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A Novel Method for Solving Conformable-Order Partial Differential Equations with Applications to Real-Life Models

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Abstract: In this paper, an efficient methodology is employed to solve nonlinear partial differential equations of non-integer conformable order. This approach, which combines the popular Adomian decomposition method with the natural transform, can be viewed as an analytic-approximate methodology. Some conformable order partial differential equations (CPDEs) that are related to real-life phenomena such as Burgers equations and gas dynamics equation are solved using this method. Moreover, we discuss the convergence of the analytic solution that results by our proposed methodology and give an upper bound of the expected error of estimation. A nice advantage of this technique is that the exact solution is obtained in some examples. Moreover, computing the absolute errors of approximations and comparing with other methods produces very small errors with small number of iterations which indicates high accuracy. To demonstrate the effectiveness, simplicity, and correctness of our chosen method, analytical, numerical, and graphic results are provided.

Keywords: Adomian polynomial, Natural transform, Fractional derivative, Partial differential equations, Conformable derivative

1 Introduction

Because fractional calculus can be used to model dynamic systems in fields like physics, engineering, and finance, its study has attracted a lot of attention recently. [1,2,3,4,5,6]. Due to the fact that fractional operators have no unique definitions, we can find many approaches such as Riemann-Liouville operators, Caputo derivative, Grunwald-Letnikov derivative, and many other operators [7,8,9,10]. One notable development in this area is the introduction of the conformable fractional derivative, which provides a more intuitive and straightforward approach to fractional calculus. The concept of conformable operators was first suggested by Khalil et al. in 2014 [11]. It is defined as follows:

Definition 1.1. [11] Consider $\Psi : [0,\infty) \to \mathbb{R}$, and $\gamma \in (0,1]$. Then the conformable derivative (CD) of Ψ of order γ for $\tau > 0$ is

$$\left({}_{\tau}^{C}T_{0}^{\gamma}\Psi\right)(\tau) = \lim_{\varepsilon\to 0^{+}} \frac{\Psi\left(\tau+\varepsilon\tau^{1-\gamma}\right)-\Psi(\tau)}{\varepsilon}. \quad (1)$$

If ${}_{\tau}^{C}T_{0}^{\gamma}\Psi$ exists over $(0,\mathfrak{a})$, then Ψ is said to be γ-conformable differentiable on $(0,\mathfrak{a})$. If, additionally, $\lim_{\tau\to 0^{+}}\left({}_{\tau}^{C}T_{0}^{\gamma}\Psi\right)(\tau)$ exists, then we define $\left({}_{\tau}^{C}T_{0}^{\gamma}\Psi\right)(0) = \lim_{\tau\to 0^{+}}\left({}_{\tau}^{C}T_{0}^{\gamma}\Psi\right)(\tau)$.

This definition was later modified by Abdeljawad [12] as:

Definition 1.2. [12] The CD of $\Psi : [a,\infty) \to \mathbb{R}$ of order $\gamma \in (0,1]$ is defined by

$$\left({}_{\tau}^{C}T_{\mathfrak{a}}^{\gamma}\Psi\right)(\tau) = \lim_{\varepsilon\to 0^{+}} \frac{\Psi\left(\tau+\varepsilon(\tau-\mathfrak{a})^{1-\gamma}\right)-\Psi(\tau)}{\varepsilon}. \quad (2)$$

If ${}_{\tau}^{C}T_{\mathfrak{a}}^{\gamma}\Psi$ exists over $(\mathfrak{a},b)$, then $\left({}_{\tau}^{C}T_{\mathfrak{a}}^{\gamma}\Psi\right)(\mathfrak{a}) = \lim_{\tau\to\mathfrak{a}^{+}}\left({}_{\tau}^{C}T_{\mathfrak{a}}^{\gamma}\Psi\right)(\tau)$.

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A beautiful property of the γ- conformable differentiable function is its relation with classical integer order derivative which is:

$$\left({}_{\tau}^{C}T_{\mathfrak{a}}^{\gamma}(\Psi)\right)(\tau)=(\tau-\mathfrak{a})^{1-\gamma}\frac{d}{d\tau}\Psi(\tau). \quad (3)$$

The definition of conformable derivative differs from other well-known fractional definitions as it is based on the limit definition of the classical derivative. So, it satisfies several properties that make it suitable for applications, such as the product rule, chain rule, and integration by parts. Some of the nice properties of conformable derivative are in the following theorem.

