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Discontinuous Analysis

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Abstract: In this article, I investigate in detail (generalized) limit of an arbitrary (discontinuous) function, defined in terms of funcoids (funcoids are briefly considered in this work, for people unfamiliar with them). Definition of generalized limit makes it obvious to define such things as derivative of an arbitrary function, integral of an arbitrary function, sum of arbitrary series, etc. It is given a definition of non-differentiable solution of a (partial) differential equation. It's raised the question how do such solutions "look like" starting a possible big future research program. It helps to calculate series, derivatives, integrals without first checking that they exist. This theory allows you to check this once in the end of the calculation, instead of checking several times in the middle. The generalized derivatives and integrals are linear operators. For example $\int_a^b f(x)dx - \int_a^b f(x)dx = 0$ is defined and true for *every* function. This has an advantage over ("competing" with my theory) distributions theory based analysis, that for example any two functions in my analysis are multiplicible, while in distributions analysis you need to check a complex condition before multiplying two functions. This is a straightforward, "no cost" advantage over traditional ways of non-smooth analysis. Moreover, in distributions analysis not every function has a derivative, but in my analysis every function is differentiable. The advantages are further bettered by the fact that I consider (generalized) limits of any values, not for a limited class of functions. The generalized solution of one simple example differential equation is also considered.

Keywords: algebraic theory of general topology, differential equations, limit, funcoid, nonsmooth analysis.

1 Introduction

I defined *funcoid* and based on this generalized *limit of an arbitrary (even discontinuous) function* in [1].

In this article I consider generalized limits in more details.

This article is written in such a way that a reader could understand the main ideas on generalized limits without resorting to reading [1] beforehand, but to follow the proofs you need read that first.

Definition of generalized limit makes it obvious to define such things as derivative of an arbitrary function, integral of an arbitrary function, etc.

Note that generalized limit is a "composite" object, not just a simple real number, point, or "regular" vector.

The problem to be solved can be formulated as follows: extend the sets of values of limits, derivatives, integrals, etc. in such a way that each function has limits at all points (and also has derivatives, integrals, etc.), in such a way that the limits, derivatives, integrals remain linear functions (to the extended space of values).

Moreover, as I show in a theorem below, any operations on the points of the "base" space can be extended to operations on the values of generalized limits, while preserving their algebraic properties (if, for example, $x - x = 0$ in the base space, then $f'(x) - f'(x) = 0$ where f' means the generalized derivative of the function f).

The last two paragraphs explain all the conclusions that a physicist or engineer should draw from this text, and then, in essence, just the proofs of these statements follow (and also preliminary notes on differential equations).

Importance of analysis of arbitrary functions is obvious and cannot be overstated.

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2 A popular explanation of generalized limit

For an example, consider some real function f from x-axis to y-axis:

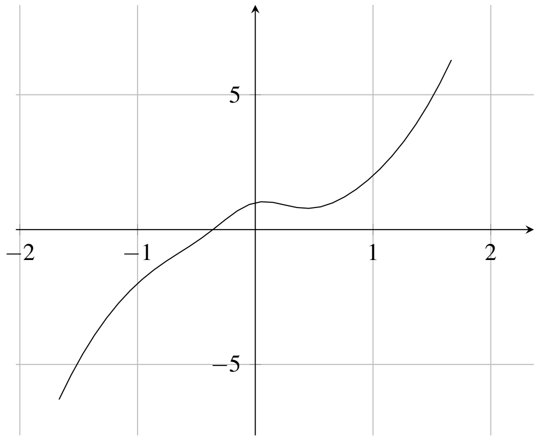

Fig. 1: Some function

Take it's infinitely small fragment (in our example, an infinitely small interval for x around zero; see below for an explanation what is infinitely small):

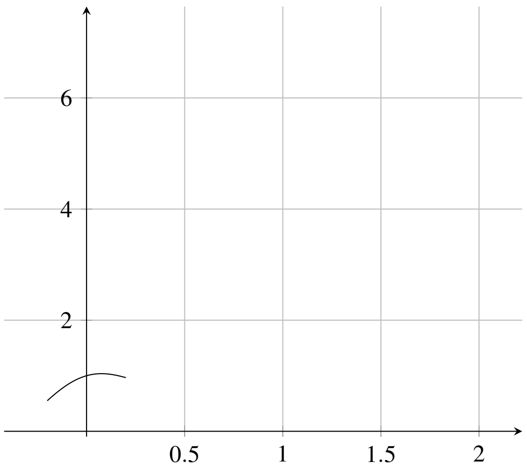

Fig. 2: An infinitely short "cut"

Next consider that with a value y replaced by an infinitely small interval like $[y-\varepsilon; y+\varepsilon]$:

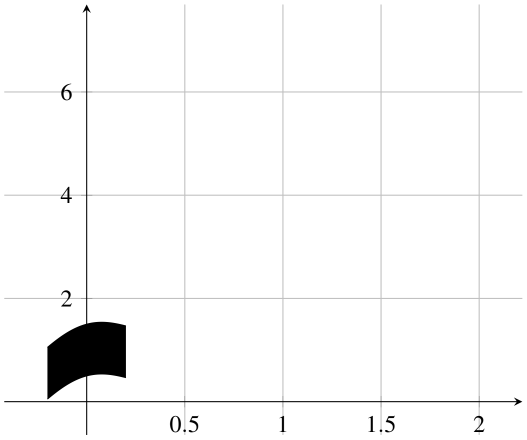

Fig. 3: An infinitely short and thick "cut"

Now we have "an infinitely thin and short strip". In fact, it is the same as an "infinitely small rectangle" (Why? So infinitely small behave, it can be counter-intuitive, but if we consider the above meditations formally, we could get this result):

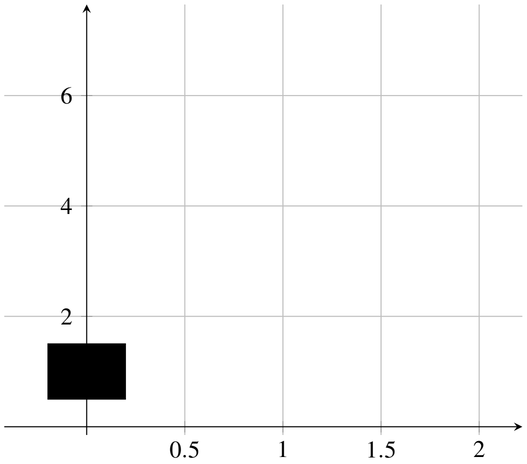

Fig. 4: An infinitely small rectangle

This infinitely small rectangle's y position uniquely characterizes the limit of our function (in our example at $x \to 0$).

If we consider the set of all rectangles we obtain by shifting this rectangle by adding an arbitrary number to x, we get

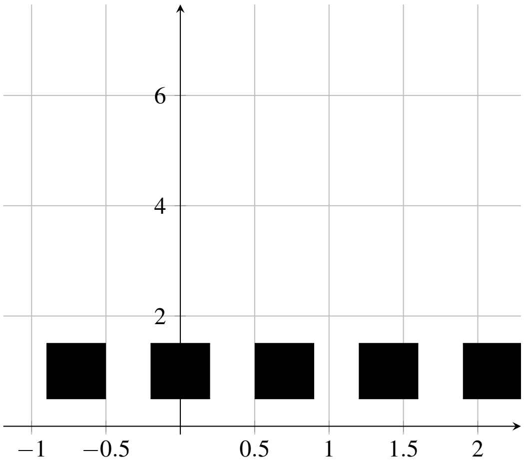

Fig. 5: An infinite set of infinitely small rectangles

Such sets one-to-one corresponds to the value of the limit of our function (at $x \to 0$): Knowing such the set, we can calculate the limit (take its arbitrary element and get its so to say y-limit point) and knowing the limit value (y), we could write down the definition of this set.

So we have a formula for *generalized limit*:

$$\lim_{x\to a} f(x) = \{\mathrm{X} \circ f|_{\Delta(a)} \circ r \mid r \in G\}$$

where G is the group of all horizontal shifts of our space $\mathbb{R}$, $f|_{\Delta(a)}$ is the function f of which we are taking limit restricted to the infinitely small interval $\Delta(a)$ around the point a, $\mathrm{X}\circ$ is "stretching" our function graph into the infinitely thin "strip" by applying a topological operation to it.

What does all this (especially "infinitely small") mean? It is filters and "funcoids" (see below for the definition).

Why do we consider all shifts of our infinitesimal rectangle? It's to make the limit independent of the horizontal shift of the function (while simultaneously shifting the point a to which x tends). Otherwise the limit would depend on the horizontal shift of the coordinate system, what is unacceptable.

Note that for discontinuous functions elements of our set (our limit is a set) won't be infinitely small "rectangles" (as on the pictures), but would "touch" more than just one y value.

The interesting thing here is that we can apply the above formula to *every* function: for example to a discontinuous function, Dirichlet function, unbounded function, unbounded and discontinuous at every point function, etc. In short, the generalized limit is defined for *every* function. We have a definition of limit for every function, not only a continuous function!

And it works not only for real numbers. It would work for example for any function between two topological vector spaces (a vector space with a topology).

Hurrah! Now we can define derivative and integral of *every* function.

3 Filters

We will denote partial orders as $\sqsubseteq$, the join as $\sqcup$, and the meet as $\sqcap$. I denote the greatest element of an order as $\top$ and the least as $\bot$.

Definition 1. $a \asymp b \stackrel{\text{def}}{=} a \sqcap b = \bot$, $a \not\asymp b \stackrel{\text{def}}{=} a \sqcap b \neq \bot$.

Definition 2. *A filter on a set U is a nonempty subset $\mathscr{F}$ of $\mathscr{P}U$ such that both:*

1. $\forall A, B \in \mathscr{F} : A \cap B \in \mathscr{F}$;
2. $\forall A \in \mathscr{F}, B \in \mathscr{P}U : (B \supseteq A \Rightarrow B \in \mathscr{F})$.

I will denote the set of filters as $\mathfrak{F}$.

Intuitively, filters may be like infinitely small sets: Consider the filter whose elements are subsets of $\mathbb{R}$ containing every neighborhood of zero.

