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Different Variations of Hermite-Hadamard Inequality Arising from New Generalized (m,h)-harmonic Godunova-Levin Preinvex Mappings

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Abstract: The connection between generalized convexity and mathematical inequalities is deeply rooted in convex analysis and operator theory. To put the ideas of preinvexity and convexity even closer together, we might state that preinvex functions are extensions of convex functions. In this article, we give a new definition for generalized convexity, which essentially generalizes harmonic convex, Godunova-Levin, preinvex, and m-convex functions, and we named them (m,h)-harmonic Godunova and Levin preinvex functions. In light of this new definition, we developed various new refinements and bounds for Hermite-Hadamard inequality, its product and symmetric forms, along with several interesting remarks and corollaries that lead to their results with other convex mappings, specifically s-convex, tgs-convex, harmonic convex and a variety of others. In order to support the main results, a number of non-trivial examples are provided.

Keywords: Hermite-Hadamard, (h,m)-preinvex, m-convex, Godunova-Levin

1 Introduction

The concept of convexity simplifies optimization through strong theoretical guarantees and efficient algorithms, but non-convex preinvex mappings are a very important concept in dealing with a broader range of real-world problems. The following are some notable uses of convex functions in various domains. Non-convexity is introduced into neural networks by the combination of non-linear activation functions. This means that a neural network optimization problem's loss or objective function is typically non-convex in terms of network parameters; see Ref. [1]. When budget limitations are taken into account in utility theory, convex sets come into play. The collection of viable consumption bundles (given a certain budget) is frequently described as a convex one; see Ref. [2]. Some games' payout functions may be concave or convex. For example, in some coordination games, participants could have convex payoffs; see Ref. [3]. Additionally, for more details about a few other recent advancements in other fields; see Ref. [4,5,6,7,8,9].

A generalized convex function broadens the idea of convexity to encompass a wider variety of functions, making it easier to explain varied events. These diverse classes led to several academics creating the following double inequality for convex functions, which is critical for optimization. The Hermite and Hadamard inequality [10] is presented as follows:

Let $\Psi : \mathtt{K} \subset \mathrm{R} \to \mathrm{R}$ be a convex function, where $\upsilon_1, \upsilon_2 \in \mathtt{K}$ with $\upsilon_1 < \upsilon_2$. Then

$$\Psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right) \leq \frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2}\Psi(\nu)\,\mathrm{d}\nu \leq \frac{\Psi(\upsilon_1)+\Psi(\upsilon_2)}{2}. \tag{1}$$

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This inequality is generalized and enhanced in several ways, such as authors employing multiple types of convex classes, fractional operators, fuzzy integral operators, different forms of order relations, as well as a stochastic approach. Since preinvex functions are generalization of convex type mappings recently various academics discuss preinvex mappings and developed several inequalities. In [11], authors used quantum integrals to create new generalized Hadamard inequality on two dimensional plane. Authors in [12] developed several new inequalities by using preinvex Godunova-Levin functions with some intriguing characteristics. Lai et al. [13] employed the fractional integral operator to produce several novel forms of Hermite-Hadamard inequality, modifying inclusion order setting. Imran et al. [14] use first-time differentiable mappings to create a novel class of preinvex mappings, a generalization of classical preinvex and harmonically preinvex functions. Khan et al. [15] used the fractional integral operator to generate many new enhanced and generalized forms of double inequalities. Santos et al. [16] utilized the notion of two dimensional preinvex fuzzy-valued mappings and the development of double inequality and its several new variations on coordinates. Lai et al. [17] used generalized preinvexity to create a number of novel Hermite-Hadamard type inclusions and its various new forms on coordinates. For some additional recent developments concerning this form of inequality, we direct readers; see Refs. [18,19,20,21]. Afzal et al. [22] found a similar result by loosening the bifunction restriction and establishing Hermite-Hadamard and Jensen type results with center-radius-interval order relations. In 2019, Ohud Almutairi and Adem Kiliman demonstrated the following result based on the h-Godunova and Levin function; see Ref. [23].

Theorem 1(see [23]). *Let $\Psi : [\upsilon_1, \upsilon_2] \to \mathsf{R}$. If Ψ is* h*-Godunova-Levin function and* $\mathsf{h}\left(\frac{1}{2}\right) \neq 0$, *then*

$$\frac{\mathsf{h}\left(\frac{1}{2}\right)}{2}\Psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right) \leq \frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2}\Psi(\nu)\mathsf{d}\nu$$
$$\leq [\Psi(\upsilon_1)+\Psi(\upsilon_2)]\int_0^1 \frac{\mathsf{d}\delta}{\mathsf{h}(\delta)}.$$

Afzal et al. [24] conducted a recent study using fractional caputo fabrizio integral operators and generated several novel varieties of double inequalities, as well as applications to special means.

Theorem 2(see [24]). *Let $\Psi : [\upsilon_1, \upsilon_2] \to \mathsf{R}$ be* h*-Godunova-Levin mapping defined on $[\upsilon_1, \upsilon_2]$ and $\sigma \in (0,1)$, then we have*

$$\frac{\mathsf{h}\left(\frac{1}{2}\right)}{2}\Psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right)$$
$$\leq \frac{\mathbf{B}(\sigma)}{\sigma(\upsilon_2-\upsilon_1)}\left[\left({}^{\mathsf{CF}}_{\upsilon_1}\mathsf{I}^{\sigma}\Psi\right)(\mathsf{s})+\left({}^{\mathsf{CF}}\mathsf{I}^{\sigma}_{\upsilon_2}\Psi\right)(\mathsf{s})-\frac{2(1-\sigma)}{\mathbf{B}(\sigma)}\Psi(\mathsf{s})\right]$$
$$\leq (\Psi(\upsilon_1)+\Psi(\upsilon_2))\int_0^1\frac{\mathsf{d}\delta}{\mathsf{h}(\delta)}. \quad (2)$$

The term *m*-convexity which lies in between regular convexity and star-shaped functions, is defined by the author in [25]. We would like to emphasise the distinction between m-convex functions and harmonic h-Godunova-Levin preinvex functions as two distinct classes of convex functions. To tie these concepts together, a new class of generalized convex functions should be introduced. Specifically, we take motivation from the results of Godunova-Levin preinvex [23], *m*-convexity [25], and harmonic Godunova-Levin [26], and constructed a new class by taking into account these three notions, and developed various new bounds and refinements of Hermite-Hadamard inequalities. In [27], authors used operator m-convex type mappings and discussed several new bounds for different types of inequalities; in [28], authors used m-convex functions and developed trapezoid type inequalities; in [29], authors used Godunova-Levin preinvex functions and developed different types of inequalities with applications; and in [30], authors used Godunova-Levin functions and developed several interesting results related to these outcomes. For several other recent results relating to developed outcomes, please see [31,32,33,34,35,36].

