

Application of Radio Mean Labelling in Communication Networks and Fault-Tolerant Facility Graphs

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Abstract: In a connected graph Ω , a bijective function h from vertex set $W(\Omega)$ to natural numbers \mathbb{N} is defined as a radio mean labelling; for any distinct nodes z and w in Ω , the inequality—where the distance between nodes added to the ceiling of their height averages is greater than or equal to one plus the diameter of Ω —holds. This study investigates radio mean labelling within a theta graph, including variations formed by vertex fusion in cycle T_a and central vertex substitution, supported by examples. The present study examines its application in designing fault-tolerant systems using facility graphs. This concept aids in understanding radio mean labelling's application in fault-tolerant systems with facility graphs, enhancing reliability and resource utilization. By analysing communication networks and facility graphs, it can be demonstrated how radio mean labelling improves system reliability and resource usage.

Keywords: Dalgaard-Strulik model, energy, economic growth, time delay, limit cycle

1 Introduction

Graph theory is a rapidly expanding field with applications in various domains of mathematics, science, and technology. It is extensively employed in biological chemistry, chemical studies, communication setups, encoding principles, informatics (particularly algorithms and computing), and operational research (e.g. timetabling). Moreover, graph theory is employed in encoding principles, X-ray diffraction analysis, radar, celestial studies, electronic layouts, communication grid allocation, and data administration, among other domains [1]. It is relevant to computer science and intersects with various engineering disciplines and geographic analyses. A. A. C. Aanisha & Manoharan (2024) examined the idea of mean labelling of the human body and graphical illustrations of blood circulation within the circulatory systems of both the heart and kidney. A. A. C. Aanisha &

Manoharan (2024) determined that the circular ladder graph CL_s when s is even, the subdivision of the star $SK_{1,s}$, the bistar graph $B_{s,s}$ when s is odd, star the start graph $K_{1,s}$ when s is even ESDC graphs using Python [2]. It encompasses the study of distance and time within the framework of road network optimisation, employing relational strategies for a thorough evaluation [3]. In biological networks, nodes represent biomolecular entities, such as chromosomes, proteins, or metabolites, whereas edges denote the interactive, physical, or chemical connections between these biomolecules. Graph theory is applied in transcriptional regulatory and metabolic networks, where it facilitates analysis and understanding. Additionally, graph theory is instrumental in studying protein-protein interaction (PPI) networks, allowing for the characterisation of drug-target interactions and drug-target partnerships [4].

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Graph labelling entails allocating integers to the vertices, edges, or both of a graph under specific circumstances. This concept was inspired by practical issues, and labelled graphs are valuable mathematical models for numerous applications. These involve encoding theory, which assists in crafting different kinds of codes, such as sync-set codes, missile navigation codes, and convolutional codes with optimal self-correlation characteristics [5]. It has demonstrated the application of graph theory in various fields of Science and Engineering. These applications include Caterpillar Trees in Chemistry, timetable scheduling in academics, communication networks, and computer science [6]. The rapid advancement of graph theory, particularly in graph labelling, with numerous studies focusing on this area [7]. It emphasises both theoretical advancements and real-world uses, such as sorcery covering and edge super-sorcery marking. The document concludes that sorcery covering can be utilised in clandestine distribution schemes and that edge sorcery marking can be implemented in gauge prototypes. [8] highlights the importance of developing efficient mobile communication networks (MCN) due to the increasing number of GSM users. As the subscriber base grows, effective frequency spectrum planning becomes crucial, particularly in densely populated areas. This study explores the use of graph labelling, specifically 2-odd labelling, to address challenges in communication networks and enhance connectivity and optimisation. Numerous other labelling methods exist in graph theory, such as sum divisor cordial labelling (SDC labelling), Edge Sum-Divisor Cordial Labelling [2], prime anti-magic labelling [9], Odd mean labelling [10], K^{th} -prime labelling [11], radio mean labelling [12]. The concept of divided square DC labelling in theta-related graphs, such as theta graph T_a , the fusion of any two vertices in a cycle of theta graph T_a , and the switching of a central vertex in the Theta graph T_a , and also implemented it using Python programming [13]. The concept of radio labelling for graphs has been studied widely [14]. This idea is inspired by the constraints involved in assigning channel frequencies to radio transmitters. Inspired by radio labelling, current modelling addresses the conception of radio mean labelling for Ω [15]. For a connected graph Ω , a one-to-one function h from the vertex set $W(\Omega)$ to the natural numbers \mathbb{N} is called a radio mean labelling if, for any distinct nodes z and w in Ω , the inequality is the distance between nodes z and w added to the ceiling of the average of their heights, which is greater than or equal to one plus the diameter of Ω holds for any two distinct nodes z and w in Ω , the condition must be satisfied, where the highest numeral is allocated to any node of Ω . The radio mean number of Ω , symbolised as $rmm(\Omega)$, is the minimum numerical value of $rmm(h)$ in general radio mean labellings f of Ω [12]. This study applies radio mean labelling to optimise channel allocation and enhance fault tolerance in networks and facility graphs. This reduces interference and improves system reliability.