Theorem 1.1. [11] Let $\gamma\in(0,1)$, ψ,P be γ-differentiable functions at $t>0$, and $\mathfrak{a},\mathfrak{b}$, are real constants, then

1. ${}_{\tau}^{C}T_{0}^{\gamma}(\mathfrak{a}\psi+\mathfrak{b}P)=\mathfrak{a}\,{}_{\tau}^{C}T_{0}^{\gamma}(\psi)+\mathfrak{b}\,{}_{\tau}^{C}T_{0}^{\gamma}(P)$
2. ${}_{\tau}^{C}T_{0}^{\gamma}(\tau^{n})=n\tau^{n-\gamma}, \quad \forall n\in\mathbb{R}$
3. ${}_{\tau}^{C}T_{0}^{\gamma}(\psi)=0,$ for all constant functions $\psi(\tau)=\lambda$
4. ${}_{\tau}^{C}T_{0}^{\gamma}(\psi P)=\psi\,{}_{\tau}^{C}T_{0}^{\gamma}(P)+P(\tau)\,{}_{\tau}^{C}T_{0}^{\gamma}(\psi)$
5. ${}_{\tau}^{C}T_{0}^{\gamma}\left(\frac{\psi}{P}\right)=\frac{P(\tau)\,{}_{\tau}^{C}T_{0}^{\gamma}(\psi)-\psi\,{}_{\tau}^{C}T_{0}^{\gamma}(P)}{P^{2}(\tau)}.$

As a result, it has been used in various applications that are related to the study of fractional differential equations (FDEs). For instance, Abdeljawad explored its properties and its applications in solving FDEs [12], Segi Rahmat introduced a new definition of conformable fractional derivative on arbitrary time scales and investigated its properties [13], and three-dimensional conformable order Lotka-Volterra system that models population dynamics was studied in [14,15]. Various studies and applications of the CD can be found in [16,17,18,19,20].

On the other hand, most of the FDEs that result from modeling dynamic systems have no exact solutions and finding approximate solutions is still not an easy task. Hence, many mathematicians devoted their research on modifying analytical and numerical techniques to fit FDEs [21]. One of the promising methods in this domain is the natural decomposition method (NDM), which facilitates finding the analytical solution of FDEs especially nonlinear partial differential equations (PDEs) [22,23,24]. This methodology is a combination between the well-known Adomian decomposition method (ADM) and the natural transform. The ADM is popular for its precision and effectiveness in handling nonlinear problems without requiring perturbation or linearization, making it a valuable tool in the analysis of complex systems. For more details about the ADM, its origin and advantages. Reader can refer to [25,26].

Moreover, the natural transform is one of several integral transforms such as Laplace, Sumudu, and Mellin transforms [27,28,29]. It was suggested by Khan in 2008 [29] and was first employed to solve fluid flow problem and Maxwell's equations [29,30]. Recently, this transform was generalized to double natural transform to be used to solve telegraphs, wave and partial integro-differential equations [27].

In this work, we extend the classical NDM to solve various conformable-order PDEs (CPDEs), so we call the method conformable natural decomposition method (CNDM). This method enhances the classical decomposition techniques by incorporating conformable fractional derivatives, which provide a more intuitive framework for analyzing dynamic systems. We explore the application of the CNDM to get solutions of nonlinear responsive gas dynamics equations [31], the KdV-Burgers equation [32], a modified nonlinear Burgers equation [33], and Schrödinger equation [37]. These equations are pivotal in understanding various physical processes, including fluid dynamics and wave propagation. The KdV-Burgers equation, for instance, models the evolution of waves in shallow water and incorporates effects of viscosity. Through the CNDM, the problem is decomposed into simpler components, and then an approximate, or may be exact, solutions can be obtained.

This paper consists out of four sections including the introduction. Section 2 presents the basics of the conformable natural transform. In Section 3, we introduce the methodology of the CNDM then we summarize its main steps in an organized algorithm in the same section. Numerical Applications are presented in Section 4 in which we apply our technique to solve three CPDE's that play vital roles in real life models. Finally, the paper ends with the conclusion section.

2 Basics of Conformable Natural Transform

In this Section, we present the definition of the conformable natural transform (CNT), some of its properties, and its relation with conformable Laplace transform (CLT) which plays a central role in the CNDM. However, these definitions require the following set of functions [34]:

$$A=\left\{\psi(\tau)\ :\ \exists M,\varepsilon_1,\varepsilon_2>0 \text{ with } \quad |\psi(\tau)|<Me^{\frac{|\tau|}{\varepsilon_j}} \quad \text{if} \quad \tau\in(-1)^{j}*[0,\infty),\ j\in\{1,2\}\right\}.$$

Before of introducing the main concepts of CNT, we give a quick review for the classical natural transform.

In fact, the natural transform $\mathcal{N}[\psi(\tau)]$ of the function $\psi(\tau)$ for $\tau\in\mathbb{R}$ is defined in [29,30,34] as:

$$\mathcal{N}[\psi(\tau)]=\Psi(s,u)=\int_{-\infty}^{\infty}e^{-s\tau}\psi(u\tau)d\tau, \quad s,u\in(-\infty,\infty). \quad (4)$$

If we restrict $s,u\in(0,\infty)$, then its notation will be

$$\mathcal{N}^+[\psi(\tau)] = \Psi^+(s,u) = \int_0^\infty e^{-s\tau}\psi(u\tau)d\tau, \quad s,u \in (0,\infty). \tag{5}$$

To notice the relation between this transform and some well-known integral transform, we recall the definitions of Laplace and Sumudu transforms that are respectively defined as:

$$\mathcal{L}[\psi(\tau)] = \int_0^\infty e^{-s\tau}\psi(\tau)\,d\tau \tag{6}$$

and

$$\mathrm{S}[\psi(\tau)] = \int_0^\infty e^{-\tau}\psi(u\tau)d\tau. \tag{7}$$

Note that when $s = 1$ in Eq (5), it becomes the Sumudu transform that is given in Eq (7), while when $u = 1$, then Eq (5) reduced to the Laplace transform in Eq (6).

