1.0t^5(0.013654z - 0.16337) - 0.1.$

$\overline{u}(t;z) = 0.1 - 1.0t(0.010013z - 1.0101) - 1.0t^2(0.1011z - 0.10134) - 1.0t^4(0.066519z - 0.06277) - 1.0t^3(0.013075z - 0.34469) - 1.0t^6(0.050839z - 0.068121) - 1.0t^5(0.014535z - 0.16425) - 0.1z.$

The results and accuracy of 6th -degree of the approximate solutions of Eq. (4.30) at $t = 1$ via BSM for different values of fuzzy level sets $0 \le z \le 1$ compared with the exact solutions of Eq. (4.33) are illustrated in Tables 5 and 6, and summarized in Figures 5 and 6 as follows:

Table 5: Approximate and exact values for the lower solutions of degree-6 BSM for Eq. (19) at $\boldsymbol{t = 1}$.

z	BSM	Exact Solution	$\underline{E}(t;z)$of BSM
0	1.218164070340774	1.261016102725077	4.285203238430246e-02
0.2	1.268567970093342	1.313727034166937	4.515906407359527e-02
0.4	1.321277053998527	1.369441041285457	4.816398728692994e-02
0.6	1.376477156444859	1.428422288413148	5.194513196828887e-02
0.8	1.434366512532448	1.490966858599166	5.660034606671838e-02
1	1.495157256826751	1.557407724654902	6.225046782815147e-02

Table 6: Approximate and exact values for the upper solutions of degree-6 BSM for Eq. (19) at $\boldsymbol{t = 1}$.

z	BSM	Exact Solution	$\overline{E}(t;z)$ of BSM
0	1.851242561942723	1.963150263095175	1.119077011524523e-01
0.2	1.772156736487454	1.870452269733388	9.829553324593410e-02
0.4	1.697302243903226	1.784124258798744	8.682201489551811e-02
0.6	1.626370702010919	1.703531445981465	7.716074397054640e-02
0.8	1.559077029192611	1.628120679100946	6.904364990833511e-02
1	1.495157256826751	1.557407724654902	6.225046782815147e-02

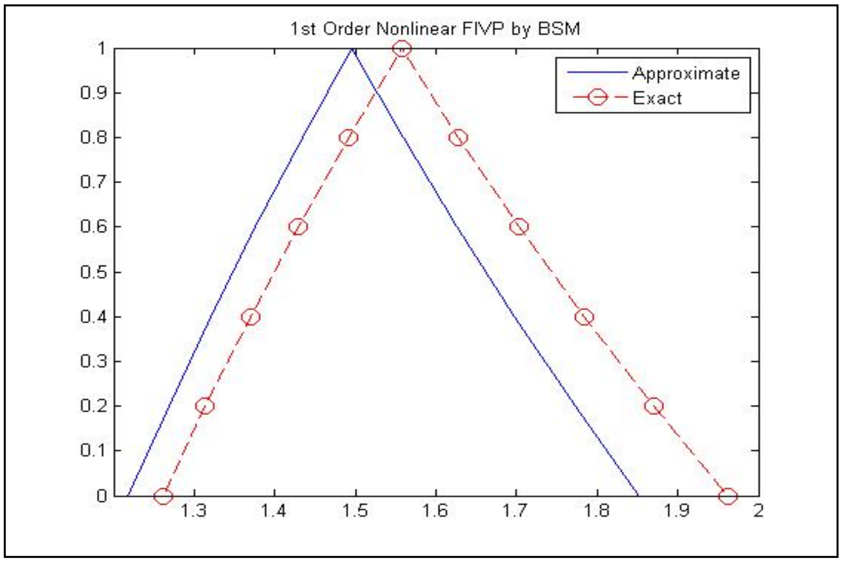

Fig. 5: Approximate and exact solutions of Eq. (19) at $t = 1$ and degree-6 BSM for all $z \in [0,1]$.

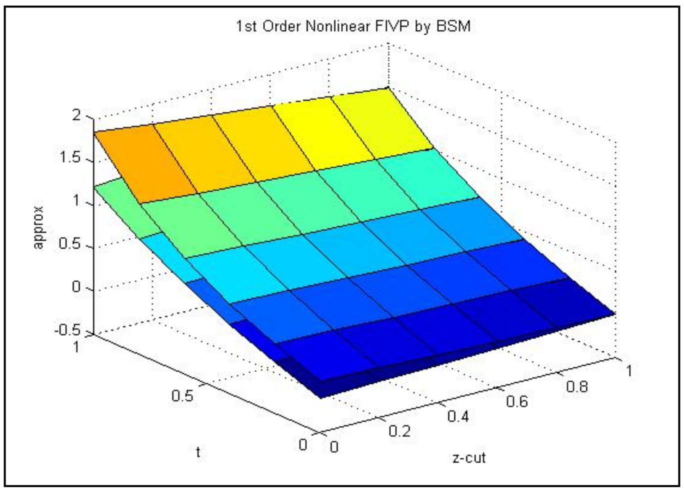

Fig. 6: Approximate solutions of Eq. (19) at $t \in [0,1]$ and degree-6 BSM for all $z \in [0,1]$