The work is organized into four sections, starting with a brief introduction and preliminary discussion of the topic. In section 3, we derive numerous novel forms of Hermite-Hadamard inequalities using our newly established convex mappings. Ultimately, we provide a clear conclusion and some potential directions in section 4.

2 Preliminaries

This section initially recalls some necessary known definitions from which we take motivation and give some new class of generalized convex mappings. Next, we define a bifunction that is used in various definitions and plays a very significant role in demonstrating the distinguishing feature between convex and preinvex mappings. Let $\mathtt{K} \subset \mathsf{R}$ and $\phi(,\cdot) : \mathtt{K}\times\mathtt{K} \longrightarrow \mathsf{R}$ be a continous bifunction.

Definition 1(see [25]). *A set $\mathtt{K}$ is considered to be m-convex in relation to the some fixed constant $m \in [0,1]$, if*

$$(1-\eta)\upsilon_1+\eta\upsilon_2 \in \mathtt{K},$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$ and $\eta \in [0,1]$.

Definition 2(see [38]). *The set $\mathtt{K}$ is considered invex in reference to the bifunction $\phi(,\cdot)$, iff*

$$\upsilon_1+\eta\phi(\upsilon_2,\upsilon_1) \in \mathtt{K},$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$ and $\eta \in [0,1]$.

Definition 3(see [38]). *The set $\mathtt{K}$ is said to be invex with respect to the bifunction $\phi(,\cdot)$, if*

$$\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)} \in \mathtt{K},$$

for all $\upsilon_1, \upsilon_2 \in K$ *and* $m \in (0,1)$.

Definition 4(see [38]). *A mappping* $\Psi : K \to R$ *is said to be m-convex, where* $m \in [0,1]$*, if*

$$\Psi((1-\eta)\upsilon_1 + \eta\upsilon_2) \leq m\Psi(\upsilon_2) + (1-\eta)\,\Psi(\upsilon_1),$$

for all $\upsilon_1, \upsilon_2 \in K$ *and* $\eta \in [0,1]$.

Definition 5(see [38]). *A mappping* $\Psi : K \to R$ *is said to be harmonic m-convex, where* $m \in (0,1]$*, if*

$$\Psi\left(\frac{\upsilon_1(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1))}{\phi(m\upsilon_2, \upsilon_1)}\right) \leq m\Psi(\upsilon_2) + (1-\eta)\,\Psi(\upsilon_1),$$

for all $\upsilon_1, \upsilon_2 \in K$ *and* $\eta \in (0,1]$.

Definition 6(see [38]). *A mappping* $\Psi : K \to R$ *defined on invex set* K *is said to be preinvex with relation to the bifunction* $\phi(,\cdot)$ *if*

$$\Psi(\upsilon_1 + \eta\phi(\upsilon_2, \upsilon_1)) \leq \eta\Psi(\upsilon_2) + (1-\eta)\,\Psi(\upsilon_1),$$

for all $\upsilon_1, \upsilon_2 \in K$ *and* $\eta \in [0,1]$.

Definition 7(see [38]). *A function* $\Psi : K \to R$ *is called to be supermultiplicative if*

$$\Psi(\upsilon_1)\Psi(\upsilon_2) \leq \Psi(\upsilon_1\upsilon_2),$$

for all $\upsilon_1, \upsilon_2 \in K$.

Definition 8(see [37]). *A mappping* $\Psi : K \to R$ *defined on invex set* K *is said to be* GL *preinvex with relation to the bifunction* $\phi(,\cdot)$ *if*

$$\Psi(\upsilon_1 + \eta\phi(\upsilon_2, \upsilon_1)) \leq \frac{\Psi(\upsilon_2)}{\eta} + \frac{\Psi(\upsilon_1)}{(1-\eta)},$$

for all $\upsilon_1, \upsilon_2 \in K$ *and* $\eta \in (0,1)$.

Definition 9(see [38]). *Let* $h : M = [0,1] \to R$. *A mappping* $\Psi : K \to R$ *defined on invex set* K *is said to be* h*-preinvex with respect to the bifunction* $\phi(,\cdot)$ *if*

$$\Psi(\upsilon_1 + \eta\phi(\upsilon_2, \upsilon_1)) \leq h(\eta)\Psi(\upsilon_2) + h(1-\eta)\,\Psi(\upsilon_1),$$

for all $\upsilon_1, \upsilon_2 \in K$ *and* $\eta \in [0,1]$.

Definition 10(see [37]). *Let* $h : M = (0,1) \to R$. *A mappping* $\Psi : K \to R$ *defined on invex set* K *is said to be* h-GL *preinvex with respect to the bifunction* $\phi(,\cdot)$ *if*

$$\Psi(\upsilon_1 + \eta\phi(\upsilon_2, \upsilon_1)) \leq \frac{\Psi(\upsilon_2)}{h(\eta)} + \frac{\Psi(\upsilon_1)}{h(1-\eta)},$$

for all $\upsilon_1, \upsilon_2 \in K$ *and* $\eta \in (0,1)$.

Definition 11(see [38]). *Let* $K \subseteq R^n$ *is said to be invex with respect to the bifunction* $\phi(,\cdot)$. *For all* $\upsilon_1, \upsilon_2 \in K$ *and* $\eta \in [0,1]$*, we have*

$$\phi(\upsilon_2, \upsilon_2 + \eta\,\phi(\upsilon_1, \upsilon_2)) = -\eta\,\phi(\upsilon_1, \upsilon_2), \quad (3)$$

and

$$\phi(\upsilon_1, \upsilon_2 + \eta\,\phi(\upsilon_1, \upsilon_2)) = (1-\eta)\,\phi(\upsilon_1, \upsilon_2). \quad (4)$$

for all $\upsilon_1, \upsilon_2 \in K$ *and* $\phi_1, \phi_2 \in [0,1]$*, and this is commonly referred to as condition C, that is*

$$\phi(\upsilon_2 + \phi_2\,\phi(\upsilon_1, \upsilon_2), \upsilon_2 + \phi_1\phi(\upsilon_1, \upsilon_2)) = (\phi_2 - \phi_1)\,\phi(\upsilon_1, \upsilon_2).$$

Since our findings are also presented in terms of special functions, we recall the very well known gamma and beta functions, respectively:

$$\Gamma(\rho) = \int_0^\infty \eta^{-t} t^{\rho-1} dt$$
$$B(\rho_1, \rho_2) = \int_0^1 t^{\rho_1-1}(1-t)^{\rho_2-1} dt = \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1+\rho_2)}, \quad \rho_1, \rho_2 > 0. \quad (5)$$

3 The main results

In this section, we first provide a novel definition that uses numerous classes of generalized convex mappings simultaneously and generalizes various earlier definitions under different scenarios.