Examples highlight the effectiveness of the fault-tolerant design. These fault-tolerant facility systems align with the principles of optimization and system reliability found in the study by [16], where efficient separation methods for colloidal nanoparticles are similarly aimed at enhancing system stability and performance. Both applications leverage graph theory to improve the effectiveness of processes, whether in communication networks or nanoparticle separation techniques, by ensuring minimal interference and optimal resource allocation.

Definitions:

1.1 [17]: For a connected graph Ω , a one-to-one function $h : W(\Omega) \rightarrow \mathbb{N}$ is called a radio labelling if, for any distinct nodes z and w in Ω , the inequality $d(z, w) + |h(z) - h(w)| \geq 1 + \text{diam}(\Omega)$ holds, where $\text{diam}(\Omega)$ is the diameter of Ω .

1.2 [18]: For a connected graph Ω , a one-to-one function $h : W(\Omega) \rightarrow \mathbb{N}$ is called a Radio Mean Labelling if, for any distinct nodes z and w in Ω , the inequality $d(z, w) + \left\lceil \frac{h(z) + h(w)}{2} \right\rceil \geq 1 + \text{diam}(\Omega)$ holds and also $d(z, w)$ represents the distance between z and w and $\text{diam}(\Omega)$ is the diameter of Ω .

1.3 [19]: A theta graph T_a is a structure consisting of two vertices with a degree of 3 that are not directly connected to each other, whereas all other vertices in the graph have a degree of 2.

1.4 [20]: Let z and w be two distinct vertices in graph Ω . A new graph Ω_1 is created by merging (or identifying) vertices u and v into a single vertex x in Ω_1 . In this new graph, every edge that is connected to either z or w in Ω is now connected to x in Ω_1 .

1.5 [20]: A vertex switching Ω_x of a graph G is created by choosing a vertex x in Ω , deleting all edges that are connected to x , and then connecting x to every vertex that was not originally connected to x in Ω .

2 Methods, Results and Discussion

Radio Mean Labelling of Theta-Related Graphs

Theorem 2.1

The Theta graph T_a is a Radio Mean graph.

Proof:

Let $\Omega = T_a$ with vertex set

$$W(\Omega) = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$$

and edge (line) set of eight elements

$$F(\Omega) = \{\{x_i, x_{i+1}\} : 1 \leq i \leq 5\} \cup \{\{x_0, x_1\}, \{x_0, x_4\}, \{x_1, x_6\}\}$$

be the line set of Ω . The number of nodes of T_a is 7, and the number of lines of T_a is 8. Here, the diameter is 3.

Define the one-to-one function $h : W(\Omega) \rightarrow \mathbb{N}$ by

$$\begin{aligned} h(x_0) &= 8, h(x_1) = 7, h(x_2) = 2, h(x_3) \\ &= 3, h(x_4) = 4, h(x_5) = 5, h(x_6) = 6 \end{aligned}$$

To prove that h is a radio mean labelling.