5.4 Degree-8 BSM

Let,

$$\begin{cases} \underline{u}(t;z) = \sum_{i=0}^{8} \underline{a_i}\ \underline{\mathrm{N}_{i,j}}(t;z), \\ \overline{\mathrm{u}}(t;z) = \sum_{i=0}^{8} \overline{a_i}\ \overline{N_{i,j}}(t;z). \end{cases} \tag{31}$$

where $0 \le z \le 1$ and $\underline{a_i}$ and $\overline{a_i}, j = 0, \ldots, 8$ are B-Spline control points need to be determined, substitute Eq. (31) into the Eq. (19) and the residual functions can be obtained, i.e.

$$\begin{cases} \underline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{8} \underline{a_i}\ \underline{\mathrm{N}_{i,j}}(t;z)\right) - \left(\sum_{i=0}^{8} \underline{a_i}\ \underline{\mathrm{N}_{i,j}}(t;z)\right)^2 - 1, \\ \overline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{8} \overline{a_i}\ \overline{N_{i,j}}(t;z)\right) - \left(\sum_{i=0}^{8} \overline{a_i}\ \overline{N_{i,j}}(t;z)\right)^2 - 1. \end{cases} \tag{32}$$

The right-hand side of Eq. (32) is a polynomial of degree-8, and therefore, the residual function can be represented in Eq. (19) as follows.

$$\begin{cases} \underline{R}(t;z) = \sum_{i=0}^{16} \underline{b_i}\ \underline{\mathrm{N}_{i,j}}(t;z), \\ \overline{\mathrm{R}}(t;z) = \sum_{i=0}^{16} \overline{b_i}\ \overline{N_{i,j}}(t;z). \end{cases} \tag{33}$$

By using Isqnonlinear command in Matlab 2022b, the unknown lower control points from the above Eq. (33) are obtained:

$\underline{a_0} = 0.11996195093888673000037670135498z - 0.52539020120684654102660715579987,$

$\underline{a_1} = 0.10995004486827475442822787954356z - 0.39230532124003664184996864605637,$

$\underline{a_2} = 0.10350230057549625328761067066807z - 0.27015933609318643071617316309 13,$

$\underline{a_3} = 0.09992147681827406868748653323 5642z - 0.15502712705265350723315975756122,$

$\underline{a_4} = 0.098846015314719565258982925115561z - 0.043740213664621559186418409126418,$

$\underline{a_5} = 0.10017063473814062646294331671015z + 0.066485846747373997245311727510853,$

$\underline{a_6} = 0.10402512223730472906702004820545z + 0.17833020703913540772056478544982,$

$\underline{a_7} = 0.11080084731307593415294832084328z + 0.29462783352096355571347885415889,$

$\underline{a_8} = 0.12123749066932892892722861688526z + 0.41867282819217782607879030365439.$

By using Isqnonlinear command in Matlab 2022b, the unknown upper control points

from the above Eq. (33) are obtained

$\overline{a_0} = -0.11080097840257963071053382009268z - 0.29462727186550807800813345238566,$

$\overline{a_1} = -0.10402508182915706780313769286295z - 0.17833019454206269771567860971118,$

$\overline{a_2} = -0.10017052670112926160417288201643z - 0.066486508816708297930908599937 53,$

$\overline{a_3} = 0.043740441679745924175026772218189 - 0.098846091914249617493837263282 33z,$

$\overline{a_4} = 0.15502727930126036781643961148802 - 0.099921477651009296683248805948097z,$

$\overline{a_5} = 0.27015860683769404859688734177325 - 0.10350212535215619347184201615164z,$

$\overline{a_6} = 0.39230544047741672253337696929 52 - 0.10995011120118697300895860280434z,$

$\overline{a_7} = 0.52539071940418113904058827756671 - 0.11996203857004034132316405703023z,$

$\overline{a_8} = 0.67465152152886465053427400562214 - 0.13474120266737976692184020066634z.$

From the above control points the approximate solutions function of BSM obtained via Matlab 2022b for all $z \in [0,1]$ is given below:

$$\underline{u}(t;z) = 0.1z - 1.0t(0.0099998z - 1.01) + t^6(0.03909z - 0.041149) + t^2(0.101z - 0.10101) + t^4(0.068367z - 0.068155) - 1.0t^7(0.009941z - 0.072321) - 1.0t^5(0.011295z - 0.14341) - 1.0t^3(0.013446z - 0.34685) + t^8(0.024521z - 0.01745) - 0.1.$$

$$\overline{u}(t;z) = 0.1 - 1.0t(0.0099995z - 1.01) - 1.0t^4(0.068476z - 0.068688) - 1.0t^2(0.101z - 0.10099) - 1.0t^6(0.03804z - 0.035981) - 1.0t^7(0.010368z - 0.072749) - 1.0t^5(0.011177z - 0.1433) - 1.0t^3(0.013457z - 0.34686) - 1.0t^8(0.02813z - 0.035201) - 0.1z.$$