Definition 12(see [38]). *Let* $h : M = [0,1] \to R$. *A mappping* $\Psi : K \to R$ *defined on invex set* K *is said to be* (m,h)*-harmonic Godunova and Levin preinvex functions with respect to the bifunction* $\phi(,\cdot)$*, where* $m \in (0,1]$. *Then*

$$\Psi\left(\frac{\upsilon_1(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1))}{\phi(m\upsilon_2, \upsilon_1)}\right) \leq \frac{\Psi(\upsilon_2)}{h(\eta)} + \frac{\Psi(\upsilon_1)}{h(1-\eta)},$$

for all $\upsilon_1, \upsilon_2 \in K$ *and* $\eta \in (0,1)$.

The class of all harmonic (m,h)-GL preinvex is denoted by **HPGL**$((m,h),[\upsilon_1,\upsilon_2],R^+)$.

Remark. –Setting $m = 1$ and $\phi(m\upsilon_2, \upsilon_1)) = m\upsilon_2 - \upsilon_1$, then Definition 12 recovers Definition 2.7 in [39].
–Setting $m = 1$, $\phi(m\upsilon_2, \upsilon_1)) = m\upsilon_2 - \upsilon_1$ and $h(\eta) = \frac{1}{\eta}$, then Definition 12 recovers Definition 2.1 in [40].
–Setting $m = 1$ and $h(\eta) = \frac{1}{\eta}$, then Definition 12 recovers Definition 2.4 in [41].

Now we have demonstrated that two similarly ordered (m,h)-harmonic Godunova and Levin preinvex functions again belong to that class, which is **HPGL**$((m,h),[\upsilon_1,\upsilon_2],R^+)$.

Lemma 1. *Let Ψ and ζ be two similarly ordered harmonic m-preinvex Godunova-Levin functions. If $\left[\frac{1}{\mathrm{h}(1-\eta)}+\frac{m}{\mathrm{h}(\eta)}\right] \leq 1$, then the product of these two functions $\Psi\zeta$ is again a (m,h)-harmonic Godunova and Levin preinvex function.*

Proof. Let Ψ and ζ be two (m,h)-harmonic Godunova and Levin preinvex functions. Then

$$\begin{aligned}
&\Psi\left(\frac{\mathrm{k}(\mathrm{k}+\phi(mn,\mathrm{k}))}{\mathrm{k}+(1-\eta)\phi(mn,\mathrm{k})}\right)\zeta\left(\frac{\mathrm{k}(\mathrm{k}+\phi(mn,\mathrm{k}))}{\mathrm{k}+(1-\eta)\phi(mn,\mathrm{k})}\right)\\
&\leq \frac{\Psi(\mathrm{k})}{\mathrm{h}(1-\eta)}+\frac{m\Psi(\mathrm{n})}{\mathrm{h}(\eta)}\left[\frac{\zeta(\mathrm{k})}{\mathrm{h}(1-\eta)}+\frac{m\zeta(\mathrm{n})}{\mathrm{h}(\eta)}\right]\\
&=\left(\frac{\Psi(\mathrm{k})}{\mathrm{h}(1-\eta)}\right)\left(\frac{\zeta(\mathrm{k})}{\mathrm{h}(1-\eta)}\right)+\frac{m\Psi(\mathrm{k})\zeta(\mathrm{n})}{\mathrm{h}(1-\eta)\mathrm{h}(\eta)}\\
&+\frac{m\Psi(\mathrm{n})\zeta(\mathrm{k})}{\mathrm{h}(1-\eta)\mathrm{h}(\eta)}+\frac{m^2\Psi(\mathrm{n})\zeta(\mathrm{n})}{\mathrm{h}^2(t)}\\
&=\frac{\Psi(\mathrm{k})\zeta(\mathrm{k})}{[\mathrm{h}(1-\eta)]^2}+\frac{m}{\mathrm{h}(\eta)\mathrm{h}(1-\eta)}(\Psi(\mathrm{k})\zeta(\mathrm{n})+\Psi(\mathrm{n})\zeta(\mathrm{k}))\\
&+\frac{m^2}{[\mathrm{h}(\eta)]^2}\Psi(\mathrm{n})\zeta(\mathrm{n})\\
&\leq \frac{\Psi(\mathrm{k})\zeta(\mathrm{k})}{[\mathrm{h}(1-\eta)]^2}+\frac{m}{\mathrm{h}(\eta)\mathrm{h}(1-\eta)}(\Psi(\mathrm{k})\zeta(\mathrm{k})+\Psi(\mathrm{n})\zeta(\mathrm{n}))\\
&+\frac{m^2}{[\mathrm{h}(\eta)]^2}\Psi(\mathrm{n})\zeta(\mathrm{n})\\
&=\left[\frac{\Psi(\mathrm{k})\zeta(\mathrm{k})}{\mathrm{h}(1-\eta)}+\frac{m\Psi(\mathrm{n})\zeta(\mathrm{n})}{\mathrm{h}(\eta)}\right]\left[\frac{1}{\mathrm{h}(1-\eta)}+\frac{m}{\mathrm{h}(\eta)}\right].
\end{aligned}$$

Here

$$\left[\frac{1}{\mathrm{h}(1-\eta)}+\frac{m}{\mathrm{h}(\eta)}\right]\leq 1$$

This implies:

$$\begin{aligned}
&\Psi\left(\frac{\mathrm{k}(\mathrm{k}+\phi(mn,\mathrm{k}))}{\mathrm{k}+(1-\eta)\phi(mn,\mathrm{k})}\right)\zeta\left(\frac{\mathrm{k}(\mathrm{k}+\phi(mn,\mathrm{k}))}{\mathrm{k}+(1-\eta)\phi(mn,\mathrm{k})}\right)\\
&\leq \frac{\Psi(\mathrm{k})\zeta(\mathrm{k})}{\mathrm{h}(1-\eta)}+\frac{m\Psi(\mathrm{n})\zeta(\mathrm{n})}{\mathrm{h}(\eta)}.
\end{aligned}$$

Hence proved.