Many useful characteristics of the natural transforms can be found in [34]. We are interested about the linear property which can be stated as follows.

Let $\psi(\tau)$ and $\beta(\tau)$ be in the set A and a,b be constants. Then

$$\mathcal{N}^+[a\psi(\tau)\pm b\beta(\tau)] = a\mathcal{N}^+[\psi(\tau)]\pm b\mathcal{N}^+[\beta(\tau)].$$

Now, we introduce the main concepts that are related to the CNDM.

Definition 2.1. [12] The conformable Laplace transform (CLT) for a function $\psi : [0,\infty) \to \mathbb{R}$ at $\tau > 0$ is defined by

$$\mathcal{L}_\alpha[\psi(\tau)] = \Psi_\alpha(s) = \int_0^\infty e^{\frac{-s\tau^\alpha}{\alpha}}\psi(\tau)\tau^{\alpha-1}d\tau, 0<\alpha\le 1. \tag{8}$$

Obviously, when $\alpha = 1$, Eq. (8) is reduced to the well-known Laplace transform.

A useful relationship between CLT $\Psi_\alpha(s) = \mathcal{L}_\alpha[\psi(\tau)]$ and the classical Laplace transform $\Psi(s) = \mathcal{L}[\psi(\tau)]$ is as follows [12]:

$$\Psi_\alpha(s) = \mathcal{L}\left[\psi(\alpha\tau)^{\frac{1}{\alpha}}\right]. \tag{9}$$

Some of the important rules in solving FDE using CLT are the following [13,35].

(i) $\mathcal{L}_\alpha\left({}^C_\tau T^\alpha_0\psi(\tau)\right) = s\Psi_\alpha(s) - \psi(0)$,

(ii) $\mathcal{L}_\alpha\left({}^C_\tau I^\alpha_0\psi(\tau)\right) = \frac{\Psi_\alpha(s)}{s}$, $s > 0$,

where ${}^C_\tau I^\alpha_0$ is the conformable integral operator of order α that is defined for $\psi : [0,\infty) \to \mathbb{R}$ by the formula

$$\left({}^C_\tau I^\alpha_0\psi\right)(\tau) = \int_0^\tau \zeta^{\alpha-1}\psi(\zeta)d\zeta.$$

Definition 2.2. [36] Let $\psi(\tau) \in A, 0 < \alpha \le 1$, then the CNT is defined by

$${}^C_\tau\mathcal{N}^+_\alpha\{\psi(\tau)\} = \frac{1}{u}\int_0^\infty e^{\frac{-s\tau^\alpha}{u\alpha}}\psi(\tau)\tau^{\alpha-1}\,d\tau, \quad u>0, s>0$$

provided the integral exists.

Some of the most useful properties related to the CNT are listed in the following theorem.

Theorem 2.1. [36] Let $\psi : [0,\infty) \to \mathbb{R}$ such that ${}^C_\tau\mathcal{N}^+_\alpha\{\psi(\tau)\} = \Psi_{N_\alpha}(s,u)$ is the CNT and $\Psi_\alpha(s) = \mathcal{L}_\alpha(\psi(\tau))$ is the CLT, and $0 \le \alpha \le 1$. Then

a. $\Psi_{N_\alpha}(s,u) = \mathcal{N}^+\left\{\psi(\alpha\tau)^{\frac{1}{\tau}}\right\}$,

b. ${}^C_\tau\mathcal{N}^+_\alpha\{\psi(\tau)\} = \frac{1}{u}\Psi_\alpha\left(\frac{s}{u}\right)$,

c. ${}^C_\tau\mathcal{N}^+_\alpha\left\{{}^C_\tau T^\alpha_0\psi(\tau)\right\} = \frac{s}{u}\Psi_{N_\alpha}(s,u) - \frac{\psi(\mathbf{0})}{u}$,

d. If $\psi^{(n)}(\tau)$ exists for $n\in\mathbb{N}$ and $0<\alpha<1$, then

$${}^C_\tau\mathcal{N}^+_\alpha\left\{{}^C_\tau T^{n\alpha}_0\psi(\tau)\right\} = \frac{s^n}{u^n}\Psi_{N_\alpha}(s,u) - \frac{s^{n-1}}{u^n}\psi(0).$$

e. For $\alpha\in(n-1,n]$ with $n\in\mathbb{N}$, we have

$${}^C_\tau\mathcal{N}^+_\alpha\left\{{}^C_\tau T^\alpha_0\psi(\tau)\right\} = \frac{s^\alpha}{u^\alpha}\Psi_{N_\alpha}(s,u) - \sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}\left({}^C_\tau T^{k-1}_0\psi\right)(0).$$

The CNT for basic functions are collected in the following theorem.