The results and accuracy of 8th-degree approximate solutions of Eq. (19) at $t = 1$ via BSM compared with the exact solutions for different values of fuzzy level sets $0 \leq z \leq 1$ are illustrated in Tables 7 and 8, and summarized in Figures 7 and 8 as follows:

Table 7: Approximate and exact values for the lower solutions of degree-8 BSM for Eq. (19) at $\boldsymbol{t = 1}$.

z	BSM	Exact Solution	$\underline{E}(t;z)$ of BSM
0	1.244828089116088	1.261016102725077	1.618801360898825e-02
0.2	1.296765524932605	1.313727034166937	1.696150923433248e-02
0.4	1.351331101643480	1.369441041285457	1.810993964197705e-02
0.6	1.408748317167920	1.428422288413148	1.967397124522785e-02
0.8	1.469258501844258	1.490966858599166	2.170835675490879e-02
1	1.533123764163785	1.557407724654902	2.428396049111781e-02

Table 8: Approximate and exact values for the upper solutions of degree-8 BSM for Eq. (19) at $\boldsymbol{t = 1}$.

z	BSM	Exact Solution	$\overline{E}(t;z)$ of BSM
0	1.913765920181525	1.963150263095175	4.938434291365068e-02
0.2	1.828262802808858	1.870452269733388	4.218946692453041e-02

0.4	1.747841004904952	1.784124258798744	3.628325389379228e-02
0.6	1.672088333446171	1.703531445981465	3.144311253529475e-02
0.8	1.600629723715657	1.628120679100946	2.749095538528934e-02
1	1.533123764163785	1.557407724654902	2.428396049111781e-02

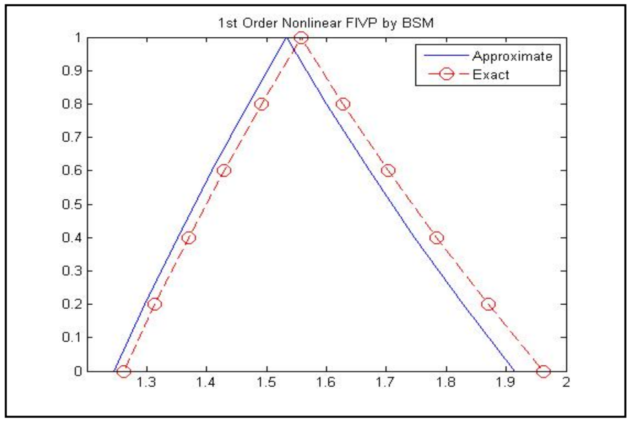

Fig. 7: Approximate and exact solutions of Eq. (19) at $t = 1$ and degree-8 BSM for all $z \in [0,1]$.

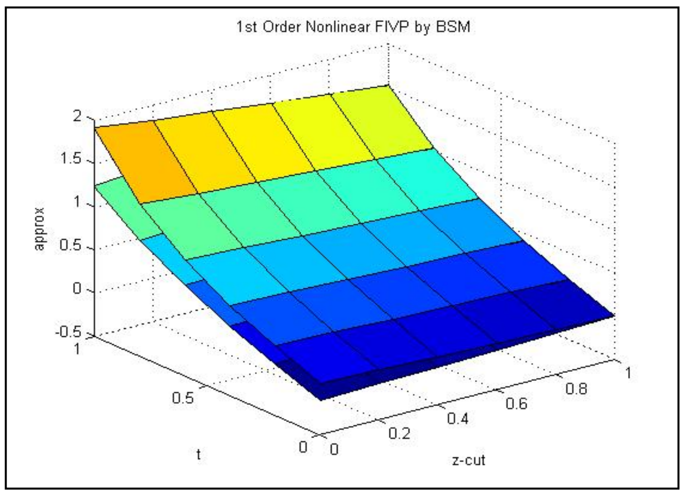

Fig. 8: Approximate solutions of Eq. (19) at $t \in [0,1]$ and degree-8 BSM for all $z \in [0,1]$.

From the aforementioned tables and figures, it is evident that BCPM achieves better results compared to BSM, at an equivalent number of control points, in terms of accuracy relative to the exact solution of Eq. (19). The validity of Theorem 1 is demonstrated by increasing the number of control points, which enhances the accuracy of the answer. The tables and figures presenting the approximate answer through BCPM and BSM are represented as the aforementioned triangular fuzzy number.