3.1 Some new variants of Hermte-Hadamard inequalities

In this section, we obtain Hermite-Hadamard inequalities for (m,h)-harmonic Godunova and Levin preinvex functions. Throughout this section, we take $\mathrm{I}_\phi=[\upsilon_1,\upsilon_1+\phi(m\upsilon_2,\upsilon_1)]$ unless otherwise specified, where $\upsilon_1<\upsilon_1+\phi(m\upsilon_2,\upsilon_1)$.

Theorem 3. *Let $\Psi:\mathrm{I}_\phi\subseteq\mathrm{R}^+\to\mathrm{R}$ be (m,h)-harmonic Godunova and Levin preinvex function, where $m\in[0,1]$. If $\Psi\in\mathrm{L}\,[\upsilon_1,\upsilon_1+\phi(m\upsilon_2,\upsilon_1)]$ (space of Lebesgue integrable functions), then*

$$\begin{aligned}
&\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}\frac{\Psi(\mathrm{k})}{\mathrm{k}^2}d\mathrm{k}\\
&\leq[\Psi(\upsilon_1)+m\Psi(\upsilon_2)]\int_0^1\frac{1}{\mathrm{h}(\eta)}d\eta.
\end{aligned}$$

Proof. Let Ψ be (m,h)-harmonic Godunova and Levin preinvex function, where $m\in[0,1]$. Then we have

$$\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)\leq\frac{\Psi(\upsilon_1)}{\mathrm{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathrm{h}(\eta)}.$$

Integrating over $\eta\in[0,1]$, we obtain:

$$\begin{aligned}
&\int_0^1\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta\\
&\leq\Psi(\upsilon_1)\int_0^1\frac{1}{\mathrm{h}(1-\eta)}d\eta+m\Psi(\upsilon_2)\int_0^1\frac{1}{\mathrm{h}(\eta)}d\eta. \quad (6)
\end{aligned}$$

Consider and make use of some change in variables

$$\int_0^1\frac{1}{\mathrm{h}(1-\eta)}d\eta,$$

for instance let

$$\begin{aligned}
1-\eta&=\mathrm{k}\\
d(1-\eta)&=d(\mathrm{k})\\
(0-1)d\eta&=d\mathrm{k}\\
d\eta&=-d\mathrm{k}.
\end{aligned}$$

If $\eta=1$ then $\mathrm{k}=0$ and if $\eta=0$, $\mathrm{k}=1$ this further implies that:

$$\int_0^1\frac{1}{\mathrm{h}(1-\eta)}d\eta=\int_1^0\frac{1}{\mathrm{h}(\mathrm{k})}(-1)d\mathrm{k}=\int_0^1\frac{1}{\mathrm{h}(\mathrm{k})}d\mathrm{k},$$

$$\int_0^1\frac{1}{\mathrm{h}(\mathrm{k})}d\mathrm{k}=\int_0^1\frac{1}{\mathrm{h}(\eta)}d\eta. \quad (7)$$

Taking into account equation (6), we have

$$\begin{aligned}
&\int_0^1\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta\\
&\leq\Psi(\upsilon_1)\int_0^1\frac{1}{\mathrm{h}(\eta)}d\eta+m\Psi(\upsilon_2)\int_0^1\frac{1}{\mathrm{h}(\eta)}d\eta.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_0^1\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta\\
&\leq[\Psi(\upsilon_1)+m\Psi(\upsilon_2)]\int_0^1\frac{1}{\mathrm{h}(\eta)}d\eta. \quad (8)
\end{aligned}$$

Now let's integrate the left hand side of the inequality (8) for instance we take

$$\int_0^1 \Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta.$$

Let and do simple calculations

$$\mathrm{k}=\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)},$$

$$d\mathrm{k}=d\left[\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right],$$

$$d\mathrm{k}=\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))\,d\left[\frac{1}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right],$$

$$d\mathrm{k}=\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)) \times d\left[\frac{1}{a+\phi(m\upsilon_2,\upsilon_1)-\eta\phi(m\upsilon_2,\upsilon_1)}\right],$$

$$d\mathrm{k}=\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))\,(-1)(a+\phi(m\upsilon_2,\upsilon_1) -\eta\phi(m\upsilon_2,\upsilon_1))^{-2}\,d(a+\phi(m\upsilon_2,\upsilon_1)-\eta\phi(m\upsilon_2,\upsilon_1)),$$

$$d\mathrm{k}=\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))\,(-1)(a+\phi(m\upsilon_2,\upsilon_1) -\eta\phi(m\upsilon_2,\upsilon_1))^{-2}\,(0-1\phi(m\upsilon_2,\upsilon_1)),$$

$$d\mathrm{k}=\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))\times\phi(m\upsilon_2,\upsilon_1)}{(\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1))^2}d\eta.$$

Multiply and divide by $\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))$ in above equation, we have

$$d\mathrm{k}=\left[\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))\times\phi(m\upsilon_2,\upsilon_1)}{(\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1))^2} \times\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}\right]d\eta,$$

$$d\mathrm{k}=\left[\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{(\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1))}\right]^2\times\frac{\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}d\eta.$$

As

$$\mathrm{k}=\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{(\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1))},$$

it follows that

$$d\mathrm{k}=\mathrm{k}^2\times\frac{\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}d\eta,$$

$$\frac{1}{\mathrm{k}^2}\times\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}d\mathrm{k}=d\eta,$$

if $\eta=1$ then

$$\mathrm{k}=\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{(\upsilon_1+(1-1)\phi(m\upsilon_2,\upsilon_1))},$$

$$\mathrm{k}=\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{(\upsilon_1+(0)\phi(m\upsilon_2,\upsilon_1))},$$

$$\mathrm{k}=\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1},$$

$$\mathrm{k}=\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)). \tag{9}$$

Now if $\eta=0$, then we have

$$\mathrm{k}=\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{(\upsilon_1+(1-0)\phi(m\upsilon_2,\upsilon_1))},$$

$$\mathrm{k}=\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))},$$

$$\mathrm{k}=\upsilon_1. \tag{10}$$

Substituting the integral value into equation (8), we get:

$$\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}\frac{\Psi(\mathrm{k})}{\mathrm{k}^2}d\mathrm{k} \leq[\Psi(\upsilon_1)+m\Psi(\upsilon_2)]\int_0^1\frac{1}{\mathrm{h}(\eta)}d\eta.$$

This completes the proof.