Theorem 2.2. [36] Let $a,c\in\mathbb{R}$ and $0<\alpha\le 1$. Then

1. ${}^C_\tau\mathcal{N}^+_\alpha\{1\} = \frac{u^\alpha}{s^{\alpha+1}}$,

2. ${}^C_\tau\mathcal{N}^+_\alpha\left\{\frac{\tau^\alpha}{\alpha}\right\} = \frac{u^{\alpha+1}}{s^{\alpha+2}}$,

3. ${}^C_\tau\mathcal{N}^+_\alpha\left\{\frac{\tau^{n-1}}{(n-1)!}\right\} = \frac{u^{\alpha+n-1}}{s^{\alpha+2}}$, $\quad n=0,1,2,\ldots$,

4. ${}^C_\tau\mathcal{N}^+_\alpha\left\{\frac{\tau^{n-1}}{\Gamma(n)}\right\} = \frac{u^{\alpha+n-1}}{s^{\alpha+2}}$, $\quad n<0,\ n\notin\mathbb{Z}$,

5. ${}^C_\tau\mathcal{N}^+_\alpha\left\{e^{\frac{a\tau^\alpha}{\alpha}}\right\} = \frac{u^\alpha}{s^\alpha(s-au)}$,

6. ${}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\left\{\sin\left(a\frac{\tau^{\alpha}}{\alpha}\right)\right\} = \frac{au}{s^2+a^2u^2}, \quad s>0,$

7. ${}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\left\{\cos\left(a\frac{\tau^{\alpha}}{\alpha}\right)\right\} = \frac{s}{s^2+a^2u^2}, \quad s>0,$

8. ${}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\left\{\sinh\left(a\frac{\tau^{\alpha}}{\alpha}\right)\right\} = \frac{au}{s^2-a^2u^2}, \quad s>|au|,$

9. ${}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\left\{\cosh\left(a\frac{\tau^{\alpha}}{\alpha}\right)\right\} = \frac{s}{s^2-a^2u^2}, \quad s>|au|.$

3 Description of the CNDM for Solving CPDE's

In this section, we present the basic idea of the CNDM which is a combination of CNT and the well-known Adomian decomposition method. We describe it for solving nonlinear CPDE's in which the simplicity of the CNDM will be clear since all nonlinear terms are easily to be handled with the Adomian polynomials.

Consider the general nonlinear nonhomogeneous CPDE of the form:

$$\begin{aligned}{}^{C}_{\tau}T^{\alpha}_{0}\psi(\xi,\tau)+K(\xi,\tau,\psi)+M(\xi,\tau,\psi)=\mathrm{Z}(\xi,\tau),\\ 0<\alpha\leq 1, \quad (\xi,\tau)\in[a,b]\times[0,T)\end{aligned} \tag{10}$$

subject to initial condition (IC):

$$\psi(\xi,0)=g(\xi), \tag{11}$$

where K is a linear differential operator, M represents the general nonlinear differential operator, and $\mathrm{Z}(\xi,\tau)$ is the source term.

To apply the CNDM, we first take the CNT to both sides of Eq (10), so we get

$$\begin{aligned}{}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\left\{{}^{C}_{t}T^{\alpha}_{0}\psi(\xi,\tau)\right\}+{}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\left\{K(\xi,\tau,\psi(\xi,\tau))\right\}\\ +{}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\left\{M(\xi,\tau,\psi(\xi,\tau))\right\}\\ ={}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\left\{\mathrm{Z}(\xi,\tau)\right\}.\end{aligned} \tag{12}$$

Applying the basic rules of the CNT, we get

$$\begin{aligned}\frac{s}{u}\Psi_{N_{\alpha}}(\xi,s,u)-\frac{\psi(\xi,0)}{u}+{}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\left\{K(\xi,\tau,\psi(\xi,\tau))\right\}\\ +{}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\left\{M(\xi,\tau,\psi(\xi,\tau))\right\}\\ ={}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\left\{\mathrm{Z}(\xi,\tau)\right\}.\end{aligned} \tag{13}$$

Using the IC in Eq (11) and rearranging the terms of Eq (13), we obtain

$$\begin{aligned}\Psi_{N_{\alpha}}(\xi,s,u)=\frac{g(\xi)}{s}-\frac{u}{s}\left\{{}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\{K(\xi,\tau,\psi(\xi,\tau))\}\right\}\\ -\frac{u}{s}\left\{{}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\{M(\xi,\tau,\psi(\xi,\tau))\}\right\}\\ +\frac{u}{s}\left\{{}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\{\mathrm{Z}(\xi,\tau)\}\right\}.\end{aligned}$$

Now, we take the inverse conformable natural transform (ICNT), ${}^{C}_{\tau}\mathscr{N}^{-1}_{\alpha}$, so we have

$$\begin{aligned}\psi(\xi,\tau)=H(\xi,\tau)-{}^{C}_{\tau}\mathscr{N}^{-1}_{\alpha}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\{K(\xi,\tau,\psi)\}\right\}\\ -{}^{C}_{\tau}\mathscr{N}^{-1}_{\alpha}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\{M(\xi,\tau,\psi)\}\right\},\end{aligned} \tag{14}$$

where $H(\xi,\tau)={}^{C}_{\tau}\mathscr{N}^{-1}_{\alpha}\left\{\frac{g(\xi)}{s}+\frac{u}{s}\left\{{}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\{\mathrm{Z}(\xi,\tau)\}\right\}\right\}$.