I will call the *principal filter* corresponding to set A the filter

$$\uparrow A = \left\{ \frac{X}{X \in \mathscr{P}Y, X \supseteq A} \right\}.$$

I will order filters *reversely* to set theoretic inclusion:

$$\mathscr{A} \sqsubseteq \mathscr{B} \Leftrightarrow \mathscr{A} \supseteq \mathscr{B}.$$

It is easy to prove that filters on a set form a lattice. For filters $\mathscr{A}$, $\mathscr{B}$:

1. $\mathscr{A} \sqcap \mathscr{B} = \left\{ \frac{A \cap B}{A \in \mathscr{A}, B \in \mathscr{B}} \right\}$;
2. $\mathscr{A} \sqcup \mathscr{B} = \mathscr{A} \cap \mathscr{B}$.

In [?] we also prove that the set of filters is an atomic (and moreover atomistic) lattice. We will denote atoms (traditionally called *maximal filters*) under a filter $\mathscr{A}$ as atoms $\mathscr{A}$.

Moreover, filters form a complete lattice. For a set S of filters:

1. $\bigsqcap S = \left\{ \frac{A_0 \cap \cdots \cap A_n}{\mathscr{A} \in S, n \in \mathbb{Z}_+, \forall i=0,\ldots,n: A_i \in \mathscr{A}} \right\}$;
2. $\bigsqcup S = \bigcap S$.

4 Funcoids

I will reprise (without proofs, that you are able to easily fill in by yourself) several equivalent definitions of funcoid from [?]:

Binary relation δ between two sets (source and destination of the funcoid), conforming to the axioms:

1. not $\emptyset \;\delta\; X$
2. not $X \;\delta\; \emptyset$
3. $I \cup J \;\delta\; K \iff I \;\delta\; K \wedge J \;\delta\; K$
4. $K \;\delta\; I \cup J \iff K \;\delta\; I \wedge K \;\delta\; J$

Pair of functions (α, β) between the sets of filters filters on some two sets (source and destination of the funcoid), conforming to the formula:

$$\alpha(\mathscr{X}) \sqcap \mathscr{Y} \neq \bot \iff \beta(\mathscr{Y}) \sqcap \mathscr{X} \neq \bot.$$

*Remark.*Funcoid (α,β) is determined by the value of α (or value of β).

A function Δ from the set of subsets of some set (source of the funcoid) to the set of filters on some set (destination of the funcoid), conforming to the axioms:

1. $\Delta(\emptyset)=\bot$
2. $\Delta(X\sqcup Y)=\Delta(X)\sqcup\Delta(Y)$

Note that we define things to have the equations:

1.

$$X\,[f]^*\,Y \Leftrightarrow X\ \delta\ Y \Leftrightarrow \uparrow X\,[(\alpha,\beta)]\uparrow Y \Leftrightarrow \alpha(X)\sqcap Y\neq\bot \Leftrightarrow \beta(Y)\sqcap X\neq\bot$$

2. $\langle f\rangle^* X=\Delta X=\alpha\uparrow X$

I will call *endofuncoid* a funcoid whose source and destination are the same.

Funcoids form a semigroup (or semicategory, dependently on the exact axioms) with the operation defined by the formula:

$$(\alpha_1,\beta_1)\circ(\alpha_0,\beta_0)=(\alpha_1\circ\alpha_0,\beta_0\circ\beta_1).$$

We denote $\langle(\alpha,\beta)\rangle=\alpha$ and $(\alpha,\beta)^{-1}=(\beta,\alpha)$.

Funcoids also form a poset which is a complete lattice (see [**?**]), with the order

$$(\alpha_0,\beta_0)\sqsubseteq(\alpha_1,\beta_1)\Leftrightarrow\alpha_0\sqsubseteq\alpha_1\wedge\beta_0\sqsubseteq\beta_1.$$

Also funcoids are a generalization of binary relations. I will denote the funcoid corresponding to a binary relation f as $\uparrow f$, the defining formulas are

$$\langle\uparrow f\rangle\uparrow X=\uparrow(f[X]),\quad X\,[\uparrow f]\,Y\Leftrightarrow(X\times Y)\cap f\neq\emptyset.$$

Funcoids are a generalization of topological spaces. For every topological space it can be constructed a funcoid:

$$\left(\mathscr{A}\mapsto\bigsqcap\left\{\frac{\uparrow X}{X\in\mathscr{A},X\text{ is an open set}}\right\},\ \bigsqcup\left\{\frac{\uparrow X}{X\in\mathscr{A},X\text{ is a closed set}}\right\}\right)$$

or its reverse

$$\left(\bigsqcup\left\{\frac{\uparrow X}{X\in\mathscr{A},X\text{ is a closed set}}\right\},\ \mathscr{A}\mapsto\bigsqcap\left\{\frac{\uparrow X}{X\in\mathscr{A},X\text{ is an open set}}\right\}\right)$$

It is easy to check that the above is a funcoid.

Funcoids are an obvious a generalization of proximity spaces (see [**?**]).

Restricted identity funcoid generalizes an identity function:

$$\mathrm{id}_{\mathscr{A}}=(\mathscr{X}\mapsto\mathscr{A}\sqcap\mathscr{X},\mathscr{X}\mapsto\mathscr{A}\sqcap\mathscr{X}).$$

Restricting a funcoid f to a filter $\mathscr{A}$ is $f|_{\mathscr{A}}=f\circ\mathrm{id}_{\mathscr{A}}$. (This generalizes restricting a function to a set.)

In [**?**] we also have a funcoid called *funcoidal product* (generalizing Cartesian product of two sets) of two filters $\mathscr{A}$ and $\mathscr{B}$ is defined as:

$$\mathscr{A}\times^{\mathsf{FCD}}\mathscr{B}=\left(\mathscr{X}\mapsto\left(\begin{cases}\mathscr{B} & \text{if }\mathscr{X}\not\asymp\mathscr{A}\\ \uparrow\bot & \text{if }\mathscr{X}\asymp\mathscr{A}\end{cases}\right),\ \mathscr{Y}\mapsto\left(\begin{cases}\mathscr{A} & \text{if }\mathscr{Y}\not\asymp\mathscr{B}\\ \uparrow\bot & \text{if }\mathscr{Y}\asymp\mathscr{B}\end{cases}\right)\right).$$

It's easy to check that this is really a funcoid.

Also, $\operatorname{im} f\overset{\text{def}}{=}\langle f\rangle\top$, $\operatorname{dom} f\overset{\text{def}}{=}\operatorname{im} f^{-1}$ for a funcoid f.

5 Limit for funcoids

The following is a straightforward generalization of the well known concepts of adherent point of a set (more generally a cluster point of a filter), a limit point of a filter, and limit of a function in a topological space.

*Note 1.*Due to an unfortunate choice of terminology, limit point of a filter is *not* a generalization of a limit point of a set. Limit point for a set isn't a beautiful term and we won't use it (in this work), so by *limit point* we will always mean a limit point of a filter.

So, generalizing the corresponding concepts for topological spaces:

Let d be a funcoid.

Definition 3.*I will denote lattice operations as* $\sqcup$ *and* $\sqcap$, *the least and the greatest elements as* $\bot$ *and* $\top$ *correspondingly.*

Definition 4.$\operatorname{Cor}\mathscr{F}=\bigcap\mathscr{F}$, *that is it's the greatest set that is below every element of the filter* $\mathscr{F}$.

Definition 5.*The set of* adherent points *of* $\mathscr{A}$ *is* $\operatorname{Cor}\langle d^{-1}\rangle\mathscr{A}$ *or what is the same* $\left\{\frac{x}{\langle d\rangle\{x\}\not\asymp\mathscr{A}}\right\}$.

Proposition 1.*There exists a (unique) funcoid* A *such that* $\langle\mathrm{A}\rangle X$ *is exactly the set of adherent points of* X *for every argument set* X, *provided that* $\langle d^{-1}\rangle X$ *is a principal filter.*

*Proof.*Obvious.

Definition 6. Limit point *of a filter* $\mathscr{A}$ *on* $\mathrm{Dst}\,X$ *is an* $x\in\mathrm{Dst}\,\mathscr{A}$ *such that* $\langle d\rangle^*\{x\}\sqsupseteq\mathscr{A}$.

Proposition 2.*If* d *is reflexive, then there exists a (unique) "dual funcoid" (a pointfree funcoid, see [?])* $\mathrm{L}:\mathrm{Ob}\,d\to(\mathrm{Ob}\,d)^{\mathrm{dual}}$ *such that* $\langle\mathrm{L}\rangle X$ *is exactly the set of limit points of* X *for every argument set* X.

*Proof.*The set of limit points of the empty set is the maximal set.

The set of limit points of $A\cup B$ (for sets A, B) is the set of points x such that $\langle d\rangle\{x\}\sqsupseteq A\cup B$ that is $\langle d\rangle\{x\}\sqsupseteq A$ and $\langle d\rangle\{x\}\sqsupseteq B$ that is the intersection of the set of limit points of A and B.

Thus the set of limit points is a component of such a pointfree funcoid.

Proposition 3.L *and* A *coincide on ultrafilters.*

*Proof.*Because $\langle d\rangle^*\{x\}\not\asymp a\Leftrightarrow\langle d\rangle^*\{x\}\sqsupseteq a$ for an ultrafilter a.

We have shown that concepts of both limit points and adherent points are essentially funcoids. In traditional general topology limit of a function is defined using limit points of a filter. We will generalize it to limit regarding an arbitrary funcoid (in place of the funcoid describing limit points). We will call this arbitrary funcoid *the point funcoid* and denote it X.

Definition 7. Limit *of a funcoid f is the filter*

$$\lim f=\operatorname{im}(\mathrm{X}\circ f).$$

Definition 8. Limit *of a funcoid f at a filter* $\mathscr{X}$ *is the filter*

$$\lim_{\mathscr{X}} f=\lim f|_{\mathscr{X}}=\langle\mathrm{X}\circ f\rangle\mathscr{X}.$$

*Remark.*If $\mathrm{X}=\mathrm{L}$, then the limit is either an one-element or the empty set (“no limit” in traditional topology).

In [?] limit for a funcoid f was defined this way: f *tends* to filter $\mathscr{A}$ ($f\to\mathscr{A}$) regarding a funcoid X on a filter $\mathscr{X}$ iff

$$\operatorname{im} f\sqsubseteq\langle\mathrm{X}\rangle\mathscr{A}.$$

$\lim f$ is such a point that f tends to $\lim f$.

Proposition 4.*That definition from [?] coincides with our above definition, if* $\mathrm{X}=\mathrm{L}$.

*Proof.*In this proof $\lim f$ will mean our definition, *not* the definition from [?]. We need to prove that $\lim f=\left\{\frac{x}{\operatorname{im} f\sqsubseteq\langle\mathrm{X}\rangle^*\{x\}}\right\}$.