Example 1. Consider $[\upsilon_1,\upsilon_1+\phi(m\upsilon_2,\upsilon_1)]=[1,2]$, $\phi(m\upsilon_2,\upsilon_1)=m\upsilon_2-2\upsilon_1, m=1, \upsilon_2=3, a=1, \mathrm{h}(\eta)=\frac{1}{\eta}$. Let $\vartheta:[\upsilon_1,\upsilon_1+\phi(m\upsilon_2,\upsilon_1)]\to\mathrm{R}^+$ is defined as

$$\Psi(\sigma)=2-\frac{1}{\sigma},\quad \sigma\in[1,2].$$

First of all we show that above defined function is belong to the class of $\mathbf{HPGL}((\mathrm{h},m),[\upsilon_1,\upsilon_2],\mathrm{R}^+)$.

Consider

$$\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)\leq\frac{\Psi(\upsilon_1)}{\mathrm{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathrm{h}(\eta)},$$

it follows that

$$2-\frac{1}{\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}}\leq\frac{2-\frac{1}{\upsilon_1}}{\frac{1}{1-\eta}}+m\frac{2-\frac{1}{\upsilon_2}}{\frac{1}{\eta}}$$

$$1.25<1.33.$$

As the inequality holds, the provided function belongs to $\mathbf{HPGL}((\mathrm{h},m),[\upsilon_1,\upsilon_2],\mathrm{R}^+)$.
Now consider

$$\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}\frac{\Psi(\mathrm{k})}{\mathrm{k}^2}d\mathrm{k} =\frac{2}{1}\int_1^2\left(\frac{-1}{\mathrm{k}}+2\right)\frac{1}{\mathrm{k}^2}d\mathrm{k}=\frac{5}{4}$$

and

$$[\Psi(\upsilon_1)+m\Psi(\upsilon_2)]\int_0^1 \frac{1}{\mathsf{h}(\eta)}d\eta = \left(1+\frac{5}{3}\right)\left[\frac{\eta^2}{2}\right]_0^1 = \frac{4}{3}.$$

Consequently, Theorem 3 is verified that is

$$\frac{5}{4} \leq \frac{4}{3}.$$

Corollary 1.
If $\phi(m\upsilon_2,\upsilon_1) = m\upsilon_2-\upsilon_1$, then Theorem 3 reduces to harmonic (m,h)-Godunova-Levin functions:

$$\frac{m\upsilon_1\upsilon_2}{m\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{m\upsilon_2}\frac{\Psi(\mathtt{k})}{\mathtt{k}^2}\,d\mathtt{k} \leq [\Psi(\upsilon_1)+m\Psi(\upsilon_2)]\int_0^1\frac{d\eta}{\mathsf{h}(\eta)}.$$

Corollary 2.
If $\phi(m\upsilon_2,\upsilon_1) = m\upsilon_2-\upsilon_1$ and $\mathsf{h}(\eta) = \eta^{\mathtt{p}}(1-\eta)^{\mathtt{q}}$ then Theorem 3 reduces to harmonic p-Godunova-Levin functions:

$$\frac{m\upsilon_1\upsilon_2}{m\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{m\upsilon_2}\frac{\Psi(\mathtt{k})}{\mathtt{k}^2}\,d\mathtt{k} \leq [\Psi(\upsilon_1)+m\Psi(\upsilon_2)]\frac{\Gamma(\mathtt{p}+1)\Gamma(\mathtt{q}+1)}{\Gamma(\mathtt{p}+\mathtt{q}+2}).$$

Remark. 1.If $\phi(m\upsilon_2,\upsilon_1)=m\upsilon_2-\upsilon_1, m=1$ and $\mathsf{h}(\eta)=1$ then Theorem 3.1 recovers Theorem 2.4 in [40].
2.If $\phi(m\upsilon_2,\upsilon_1)=m\upsilon_2-\upsilon_1, m=1$ and $\mathsf{h}(\eta)=\frac{1}{\eta}$ then Theorem 3.1 recovers Theorem 3.2 in [42].
3.If $m=1$ and $\mathsf{h}(\eta)=\frac{1}{\eta}$ then Theorem 3.1 recovers Theorem 2.1 in [41].

Theorem 4. *Let $\Psi,\zeta : \mathtt{I}_\phi \subseteq \mathsf{R}^+ \to \mathsf{R}$ be (m,h)-harmonic Godunova and Levin preinvex functions, where $m\in[0,1]$. If $\Psi \in \mathsf{L}\,[\upsilon_1,\upsilon_1+\phi(m\upsilon_2,\upsilon_1)]$, then*

$$\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}\frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathtt{k}^2}d\mathtt{k} \leq M(\upsilon_1,\upsilon_2)$$

where

$$M(\upsilon_1,\upsilon_2)=[\Psi(\upsilon_1)\zeta(\upsilon_1)+m\Psi(\upsilon_2)\zeta(\upsilon_2)] \times\int_0^1\frac{1}{[\mathsf{h}(\eta)]^2}d\eta+[m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)] \times\int_0^1\frac{1}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}d\eta. \tag{11}$$

Proof. Let Ψ, ζ be (m,h)-harmonic Godunova and Levin preinvex functions, we have:

$$\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right) \leq \frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)},$$

$$\zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right) \leq \frac{\zeta(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\zeta(\upsilon_2)}{\mathsf{h}(\eta)}.$$

Now consider

$$\begin{aligned}&\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)\times\zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)\\ &\leq\left[\frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)}\right]\times\left[\frac{\zeta(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\zeta(\upsilon_2)}{\mathsf{h}(\eta)}\right]\\ &\leq\frac{1}{[\mathsf{h}(1-\eta)]^2}\Psi(\upsilon_1)\zeta(\upsilon_1)+[m\zeta(\upsilon_1)\Psi(\upsilon_2)+m\Psi(\upsilon_1)\zeta(\upsilon_2)]\\ &\times\frac{1}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}+m^2\Psi(\upsilon_2)\zeta(\upsilon_2)\frac{1}{[\mathsf{h}(\eta)]^2}.\end{aligned} \tag{12}$$