The idea of the CNDM is based on replacing the unknown function $\psi(\xi,\tau)$ in the linear term by the infinite series

$$\psi(\xi,\tau)=\sum_{k=0}^{\infty}w_k(\xi,\tau) \tag{15}$$

and the nonlinear term by a series of the Adomian polynomials. That is

$$M(\xi,\tau,\psi(\xi,\tau))=\sum_{k=0}^{\infty}\mathscr{A}_k, \tag{16}$$

where the $\mathscr{A}_k$ are polynomials of $w_0,w_1,w_2,w_3,\ldots,w_k$ and can be calculated by the successive relations:

$$\mathscr{A}_k=\frac{1}{k!}\frac{d^k}{d\eta^k}M\left(\sum_{\iota=0}^{k}\eta^{\iota}w_{\iota}\right)\Bigg|_{\eta=0};k=0,1,2,\ldots. \tag{17}$$

Applying the general formula in Eq. (17), $\mathscr{A}_k$ can be further simplified as follows:

$$\begin{aligned}\mathscr{A}_0&=M(w_0),\\ \mathscr{A}_1&=w_1M'(w_0),\\ \mathscr{A}_2&=w_2M'(w_0)+\frac{1}{2!}w_1^2M''(w_0),\\ \mathscr{A}_3&=w_3M'(w_0)+w_1w_2M''(w_0)+\frac{1}{3!}w_1^3M'''(w_0),\\ &\vdots\end{aligned}$$

Now, we substitute Eq. (15) and Eq. (16), into Eq. (14), to get:

$$\begin{aligned}\sum_{k=0}^{\infty}w_k(\xi,\tau)=H(\xi,\tau)\\ -{}^{C}_{\tau}\mathscr{N}^{-1}_{\alpha}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}^{+}_{\alpha}\left\{M\left(\xi,\tau,\sum_{k=0}^{\infty}w_k(\xi,\tau)\right)\right.\right.\\ \left.\left.+\sum_{k=0}^{\infty}\mathscr{A}_k\right\}\right\}\end{aligned} \tag{18}$$

which produces the recurrence relations:

$$
\begin{aligned}
&w_0(\xi,\tau)=H(\xi,\tau),\\
&w_{k+1}(\xi,\tau)=-{}^{C}_{\tau}\mathscr{N}_\alpha^{-1}\left\{{}^{u}_{s}{}^{C}_{\tau}\mathscr{N}_\alpha^{+}\left\{M(\xi,\tau,w_k(\xi,\tau))\right.\right.\\
&\qquad\left.\left.+\mathscr{A}_k\right\}\right\},\quad \text{for } k=0,1,2,\dots
\end{aligned} \tag{19}
$$

Hence, from the general recursive relation in Eq. (19), we can easily compute the remaining components of $\psi(\xi,\tau)$ in Eq. (15) as $w_1(\xi,\tau), w_2(\xi,\tau),\dots$.

Computing finite number of components leads to the approximate solution which can be considered as the N$^{\text{th}}$ truncated series as follows

$$
\psi_N(\xi,\tau)=\sum_{k=0}^{N} w_k(\xi,\tau). \tag{20}
$$

Surely, the large number of iterations, N, leads to more accuracy. In fact, under some assumptions, it can be shown that

$$
\lim_{N\to\infty}\psi_N(\xi,\tau)=\psi(\xi,\tau). \tag{21}
$$

This idea which is related to the convergence and the error of approximation is discussed through the following theorems.

Theorem 3.1. Assume W is a Banach space and $\|w_0(\xi,T)\|<M$ over $[a,\ \text{b}]\times[0,T]$ for some constant M. Then the series $\sum_{k=0}^{\infty} w_k(\xi,\tau)$ in Eq. (15) converges in W if $\exists\beta\in[0,1)$ with $\|w_k(\xi,\tau)\|\leqslant\beta\|w_{k-1}(\xi,\tau)\|, \forall k\in\mathbb{N}$.

Proof: First, define the j^{th} partial sum of Eq. (15), as $\sigma_j=\sum_{k=0}^{j} w_k(\xi,\tau)$.
We proceed by proving that $\{\sigma_j\}_{j=0}^{\infty}$ is Cauchy sequence in the Banach space W. To do so, notice that

$$
\begin{aligned}
\|\sigma_{j+1}-\sigma_j\| &= \left\|\sum_{k=0}^{j+1} w_k(\xi,\tau)-\sum_{k=0}^{j} w_k(\xi,\tau)\right\|\\
&=\|w_{j+1}(\xi,\tau)\|\\
&\leqslant\beta\|w_j(\xi,\tau)\| \text{ from the assumption.}
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|\sigma_{j+1}-\sigma_j\| &\le \beta\|w_j(\xi,\tau)\|\\
&\le \beta^2\|w_{j-1}(\xi,\tau)\|\\
&\ \ \vdots\\
&\le \beta^{j+1}\|w_0(\xi,\tau)\|.
\end{aligned} \tag{22}
$$