Really,

$$\lim f=\operatorname{im}(\mathrm{L}\circ f)=\langle\mathrm{L}\rangle\operatorname{im} f=\left\{\frac{x}{\langle\mathrm{X}\rangle^*\{x\}\sqsupseteq\operatorname{im} f}\right\}.$$

A funcoid f is T_1-separable iff $\neg(\uparrow\{a\}[f]\uparrow\{b\})$ for every $a\in\operatorname{Src} f$ and $b\in\operatorname{Dst} f$. A funcoid f is Hausdorff (T_2-separable) iff $f^{-1}\circ f$ is T_1-separable.

Obvious 1.*A funcoid f is* T_2*-separable iff* $a\neq b\Rightarrow\langle f\rangle\uparrow\{a\}\asymp\langle f\rangle\uparrow\{b\}$ *for every* $a,b\in\operatorname{Src} f$.

If X is Hausdorff (T_2-separable) (see [?]), then there exists no more than one $\lim f$.

6 Axiomatic generalized limit

Let X be a (fixed) funcoid. For example, $\mathrm{X}=d$ where d is some proximity or $\mathrm{X}=\mathrm{A}$ or $\mathrm{X}=\mathrm{L}$ (up to a duality).

By definition $\lim_{\mathscr{X}} f=\langle\mathrm{X}\circ f\rangle\mathscr{X}$ (for every funcoid f).

*Remark.*If $\langle\mathrm{X}\rangle y$ is an limit point (considered as an one-element set) of y and f is a function, then the above defined lim is the same as limit in traditional calculus and topology (except that it is an one-element set of points instead of a point). Empty set means “no limit”.

Let some group G (e.g. the group of all shifts on a vector space, to give an example) is fixed.

Definition 9. Axiomatic generalized limit *is a two-arguments function* $(f,\mathscr{X})\mapsto\operatorname{xlim}_{\mathscr{X}} f$ *from the set* $\mathsf{FCD}(A,B)\times\mathfrak{F}(A)$ *to the set of functions defined on filters* $\mathscr{C}$ *such that exists* $r\in G$ *such that* $\mathscr{C}\sqsubseteq\langle r\rangle\mathscr{X}$ *by the formula:*

$$(\operatorname{xlim}_{\mathscr{X}} f)\mathscr{C}=\lim_{\langle r^{-1}\rangle\mathscr{C}} f.$$

*Remark.*Its meaning is that $\lim_{\mathscr{C}} f$ can be restored from $\operatorname{xlim}_{\mathscr{X}} f$.

Proposition 5.*To describe an axiomatic generalized limit, it's enough to define it on ultrafilters.*

*Proof.*Easily follows from the fact [?] that a funcoid is described by its values on ultrafilters.

Thus axiomatic generalized limit gives a detailed behavior of a function at a filter (its limit at every its atomic subfilter).

Obvious 2.$\operatorname{xlim}_{\mathscr{X}} f=\operatorname{xlim}_{\langle r\rangle\mathscr{X}}(f\circ r^{-1})$ *for every* $r\in G$.

Theorem 1.$\lim_{\mathscr{X}} f$ *can be restored knowing* $\operatorname{xlim}_{\mathscr{X}} f$.

Proof.

$$\lim_{\mathscr{X}} f=\langle\mathrm{X}\circ f\rangle\mathscr{X}=\bigsqcup_{x\in\operatorname{atoms}\mathscr{X}}\langle\mathrm{X}\circ f\rangle x=\bigsqcup_{x\in\operatorname{atoms}\mathscr{X}}\lim x=(\operatorname{xlim}_{\mathscr{X}} f)x.$$

Let y be an arbitrary point of the space $\operatorname{Dst}\mathrm{X}$. Consider the constant function f whose value is this y. Then the first axiom above determines the $\operatorname{xlim}_{\mathscr{C}} f$ for every filter $\mathscr{C}$.

I will denote $\operatorname{xlim}_{\mathscr{C}} f=\tau(y)$.

*Remark.*The above easily generalizes for y being a set of points.

Obvious 3.τ *is an injection, if* $\operatorname{Src}\mathrm{X}$ *is a non-empty set and* $\langle\mathrm{X}\rangle$ *is an injection on one-element sets.*

Corollary 1.τ *is an injection, if* $\operatorname{Src}\mathrm{X}$ *is a non-empty set and* $\mathrm{X}=\mathrm{A}$ *and* X *is Hausdorff.*

In other words, on Hausdorff topologies the set of singularities with non-empty domains is an extension of the set of points $\operatorname{Dst}\mathrm{X}$ (up to a bijection).

7 Generalized limit

7.1 The definition of generalized limit

In [?] generalized limit is defined like the formula:

$$\operatorname{xlim} f = \left\{ \frac{\mathrm{X} \circ f \circ \uparrow r}{r \in G} \right\}. \quad (1)$$

We suppose:

Let μ and X be endofuncoids (on sets $\operatorname{Ob}\mu$, $\operatorname{Ob}\mathrm{X}$). Let G be a transitive permutation group on $\operatorname{Ob}\mu$.

We require that μ and every $r \in G$ commute, that is

$$\mu \circ \uparrow r = \uparrow r \circ \mu. \quad (2)$$

We require for every $y \in \operatorname{Ob}\mathrm{X}$

$$\mathrm{X} \sqsupseteq \langle \mathrm{X} \rangle \uparrow \{y\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \uparrow \{y\}. \quad (3)$$

Proposition 6. *Formula (3) follows from* $\mathrm{X} \sqsupseteq \mathrm{X} \circ \mathrm{X}^{-1}$.

Proof. Let $\mathrm{X} \sqsupseteq \mathrm{X} \circ \mathrm{X}^{-1}$. Then

$$\begin{aligned} \langle \mathrm{X} \rangle \uparrow \{y\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \uparrow \{y\} &= \\ \mathrm{X} \circ (\uparrow \{y\} \times^{\mathsf{FCD}} \uparrow \{y\}) \circ \mathrm{X}^{-1} &= \\ \mathrm{X} \circ \uparrow (\{y\} \times \{y\}) \circ \mathrm{X}^{-1} &\sqsubseteq \\ \mathrm{X} \circ 1 \circ \mathrm{X}^{-1} &= \\ \mathrm{X} \circ \mathrm{X}^{-1} &\sqsubseteq \mathrm{X}. \end{aligned}$$

(Here 1 is the identity element of the semigroup of endofuncoids.)

So we have (generalized) limits of arbitrary functions acting from $\operatorname{Ob}\mu$ to $\operatorname{Ob}\mathrm{X}$. (The functions in consideration are not required to be continuous.)

Remark. Most typically G is the group of translations of some topological vector space[1]. So in particular we have defined limit of an arbitrary function acting from a vector topological space to a topological space.

7.2 Injection from the set of points to the set of all generalized limits

The function τ will define an injection from the set of points of the space X ("numbers", "points", or "vectors") to the set of all (generalized) limits (i.e. values which $\operatorname{xlim}_x f$ may take).

Definition 10.

$$\tau(y) \stackrel{\text{def}}{=} \left\{ \frac{\langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \uparrow \{y\}}{x \in D} \right\}.$$

[1] I remind that every Banach space, every normed space, and every Hilbert space is a vector topological space.

Proposition 7.

$$\tau(y) = \left\{ \frac{(\langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \uparrow \{y\}) \circ \uparrow r}{r \in G} \right\}$$

for every (fixed) $x \in D$.

Proof.

$$\begin{aligned} (\langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \uparrow \{y\}) \circ \uparrow r &= \\ \langle \uparrow r^{-1} \rangle \langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \uparrow \{y\} &= \\ \langle \mu \rangle \langle \uparrow r^{-1} \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \uparrow \{y\} &= \\ \langle \mu \rangle \uparrow \{r^{-1} x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \uparrow \{y\} &\in \\ \left\{ \frac{\langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \uparrow \{y\}}{x \in D} \right\}&. \end{aligned}$$

Reversely

$$\langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \uparrow \{y\} = (\langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \uparrow \{y\}) \circ \uparrow e$$

where e is the identify element of G.

Proposition 8.

$$\tau(y) = \operatorname{xlim}(\langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \uparrow \{y\})$$

(for every x). Informally: Every $\tau(y)$ is a generalized limit of a constant function.

Proof.

$$\begin{aligned} &\operatorname{xlim}(\langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \uparrow \{y\}) = \\ &\left\{ \frac{\mathrm{X} \circ (\langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \uparrow \{y\}) \circ \uparrow r}{r \in G} \right\} = \\ &\left\{ \frac{(\langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \uparrow \{y\}) \circ \uparrow r}{r \in G} \right\} = \tau(y). \end{aligned}$$

Corollary 2. *The τ defined in this section for generalized limits "coincides" with the τ defined in the section about axiomatic generalized limits.*

In further we will use one of the definitions of continuity from [?]:

$$f \in \mathrm{C}(\mu, \mathrm{X}) \Leftrightarrow f \circ \mu \sqsubseteq \mathrm{X} \circ f.$$

Theorem 2. *If f is a function and $f|_{\langle \mu \rangle \uparrow \{x\}} \in \mathrm{C}(\mu, \mathrm{X})$ and $\langle \mu \rangle \uparrow \{x\} \sqsupseteq \uparrow \{x\}$ then $\operatorname{xlim}_x f = \tau(fx)$.*

Proof. $f|_{\langle \mu \rangle \uparrow \{x\}} \circ \mu \sqsubseteq \mathrm{X} \circ f|_{\langle \mu \rangle \uparrow \{x\}} \sqsubseteq \mathrm{X} \circ f$; thus $\langle f \rangle \langle \mu \rangle \uparrow \{x\} \sqsubseteq \langle \mathrm{X} \rangle \langle f \rangle \uparrow \{x\}$; consequently we have

$$\begin{aligned} \mathrm{X} \sqsupseteq \langle \mathrm{X} \rangle \langle f \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \langle f \rangle \uparrow \{x\} &\sqsupseteq \\ \langle f \rangle \langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \langle f \rangle \uparrow \{x\}&. \end{aligned}$$

$$\begin{aligned} \mathrm{X} \circ f|_{\langle \mu \rangle \uparrow \{x\}} &\sqsupseteq \\ (\langle f \rangle \langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \langle f \rangle \uparrow \{x\}) \circ f|_{\langle \mu \rangle \uparrow \{x\}} &= \\ (f|_{\langle \mu \rangle \uparrow \{x\}})^{-1} \langle f \rangle \langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \langle f \rangle \uparrow \{x\} &\sqsupseteq \\ \left\langle \mathrm{id}^{\mathsf{FCD}}_{\operatorname{dom} f|_{\langle \mu \rangle \uparrow \{x\}}} \right\rangle \langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \langle f \rangle \uparrow \{x\} &\sqsupseteq \\ \operatorname{dom} f|_{\langle \mu \rangle \uparrow \{x\}} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \langle f \rangle \uparrow \{x\} &= \\ \langle \mu \rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle \mathrm{X} \rangle \langle f \rangle \uparrow \{x\}&. \end{aligned}$$

$$\operatorname{im}(\mathrm{X}\circ f|_{\langle\mu\rangle\uparrow\{x\}}) = \langle\mathrm{X}\rangle\,\langle f\rangle\uparrow\{x\};$$

$$\mathrm{X}\circ f|_{\langle\mu\rangle\uparrow\{x\}} \sqsubseteq \langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\operatorname{im}(\mathrm{X}\circ f|_{\langle\mu\rangle\uparrow\{x\}}) = \langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\,\langle f\rangle\uparrow\{x\}.$$