Integrating over [0,1] we have

$$\begin{aligned}&\int_0^1\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)\ \zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta\\ &\leq\Psi(\upsilon_1)\zeta(\upsilon_1)\int_0^1\frac{1}{[\mathsf{h}(1-\eta)]^2}d\eta+\\ &[m\zeta(\upsilon_1)\Psi(\upsilon_2)+m\Psi(\upsilon_1)\zeta(\upsilon_2)]\int_0^1\frac{1}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}d\eta\\ &+m^2\Psi(\upsilon_2)\zeta(\upsilon_2)\int_0^1\frac{1}{[\mathsf{h}(\eta)]^2}d\eta.\end{aligned} \tag{13}$$

Using equations (7), (9), and (10), we obtain the needed result

$$\begin{aligned}&\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}\frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathtt{k}^2}d\mathtt{k}\\ &\leq[\Psi(\upsilon_1)\zeta(\upsilon_1)+m\Psi(\upsilon_2)\zeta(\upsilon_2)]\int_0^1\frac{1}{[\mathsf{h}(\eta)]^2}d\eta+[m\Psi(\upsilon_1)\zeta(\upsilon_2)\\ &+m\Psi(\upsilon_2)\zeta(\upsilon_1)]\int_0^1\frac{1}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}d\eta.\end{aligned} \tag{14}$$

Example 2. Consider $[\upsilon_1,\upsilon_1+\phi(m\upsilon_2,\upsilon_1)] = [1,2]$, $\phi(m\upsilon_2,\upsilon_1)=m\upsilon_2-2\upsilon_1, m=1, \upsilon_2=3, a=1, \mathsf{h}(\eta)=\frac{1}{\eta}$. Let $\Psi,\zeta : [\upsilon_1,\upsilon_1+\phi(m\upsilon_2,\upsilon_1)] \to \mathsf{R}^+$ are defined as

$$\Psi(\sigma)=\frac{-1}{\sigma}+1 \text{ and } \quad \Psi(\sigma)=\frac{1}{\sigma}+1, \quad \sigma\in[1,2].$$

Consider

$$\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}\frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathtt{k}^2}d\mathtt{k} = \frac{2}{1}\int_1^2\left(\frac{1}{\mathtt{k}}+1\right)\left(\frac{-1}{\mathtt{k}}+1\right)\frac{1}{\mathtt{k}^2}d\mathtt{k}=\frac{5}{12},$$

and

$$\begin{aligned}&[\Psi(\upsilon_1)\zeta(\upsilon_1)+m\Psi(\upsilon_2)\zeta(\upsilon_2)]\int_0^1\frac{1}{[\mathsf{h}(\eta)]^2}d\eta\\ &+[m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)]\int_0^1\frac{1}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}d\eta\\ &=\frac{10}{9}.\end{aligned}$$

Consequently, Theorem 4 is verified that is

$$\frac{5}{12}\leq\frac{10}{9}.$$

Corollary 3. *If $\phi(m\upsilon_2,\upsilon_1) = m\upsilon_2 - \upsilon_1$, then Theorem 4 reduces to m-harmonic convex:*

$$\frac{m\upsilon_1\upsilon_2}{m\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{m\upsilon_2}\frac{\Psi(\mathrm{k})\zeta(\mathrm{k})}{\mathrm{k}^2}d\mathrm{k} \leq [\Psi(\upsilon_1)\zeta(\upsilon_1)+m\Psi(\upsilon_2)\zeta(\upsilon_2)]\int_0^1\frac{1}{[\mathrm{h}(\eta)]^2}d\eta + [m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)]\int_0^1\frac{1}{\mathrm{h}(\eta)\mathrm{h}(1-\eta)}d\eta.$$

Corollary 4. *If $\phi(m\upsilon_2,\upsilon_1) = m\upsilon_2 - \upsilon_1$ and $\mathrm{h}(\eta) = \eta^{-\mathrm{s}}$, then Theorem 4 reduces to s-harmonic convex:*

$$\frac{m\upsilon_1\upsilon_2}{m\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{m\upsilon_2}\frac{\Psi(\mathrm{k})\zeta(\mathrm{k})}{\mathrm{k}^2}d\mathrm{k} \leq \frac{[\Psi(\upsilon_1)\zeta(\upsilon_1)+m\Psi(\upsilon_2)\zeta(\upsilon_2)]}{2\mathrm{s}+1} + [m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)]\frac{\Gamma(\mathrm{s}+1)\Gamma(\mathrm{s}+1)}{\Gamma(2\mathrm{s}+2)}.$$

Corollary 5. *If $\phi(m\upsilon_2,\upsilon_1) = m\upsilon_2 - \upsilon_1$ and $\mathrm{h}(\eta) = \eta^{\mathrm{p}}(1-\eta)^{\mathrm{q}}$, then Theorem 4 reduces to tgs-harmonic convex:*

$$\frac{m\upsilon_1\upsilon_2}{m\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{m\upsilon_2}\frac{\Psi(\mathrm{k})\zeta(\mathrm{k})}{\mathrm{k}^2}d\mathrm{k} \leq [\Psi(\upsilon_1)\zeta(\upsilon_1)+m\Psi(\upsilon_2)\zeta(\upsilon_2)]\frac{\Gamma(2\mathrm{p}+1)\Gamma(2\mathrm{q}+1)}{\Gamma(2\mathrm{p}+2\mathrm{q}+3)} + [m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)]\frac{\Gamma(\mathrm{q}+\mathrm{p}+1)\Gamma(\mathrm{q}+\mathrm{p}+1)}{\Gamma(2\mathrm{p}+2\mathrm{q}+3)}.$$

Theorem 5. *Let $\Psi,\zeta : \mathrm{I}_\phi \subseteq \mathrm{R}^+ \to \mathrm{R}$ be (m,h)-harmonic Godunova and Levin preinvex function, where $m \in [0,1]$. If $\Psi,\zeta \in \mathrm{L}[\upsilon_1,\upsilon_1+\phi(m\upsilon_2,\upsilon_1)]$, then*

$$\left[\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\right]^2 \int_{\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}}^{\frac{1}{\upsilon_1}} \mathrm{h}\left(\mathrm{k}-\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}\right) \times\left[\Psi(\upsilon_1)\zeta\left(\frac{1}{\mathrm{k}}\right)+\zeta(\upsilon_1)\Psi\left(\frac{1}{\mathrm{k}}\right)\right]d\mathrm{k} \times\left[\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\right]^2 \times\int_{\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}}^{\frac{1}{\upsilon_1}} \mathrm{h}\left(\frac{1}{\upsilon_1}-\mathrm{k}\right) \times\left[m\Psi(\upsilon_2)\zeta\left(\frac{1}{\mathrm{k}}\right)+m\zeta(\upsilon_2)\Psi\left(\frac{1}{\mathrm{k}}\right)\right]d\mathrm{k} \leq [\Psi(\upsilon_1)\zeta(\upsilon_1)+m\Psi(\upsilon_2)\zeta(\upsilon_2)]\int_0^1\frac{1}{[\mathrm{h}(\eta)]^2}d\eta + [m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)]\int_0^1\frac{1}{\mathrm{h}(\eta)\mathrm{h}(1-\eta)}d\eta + \frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}\frac{\Psi(\mathrm{k})\zeta(\mathrm{k})}{\mathrm{k}^2}d\mathrm{k}. \tag{15}$$