The diameter is 3. So, it is enough to prove that the distance between the node z and node w plus $\left\lceil \frac{h(z)+h(w)}{2} \right\rceil \geq 4$ for every pair of nodes (z, w) , where $z \neq w$.

The proof involves the following cases:

Case 1: Consider the pair (x_2, x_1) with the distance between node x_2 and node x_1 being 1. We have:

$$\begin{aligned} &\text{distance between } x_2 \text{ and } x_1 + \left\lceil \frac{h(x_2) + h(x_1)}{2} \right\rceil \\ &= 1 + \left\lceil \frac{2+7}{2} \right\rceil = 1 + 5 = 6 \geq 4 \end{aligned}$$

Case 2: Consider the pair (x_3, x_5) with the distance between node x_3 and node x_5 being 2. We have:

$$\begin{aligned} &\text{distance between } x_3 \text{ and } x_5 + \left\lceil \frac{h(x_3) + h(x_5)}{2} \right\rceil \\ &= 2 + \left\lceil \frac{3+5}{2} \right\rceil = 2 + 4 = 6 \geq 4 \end{aligned}$$

Case 3: Consider the pair (x_3, x_6) with the distance between node x_3 and node x_6 being 3. We have:

$$\begin{aligned} &\text{distance between } x_3 \text{ and } x_6 + \left\lceil \frac{h(x_3) + h(x_6)}{2} \right\rceil \\ &= 3 + \left\lceil \frac{3+6}{2} \right\rceil = 3 + 5 = 8 \geq 4 \end{aligned}$$

Hence, in all the above cases, the mean radio condition was satisfied by f .

Therefore, the Theta graph T_a is a Radio Mean Graph.

Hence, proved.

Example 2.2:

Radio mean labelling of the theta graph T_a is shown in Figure 1.

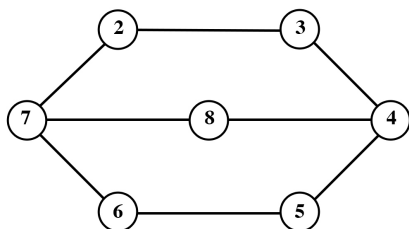


Fig. 1: Radio Mean Labelling of theta graph T_a

Theorem 2.3

The fusion of any two vertices in a cycle of T_a is a Radio Mean graph.

Proof:

Let T_a be a Theta graph with center x_0 . Let $W(T_a) = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$ be the node set and $F(T_a) = \{\{x_i, x_{i+1}\} : 1 \leq i \leq 5; \{x_0, x_1\}, \{x_0, x_4\}, \{x_1, x_6\}\}$ be the line set of T_a . The number of nodes of T_a is 7, and the number of lines of T_a is 8.

Let Ω be a graph obtained by the fusion of two vertices x_2 and x_3 in the cycle of T_a , and we call it vertex x_2 . The number of nodes of Ω is 6, and the number of lines of Ω is 7. Here, the diameter is 2.

Define the one-to-one function $h : W(\Omega) \rightarrow \mathbb{N}$ by:

$$\begin{aligned} h(x_0) &= 6, h(x_1) = 5, h(x_2) = 1, h(x_4) \\ &= 2, h(x_5) = 3, h(x_6) = 4 \end{aligned}$$

To prove that h is a radio mean labelling.

The diameter is 2. So, it is enough to prove that the distance between the node z and node w plus $\left\lceil \frac{h(z)+h(w)}{2} \right\rceil \geq 3$ for every pair of nodes (z, w) , where $z \neq w$.