So $\mathrm{X}\circ f|_{\langle\mu\rangle\uparrow\{x\}} = \langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\,\langle f\rangle\uparrow\{x\}$.
Thus

$$\operatorname{xlim}_x f = \left\{\frac{(\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\,\langle f\rangle\uparrow\{x\})\circ\uparrow r}{r\in G}\right\} = \tau(fx).$$

Remark. Without the requirement of $\langle\mu\rangle\uparrow\{x\}\sqsupseteq\uparrow\{x\}$ the last theorem would not work in the case of removable singularity.

Theorem 3. *Let* $\mathrm{X}\sqsubseteq\mathrm{X}\circ\mathrm{X}$. *If* $f|_{\langle\mu\rangle\uparrow\{x\}}\xrightarrow{\mathrm{X}}\uparrow\{y\}$ *then* $\operatorname{xlim}_x f = \tau(y)$.

Proof. $\operatorname{im} f|_{\langle\mu\rangle\uparrow\{x\}}\sqsubseteq\langle\mathrm{X}\rangle\uparrow\{y\}$; $\langle f\rangle\,\langle\mu\rangle\uparrow\{x\}\sqsubseteq\langle\mathrm{X}\rangle\uparrow\{y\}$;

$$\begin{aligned}&\mathrm{X}\circ f|_{\langle\mu\rangle\uparrow\{x\}}\sqsupseteq\\&(\langle\mathrm{X}\rangle\uparrow\{y\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\uparrow\{y\})\circ f|_{\langle\mu\rangle\uparrow\{x\}} =\\&\left\langle(f|_{\langle\mu\rangle\uparrow\{x\}})^{-1}\right\rangle\langle\mathrm{X}\rangle\uparrow\{y\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\uparrow\{y\} =\\&\left\langle\mathrm{id}^{\mathsf{FCD}}_{\langle\mu\rangle\uparrow\{x\}}\circ f^{-1}\right\rangle\langle\mathrm{X}\rangle\uparrow\{y\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\uparrow\{y\}\sqsupseteq\\&\left\langle\mathrm{id}^{\mathsf{FCD}}_{\langle\mu\rangle\uparrow\{x\}}\circ f^{-1}\right\rangle\langle f\rangle\,\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\uparrow\{y\} =\\&\left\langle\mathrm{id}^{\mathsf{FCD}}_{\langle\mu\rangle\uparrow\{x\}}\right\rangle\langle f^{-1}\circ f\rangle\,\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\uparrow\{y\}\sqsupseteq\\&\left\langle\mathrm{id}^{\mathsf{FCD}}_{\langle\mu\rangle\uparrow\{x\}}\right\rangle\left\langle\mathrm{id}^{\mathsf{FCD}}_{\langle\mu\rangle\uparrow\{x\}}\right\rangle\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\uparrow\{y\} =\\&\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\uparrow\{y\}.\end{aligned}$$

On the other hand,

$$f|_{\langle\mu\rangle\uparrow\{x\}}\sqsubseteq\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\uparrow\{y\};$$

$$\mathrm{X}\circ f|_{\langle\mu\rangle\uparrow\{x\}}\sqsubseteq\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\,\langle\mathrm{X}\rangle\uparrow\{y\}\sqsubseteq\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\uparrow\{y\}.$$

So $\mathrm{X}\circ f|_{\langle\mu\rangle\uparrow\{x\}} = \langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\uparrow\{y\}$.

$$\operatorname{xlim}_x f = \left\{\frac{\mathrm{X}\circ f|_{\langle\mu\rangle\uparrow\{x\}}\circ\uparrow r}{r\in G}\right\} = \left\{\frac{(\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\uparrow\{y\})\circ\uparrow r}{r\in G}\right\} = \tau(y).$$

Corollary 3. *If* $\lim^{\mathrm{X}}_{\langle\mu\rangle\uparrow\{x\}} f = y$ *then* $\operatorname{xlim}_x f = \tau(y)$ *(provided that* $\mathrm{X}\sqsubseteq\mathrm{X}\circ\mathrm{X}$*).*

We have injective τ if $\langle\mathrm{X}\rangle\uparrow\{y_1\}\sqcap\langle\mathrm{X}\rangle\uparrow\{y_2\} = \bot^{\mathfrak{F}(\operatorname{Ob}\mu)}$ for every distinct $y_1,y_2\in\operatorname{Ob}\mathrm{X}$ that is if X is T_2-separable.

7.3 Hausdorff and Kolmogorov funcoids

Definition 11. *A funcoid f is* Kolmogorov *when $\langle f\rangle\uparrow\{x\}\neq\langle f\rangle\uparrow\{y\}$ for every distinct points $x,y\in\operatorname{dom} f$.*

Definition 12. Limit $\lim\mathscr{F} = x$ *of a filter $\mathscr{F}$ regarding funcoid f is such a point that $\langle f\rangle\uparrow\{x\}\sqsupseteq\mathscr{F}$.*

Definition 13. Hausdorff *funcoid is such a funcoid that every proper filter on its image has at most one limit.*

Proposition 9. *The following are pairwise equivalent for every funcoid f:*

1. *f is Hausdorff.*
2. *$x\neq y\Rightarrow\langle f\rangle\uparrow\{x\}\sqcap\langle f\rangle\uparrow\{y\} = \bot$.*

Proof.

1⇒2. If 2 does not hold, then there exist distinct points x and y such that $\langle f\rangle\uparrow\{x\}\sqcap\langle f\rangle\uparrow\{y\}\neq\bot$. So x and y are both limit points of $\langle f\rangle\uparrow\{x\}\sqcap\langle f\rangle\uparrow\{y\}$, and thus f is not Hausdorff.

2⇒1. Suppose $\mathscr{F}$ is proper.

$$\langle f\rangle\uparrow\{x\}\sqsupseteq\mathscr{F}\wedge\langle f\rangle\uparrow\{y\}\sqsupseteq\mathscr{F}\Rightarrow\langle f\rangle\uparrow\{x\}\sqcap\langle f\rangle\uparrow\{y\}\neq\bot\Rightarrow x=y.$$

Corollary 4. *Every entirely defined Hausdorff funcoid is Kolmogorov.*

Remark. It is enough to be “almost entirely defined” (having nonempty value everywhere except of one point).

Obvious 4. *For a complete funcoid induced by a topological space this coincides with the traditional definition of a Hausdorff topological space.*

8 Generalized limit vs axiomatic generalized limit

I will call *singularities* the set of generalized limits of the form $\operatorname{xlim}_{\langle\mu\rangle\uparrow\{x\}} f$ where f is an entirely defined funcoid and x ranges all points of $\operatorname{Ob}\mu$.

I will call *axiomatic singularities* the set of axiomatic generalized limits of the form $\operatorname{xlim}_{\langle\mu\rangle\uparrow\{x\}} f$ where f is an entirely defined funcoid and x ranges all points of $\operatorname{Ob}\mu$.

Switching back and forth between generalized limits and what I call F-singularities:

$$\Phi f = \left\{\frac{(\operatorname{dom} F, F)}{F\in f}\right\};$$
$$\Psi f = \operatorname{im} f.$$

Proposition 10. *Let the funcoid μ is Kolmogorov and X is entirely defined. Then Φ is an injection from the set of singularities to the set of monovalued functions.*

Proof. That it's an injection is obvious.

We need to prove that $\operatorname{dom} F_0 \neq \operatorname{dom} F_1$ for each $F_0, F_1 \in f$ such that $F_0 \neq F_1$. Really, $F_0 = \mathrm{X} \circ f|_{\langle\mu\rangle\uparrow\{x_0\}} \circ \uparrow r_0$ for $x_0 \in \operatorname{Ob}\mu$, $r_0 \in G$. We have $\operatorname{dom} F_0 = \operatorname{dom} f|_{\langle\mu\rangle\uparrow\{x_0\}} = \langle\mu\rangle\uparrow\{x_0\}$. Similarly $\operatorname{dom} F_1 = \langle\mu\rangle\uparrow\{x_1\}$ for some $x_1 \in \operatorname{Ob}\mu$. Thus $\operatorname{dom} F_0 \neq \operatorname{dom} F_1$ because otherwise $x_0 = x_1$ and so $r_0 \neq r_1$,

$$\operatorname{dom} F_0 = \left\langle \uparrow r_0^{-1} \right\rangle \langle\mu\rangle \uparrow\{x_0\} = \langle\mu\rangle \left\langle \uparrow r_0^{-1} \right\rangle \uparrow\{x_0\} \neq \text{(Kolmogorov property)} \neq \langle\mu\rangle \left\langle \uparrow r_1^{-1} \right\rangle \{x_0\} = \left\langle \uparrow r_1^{-1} \right\rangle \langle\mu\rangle \{x_0\} = \operatorname{dom} F_1,$$

contradiction.

It remains to prove that $(f, \mathscr{X}) \mapsto \Phi(\operatorname{xlim}_{\mathscr{X}} f)$ conforms to the axioms.

The second axiom is obvious.