Example 5.2 [26]: This example will apply the approach to a nonlinear first-order Riccati FIVP that lacks an exact solution. This test example aims to demonstrate the efficacy of the methods offered for determining solutions and accuracy without requiring the exact solution of the equation, followed by a comparative illustration with HPM about accuracy.

$$\begin{cases} \tilde{u}'(t) = \tilde{u}(t)^2 + t^2, \ t \in [0,1]. \\ \tilde{u}(0) = [0.1z - 0.1, 0.1 - 0.1z], \quad z \in [0,1]. \end{cases} \tag{34}$$

Section 2 states that Equation (34) can be expressed in defuzzification format.

$$\begin{cases} \underline{u}'(t;z) = \underline{u}^2(t;z) + \underline{t}^2, \ \underline{u}(t;z) = 0.1z - 0.1, \\ \overline{u}'(t;z) = \overline{u^2}(t;z) + \overline{t^2}, \ \overline{u}(t;z) = 0.1 - 0.1z. \end{cases} \tag{35}$$

For numerical implementation, we consider the approximate solutions using BCPM as given section 3 of degree-11 (m = 11); the detailed results are as follows.

BCPM Solution

5.5 Degree-11 BCPM

Let

$$\begin{cases} \underline{u}(t;z) = \sum_{i=0}^{11} \underline{a_i}\, \underline{B_i^{11}}(t;z), \\ \overline{u}(t;z) = \sum_{i=0}^{11} \overline{a_i}\, \overline{B_i^{11}}(t;z), \end{cases} \tag{36}$$

where $0 \le z \le 1$ and $\underline{a_i}$ and $\overline{a_i}, i = 0, \dots, 11.$ are the Bézier control points that need to be determined, substitute Eq. (36) into Eq. (34), and the residual functions can be obtained, i.e.

$$\begin{cases} \underline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{11} \underline{a_i}\, \underline{B_i^{11}}(t;z)\right) - \left(\sum_{i=0}^{11} \underline{a_i}\, \underline{B_i^{11}}(t;z)\right)^2 - t^2, \\ \overline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{11} \overline{a_i}\, \overline{B_i^{11}}(t;z)\right) - \left(\sum_{i=0}^{11} \overline{a_i}\, \overline{B_i^{11}}(t;z)\right)^2 - t^2. \end{cases} \tag{37}$$

The right-hand side of Eq. (37) is a polynomial of degree-11, and therefore, the residual function can be represented in Eq. (34) as follows.

$$\begin{cases} \underline{R}(t;z) = \sum_{i=0}^{22} \underline{b_i}\, \underline{B_i^{22}}(t;z), \\ \overline{R}(t;z) = \sum_{i=0}^{22} \overline{b_i}\, \overline{B_i^{22}}(t;z). \end{cases} \tag{38}$$

The unknown lower control points from the aforementioned Eq. (38) are obtained by utilizing the Isqnonlinear command in Matlab 2022b.

$\underline{a_0} = 0.1z - 0.1,$

$\underline{a_1} = 0.09909091229158141500565903925235z - 0.09909090248350341245675565460 2603,$

$\underline{a_2} = 0.098199965351322929252297910807101z - 0.098200071321845383565118936530 78,$

$\underline{a_3} = 0.097326839446153723400811941246502z - 0.095306109574385691751707838648144,$

$\underline{a_4} = 0.096520309336416712508999182773 55z - 0.088441117518465936497840118590 83,$

$\underline{a_5} = 0.095879775018524349139426021793042z - 0.075674308888676988793520195031306,$

$\underline{a_6} = 0.095536791509515689435616536684392z - 0.055138324915008481152955965853835,$

$\underline{a_7} = 0.095680655035165290689391781597806z - 0.024917992208297320283483244907075,$

$\underline{a_8} = 0.096518498697201055014360804307216z + 0.016988112353739197313284847723 48,$

$\underline{a_9} = 0.098381280977864374315267070869595z + 0.073060724306077559631056317357434,$

$\underline{a_{10}} = 0.10170049582014717426403649369604z + 0.14647115095180940347319165084627,$

$\underline{a_{11}} = 0.10749514868575502091196938181383z + 0.24273669562876035166176791335602.$

Also, the unknown upper control points from the aforementioned Eq. (38) are obtained through the use of the Isqnonlinear command in Matlab 2022b.