The proof involves the following cases:

Case 1: Consider the pair (x_2, x_4) with the distance between node x_2 and node x_4 being 1. We have:

$$\begin{aligned} &\text{distance between } x_2 \text{ and } x_4 + \left\lceil \frac{h(x_2) + h(x_4)}{2} \right\rceil \\ &= 1 + \left\lceil \frac{1+2}{2} \right\rceil = 1 + 2 = 3 \geq 3 \end{aligned}$$

Case 2: Consider the pair (x_2, x_5) with the distance between node x_2 and node x_5 being 2. We have:

$$\begin{aligned} &\text{distance between } x_2 \text{ and } x_5 + \left\lceil \frac{h(x_2) + h(x_5)}{2} \right\rceil \\ &= 2 + \left\lceil \frac{1+3}{2} \right\rceil = 2 + 2 = 4 \geq 3 \end{aligned}$$

Case 3: Consider the pair (x_2, x_6) with the distance between node x_2 and node x_6 being 2. We have:

$$\begin{aligned} &\text{distance between } x_2 \text{ and } x_6 + \left\lceil \frac{h(x_2) + h(x_6)}{2} \right\rceil \\ &= 2 + \left\lceil \frac{1+4}{2} \right\rceil = 2 + 3 = 5 \geq 3 \end{aligned}$$

Hence, in all the above cases, the radio mean condition is satisfied by f .

Therefore, the fusion of any two vertices in a cycle of T_a is a Radio Mean Graph.

Hence, proved.

Example 2.4

The Radio Mean labelling of the fusion of any two vertices in a cycle of T_a is shown in Figure 2.

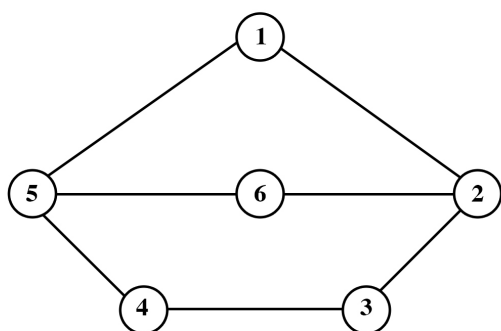


Fig. 2: Radio Mean Labelling of fusion of any two vertices in a cycle of theta graph T_a

Theorem 2.5

The switching of a central vertex in the theta graph T_a is a Radio Mean graph.

Proof:

Let T_a be a Theta graph with center x_0 . Let $W(T_a) = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$ be the node set and

$$F(T_a) = \{\{x_i, x_{i+1}\} : 1 \leq i \leq 5; \{x_0, x_1\}, \{x_0, x_4\}, \{x_1, x_6\}\}$$

be the line set of T_a . The number of nodes of T_a is 7 and the number of lines of T_a is 8. Let Ω be a graph obtained from T_a after switching the central vertex x_0 .

Define the one-to-one function $h : W(\Omega) \rightarrow \mathbb{N}$ by

$$\begin{aligned} h(x_0) &= 8, h(x_1) = 7, h(x_2) = 2, h(x_3) \\ &= 3, h(x_4) = 4, h(x_5) = 5, h(x_6) = 6 \end{aligned}$$

To prove h is a radio mean labelling.

Here, the diameter is 3. So, it is enough to prove that the distance between node z and node w plus $\left\lceil \frac{h(z)+h(w)}{2} \right\rceil \geq 4$ for every pair of nodes (z, w) , where $z \neq w$. Now, the proof involves the following cases.

Case 1: Consider the pair (x_2, x_3) with distance between node x_2 and node x_3 being 1:

$$\begin{aligned} \text{distance between } x_2 \text{ and } x_3 + \left\lceil \frac{h(x_2)+h(x_3)}{2} \right\rceil \\ = 1 + \left\lceil \frac{2+3}{2} \right\rceil = 1 + 3 = 4 \geq 4 \end{aligned}$$

Case 2: Consider the pair (x_2, x_4) with distance between node x_2 and node x_4 being 2:

$$\begin{aligned} \text{distance between } x_2 \text{ and } x_4 + \left\lceil \frac{h(x_2)+h(x_4)}{2} \right\rceil \\ = 2 + \left\lceil \frac{2+4}{2} \right\rceil = 2 + 3 = 5 \geq 4 \end{aligned}$$