It remains to prove that

$$\langle \Phi(\operatorname{xlim}_{\mathscr{X}} f) \rangle \mathscr{C} = \lim_{\mathscr{C}} f.$$

Really, $\Phi(\operatorname{xlim}_{\mathscr{X}} f)$ is equal to an $F \in \operatorname{xlim}_{\mathscr{X}} f$ such that $\operatorname{dom} F = \mathscr{X}$. So $F = \mathrm{X} \circ f$.

$$\langle \Phi(\operatorname{xlim}_{\mathscr{X}} f) \rangle \mathscr{C} = \langle \mathrm{X} \circ f \rangle \mathscr{C} = \langle \mathrm{X} \circ f \rangle \mathscr{C} = \lim_{\mathscr{C}} f.$$

Theorem 4. *The following are mutually inverse bijections between generalized limits and axiomatic generalized limits on a filter $\mathscr{X}$ such that $\langle r \rangle \mathscr{X} \asymp \mathscr{X}$ for all $r \in G \setminus \{e\}$:*

1. *Let y be a generalized limit. The corresponding axiomatic generalized limit maps $\mathscr{C}$ to $\langle f \rangle \mathscr{C}$ whenever $f \in y$ and $\mathscr{C} \sqsubseteq \operatorname{dom} f$ and is undefined if there is no such f. (Remark: obviously $y \neq \emptyset$.)*
2. *Let u be an axiomatic generalized limit. The corresponding generalized limit y is a set of funcoids (of suitable source and destination) such that*

$$q \in y \Leftrightarrow \langle q \rangle \mathscr{C} = \begin{cases} u\mathscr{C} & \text{if } \mathscr{C} \in \operatorname{dom} u; \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. First, we need to prove that the mapping from $\mathscr{C}$ to $\langle f \rangle \mathscr{C}$ is monovalued. For $\mathscr{C} = \emptyset$ it's obvious. It's enough to show that $f_0 = f_1$ if $f_0, f_1 \in y$ and $\mathscr{C} \sqsubseteq \operatorname{dom} f_0 \wedge \mathscr{C} \sqsubseteq \operatorname{dom} f_1$. But for $\mathscr{C} \neq \emptyset$ it follows from the fact that $\operatorname{dom} f_0 \asymp \operatorname{dom} f_1$ for $f_0 \neq f_1$ as directly follow from theorem conditions.

Next we need to show that our functions are mutually inverse.

Let y_0 be a generalized limit. Let u be the axiomatic generalized limit corresponding to it by 1. Let y_1 be the generalized limit corresponding to it by 2. Then

$$q \in y_1 \Leftrightarrow \forall \mathscr{C} : \langle q \rangle \mathscr{C} = \begin{cases} \langle f \rangle \mathscr{C} & \text{if } \mathscr{C} \sqsubseteq \operatorname{dom} f; \\ \emptyset & \text{otherwise;} \end{cases} \Leftrightarrow q = f$$

for some $f \in y_0$. That is, if $q \in y_1$ then $q = f$ for some $f \in y_0$ and if $q = f$ for some $f \in y_0$ then $f \in y_1$. So $y_0 = y_1$.

Let now u_0 be an axiomatic generalized limit. Let y be the generalized limit corresponding to it by by 2. Let u_1 be the axiomatic generalized limit corresponding to it by 1. Then u_1 maps $\mathscr{C}$ to $\langle f \rangle \mathscr{C}$ whenever $f \in y$ that is whenever

$$\langle f \rangle \mathscr{C} = \begin{cases} u_0 \mathscr{C} & \text{if } \mathscr{C} \in \operatorname{dom} u_0; \\ \emptyset & \text{otherwise.} \end{cases}$$

and $\mathscr{C} \sqsubseteq \operatorname{dom} f$ and it undefined if there is no such f. In other words, u_1 maps $\mathscr{C}$ to $u_0\mathscr{C}$ if $\mathscr{C} \in \operatorname{dom} u_0$ (the case "otherwise" is not to be considered because $\operatorname{dom} u_1 = \operatorname{dom} u_0$). So $u_1 = u_0$.

So if we define a function on the set of functions whose values are funcoids, we automatically define (as this injection preimage) a function on the set of singularities. Let's do it.

Let φ be a (possibly multivalued) multiargument function.

9 Operations on generalized limits

9.1 Applying functions to functions

As usually in calculus:

Let φ is an arbitrary multiargument function.

Definition 14. $(\varphi f)x = \varphi(\lambda i \in D : f_i x)$ *for an indexed (by $\operatorname{dom}\varphi$) family f of functions of the same domain D to domains of arguments of φ.*

9.2 Applying functions to sets

Definition 15. $\varphi X = \langle\varphi\rangle^* \prod X$ *for a family X of sets, where each X_i is an element of the domain of i-th argument of φ.*

Obvious 5. $\varphi(\lambda i \in D : \{x_i\}) = \{\varphi x\}$.

9.3 Applying functions to filters

This is an advanced section requiring studying reloids and pointfree funcoids from [?].

Definition 16. $\varphi x = \langle\varphi\rangle \prod^{\mathsf{RLD}}_{X \in \prod x} X$ *for a family x of atomic filters.*

Proposition 11. *φ can be continued to a pointfree funcoid.*

Proof. We need to prove (Theorem 1549 in [?]):

$$\langle\varphi\rangle \prod^{\mathsf{RLD}}_{X \in \prod a} X \sqsubseteq \bigsqcap \left\{ \frac{\bigsqcup \left\langle x \mapsto \langle\varphi\rangle \prod^{\mathsf{RLD}}_{X \in \prod x} X \right\rangle^* \operatorname{atoms} \mathscr{X}}{\mathscr{X} \in \operatorname{up} a} \right\}.$$

Indeed,

$$\bigsqcup\left\langle x\mapsto\langle\varphi\rangle\prod^{\mathsf{RLD}}_{X\in\prod x}X\right\rangle^{*}\operatorname{atoms}\mathscr{X}=$$
$$\langle\varphi\rangle\bigsqcup\left\langle x\mapsto\prod^{\mathsf{RLD}}_{X\in\prod x}X\right\rangle^{*}\operatorname{atoms}\mathscr{X}.$$

By Theorem 1875 in [?]:

$$\bigsqcup\left\langle x\mapsto\prod^{\mathsf{RLD}}_{X\in\prod x}X\right\rangle^{*}\operatorname{atoms}\mathscr{X}=\prod^{\mathsf{RLD}}_{X\in\prod\mathscr{X}}X.$$

Therefore,

$$\bigsqcup\left\langle x\mapsto\prod^{\mathsf{RLD}}_{X\in\prod a}X\right\rangle^{*}\operatorname{atoms}\mathscr{X}\sqsupseteq\prod^{\mathsf{RLD}}_{X\in\prod a}\mathscr{X}.$$

Thus, the thesis follows.

9.4 Applying functions to funcoids

Definition 17. *For a family f of funcoids having a common source set and filter on this set $\mathscr{X}$*

$$(\varphi f)\mathscr{X}=\varphi\left(\lambda i\in\operatorname{dom}f:\langle f_i\rangle\mathscr{X}\right).$$

Proposition 12. *It is a component of a funcoid.*

Proof. As composition of two components of pointfree funcoids:
$\varphi(f_0,\dots,f_n)=\varphi\circ(\mathscr{X}\mapsto(\lambda i\in\operatorname{dom}f:\langle f_i\rangle\mathscr{X}))$.
Note that $\mathscr{X}\mapsto(\lambda i\in\operatorname{dom}f:\langle f_i\rangle\mathscr{X})$ is a component of a pointfree funcoid because

$$\begin{aligned}&\mathscr{Y}\not\asymp(\lambda i\in\operatorname{dom}f:\langle f_i\rangle\mathscr{X})\Leftrightarrow\\&\exists i\in\operatorname{dom}f:\mathscr{Y}_i\not\asymp\langle f_i\rangle\mathscr{X}\Leftrightarrow\\&\exists i\in\operatorname{dom}f:\mathscr{X}\not\asymp\left\langle f_i^{-1}\right\rangle\mathscr{Y}_i\Leftrightarrow\\&\mathscr{X}\not\asymp\left(\lambda i\in\operatorname{dom}f:\left\langle f_i^{-1}\right\rangle\mathscr{Y}_i\right)=\\&\mathscr{X}\not\asymp\left(\mathscr{Y}\mapsto\left(\lambda i\in\operatorname{dom}f:\left\langle f_i^{-1}\right\rangle\mathscr{Y}_i\right)\right)\mathscr{Y}.\end{aligned}$$

Proposition 13. *Applying to funcoids is consistent with applying to functions.*

Proof. Consider values on principal atomic filters.

9.5 Applying to axiomatic generalized limits

Definition 18. *Define applying finitary (multivalued) functions φ to and indexed family x of axiomatic generalized limits of the same domain D (and probably different destination sets) as*

$$\varphi x=\langle\mathrm{X}\rangle\circ\varphi\circ x.$$

(Here φ is considered as a function on filters, x is considered as a function on indexed families of functions.)

Proposition 14. *If $\mathrm{X}\circ\mathrm{X}\sqsupseteq\mathrm{X}$ and reflexive and X commutes with φ in some argument k, then*

$$\varphi x=\phi\circ x.$$

Proof. $\varphi\circ x\sqsubseteq\varphi x$ because X is reflexive.
$x_i=\langle\mathrm{X}\circ f_i\rangle$ for some funcoid f_i.
$x_i\mathscr{X}=\langle\mathrm{X}\rangle\langle f_i\rangle\mathscr{X}$.

$$\begin{aligned}&\varphi\circ x=\\&\mathscr{C}\mapsto\varphi(\lambda i\in\operatorname{dom}x:\langle\mathrm{X}\rangle\langle f_i\rangle\mathscr{C})=\\&\mathscr{C}\mapsto\langle\mathrm{X}\rangle\circ\varphi\circ(\lambda i\in\operatorname{dom}x:\langle\mathrm{X}\rangle\langle f_i\rangle\mathscr{C})=\\&\langle\mathrm{X}\rangle\,\varphi(\mathscr{C}\mapsto\lambda i\in\operatorname{dom}x:\langle\mathrm{X}\rangle\langle f_i\rangle\mathscr{C})\sqsupseteq\\&\varphi\left(\mathscr{C}\mapsto\lambda i\in\operatorname{dom}x:\begin{cases}\langle\mathrm{X}\rangle\langle f_i\rangle\mathscr{C}&\text{if }i\neq k\\\langle\mathrm{X}\rangle\langle\mathrm{X}\rangle\langle f_i\rangle\mathscr{C}&\text{if }i=k\end{cases}\right)\sqsupseteq\\&\varphi\left(\mathscr{C}\mapsto\lambda i\in\operatorname{dom}x:\langle\mathrm{X}\rangle\langle f_i\rangle\mathscr{C}\right)=\\&\langle\mathrm{X}\rangle\circ\varphi\circ(\mathscr{C}\mapsto\lambda i\in\operatorname{dom}x:\langle\mathrm{X}\rangle\langle f_i\rangle\mathscr{C})=\\&\mathscr{C}\mapsto\langle\mathrm{X}\rangle\,\varphi\left(\lambda i\in\operatorname{dom}x:\langle\mathrm{X}\rangle\langle f_i\rangle\mathscr{C}\right)=\langle\mathrm{X}\rangle\circ\varphi\circ x=\varphi x.\end{aligned}$$

Proposition 15. *The conditions of the previous propositions hold for A and for L if they hold for d. More exactly:*
Let d is transitive ($d\circ d\sqsubseteq d$) and reflexive. Then:

1. *Let A is transitive ($\mathrm{A}\circ\mathrm{A}\sqsubseteq\mathrm{A}$) and reflexive.*
2. *Let L is transitive ($\mathrm{L}\circ\mathrm{L}\sqsubseteq\mathrm{L}$) and reflexive.*

Proof.