$\overline{a_0} = 0.1 - 0.1z,$

$\overline{a_1} = 0.10090910753635794205873565942966 - 0.10090909772827993950983227477991z,$

$\overline{a_2} = 0.10183618391461186913904413131604 - 0.10183628988513430957407734922526z,$

$\overline{a_3} = 0.10480352156875738511843110245536 - 0.10278279169698939510269042330037z,$

$\overline{a_4} = 0.11187648172100099486225843747889 - 0.10379728990305017721773594985279z,$

$\overline{a_5} = 0.12519819890476233115172988163977 - 0.10499273277491501243918747832 14z,$

$\overline{a_6} = 0.14691852808528260698750500523602 - 0.10652006149077539176595053049823z,$

$\overline{a_7} = 0.17939781977832147674689622363076 - 0.10863515695145350981043463889364z,$

$\overline{a_8} = 0.22512128723653082595923535791371 - 0.11161467618559059444827141760 47z,$

$\overline{a_9} = 0.28740078925844142476719866863277 - 0.11595878397449951857645089603466z,$

$\overline{a_{10}} = 0.37048091072667010958952005239553 - 0.12230926395471355960786752348213z$,

$\overline{a_{11}} = 0.48255938899984057721681551811344 - 0.13232754468532520464307822294359z$.

The approximate solution function of BCPM derived from the aforementioned Bézier control points, calculated using Matlab 2022b for every $z \in [0,1]$, is presented below:

$\underline{u}(t;z) = 0.1z - 1.0t^6(0.0092404z + 0.019726) - 1.0t(0.01z - 0.01) - 1.0t^8(0.028645z + 0.064452) + t^7(0.021139z + 0.059484) + t^5(0.0007072z + 0.0075693) + t^9(0.027199z + 0.056563) + t^{11}(0.0035108z + 0.007358) - 1.0t^{10}(0.014322z + 0.028931) + t^4(0.016201z - 0.017654) + t^2(0.00099774z - 0.0010046) - 1.0t^3(0.000052756z - 0.33353) - 0.1.$

$\overline{u}(t;z) = t^6(0.019893z - 0.04886) - 1.0t(0.01z - 0.01) - 1.0t^4(0.015666z - 0.014213) - 0.1z + t^8(0.062799z - 0.1559) - 1.0t^2(0.00099519z - 0.00098829) - 1.0t^3(0.00020051z - 0.33368) - 1.0t^7(0.04497z - 0.12559) - 1.0t^{11}(0.0072722z - 0.018141) - 1.0t^9(0.058461z - 0.14222) + t^{10}(0.030303z - 0.073556) - 1.0t^5(0.007757z - 0.016033) + 0.1.$

Since Eq. (34) is without an exact solution we define the following residual error in order to detect the accuracy of degree-11 BCPM solution such that:

$$\tilde{R}e(t;z) = \tilde{u}'(t;z) - \tilde{u}(t;z)^2 - t^2. \quad (39)$$

The results and accuracy of 11th-degree of the approximate solutions of Eq. (38) at $t = 0.5$ via BCPM for different values of fuzzy level sets $z \in [0,1]$ compared with the HPM [26] of degree-11 of the polynomial are illustrated in Tables 9 and 10, and summarized in Figures 9 and 10 as follows:

Table 9: Comparison between the lower solutions for the accurate and the approximate solutions of degree-11 BCPM for Eq. (34) at $\boldsymbol{t = 0.5}$.

z	BCPM	$\underline{Re}(t,z)$ of BCPM	$\underline{Re}(t,z)$ of HPM
0	-5.443611192657533e-02	4.146174754345426e-09	$1.6306788234166092e-06$
0.2	-3.593235842338293e-02	4.426524844428663e-09	$1.0343067185503152e-06$
0.4	-1.706857934168008e-02	4.740890378522574e-09	$6.334864369739179e-07$
0.6	2.165836012498687e-03	5.099018973038876e-09	$3.513160492518707e-07$
0.8	2.178191942834261e-02	5.507623297161546e-10	$1.349085781421344e-07$
1	4.179114488897558e-02	5.976517283436464e-10	$5.463860805465792e-08$

Table 10: Comparison between the upper solutions for the accurate and the approximate solutions of degree-11 BCPM for Eq. (34) at $\boldsymbol{t = 0.5}$.

z	BCPM	$\overline{Re}(t;z)$ of BCPM	$\overline{Re}(t;z)$ of HPM
0	1.481698847321594e-01	9.774196260885051e-09	$1.994205467870102e-06$
0.2	1.260057161648537e-01	8.747391047274743e-09	$1.2797120836904874e-06$
0.4	1.042995259730579e-01	7.881663800228169e-09	$8.091259636466841e-07$
0.6	8.303726450051129e-02	7.146466742751253e-09	$4.871648949522633e-07$
0.8	6.220545095156271e-02	6.517806192320647e-10	$2.504206709219581e-07$
1	4.179114488897558e-02	5.976517283436464e-10	$5.463860805465792e-08$