Case 3: Consider the pair (x_2, x_5) with distance between node x_2 and node x_5 being 3:

$$\begin{aligned} \text{distance between } x_2 \text{ and } x_5 + \left\lceil \frac{h(x_2)+h(x_5)}{2} \right\rceil \\ = 3 + \left\lceil \frac{2+5}{2} \right\rceil = 4 + 5 = 9 \geq 4 \end{aligned}$$

Case 4: Consider the pair (x_2, x_6) with distance between node x_2 and node x_6 being 2:

$$\begin{aligned} \text{distance between } x_2 \text{ and } x_6 + \left\lceil \frac{h(x_2)+h(x_6)}{2} \right\rceil \\ = 2 + \left\lceil \frac{2+6}{2} \right\rceil = 2 + 4 = 6 \geq 4 \end{aligned}$$

Case 5: Consider the pair (x_2, x_1) with distance between node x_2 and node x_1 being 1:

$$\begin{aligned} \text{distance between } x_2 \text{ and } x_1 + \left\lceil \frac{h(x_2)+h(x_1)}{2} \right\rceil \\ = 1 + \left\lceil \frac{2+7}{2} \right\rceil = 1 + 5 = 6 \geq 4 \end{aligned}$$

Case 6: Consider the pair (x_2, x_0) with distance between node x_2 and node x_0 being 2:

$$\begin{aligned} \text{distance between } x_2 \text{ and } x_0 + \left\lceil \frac{h(x_2)+h(x_0)}{2} \right\rceil \\ = 2 + \left\lceil \frac{2+8}{2} \right\rceil = 2 + 5 = 7 \geq 4 \end{aligned}$$

Case 7: Consider the pair (x_3, x_4) with distance between node x_3 and node x_4 being 1:

$$\begin{aligned} \text{distance between } x_3 \text{ and } x_4 + \left\lceil \frac{h(x_3)+h(x_4)}{2} \right\rceil \\ = 1 + \left\lceil \frac{3+4}{2} \right\rceil = 1 + 4 = 5 \geq 4 \end{aligned}$$

Case 8: Consider the pair (x_3, x_5) with distance between node x_3 and node x_5 being 2:

$$\begin{aligned} \text{distance between } x_3 \text{ and } x_5 + \left\lceil \frac{h(x_3)+h(x_5)}{2} \right\rceil \\ = 2 + \left\lceil \frac{3+5}{2} \right\rceil = 2 + 4 = 6 \geq 4 \end{aligned}$$

Case 9: Consider the pair (x_3, x_6) with distance between node x_3 and node x_6 being 3:

$$\text{distance between } x_3 \text{ and } x_6 + \left\lceil \frac{h(x_3)+h(x_6)}{2} \right\rceil$$

$$= 3 + \left\lceil \frac{3+6}{2} \right\rceil = 3 + 5 = 8 \geq 4$$

Case 10: Consider the pair (x_3, x_1) with distance between node x_3 and node x_1 being 2:

$$\begin{aligned} &\text{distance between } x_3 \text{ and } x_1 + \left\lceil \frac{h(x_3) + h(x_1)}{2} \right\rceil \\ &= 2 + \left\lceil \frac{3+7}{2} \right\rceil = 2 + 5 = 7 \geq 4 \end{aligned}$$

Case 11: Consider the pair (x_3, x_0) with distance between node x_3 and node x_0 being 2:

$$\begin{aligned} &\text{distance between } x_3 \text{ and } x_0 + \left\lceil \frac{h(x_3) + h(x_0)}{2} \right\rceil \\ &= 2 + \left\lceil \frac{3+8}{2} \right\rceil = 2 + 6 = 8 \geq 4 \end{aligned}$$

Case 12: Consider the pair (x_4, x_5) with distance between node x_4 and node x_5 being 1:

$$\begin{aligned} &\text{distance between } x_4 \text{ and } x_5 + \left\lceil \frac{h(x_4) + h(x_5)}{2} \right\rceil \\ &= 1 + \left\lceil \frac{4+5}{2} \right\rceil = 1 + 5 = 6 \geq 4 \end{aligned}$$