Proof. Let Ψ and ζ be (m,h)-harmonic Godunova and Levin preinvex functions, we have:

$$\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right) \leq \frac{\Psi(\upsilon_1)}{\mathrm{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathrm{h}(\eta)},$$

$$\zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right) \leq \frac{\zeta(\upsilon_1)}{\mathrm{h}(1-\eta)}+\frac{m\zeta(\upsilon_2)}{\mathrm{h}(\eta)}.$$

It follows that

$$\frac{\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)}{\frac{\Psi(\upsilon_1)}{\mathrm{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathrm{h}(\eta)}} \leq 1, \tag{16}$$

$$\frac{\zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)}{\frac{\zeta(\upsilon_1)}{\mathrm{h}(1-\eta)}+\frac{m\zeta(\upsilon_2)}{\mathrm{h}(\eta)}} \leq 1. \tag{17}$$

Multiplying the above equations (16) and (17), we get

$$\left[\frac{\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)}{\frac{\Psi(\upsilon_1)}{\mathrm{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathrm{h}(\eta)}}\right]\left[\frac{\zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)}{\frac{\zeta(\upsilon_1)}{\mathrm{h}(1-\eta)}+\frac{m\zeta(\upsilon_2)}{\mathrm{h}(\eta)}}\right] \leq 1$$

$$\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)\left[\frac{\zeta(\upsilon_1)}{\mathrm{h}(1-\eta)}+\frac{m\zeta(\upsilon_2)}{\mathrm{h}(\eta)}\right] + \zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)\left[\frac{\Psi(\upsilon_1)}{\mathrm{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathrm{h}(\eta)}\right] \leq \left[\frac{\Psi(\upsilon_1)}{\mathrm{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathrm{h}(\eta)}\right]\left[\frac{\zeta(\upsilon_1)}{\mathrm{h}(1-\eta)}+\frac{m\zeta(\upsilon_2)}{\mathrm{h}(\eta)}\right] + \left[\zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)\right] \times\left[\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)\right]. \tag{18}$$

By simplifying equation (18), we have

$$\frac{\zeta(\upsilon_1)}{\mathrm{h}(1-\eta)}\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right) + \frac{m\zeta(\upsilon_2)}{\mathrm{h}(\eta)}\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right) + \frac{\Psi(\upsilon_1)}{\mathrm{h}(1-\eta)}\zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right) + \frac{m\Psi(\upsilon_2)}{\mathrm{h}(\eta)}\zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right) \leq \Psi(\upsilon_1)\zeta(\upsilon_1)\frac{1}{[\mathrm{h}(1-\eta)]^2} + [m\Psi(\upsilon_1)\zeta(\upsilon_2) + m\Psi(\upsilon_2)\zeta(\upsilon_1)]\frac{1}{\mathrm{h}(\eta)\mathrm{h}(1-\eta)} + \frac{m^2\Psi(\upsilon_2)\zeta(\upsilon_2)}{[\mathrm{h}(\eta)]^2} + \Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)$$

$$\times \zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right).$$

Integrating over [0,1], we have

$$\zeta(\upsilon_1)\int_0^1 \frac{1}{\mathsf{h}(1-\eta)}\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta$$
$$+m\zeta(\upsilon_2)\int_0^1 \frac{1}{\mathsf{h}(\eta)}\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta$$
$$+\Psi(\upsilon_1)\int_0^1 \frac{1}{\mathsf{h}(1-\eta)}\zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta$$
$$+\int_0^1 m\Psi(\upsilon_2)\frac{1}{\mathsf{h}(\eta)}\zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta$$
$$\leq [\Psi(\upsilon_1)\zeta(\upsilon_1)+m^2\Psi(\upsilon_2)\zeta(\upsilon_2)]\int_0^1 \frac{1}{[\mathsf{h}(\eta)]^2}d\eta$$
$$+[m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)]\int_0^1 \frac{1}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}d\eta$$
$$+\int_0^1 \Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)$$
$$\times\zeta\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta. \quad (19)$$

Now consider

$$\int_0^1 \frac{1}{\mathsf{h}(1-\eta)}\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta.$$

put

$$\frac{1}{\mathtt{k}} = \frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)},$$

this implies that

$$\mathtt{k} = \frac{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))},$$

$$\mathtt{k}[\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))] = \upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1),$$
$$\mathtt{k}[\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))]-\upsilon_1 = (1-\eta)\phi(m\upsilon_2,\upsilon_1),$$
$$\frac{\mathtt{k}[\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))]-\upsilon_1}{\phi(m\upsilon_2,\upsilon_1)} = (1-\eta),$$
$$d\left[\frac{\mathtt{k}[\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))]-\upsilon_1}{\phi(m\upsilon_2,\upsilon_1)}\right] = d(1-\eta),$$
$$\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}d\mathtt{k} = -d\eta,$$
$$-\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}d\mathtt{k} = d\eta,$$

if $\eta = 0$, then

$$\mathtt{k} = \frac{\upsilon_1+(1-0)\phi(m\upsilon_2,\upsilon_1)}{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))} = \frac{1}{\upsilon_1},$$

if $\eta = 1$, then we have

$$\mathtt{k} = \frac{\upsilon_1+(1-1)\phi(m\upsilon_2,\upsilon_1)}{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))} = \frac{1}{\upsilon_1+\phi(mb,\upsilon_1)}.$$