1. Reflexivity is obvious. Prove that $\mathrm{A}\circ\mathrm{A}\sqsubseteq\mathrm{A}$. Really, $\mathrm{A}\circ\mathrm{A}=(c\circ d^{-1})\circ(c\circ d^{-1})\sqsubseteq c\circ d^{-1}\circ d^{-1}\sqsubseteq c\circ d^{-1}=\mathrm{A}$ where c is the "core part".
2. Reflexivity is obvious. Prove that $\mathrm{L}\circ\mathrm{L}\sqsubseteq\mathrm{L}$. Really,

$$\begin{aligned}&\langle L\rangle\langle L\rangle\mathscr{A}=\langle L\rangle\left\{\frac{x}{\langle d\rangle^{*}\{x\}\sqsupseteq\mathscr{A}}\right\}=\\&\langle L\rangle\left\{\frac{x}{\langle d\rangle^{*}\{x\}\not\asymp\mathscr{A}}\right\}=\\&\langle L\rangle\left\{\frac{x}{\{x\}\not\asymp\langle d^{-1}\rangle\mathscr{A}}\right\}\sqsubseteq\langle L\rangle\left\langle d^{-1}\right\rangle\mathscr{A}=\\&\left\{\frac{x}{\langle d\rangle^{*}\{x\}\sqsupseteq\langle d^{-1}\rangle\mathscr{A}}\right\}\sqsubseteq\\&\left\{\frac{x}{\langle d\rangle^{*}\{x\}\sqsupseteq\mathscr{A}}\right\}=\langle L\rangle\mathscr{A}.\end{aligned}$$

Definition 19. *Applying to singularities: $\varphi x=\Psi f(\Phi\circ x)$ (applicable only if limits x_i are taken on filters that are equal up to $\langle r\rangle$ for $r\in G$).*

Theorem 5. *If φ is continuous regarding X in each argument and $\operatorname{dom}f_0=\dots=\operatorname{dom}f_n=\Delta$ and $\mathrm{X}\circ\mathrm{X}\sqsubseteq\mathrm{X}$, then for singularities*

$$\lim\varphi(f_0,\dots,f_n)=\varphi(\lim f_0,\dots,\lim f_n).$$

*Proof.*We will prove instead

$$\varphi(\Phi\lim f_0,\ldots,\Phi\lim f_n) = \Phi\lim\varphi(f_0,\ldots,f_n).$$

Equivalently transforming:

$$\lambda\Delta\in D:\langle\mathrm{X}\rangle\,\varphi((\Phi\lim f_0)\Delta,\ldots,(\Phi\lim f_n)\Delta) = \Phi\lim\varphi(f_0,\ldots,f_n);$$

$\mathrm{X}\circ\varphi(\mathrm{X}\circ f_0\circ r,\ldots,\mathrm{X}\circ f_n\circ r) = \mathrm{X}\circ\varphi(f_0,\ldots,f_n)\circ r;$
$\mathrm{X}\circ\varphi(\mathrm{X}\circ f_0,\ldots,\mathrm{X}\circ f_n) = \mathrm{X}\circ\varphi(f_0,\ldots,f_n);$
Obviously,
$\mathrm{X}\circ\varphi(\mathrm{X}\circ f_0,\ldots,\mathrm{X}\circ f_n)\sqsupseteq\mathrm{X}\circ\varphi(f_0,\ldots,f_n).$
Reversely, applying continuity $n+1$ times, we get:

$$\mathrm{X}\circ\varphi(\mathrm{X}\circ f_0,\ldots,\mathrm{X}\circ f_n)\sqsubseteq\underbrace{\mathrm{X}\circ\mathrm{X}}_{n+1\text{ times}}\circ\varphi(f_0,\ldots,f_n)\sqsubseteq\mathrm{X}\circ\varphi(f_0,\ldots,f_n).$$

So $\mathrm{X}\circ\varphi(\mathrm{X}\circ f_0,\ldots,\mathrm{X}\circ f_n) = \mathrm{X}\circ\varphi(f_0,\ldots,f_n)$.

Proposition 16.*If φ is continuous regarding X in each argument, then*

$$\varphi(\lim f_0|_\Delta,\ldots,\lim f_n|_\Delta) = \lim\varphi(f_0|_\Delta,\ldots,f_n|_\Delta) = \lim_\Delta\varphi(f_0,\ldots,f_n)$$

for funcoids f_0, ..., f_n,

*Proof.*The first equality follows from the above.
It remains to prove

$$\varphi(f_0|_\Delta,\ldots,f_n|_\Delta) = (\varphi(f_0,\ldots,f_n))|_\Delta.$$

Equivalently transforming,
$\langle\varphi(f_0|_\Delta,\ldots,f_n|_\Delta)\rangle\,\mathscr{X} = \langle(\varphi(f_0,\ldots,f_n))|_\Delta\rangle\,\mathscr{X};$

$$\varphi\left(\langle f_0|_\Delta\rangle\,\mathscr{X},\ldots,\langle f_n|_\Delta\rangle\,\mathscr{X}\right) = \langle\varphi(f_0,\ldots,f_n)\rangle\,(\Delta\sqcap\mathscr{X});$$

$$\varphi\left(\langle f_0\rangle\,(\Delta\sqcap\mathscr{X}),\ldots,\langle f_n\rangle\,(\Delta\sqcap\mathscr{X})\right) = \varphi\left(\langle f_0\rangle\,(\Delta\sqcap\mathscr{X}),\ldots,\langle f_n\rangle\,(\Delta\sqcap\mathscr{X})\right)(\Delta\sqcap\mathscr{X}).$$

Theorem 6.*Let Δ be a filter on μ. Let S be the set of all functions $p\in\mathsf{FCD}(\operatorname{Ob}\mu,\operatorname{Ob}\mathrm{X})$ such that $\operatorname{dom}p=\Delta$. Let f, g be finitary multiargument functions on $\operatorname{Ob}\mathrm{X}$. Let J be an index set. Let $k\in J^{\operatorname{dom}P}$, $l\in J^{\operatorname{dom}Q}$. Then*

$$\forall x\in(\operatorname{Ob}\mathrm{X})^J: f(\lambda i\in\operatorname{dom}f: x_{k_i}) = g(\lambda i\in\operatorname{dom}g: x_{l_i})$$

implies

$$\forall x\in(\langle\lim\rangle^* S)^J: f(\lambda i\in\operatorname{dom}f: x_{k_i}) = g(\lambda i\in\operatorname{dom}g: x_{l_i}),$$

provided that f and g are continuous regarding X in each argument.

*Remark.*This theorem implies that if $\operatorname{Ob}\mathrm{X}$ is a group, ring, vector space, etc., then $\langle\lim\rangle^* S$ is also accordingly a group, ring, vector space, etc.

*Proof.*Every $x_{j_i} = \lim_\Delta t$ for some function t.
By proved above,

$$f(\lambda i\in\operatorname{dom}f: x_{k_i}) = \lim_\Delta f(\lambda i\in\operatorname{dom}f: t_{k_i}).$$

It's enough to prove

$$f(\lambda i\in\operatorname{dom}f: t_{k_i}) = g(\lambda i\in\operatorname{dom}f: t_{l_i}).$$

But that's trivial.

*Conjecture 1.*The above theorem stays true if S is instead a set of limits of monovalued funcoids[2].

9.6 Applications

Having generalized limit, we can in an obvious way define derivative of an arbitrary function.

We can also define definite integral of an arbitrary function (I remind that integral is just a limit on a certain filter). The result may differ dependently on whether we use Riemann and Lebesgue integrals.

From above it follows that my generalized derivatives and integrals are linear operators.

10 Equivalence of different generalized limits

Proposition 17.*Axiomatic generalized limits of monovalued funcoids for $\mathrm{X}=\mathrm{A}$ and $\mathrm{X}=\mathrm{L}$ coincide on ultrafilters.*

*Proof.*Follow from the facts that the image of an ultrafilter by an atomic funcoid is an ultrafilter and that L and A coincide on ultrafilters.

*Question 1.*Under which conditions the algebras of all functions on the set of of all possible values of axiomatic generalized limits between two fixed sets A and B induced (as described above) by functions $A\to B$ for generalized limits are pairwise isomorphic (with an obvious bijection) for:

1. $\mathrm{X}=\mathrm{A}$;
2. $\mathrm{X}=\mathrm{L}$.

Moreover, if d is T_2-separable, they are also isomorphic to the case $\mathrm{X}=d\circ d^{-1}$ (remark: for a pretopology d, it's the proximity of two sets being near if they have intersecting closures). The isomorphism is composing every element F of a value of an axiomatic generalized limit with d on the left ($F\mapsto d\circ F$).

Also let question how to generalize the above for functions φ between *different* kinds of singularities is also not yet settled.

[2] A *monovalued funcoid* is a funcoid f such that $f\circ f^{-1}\sqsubseteq 1$, where 1 is the identity of the semigroup of funcoids.

11 Hierarchy of singularities

Above we have defined (having fixed endofuncoids μ and X) for every set of "points" $R = \mathrm{Ob}\,X$ its set of singularities $\mathrm{SNG}(R)$.

We can further consider

$$\mathrm{SNG}(\mathrm{SNG}(R)), \mathrm{SNG}(\mathrm{SNG}(\mathrm{SNG}(R))),$$

etc. If we try to put our generalized derivative into say the differential equation $h \circ f' = g \circ f$ on real numbers, we have a trouble: The left part belongs to the set of functions to $\mathrm{SNG}(Y)$ and the right part to the set of functions to Y, where Y is the set of solutions. How to equate them? If Y would be just $\mathbb{R}$ we would take the left part of the type $\mathrm{SNG}(\mathbb{R})$ and equate them using the injection τ defined above. But stop, it does not work: if the left part is of $\mathrm{SNG}(\mathbb{R})$ then the right part, too. So the left part would be $\mathrm{SNG}(\mathrm{SNG}(\mathbb{R}))$, etc. infinitely.