Case 13: Consider the pair (x_4, x_6) with distance between node x_4 and node x_6 being 2:

$$\begin{aligned} &\text{distance between } x_4 \text{ and } x_6 + \left\lceil \frac{h(x_4) + h(x_6)}{2} \right\rceil \\ &= 2 + \left\lceil \frac{4+6}{2} \right\rceil = 2 + 5 = 7 \geq 4 \end{aligned}$$

Case 14: Consider the pair (x_4, x_1) with distance between node x_4 and node x_1 being 2:

$$\begin{aligned} &\text{distance between } x_4 \text{ and } x_1 + \left\lceil \frac{h(x_4) + h(x_1)}{2} \right\rceil \\ &= 2 + \left\lceil \frac{4+7}{2} \right\rceil = 2 + 6 = 8 \geq 4 \end{aligned}$$

Case 15: Consider the pair (x_4, x_0) with distance between node x_4 and node x_0 being 1:

$$\begin{aligned} &\text{distance between } x_4 \text{ and } x_0 + \left\lceil \frac{h(x_4) + h(x_0)}{2} \right\rceil \\ &= 1 + \left\lceil \frac{4+8}{2} \right\rceil = 1 + 6 = 7 \geq 4 \end{aligned}$$

Case 16: Consider the pair (x_5, x_6) with distance between node x_5 and node x_6 being 1:

$$\text{distance between } x_5 \text{ and } x_6 + \left\lceil \frac{h(x_5) + h(x_6)}{2} \right\rceil$$

$$= 1 + \left\lceil \frac{5+6}{2} \right\rceil = 1 + 6 = 7 \geq 4$$

Case 17: Consider the pair (x_5, x_1) with distance between node x_5 and node x_1 being 2:

$$\begin{aligned} &\text{distance between } x_5 \text{ and } x_1 + \left\lceil \frac{h(x_5) + h(x_1)}{2} \right\rceil \\ &= 2 + \left\lceil \frac{5+7}{2} \right\rceil = 2 + 6 = 8 \geq 4 \end{aligned}$$

Case 18: Consider the pair (x_5, x_0) with distance between node x_5 and node x_0 being 2:

$$\begin{aligned} &\text{distance between } x_5 \text{ and } x_0 + \left\lceil \frac{h(x_5) + h(x_0)}{2} \right\rceil \\ &= 2 + \left\lceil \frac{5+8}{2} \right\rceil = 2 + 7 = 9 \geq 4 \end{aligned}$$

Case 19: Consider the pair (x_6, x_1) with distance between node x_6 and node x_1 being 1:

$$\begin{aligned} &\text{distance between } x_6 \text{ and } x_1 + \left\lceil \frac{h(x_6) + h(x_1)}{2} \right\rceil \\ &= 1 + \left\lceil \frac{6+7}{2} \right\rceil = 1 + 7 = 8 \geq 4 \end{aligned}$$

Case 20: Consider the pair (x_6, x_0) with distance between node x_6 and node x_0 being 2:

$$\begin{aligned} &\text{distance between } x_6 \text{ and } x_0 + \left\lceil \frac{h(x_6) + h(x_0)}{2} \right\rceil \\ &= 2 + \left\lceil \frac{6+8}{2} \right\rceil = 1 + 7 = 8 \geq 4 \end{aligned}$$

Case 21: Consider the pair (x_1, x_0) with distance between node x_1 and node x_0 being 1:

$$\begin{aligned} &\text{distance between } x_1 \text{ and } x_0 + \left\lceil \frac{h(x_1) + h(x_0)}{2} \right\rceil \\ &= 1 + \left\lceil \frac{7+8}{2} \right\rceil = 1 + 8 = 9 \geq 4 \end{aligned}$$

Hence, by all the above cases, the radio mean condition is satisfied by f .