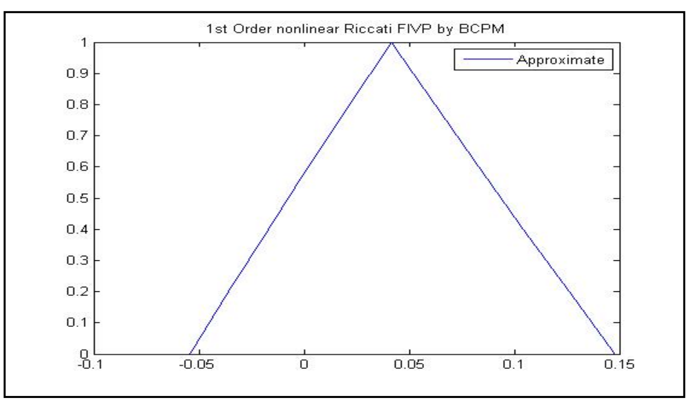

Fig. 9: Approximate solutions of Eq. (34) at $t = 0.5$ and degree-11 BCPM for all $z \in [0,1]$.

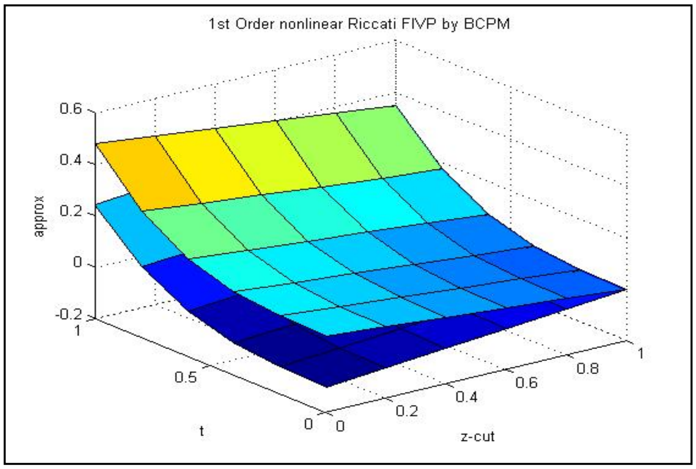

Fig. 10: Approximate solutions of Eq. (34) at $t \in [0,1]$ and degree-11 BCPM for all $z \in [0,1]$.

Clearly, from the above Tables 9 and 10 BCPM approximate solutions of degree-11 of Eq. (34) at $t = 0.5$ more accurate solutions than HPM [26] for different values of fuzzy level sets $z \in [0,1]$. Moreover, the displayed results in Figures 9 and 19, that showed that the BCPM degree-11 solutions were in the form of triangular fuzzy number. For numerical implementation, we consider the approximate solutions using BSM as given in Eq. (16) of degree-11 (m = 11) the detailed results are as follows.

BSM Solution

5.6. Degree-11 BSM

Let,

$$\begin{cases} \underline{u}(t;z) = \sum_{i=0}^{11} \underline{a_i}\ \underline{N_{i,j}}(t;z), \\ \overline{u}(t;z) = \sum_{i=0}^{11} \overline{a_i}\ \overline{N_{i,j}}(t;z). \end{cases} \tag{40}$$

where $z \in [0,1]$ and $\underline{a_i}$ and $\overline{a_i}, j = 0, \dots, 11$ are the B-Spline control points that need to be determined, substitute Eq. (40) into Eq. (34) and the residual functions can be obtained, i.e.

$$\begin{cases} \underline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{11} \underline{a_i}\ \underline{N_{i,j}}(t;z)\right) - \left(\sum_{i=0}^{11} \underline{a_i}\ \underline{N_{i,j}}(t;z)\right)^2 - t^2, \\ \overline{R}(t;z) = \frac{d}{dt}\left(\sum_{i=0}^{11} \overline{a_i}\ \overline{N_{i,j}}(t;z)\right) - \left(\sum_{i=0}^{11} \overline{a_i}\ \overline{N_{i,j}}(t;z)\right)^2 - t^2. \end{cases} \tag{41}$$

The right-hand side of Eq. (41) is a polynomial of degree-11, and therefore, the residual function can be represented in the form of Eq. (34) as follows

$$\begin{cases} \underline{R}(t;z) = \sum_{i=0}^{22} \underline{b_i}\ \underline{N_{i,j}}(t;z), \\ \overline{R}(t;z) = \sum_{i=0}^{22} \overline{b_i}\ \overline{N_{i,j}}(t;z). \end{cases} \tag{42}$$

By using Isqnonlinear command in Matlab 2022b, the unknown lower control points from the above Eq. (42) are obtained:

$\underline{a_0} = 0.10461373912403360009193420410156z - 0.12434370932169258594512939453125,$

$\underline{a_1} = 0.10337736919120153744611201318548z - 0.11170438497040113823288720595883,$

$\underline{a_2} = 0.10232561046522301029249035764224z - 0.10451689226361489393202219844 17,$

$\underline{a_3} = 0.10136920118155871850529337052649z - 0.10118140867393006498975438489651,$