So we need to consider the entire set (*supersingularities*)

$$\mathrm{SUPER}(R) = R \cup \mathrm{SNG}(R) \cup \mathrm{SNG}(\mathrm{SNG}(R)) \cup \ldots$$

But what is the limit (and derivative) on this set? And how to perform addition, subtraction, multiplication, division, etc. on this set?

Finitary functions on the set $\mathrm{SUPER}(R)$ are easy: just apply τ to arguments belonging to "lower" parts of the hierarchy of singularities a finite number of times, to make them to belong to the same singularity level (the biggest singularity level of all arguments).

Instead of generalized limit, we will use "regular" limit but on the set $\mathrm{SUPER}(R)$ (which below we will make into a funcoid) rather than on the set R.

See? We have a definition of (finite) differential equations (even partial differential equations) for discontinuous functions. It is just a differential equation on the ring $\mathrm{SUPER}(R)$ (if R is a ring).

What nondifferentiable solutions of such equations do look like? No idea (except that below a simple special case is considered)! Do they contain singularities of higher levels of the above hierarchy? What about singularities in our sense at the center of a black hole (that contain "lost" information)? We have something intriguing to research.

12 Funcoid of singularities

I remind that for funcoid X the relation $[X]^*$ can be thought as generalized nearness.

We will extend $[X]^*$ from the set R of points to the set of funcoids from a (fixed) set A to R having the same domain (or empty domain):

$$y_0 \,[X]^*\, y_1 \Leftrightarrow \forall x \in \operatorname{atoms}\operatorname{dom} y_0 : \langle y_0 \rangle x \,[X]^*\, \langle y_1 \rangle x$$

where $\operatorname{atoms}\operatorname{dom} y_0$ is the set of ultrafilters over the filter $\operatorname{dom} y_0$.

The above makes X a pointfree funcoid (as defined in [**?**]) on this set of funcoids:

Proof. Because funcoids are isomorphic to filters on certain boolean lattice, it's enough to prove:

$$\neg(\bot \,[X]^*\, y_1), \quad \neg(y_0 \,[X]^*\, \bot),$$
$$i \sqcup j \,[X]^*\, y_1 \Leftrightarrow i \,[X]^*\, y_1 \vee j \,[X]^*\, y_1,$$
$$y_0 \,[X]^*\, i \sqcup j \Leftrightarrow y_0 \,[X]^*\, i \vee y_0 \,[X]^*\, j.$$

The first two formulas are obvious. Let's prove the third (the fourth is similar):

$$\begin{aligned}
& i \sqcup j \,[X]^*\, y_1 \Leftrightarrow \\
& \quad \forall x \in \operatorname{atoms}\operatorname{dom}(i \sqcup j) : \langle y_0 \rangle x \,[X]^*\, \langle y_1 \rangle x \Leftrightarrow \\
& \quad \forall x \in \operatorname{atoms}(\operatorname{dom} i \sqcup \operatorname{dom} j) : \langle y_0 \rangle x \,[X]^*\, \langle y_1 \rangle x \Leftrightarrow \\
& \quad \forall x \in \operatorname{atoms}\operatorname{dom} i \cup \operatorname{atoms}\operatorname{dom} j : \\
& \quad \langle y_0 \rangle x \,[X]^*\, \langle y_1 \rangle x \Leftrightarrow \\
& \quad \forall x \in \operatorname{atoms}\operatorname{dom} i : \langle y_0 \rangle x \,[X]^*\, \langle y_1 \rangle x \vee \\
& \quad \forall x \in \operatorname{atoms}\operatorname{dom} j : \langle y_0 \rangle x \,[X]^*\, \langle y_1 \rangle x \Leftrightarrow \\
& \quad i \,[X]^*\, y_1 \vee j \,[X]^*\, y_1.
\end{aligned}$$

We will define two singularities being "near" in terms of F-singularities (that are essentially the same as singularities):

Two F-singularities y_0, y_1 are near iff there exist two elements of y_0 and y_1 correspondingly such that $\operatorname{dom} y_0 = \operatorname{dom} y_1$ and every $Y_0 \in y_0$, $Y_1 \in y_1$ are near.

Let's prove it defines a funcoid on the set of F-singularities:

Proof. Not $\emptyset \,[X]^*\, X$ and not $X \,[X]^*\, \emptyset$ are obvious.

It remains to prove for example

$$I \cup J \,[X]^*\, K \Leftrightarrow I \,[X]^*\, K \vee J \,[X]^*\, K,$$

but that's obvious.

13 Funcoid of supersingularities

It remains to define the funcoid of supersingularities.

Let y_0, y_1 be sets of supersingularities.

We will define y_0 and y_1 to be near iff there exist natural n, m such that

$$\tau^n[y_0 \cap \mathrm{SNG}^m(R)] \,[X]^*\, \tau^m[y_1 \cap \mathrm{SNG}^n(R)].$$

Remark. In this formula both the left and the right arguments of $[X]^*$ belong to $\mathrm{SNG}^{n+m}(R)$.

Let's prove that the above formula really defines a funcoid:

Proof. We need to show

$$\neg(\emptyset\,[\mathrm{X}]^*\,y_1),\quad \neg(y_0\,[\mathrm{X}]^*\,\emptyset),$$
$$i\cup j\,[\mathrm{X}]^*\,y_1 \Leftrightarrow i\,[\mathrm{X}]^*\,y_1 \vee j\,[\mathrm{X}]^*\,y_1,$$
$$y_0\,[\mathrm{X}]^*\,i\cup j \Leftrightarrow y_0\,[\mathrm{X}]^*\,i \vee y_0\,[\mathrm{X}]^*\,j.$$

The first two formulas are obvious. Let's prove the third (the fourth is similar):

$$\begin{aligned}
&i\cup j\,[\mathrm{X}]^*\,y_1 \Leftrightarrow\\
&\exists n,m\in\mathbb{N}:\\
&\tau^n[(i\cup j)\cap \mathrm{SNG}^m(R)]\,[\mathrm{X}]^*\,\tau^m[y_1\cap \mathrm{SNG}^n(R)] \Leftrightarrow\\
&\exists n,m\in\mathbb{N}:\\
&\tau^n[(i\cap \mathrm{SNG}^m(R))\cup(j\cap \mathrm{SNG}^m(R)))]\,[\mathrm{X}]^*\\
&\tau^m[y_1\cap \mathrm{SNG}^n(R)] \Leftrightarrow\\
&\exists n,m\in\mathbb{N}:\\
&\tau^n[i\cap \mathrm{SNG}^m(R)]\cup\tau^n[j\cap \mathrm{SNG}^m(R)))]\,[\mathrm{X}]^*\\
&\tau^m[y_1\cap \mathrm{SNG}^n(R)] \Leftrightarrow\\
&\exists n,m\in\mathbb{N}:\\
&(\tau^n[i\cap \mathrm{SNG}^m(R)]\,[\mathrm{X}]^*\,\tau^m[y_1\cap \mathrm{SNG}^n(R)]\vee\\
&\tau^n[j\cap \mathrm{SNG}^m(R)]\,[\mathrm{X}]^*\,\tau^m[y_1\cap \mathrm{SNG}^n(R)]) \Leftrightarrow\\
&\exists n,m\in\mathbb{N}:\\
&\tau^n[i\cap \mathrm{SNG}^m(R)]\,[\mathrm{X}]^*\,\tau^m[y_1\cap \mathrm{SNG}^n(R)]\\
&\exists n,m\in\mathbb{N}:\\
&\tau^n[j\cap \mathrm{SNG}^m(R)]\,[\mathrm{X}]^*\,\tau^m[y_1\cap \mathrm{SNG}^n(R)] \Leftrightarrow\\
&i\,[\mathrm{X}]^*\,y_1 \vee j\,[\mathrm{X}]^*\,y_1.
\end{aligned}$$

14 Derivative

We will denote $\mathrm{xlim}_{x\to a} f(x) = \mathrm{xlim}_{\langle\mu\rangle^*\{a\}\setminus a} f$.

Definition 20. *We define* (generalized) derivative *as*

$$f'(x) = \mathrm{xlim}_{t\to x}\frac{f(t)-f(x)}{t-x}.$$

(It is defined for left and right modules.)

Proposition 18. *For this generalized derivative*

$$(g\circ f)' = (g'\circ f)\cdot f'$$

if f is continuous.

Proof.

$$\begin{aligned}
(g\circ f)'x = \mathrm{xlim}_{t\to x}\left(\frac{gf(t)-gf(x)}{t-x}\right) &=\\
\mathrm{xlim}_{t\to x}\left(\frac{gft-gfx}{ft-fx}\cdot\frac{ft-fx}{t-x}\right) &=\\
\mathrm{xlim}_{t\to x}\frac{gft-gfx}{ft-fx}\cdot \mathrm{xlim}_{t\to x}\frac{ft-fx}{t-x} &=\\
\mathrm{xlim}_{t\to fx}\left(\left(y\mapsto\frac{gy-gfx}{y-fx}\right)\circ f\right)x\cdot&\\
\mathrm{xlim}_{t\to x}\frac{ft-fx}{t-x} &=\\
\left(\left(\mathrm{xlim}_{t\to fx}\frac{gy-gfx}{y-fx}\right)\circ f\right)x\cdot f'x = (g'\circ f)x\cdot f'x.&
\end{aligned}$$

The case $ft-fx=0$ remains as an exercise to the reader.

Proposition 19.

$$f' = x\mapsto \mathrm{xlim}\left(\left(t\mapsto\frac{f(t)-f(x)}{t-x}\right)\circ \mathrm{id}_{\langle\mu\rangle x\setminus\{x\}}\right).$$

Proof.

$$\begin{aligned}
f'(x) = \mathrm{xlim}\left(\left(t\mapsto\frac{f(t)-f(x)}{t-x}\right)\Big|_{\langle\mu\rangle x\setminus\{x\}}\right) &=\\
\mathrm{xlim}\left(\left(t\mapsto\frac{f(t)-f(x)}{t-x}\right)\circ \mathrm{id}_{\langle\mu\rangle x\setminus\{x\}}\right);&
\end{aligned}$$

$$f' = x\mapsto \mathrm{xlim}\left(\left(t\mapsto\frac{f(t)-f(x)}{t-x}\right)\circ \mathrm{id}_{\langle\mu\rangle x\setminus\{x\}}\right).$$

More generally,

Definition 21. *Let f be a function from a real vector space A to a real vector space B. Let moreover function space $A\to B$ be conforming to conditions for generalized limit that is have a funcoid* X *such as*

$$\forall y\in B: \mathrm{X}\sqsupseteq\langle\mathrm{X}\rangle\uparrow\{y\}\times^{\mathsf{FCD}}\langle\mathrm{X}\rangle\uparrow\{y\}.$$

Then Porton derivative

$$f'(x) = \mathrm{xlim}_{r\to 0}\left(h\mapsto\frac{f(x+rh)-f(x)}{r}\right).$$

Remark. Note that Porton derivative is a function of vector h.