Thus, the switching of a central vertex in the theta graph T_a is a Radio Mean Graph.

Hence, proved.

Example 2.6 The radio mean labelling of a switching of a central vertex in the theta graph T_a is shown in Figure 3.

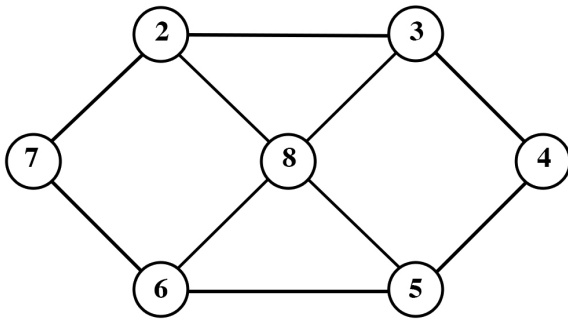


Fig. 3: Radio Mean Labelling of a switching of a central vertex in the theta graph T_a

2.1 Application of Radio Mean Labelling in Communication Networks

In communication networks, efficient channel assignment is crucial for minimising interference between transmitters. This problem can be effectively addressed using graph-labelling techniques. A communication network can be modelled as a graph $\Omega = (W(\Omega), F(\Omega))$, where $W(\Omega)$ represents the set of transmitters (or stations), and $F(\Omega)$ represents the connections between these stations. The distance between any two nodes z and w in Ω is characterised by the extent of the briefest journey between them. The width of the chart, indicated as $\text{span}(G)$, is the supreme separation between any duo of nodes in Ω .

Radio mean labelling is an effective method for assigning natural numbers (channel numbers) to the nodes of G to minimise interference. The function $h : W(\Omega) \rightarrow \mathbb{N}$ must satisfy the condition:

distance between node z and node w +

$$\left\lceil \frac{h(z) + h(w)}{2} \right\rceil \geq 1 + \text{diam}(\Omega)$$

for all distinct nodes z and w in Ω . The radio mean number of a labelling h , denoted $\text{rnm}(h)$, is the highest numeral allocated to any node of Ω . The radio mean number of Ω , symbolised as $\text{rnm}(\Omega)$, is the minimum numerical value of $\text{rnm}(h)$ in general radio mean labelling f for Ω . To practically apply radio mean labelling in a communication network, each station is assigned a channel number. These channel numbers must be spaced according to the radio mean labelling rules to minimise interference.

Consider a simple example of a network with four stations (A, B, C, and D) forming a square (cycle graph C_4). Graph G is represented as follows:

$$\Omega = (W(\Omega), F(\Omega)),$$

where $W(\Omega) = \{A, B, C, D\}$ and $F(\Omega) = \{(A, B), (B, C), (C, D), (D, A), (A, C), (B, D)\}$.

The distances between the stations are:

Distance between station A and station B

= Distance between station B and station C

= Distance between station C and station D

= Distance between station D and station A = 1,

and

Distance between station A and station C

= Distance between station B and station D = 2.

The diameter of the graph is $\text{diam}(G) = 2$.

Labels $h(A)$, $h(B)$, $h(C)$, and $h(D)$ are assigned such that the condition is satisfied:

$$\text{distance between } z \text{ and } w + \left\lceil \frac{h(z) + h(w)}{2} \right\rceil \geq 3.$$

One possible assignment is:

$$h(A) = 1, h(B) = 5, h(C) = 9, h(D) = 13.$$

Verifying the condition for all pairs:

For (A, B):

$$\text{distance between station A and station B} + \left\lceil \frac{h(A) + h(B)}{2} \right\rceil$$

$$= 1 + \left\lceil \frac{1+5}{2} \right\rceil = 1 + 3 = 4 \geq 3.$$

For (A, C):

$$\text{distance between station A and station C} + \left\lceil \frac{h(A) + h(C)}{2} \right\rceil$$

$$= 2 + \left\lceil \frac{1+9}{2} \right\rceil = 2 + 5 = 7 \geq 3.$$

For (A, D):

$$\text{distance between station A and station D} + \left\lceil \frac{h(A) + h(D)}{2} \right\rceil$$

$$= 1 + \left\lceil \frac{1+13}{2} \right\rceil = 1 + 7 = 8 \geq 3.$$

For (B, C):

$$\text{distance between station B and station C} + \left\lceil \frac{h(B) + h(C)}{2} \right\rceil$$

$$= 1 + \left\lceil \frac{5+9}{2} \right\rceil = 1 + 7 = 8 \geq 3.$$

For (B,D):

$$\begin{aligned} \text{distance between station B and station D} &+ \left\lceil \frac{h(B)+h(D)}{2} \right\rceil \\ &= 2 + \left\lceil \frac{5+13}{2} \right\rceil = 2 + 9 = 11 \geq 3. \end{aligned}$$

For (C,D):

$$\begin{aligned} \text{distance between station C and station D} &+ \left\lceil \frac{h(C)+h(D)}{2} \right\rceil \\ &= 1 + \left\lceil \frac{9+13}{2} \right\rceil = 1 + 11 = 12 \geq 3. \end{aligned}$$

The labelling $h(A) = 1, h(B) = 5, h(C) = 9, h(D) = 13$ satisfies the condition for all pairs of nodes. Thus, $r_{mn}(f) = 13$. The given illustration depicts the following graph figure 4 with radio mean labelling based on the above example:

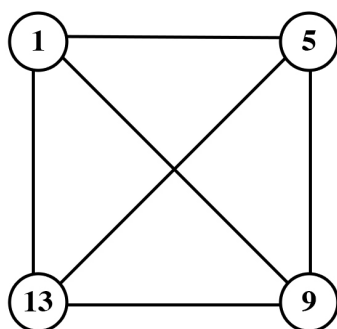


Fig. 4: Radio Mean Labelling of $h(A) = 1, h(B) = 5, h(C) = 9, \text{ and } h(D) = 13$

By following these steps, radio mean labelling can be effectively applied to any communication network to minimise interference and ensure an optimal channel assignment. This method provides a systematic approach to managing communication channels, which is crucial for the efficient functioning of modern communication networks.

2.2 Application of Radio Mean Labelling in Designing Fault-Tolerant Systems with Facility Graphs

Facility graphs model the interconnections between system facilities, both hardware and software, in fault-tolerant systems. These graphs help optimise the system performance and reliability by illustrating the

facility relationships and resource sharing. Radio mean labelling assigns unique channel numbers to minimise interference between facilities and enhance fault tolerance. The labelling process ensures that the distance between facilities, combined with their assigned labels, satisfies specific conditions to avoid conflicts. This approach optimises resource utilisation and improves communication efficiency. Ultimately, this supports the development of resilient and efficient fault-tolerant systems.

3 Conclusion

In this study, it has been demonstrated how the concept of radio mean labelling is applied in the theta graph T_a , along with its variations formed by the fusion of any two vertices in a cycle of the theta graph T_a and the switching of a central vertex in the Theta graph T_a with some examples. Additionally, this study analysed its application in designing Fault-Tolerant Systems with Facility Graphs. It is applicable in communication networks, giving a simple example through some stations.

Author Contributions

- A.V. - Conceptualization, resources, writing—original draft preparation, and writing—review and editing.
- H.K. - Conceptualization, methodology, validation, investigation, and data curation.
- A.A. - Methodology, software, formal analysis, investigation, and visualization.
- M.R. - Methodology, software, formal analysis, and validation.
- V.S. - Methodology, formal analysis, resources, and supervision, while also contributing to the writing—original draft preparation and visualization.
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Conflicts of Interest

The authors declare no conflict of interest.

Transparency

The authors confirm that the manuscript is an honest, accurate, and transparent account of the study; that no vital features of the study have been omitted; and that any discrepancies from the study as planned have been explained. This study followed all ethical practices during writing.

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