$\underline{a_4} = 0.10045048944896603493326381340012z - 0.10013741439313007808298294776 25,$

$\underline{a_5} = 0.09954145184310811411698693973448 8z - 0.09985452706475533846131042992055 9,$

$\underline{a_6} = 0.09864213424364412508005273139133 4z - 0.09882992608896914998872773594484 9$

$\underline{a_7} = 0.09777916190667348961262916873238 3z - 0.09558788184458835712753455027268 5,$

$\underline{a}_8 = 0.097004316683519012376457624213799z - 0.088677302031786406355529095435486,$

$\underline{a}_9 = 0.096393494740771273887602887953108z - 0.076663521291003566826560700064874,$

$\underline{a}_{10} = 0.096046649398322025970031745600863z - 0.058110745847458178392486161101 26,$

$\underline{a}_{11} = 0.095716190046649398322025970026291z - 0.050110745847458178392486161579 51.$

By using Isqnonlinear command in Matlab 2022b, the unknown upper control points from the above Eq. (42) are obtained:

$\overline{a_0} = 0.076663523504805652919458225369453 - 0.096393493810639441221610468346626z,$

$\overline{a_1} = 0.088677300911446960141049089543 23 - 0.097004316702325593668732039986935z,$

$\overline{a_2} = 0.095587879956754454036094159619097 - 0.097779161736570377461141845287784z,$

$\overline{a_3} = 0.098829926525536140879957258675859 - 0.098642134001757211803074198996 91z,$

$\overline{a_4} = 0.099854526912154531470555696159863 - 0.099541451860264293372004829052457z,$

$\overline{a_5} = 0.100137414353663967148122537764 72 - 0.100450489579303123277576048622z,$

$\overline{a_6} = 0.101181409204707306592752047436 08 - 0.101369201045985166120821929780 5z,$

$\overline{a_7} = 0.104516890254375008351139797468 93 - 0.102325610185531587492666884797 18z,$

$\overline{a_8} = 0.111704383898948780839965877476 06 - 0.103377369240087391144555795108 32z,$

$\overline{a_9} = 0.124343711763402600678496412456 3 - 0.104613738312944681840832572561 34z,$

$\overline{a_{10}} = 0.144094651570579879162892211752 48 - 0.106158748003807701376643990442 97z.$

$\overline{a_{11}} = 0.160328967888795500762455503718 2 - 0.108076654535367537279810790096 23z.$

The approximate solutions function of BSM for all $z \in [0,1]$ is obtained via Matlab 2022b from the aforementioned control points. The function is as follows:

$\underline{u}(t;z) = 0.1z - 1.0t(0.01z - 0.01) + t^{10}(0.000081552z + 0.00032404) + t^8(0.0017716z - 0.0018898) - 1.0t^5(0.0020007z - 0.0020043) + t^4(0.016676z - 0.016677) + t^2(0.001z - 0.001) - 1.0t^7(0.000031011z - 0.015859) + t^6(0.00023611z - 0.00022302) - 1.0t^3(0.00010001z - 0.33343) - 1.0t^9(0.00023127z - 0.00053947) - 0.1.$

$\overline{u}(t;z) = 0.1 - 1.0t(0.01z - 0.01) - 1.0t^5(0.0020011z - 0.0020047) - 1.0t^2(0.001z - 0.001) - 1.0t^4(0.016677z - 0.016676) - 1.0t^{10}(0.00013995z - 0.00054163) - 1.0t^8(0.0017754z - 0.0016566) - 1.0t^3(0.000099987z - 0.33343) - 1.0t^9(0.00023778z - 0.0005494) - 1.0t^7(0.000028059z - 0.015856) - 1.0t^6(0.00023402z - 0.0002472) - 0.1z.$

The results and accuracy of 11^{th}-degree of the approximate solutions of Eq. (43) at $t = 0.5$ via BSM for different values of fuzzy level sets $z \in [0,1]$ compared with the HPM [26] of degree-11 of the polynomial are illustrated in Tables 11 and 12, that summarized in Figures 11 and 12 as follows:

Table 11: Comparison between the lower solutions for the accurate and the approximate solutions of degree-11 BSM for Eq. (34) at $t = 0.5$.

z	***BSM***	$\underline{Re}(t;z)$ ***of BSM***	$\underline{Re}(t,z)$ ***of HPM***
0	-5.443612950077088e-02	3.295256632964843e-08	$1.6306788234166092e-06$
0.2	-3.593237685386455e-02	3.280720396423532e-08	$1.0343067185503152e-06$
0.4	-1.706859829838221e-02	3.447741402577843e-08	$6.334864369739179e-07$
0.6	2.165816659555925e-03	3.348481754805823e-08	$3.513160492518707e-07$
0.8	2.178189979874421e-02	3.291766879224013e-08	$1.349085781421344e-07$
1	4.179112415803880e-02	3.181846350154864e-08	$5.463860805465792e-08$