Now, for any $j,i\in\mathbb{N}$ with $j\geqslant i$, we have:

$$
\begin{aligned}
\|\sigma_j-\sigma_i\| &= \|(\sigma_j-\sigma_{j-1})+(\sigma_{j-1}-\sigma_{j-2})+\cdots\\
&\quad+(\sigma_{i+1}-\sigma_i)\|\\
&\leqslant\|\sigma_j-\sigma_{j-1}\|+\|\sigma_{j-1}-\sigma_{j-2}\|+\cdots\\
&\quad+\|\sigma_{i+2}-\sigma_{i+1}\|+\|\sigma_{i+1}-\sigma_i\|\\
&\leqslant\beta^j\|w_0(\xi,\tau)\|+\beta^{j-1}\|w_0(\xi,\tau)\|+\cdots\\
&\quad+\beta^{i+2}\|w_0(\xi,\tau)\|+\beta^{i+1}\|w_0(\xi,\tau)\|\\
&=\frac{\beta^{i+1}\left(1-\beta^{j-1-i}\right)}{1-\beta}\|w_0(\xi,\tau)\|\\
&\leqslant\frac{\beta^{i+1}}{1-\beta}\|w_0(\xi,\tau)\| \quad \text{since } \beta\in(0,1)\\
&<\frac{\beta^{i+1}}{1-\beta}\cdot M \quad \text{by assumption.}
\end{aligned}
$$

This implies that $\lim_{i,j\to\infty}\|\sigma_j-\sigma_i\|=0$. That is, $\{\sigma_j\}_{j=0}^{\infty}$ is Cauchy sequence. As a result, the series $\sum_{k=0}^{\infty} w_k(\xi,\tau)$ in equation (15) must converge in the Banach space W, which completes the proof.

Theorem 3.2. If $\mathscr{E}_N(\xi,\tau)$ denotes the error of approximating $\psi(\xi,\tau)$ in Eq. (15) by $\psi_N(\xi,\tau)$ in Eq(20), i.e.,

$$
\mathscr{E}_N(\xi,\tau)=\|\psi(\xi,\tau)-\psi_N(\xi,\tau)\|,
$$

then under the assumptions of Theorem 3.1, the upper bound of $\mathscr{E}_N(\xi,\tau)$ is

$$
\frac{\beta^{N+1}}{1-\beta}\|w_0(\xi,\tau)\|,\quad (\xi,\tau)\in[a,\ \text{b}]\times[0,T]. \tag{23}
$$

Proof: Using Eq (22), we have for any $N,N^*\in\mathbb{N}$ with $N^*\geqslant N$:

$$
\begin{aligned}
\|\sigma_{N^*}-\sigma_N\| &= \left\|\sum_{k=0}^{N^*} w_k(\xi,\tau)-\sum_{k=0}^{N} w_k(\xi,\tau)\right\|\\
&\leqslant\frac{\beta^{N+1}}{1-\beta}\|w_0(\xi,\tau)\|.
\end{aligned} \tag{24}
$$

But σ_N converges to $\psi(\xi,\tau)$ by Theorem 3.1. So, as $N^*\to\infty$, we may rewrite Eq (24) as

$$
\left\|\psi(\xi,\tau)-\sum_{k=0}^{N} w_k(\xi,\tau)\right\|\leqslant\frac{\beta^{N+1}}{1-\beta}\|w_0(\xi,\tau)\|.
$$

Equivalently,

$$
\mathscr{E}_N(\xi,\tau)\leqslant\frac{\beta^{N+1}}{1-\beta}\|w_0(\xi,\tau)\|,
$$

which completes the proof.

Now, we summarize the proposed CNDM to solve a given nonlinear CPDEs in the following algorithm.
Algorithm 3.1. To apply the CNDM to solve nonlinear CPDEs in Eq (10) with IC as in Eq (11), we follow the steps:

Step (i) Separate the right-hand side of the CPDE into linear and nonlinear functions.

Step (ii) Take the CNT to both sides of the equation and use the IC.

Step (iii) Make suitable manipulations and take the ICNT of both sides of the CPDE.

Step (iv) Replace the unknown function $\psi(\xi,\tau)$ in the linear terms with the infinite series $\psi(\xi,\tau)=\sum_{k=0}^{\infty} w_k(\xi,\tau)$.

Step (v) Replace the nonlinear term with the series of Adomian polynomials in Eq. (16) and calculate them using Eq. (17).

Step (vi) Get a suitable recurrence relation and apply it to get the terms of the approximate solution $\psi_N(\xi,\tau)=\sum_{k=0}^{N} w_k(\xi,\tau)$. Choose N to achieve the desired accuracy.

4 Applications of CNDM for Solving Real-Life Models

In this section, we apply our adopted technique to deal with modified forms of some well-known dynamic systems. These models are the gas dynamics equation, Burgers equations, and Schrödinger equation. For each model, we carry out the CNT methodology then present some numerical and graphical results including comparison with other methods. Fortunately, the CNDM leads to the exact solution in most of our examples unlike other methods.