So, we have

$$\int_0^1 \frac{1}{\mathsf{h}(1-\eta)}\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta$$
$$=\int_{\frac{1}{\upsilon_1}}^{\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}} \frac{1}{\mathsf{h}\left(\mathtt{k}-\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}\right)}$$
$$\times\Psi\left(\frac{1}{\mathtt{k}}\right)(-1)\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}d\mathtt{k}$$
$$=-\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\int_{\frac{1}{\upsilon_1}}^{\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}} \frac{1}{\mathsf{h}\left(\mathtt{k}-\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}\right)}\Psi\left(\frac{1}{\mathtt{k}}\right)d\mathtt{k}$$
$$=\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\int_{\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}}^{\frac{1}{\upsilon_1}} \frac{1}{\mathsf{h}\left(\mathtt{k}-\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}\right)}\Psi\left(\frac{1}{\mathtt{k}}\right)d\mathtt{k}$$
$$=\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\int_{\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}}^{\frac{1}{\upsilon_1}} \frac{1}{\mathsf{h}\left(\mathtt{k}-\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}\right)}$$
$$\times\Psi\left(\frac{1}{\mathtt{k}}\right)\zeta(\upsilon_1)d\mathtt{k}, \quad (20)$$

it follows that

$$\left[\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\right]^2$$
$$\times\int_{\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}}^{\frac{1}{\upsilon_1}} \mathsf{h}\left(\mathtt{k}-\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}\right)$$
$$\times\left[\Psi(\upsilon_1)\zeta\left(\frac{1}{\mathtt{k}}\right)+\zeta(\upsilon_1)\Psi\left(\frac{1}{\mathtt{k}}\right)\right]d\mathtt{k}$$
$$\times\left[\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\right]^2$$
$$\times\int_{\frac{1}{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)}}^{\frac{1}{\upsilon_1}} \mathsf{h}\left(\frac{1}{\upsilon_1}-\mathtt{k}\right)\left[m\Psi(\upsilon_2)\zeta\left(\frac{1}{\mathtt{k}}\right)+m\zeta(\upsilon_2)\Psi\left(\frac{1}{\mathtt{k}}\right)\right]d\mathtt{k}$$
$$\leq[\Psi(\upsilon_1)\zeta(\upsilon_1)+m\Psi(\upsilon_2)\zeta(\upsilon_2)]$$
$$\times\int_0^1 \frac{1}{[\mathsf{h}(\eta)]^2}d\eta+[m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)]\int_0^1 \frac{1}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}d\eta$$
$$+\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)} \frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathtt{k}^2}d\mathtt{k}. \quad (21)$$

This completes the proof.

Corollary 6. *If $\phi(m\upsilon_2,\upsilon_1) = m\upsilon_2-\upsilon_1$, then Theorem 5 reduces to m-harmonic convex:*

$$\left(\frac{m\upsilon_1\upsilon_2}{m\upsilon_2-\upsilon_1}\right)^2$$
$$\times\int_{\frac{1}{m\upsilon_2}}^{\frac{1}{\upsilon_1}} \mathsf{h}\left(\mathtt{k}-\frac{1}{m\upsilon_2}\right)\left[\Psi(\upsilon_1)\zeta\left(\frac{1}{\mathtt{k}}\right)+\zeta(\upsilon_1)\Psi\left(\frac{1}{\mathtt{k}}\right)\right]d\mathtt{k}$$
$$\left(\frac{m\upsilon_1\upsilon_2}{m\upsilon_2-\upsilon_1}\right)^2\int_{\frac{1}{m\upsilon_2}}^{\frac{1}{\upsilon_1}} \mathsf{h}\left(\frac{1}{\upsilon_1}-\mathtt{k}\right)$$
$$\times m\left[\Psi(\upsilon_2)\zeta\left(\frac{1}{\mathtt{k}}\right)+\zeta(\upsilon_2)\Psi\left(\frac{1}{\mathtt{k}}\right)\right]d\mathtt{k}$$
$$\leq M(\upsilon_1,\upsilon_2)+\frac{m\upsilon_1\upsilon_2}{m\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{m\upsilon_2} \frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathtt{k}^2}d\mathtt{k}.$$

4 Conclusions and future remarks

The main contribution in this article is that we propose a new definition which generalizes several well known existing definitions under different settings further to show the distinct features of our new class, we provide an interesting example that demonstrates that classical convex mappings belong in our newly proposed class, while converse is not true in general. As a result of this new definition, we have developed a number of new refinements and bounds for Hermite and Hadamard inequality, its product and its symmetric forms, together with a number of interesting remarks and corollaries relating their results to other convex mappings, including s-convex, tgs-convex, harmonic convex, etc. There are a number of non-trivial examples provided to illustrate the main results. We suggest that readers derive these results by using a following new fractional integral which defined as follows:

$$\left({}^{\mathtt{V}}\mathtt{I}_{\mathtt{a}^+}^{\phi,\Psi}\mathtt{S}\right)(\mathtt{s}) = \frac{1}{\Psi^{\phi}\Gamma(\phi)}\int_{\mathtt{a}}^{\mathtt{s}} \exp\left[\frac{\Psi-1}{\Psi}\left(\mathtt{V}(\mathtt{s})-\mathtt{V}(\mathtt{u})\right)\right] \times \left(\mathtt{V}(\mathtt{s})-\mathtt{V}(\mathtt{u})\right)^{\phi-1}\mathtt{V}'(\mathtt{u})\mathtt{S}(\mathtt{u})\,\mathtt{du}$$

and

$$\left({}^{\mathtt{V}}\mathtt{I}_{\mathtt{b}^-}^{\phi,\Psi}\mathtt{S}\right)(\mathtt{s}) = \frac{1}{\Psi^{\phi}\Gamma(\phi)}\int_{\mathtt{b}}^{\mathtt{s}} \exp\left[\frac{\Psi-1}{\Psi}\left(\mathtt{V}(\mathtt{u})-\mathtt{V}(\mathtt{s})\right)\right] \times \left(\mathtt{V}(\mathtt{u})-\mathtt{V}(\mathtt{s}))\right)^{\phi-1}\mathtt{V}'(\mathtt{u})\mathtt{S}(\mathtt{u})\,\mathtt{du}.$$

Moreover in this work, comparable inequalities may be identified for employing an integration with regard to Brownian motion, which is a highly fresh thought for readers.

$$\int_0^{\mathtt{t}} \mathtt{HdB} = \lim_{\mathtt{n}\to\infty} \sum_{[\mathtt{t_{i-1}},\mathtt{t_i}]\in\pi_{\mathtt{n}}} \zeta_{\mathtt{t_{i-1}}}\left(\mathtt{B_{t_i}}-\mathtt{B_{t_{i-1}}}\right).$$

where $\mathtt{dB}$ is Brownian motion.

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