Table 12: Comparison between the upper solutions for the accurate and the approximate solutions of degree-11 BSM for Eq. (34) at $t = 0.5$.

z	BSM	$\overline{Re}(t;z)$ *of BSM*	$\overline{Re}(t;z)$ of HPM
0	1.481698605912774e-01	3.687406741223827e-08	$1.994205467870102e-06$
0.2	1.260056923441927e-01	3.673792553400427e-08	$1.2797120836904874e-06$
0.4	1.042995035104316e-01	3.491884523560140e-08	$8.091259636466841e-07$
0.6	8.303724235485976e-02	3.548760430631652e-08	$4.871648949522633e-07$

0.8	6.220542940060618e-02	3.501388644010452e-08	$2.504206709219581e-07$
1	4.179112415803880e-02	3.181846350154864e-08	$5.463860805465792e-08$

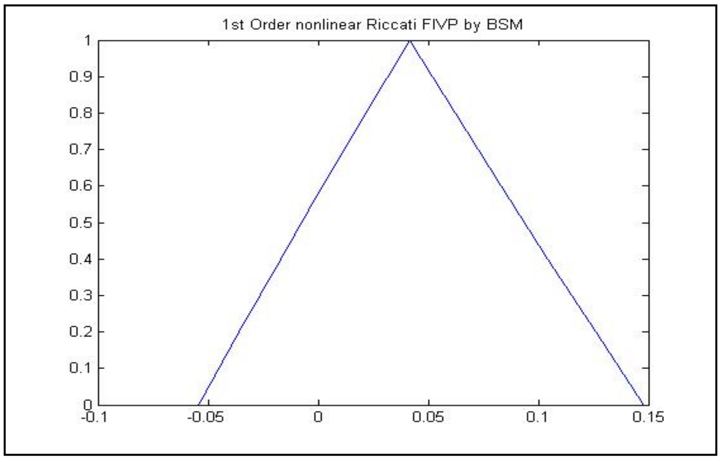

Fig. 11: Approximate solutions of Eq. (4.45) at $t = 0.5$ and degree-11 BSM for all $z \in [0,1]$.

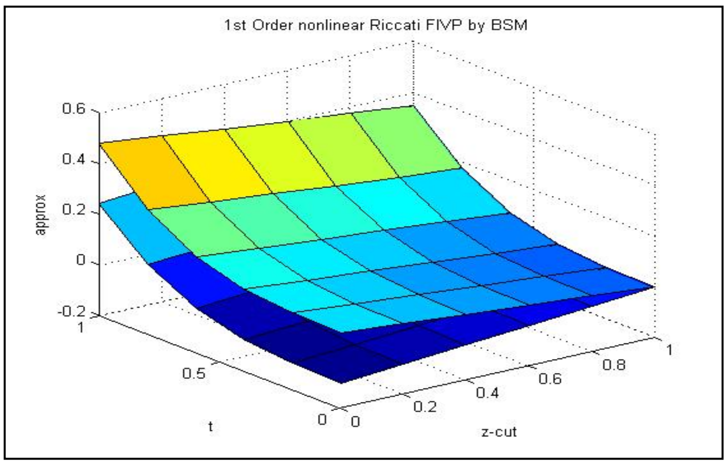

Fig. 12: Approximate solutions of Eq. (34) at $t \in [0,1]$ and degree-11 BSM for all $z \in [0,1]$.

Approximation solutions of the degree-11 solutions of Eq. (34) at $t = 0.5$ BSM better than HPM for all $z \in [0,1]$ can be seen in Tables 11 and 12, and Figures 11 and12, show that the BSM degree-11 solutions in the form triangular fuzzy number.

6 Summary

During the process of this research, we studied and utilized the newly developed methods of approximate series for BCPM and BSM. Some principles from fuzzy set theory have been utilised in the process of developing the approaches, which have been developed from the crisp domain to the fuzzy domainThe objective of this study is to get an approximate solution for the first order nonlinear FIVPs that involve ordinary differential equations and followed by an analysis of general convergence and a demonstration of the convergence theorem. FIVPs with or without a precise solution at varying degrees of series polynomial solution were effectively solved by applying the approaches, which were successful in finding solutions to first-order nonlinear problem. A comparison was also made between the solutions obtained through the use of BCPM and BSM with the same degree of the polynomial and other methods that are currently in use. In addition to this, each one of the conclusions that were acquired in terms of BCPM and BSM coincided with the features of fuzzy numbers. The properties of fuzzy numbers were satisfied by every one of the outcomes that were produced in relation to BCPM and BSM. In conclusion, based on the results of the test examples, the BCPM was able to obtain a more accurate solution than the BSM.

Conflicts of Interest Statement

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such *as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.*

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