4.1 Homogeneous gas dynamics equation

Assuming no external forces or source terms are present, the conservation of mass, momentum, and energy in a compressible gas is governed by a system of PDEs known as the homogeneous gas dynamics equations (HGDE's). Often called the Euler equations, they offer an idealized mathematical framework for explaining fluid motion [31]. A numerical example of the HGDE's is as follows.
Example 4.1 Consider the nonlinear conformable HGDE

$$ {}_{\tau}^{C}T_0^{\alpha}\psi(\xi,\tau)+\tfrac{1}{2}\frac{d}{d\xi}\psi^2(\xi,\tau)-\psi(\xi,\tau)\big(1-\psi(\xi,\tau)\big)=0, \quad 0<\alpha\le 1 \tag{25} $$

with IC:

$$ \psi(\xi,0)=e^{-\xi}. \tag{26} $$

Its exact solution is found to be

$$ \psi(\xi,\tau)=e^{-\xi+\frac{\tau^{\alpha}}{\alpha}}. \tag{27} $$

In order to obtain analytic-approximate solution for arbitrary $\alpha\in(0,1]$, we apply the CNDM as follows.

We first take the CNT both sides in Eq (25), so we get

$$ {}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left[{}_{\tau}^{C}T_0^{\alpha}\psi(\xi,\tau)\right]+{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left\{\tfrac{1}{2}\tfrac{d}{d\xi}\psi^2(\xi,\tau)\right\} -{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left\{\psi(\xi,\tau)\big(1-\psi(\xi,\tau)\big)\right\} =0 $$

which gives

$$ \frac{s}{u}\Psi_{N_{\alpha}}(\xi,s,u)-\frac{\psi(\xi,0)}{u}+{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left\{\tfrac{1}{2}\tfrac{d}{d\xi}\psi^2(\xi,\tau)\right\} -{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\{\psi(\xi,\tau)\}+{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\{\psi^2(\xi,\tau)\}=0 \tag{28} $$

where $\Psi_{N_{\alpha}}(\xi,s,u)={}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\{\psi(\xi,\tau)\}$.
Using the IC in (26) and rearranging the terms of (28), we obtain

$$ \Psi_{N_{\alpha}}(\xi,s,u)=\frac{e^{-\xi}}{s}-\frac{u}{s}{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left\{\tfrac{1}{2}\tfrac{d}{d\xi}\psi^2(\xi,\tau)+\psi^2(\xi,\tau)\right\} +\frac{u}{s}{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\{\psi(\xi,\tau)\}. $$

Now, we take the ICNT, so we get

$$ \psi(\xi,\tau)=e^{-\xi} -{}_{\tau}^{C}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\{\tfrac{1}{2}\tfrac{d}{d\xi}\psi^2(\xi,\tau)+\psi^2(\xi,\tau)\}\right\} +{}_{\tau}^{C}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\{\psi(\xi,\tau)\}\right\}. \tag{29} $$

The CNDM presents the solution of system Eq (25) in the form of infinite series as

$$ \psi(\xi,\tau)=\sum_{k=0}^{\infty} w_k(\xi,\tau) $$

while it expresses the nonlinear terms $\frac{d}{d\xi}\psi^2(\xi,\tau),\psi^2(\xi,\tau)$ in term of Adomian polynomials as follows:

$$ \frac{d}{d\xi}\psi^2(\xi,\tau)=\sum_{k=0}^{\infty}\mathscr{A}_k(\xi,\tau), $$

$$ \mathscr{A}_k=\frac{1}{k!}\frac{d^k}{d\eta^k}\frac{d}{d\xi}\left(\sum_{\delta=0}^{k}\eta^{\delta}w_{\delta}\right)^2, \quad k=0,1,2,\dots $$

and

$$\psi^2(\xi,\tau)=\sum_{k=0}^{\infty}B_k(\xi,\tau),$$
$$B_k=\frac{1}{k!}\frac{d^k}{d\eta^k}\left(\sum_{\delta=0}^{k}\eta^{\delta}w_{\delta}\right)^2,\quad k=0,1,2,\ldots$$

The first some Adomian polynomials are:

$$\begin{aligned}
\mathscr{A}_0&=w_0^2\\
\mathscr{A}_1&=2w_0w_1\\
\mathscr{A}_2&=2w_0w_2+w_1^2\\
\mathscr{A}_3&=2w_0w_2+2w_1w_2\\
B_0&=w_{0\xi}^2\\
B_1&=2w_{0\xi}w_{1\xi}\\
B_2&=2w_{0\xi}w_{2\xi}+w_{1\xi}^2\\
B_3&=2w_{0\xi}w_{2\xi}+2w_{1\xi}w_{2\xi}
\end{aligned}$$