Obvious 6. *Porton derivative*

$$\begin{aligned}
f'(x) = h\mapsto \mathrm{xlim}_{r\to 0}\left(\frac{f(x+rh)-f(x)}{r}\right) &=\\
\mathrm{xlim}_{r\to 0}(h\mapsto(r\mapsto f(x+rh))').&
\end{aligned}$$

It obviously generalizes to every (left or right) module over a division ring, where the ring has a funcoid and a group whose elements are commuting with.

Compare to Gateaux derivative [2]:

$$f'(x) = h \mapsto \lim_{r\to 0} \frac{f(x+rh) - f(x)}{r}$$

We see that Gateaux derivative is a special case of Porton derivative. Fréchet derivative, in turn, is special case of Gateaux derivative: "If F is Fréchet differentiable, then it is also Gateaux differentiable, and its Fréchet and Gateaux derivatives agree." [?]

It is well known ([?]) that Gateaux derivative may produce a non-linear dependency on h, so Porton derivative, too, may be non-linear in h.

TODO: It can be generalized for spaces without 0 (but with subtraction)?

Obvious 7. *Porton derivative $f \mapsto f'$ is a linear operator.*

Therefore, as f' is a linear function of h we can write $f' = \frac{\mathrm{d}f}{\mathrm{d}e}$ where $\mathrm{d}f = f'h$ and $\mathrm{d}e = h$.

Theorem 7. $(g \circ f)' = (g' \circ f) \cdot f'$ *if f is continuous.*

Proof.

$$\begin{aligned}(g\circ f)'x = \operatorname{xlim}_{r\to 0}(h \mapsto (r \mapsto gf(x+rh))') = \\ \operatorname{xlim}_{r\to 0}(h \mapsto (g\circ (r\mapsto f(x+rh)))') = \\ (\text{because } f\circ(r\mapsto x+rh) \text{ is continuous}) = \\ \operatorname{xlim}_{r\to 0}(h\mapsto ((g'\circ(r\mapsto f(x+rh)))\cdot \\ (r\mapsto f(x+rh))') = \\ \operatorname{xlim}_{r\to 0}(h\mapsto (g'\circ(r\mapsto f(x+rh))))\cdot \\ \operatorname{xlim}_{r\to 0}(h\mapsto (r\mapsto f(x+rh))') = \\ (\text{because } r\mapsto f(x+rh) \text{ is continuous}) = \\ g'(h\mapsto (r\mapsto f(x)))\cdot(h\mapsto (r\mapsto f(x+rh))') = \\ (g'fx)\cdot f'x.\end{aligned}$$

15 The necessary condition for minimum

Definition 22. *Let f be a funcoid from the object of an endofuncoid μ to a poset. A funcoid f has local minimum at point a when*

$$\langle f\rangle \langle \mu\rangle \{a\} \sqsubseteq \langle \geq\rangle \langle f\rangle \{a\}.$$

Obvious 8. *The above is a generalization of local minimum of a function.*

Replace in the last formula $\{a\}$ by a filter x. Equivalently transforming, $\operatorname{im}(f\circ\mu|_x) \sqsubseteq \operatorname{im}(\geq \circ f|_x)$.

If x is an ultrafilters, it further is equivalent to $f\circ\mu|_x \sqsubseteq (\geq)\circ f|_x$; $f\circ\mu|_x \sqsubseteq (\geq)\circ f$; $f \in \mathrm{C}(\mu|_x, \geq)$.

If we take $f \in \mathrm{C}(\mu|_{\mathscr{X}}, \geq)$ as the definition of f having local minimum on a filter $\mathscr{X}$, then it has a local minimum on a filter $\mathscr{X}$ iff it has local minimum on each atom of $\mathscr{X}$.

We can also fully analogous define strict local minimum replacing $\geq$ by $>$.

Obvious 9. *If a function f from a normed space to funcoid (with all conditions for axiomatic generalized limit definition of the derivative) has local minimum at point a, then*

$$f'\langle\mu\rangle\{a\} \sqsubseteq [0;+\infty[$$

(here f' is considered as an axiomatic generalized limit).

16 Example differential equation

Definition 23. *I will call a function f* pseudocontinuous *on D when*

$$\forall a \in D : \operatorname{xlim}_{\Delta\{a\}\setminus\{a\}} f = f(a).$$

16.1 Arbitrary pseudocontinuous continuations

Note that arbitrary pseudocontinuous continuations of generalized solutions of differential equations (diffeqs) are silly:

Let $A(f(x), f'(x)) = 0$ be a diffeq and let the equality is undefined at some point (e.g. contains division by zero). Let f be its solution with derivative f'. Replace the value in undefined point x of the solution by an arbitrary value y and calculate the derivative y' at this point. No need to hold $A(y, y') = 0$ at this point because the point is outside of the domain of the original solution. Then replace in our solution f the value at this point x by y and the derivative by y'. Then we have another continuation of the solution because the equality $A(f(x), f'(x)) = 0$ holds both for the point x and all other points.

Thus, we can take any solution and add one point of it with an arbitrary value. That's largely a nonsense from the practical point of view. (Why we would arbitrarily change one point of the solution?)

16.2 Solutions with pseudocontinuous derivative

So I will require for generalized solutions instead that the derivative f' is pseudocontinuous.

Next, we will consider a particular example, the diffeq $y'(x) = -1/x^2$. Let us find its continuations of generalized solutions y to the entire real line (including $x = 0$) with a y' being pseudocontinuous.

As it's well known, its solutions in the traditional sense are $y(x) = c_1 + \frac{1}{x}$ for $x < 0$ and $y(x) = c_2 + \frac{1}{x}$ for $x > 0$ where c_1, c_2 are arbitrary constants. The derivative is $y'(x) = -1/x^2$.

Remark. We could consider solutions on the space of supersingularities and it would be the same, except that we would be allowed to take c_1, c_2 arbitrary supersingularities instead of real numbers. This is because the supersingularities form a ring and thus the algorithm of solving the diffeq is the same as for the real numbers, thus producing the solutions of the same form.

Let's find the pseudocontinuous generalized derivative at zero by pseudocontinuity:

$$y'(0) = \lim_{x\to 0} \frac{1}{x^2}.$$

On the other hand, by the definition of derivative

$$y'(0) = \lim_{\varepsilon\to 0} \frac{c_i + \frac{1}{\varepsilon} - f(0)}{\varepsilon}.$$

The equality is possible only when $c_1 = c_2 = c = f(0)$.

So, finally, our solution is $y(x) = c + \frac{1}{x}$ for $x \neq 0$ and $y(0) = c$.

A thing to notice that now the solution is "whole": it exists at zero and does not split to two "branches" with independent constants. Our $y(x)$ is a real function, but the derivative has a singularity in my sense.

We considered generalized solutions with pseudocontinuous derivative. It is apparently the right way to define a class of generalized solutions. Now I will consider also several apparently wrong classes of solutions.

16.3 Pseudocontinuous generalized solutions

Let us try to require the solution of our diffeq to be pseudocontinuous instead of its derivative to be pseudocontinuous.

We already have the solution for nonzero points. For zero:

$$y'(0) = \lim_{x\to 0}(-1/x^2).$$

So the derivative:

$$y'(0) = \lim_{x\to 0} \frac{y(x) - y(0)}{x} = \lim_{x\to 0} \frac{y(x)}{x} - \lim_{x\to 0} \frac{y(0)}{x} = \lim_{x\to 0} \frac{1}{x^2} - \lim_{x\to 0} \frac{y(0)}{x}.$$

The equality is impossible.

17 Applications to General Relativity

This is a conjecture about general relativity in context of quantum gravity. If the conjecture is true and you prove it, you have serious chances to share Nobel prize with me.

Consider generalized (as per the "Generalized limit" course) solutions of Einstein equations with the requirement that they are pseudodifferentiable in timelike curves.

In spacelike curves they may be not differentiable.

I will call this theory *supersingular GR*.

By the analogy with $y'(x) = -1/x^2$ equation in the previous chapter, when it was solved without pseudodifferentiability requirement I conjecture that singularities may take arbitrary values. I further conjecture that these singularities may hold information about formation of a black hole, solving the black hole information paradox in another way than Hawking radiation.

The produces theory does not conform to observations: In my QG the space is the same for all quantum worlds. (I consider many-world interpretation a proved theory because of [3].) Therefore gravity is the same in worlds with different positions of the Sun, what is obviously wrong.

It seems my theory can be improved to become compatible with the many world-interpretation:

1. Replace GR with my "supersingular GR".
2. Quantize the resulting theory (introduce gravitons).
3. "Merge" it with QFT in curved space.
4. Replace the resulting theory with the corresponding many-worlds theory.

References

[1] V. Porton, *Algebraic General Topology*, INFRA-M, 2019. Available at https://znanium.ru/catalog/document?id=347707.

[2] Wikipedia, *Gateaux derivative*, 2022. Available at https://en.wikipedia.org/w/index.php?title=Gateaux_derivative&oldid=1086102468, online; accessed 8 July 2022.

[3] O. Lombardi, J. Ardenghi, S. Fortin, and M. Narvaja, Foundations of quantum mechanics: decoherence and interpretation, *International Journal of Modern Physics D*, vol. 20, no. 5, pp. 861–875, 2011. https://doi.org/10.1142/S0218271811019074.

Victor Porton studied in Perm State University, but didn't receive a degree because of religious discrimination of so called "sectarians" in Russia. Victor developed algebraic theory of general topology (later reducing it to ordered semicategory actions, that were discovered by him in 2019). The idea of *funcoids* was developed while being in the first year of the university. Also Victor is interested in finite and infinite expressions and subjective probabilities. Victor also works on innovative software, particularly blockchain one, such as software for inclusive science financing. Victor is also a religious thinker.