Substituting into Eq (29) implies:

$$\begin{aligned}
\sum_{k=0}^{\infty}w_k(\xi,\tau)=&\ e^{-\xi}\\
&+{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{-\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\frac{1}{2}\sum_{k=0}^{\infty}\mathscr{A}_k(\xi,\tau)\right\}\right\}\\
&+{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\sum_{k=0}^{\infty}w_k(\xi,\tau)\right\}\right\}\\
&-{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\sum_{k=0}^{\infty}B_k(\xi,\tau)\right\}\right\}
\end{aligned}$$

which produces the recurrence relations:

$$\begin{aligned}
w_0(\xi,\tau)&=e^{-\xi}\\
w_{k+1}(\xi,\tau)&=-{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\{\frac{1}{2}\mathscr{A}_k(\xi,\tau)\}\right\}\\
&\quad+{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\{w_k(\xi,\tau)\}\right\}\\
&\quad-{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\{B_k(\xi,\tau)\}\right\}.
\end{aligned}$$

Hence, for $k=0$,

$$\begin{aligned}
w_1(\xi,\tau)=&-{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\frac{1}{2}\mathscr{A}_0(\xi,\tau)\right\}\right\}\\
&+{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\{w_0(\xi,\tau)\}\right\}\\
&-{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\{B_0(\xi,\tau)\}\right\}\\
=&\ {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{e^{-2\xi}+e^{-\xi}-e^{-2\xi}\right\}\right\}\\
=&\ {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{e^{-\xi}\right\}\right\}=\frac{e^{-\xi}\tau^{\alpha}}{\alpha}.
\end{aligned}$$

Similarly, for $k=1$,

$$\begin{aligned}
w_2(\xi,\tau)=&\ {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{-\frac{1}{2}\mathscr{A}_1(\xi,\tau)\right\}\right\}\\
&+{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\{w_1(\xi,\tau)\}\right\}\\
&-{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\{B_1(\xi,\tau)\}\right\}\\
=&\ {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{2\tau e^{-2\xi}+\tau e^{-\xi}-2\tau e^{-2\xi}\right\}\right\}\\
=&\ {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\tau e^{-\xi}\right\}\right\}=\frac{e^{-\xi}\tau^{2\alpha}}{2\alpha}.
\end{aligned}$$

Repeating this process for $k=2,3,4$ produces:

$$\begin{aligned}
w_3(\xi,\tau)&=\frac{e^{-\xi}\tau^{3\alpha}}{6\alpha^3},\\
w_4(\xi,\tau)&=\frac{e^{-\xi}\tau^{4\alpha}}{24\alpha^4},\\
w_5(\xi,\tau)&=\frac{e^{-\xi}\tau^{5\alpha}}{120\alpha^5}.
\end{aligned}$$

In fact, the fifth approximate solution of Eq (25) can be taken as the fifth truncated series as

$$\psi_5(\xi,\tau)=\sum_{k=0}^{5}w_k(\tau)=e^{-\xi}+\frac{e^{-\xi}\tau^{\alpha}}{\alpha}+\frac{e^{-\xi}\tau^{2\alpha}}{2\alpha^2}+\frac{e^{-\xi}\tau^{3\alpha}}{6\alpha^3}+\frac{e^{-\xi}\tau^{4\alpha}}{24\alpha^4}+\frac{e^{-\xi}\tau^{5\alpha}}{120\alpha^5}.$$

To achieve high accuracy, we repeat the process using the Mathematica software for more iterations. We carry out the CNDM for 20^{th} iterations and get

$$\psi_{20}(\xi,\tau) = e^{-\xi}\left(1+\frac{\tau^{\alpha}}{\alpha}+\frac{\tau^{2\alpha}}{2\alpha^2}+\frac{\tau^{3\alpha}}{6\alpha^3}+\frac{\tau^{4\alpha}}{24\alpha^4}+\frac{\tau^{5\alpha}}{120\alpha^5}+\frac{\tau^{6\alpha}}{720\alpha^6}+\frac{\tau^{7\alpha}}{5040\alpha^7}+\frac{\tau^{8\alpha}}{40320\alpha^8}+\frac{\tau^{9\alpha}}{362880\alpha^9}+\frac{\tau^{10\alpha}}{3628800\alpha^{10}}+\frac{\tau^{11\alpha}}{39916800\alpha^{11}}+\frac{\tau^{12\alpha}}{479001600\alpha^{12}}+\frac{\tau^{13\alpha}}{6227020800\alpha^{13}}+\frac{\tau^{14\alpha}}{87178291200\alpha^{14}}+\frac{\tau^{15\alpha}}{1307674368000\alpha^{15}}+\frac{\tau^{16\alpha}}{20922789888000\alpha^{16}}+\frac{\tau^{17\alpha}}{355687428096000\alpha^{17}}+\frac{\tau^{18\alpha}}{6402373705728000\alpha^{18}}+\frac{\tau^{19\alpha}}{121645100408832000\alpha^{19}}+\frac{\tau^{20\alpha}}{2432902008176640000\alpha^{20}}\right).$$

In order to notice the efficiency of the proposed technique in solving nonlinear HGDE's, we use Mathematica 13 to get some tabulated and graphical results for $\alpha \in \{1, 0.95, 0.85, 0.75, 0.65\}$ over the domain $\tau \in [0,5]$. Moreover, the accuracy of CNTDM is revealed through Table 1 which shows small absolute errors for different values of the conformable orders. The effect of conformable derivative to the solution of nonlinear HGDE is obvious in Figure 1.

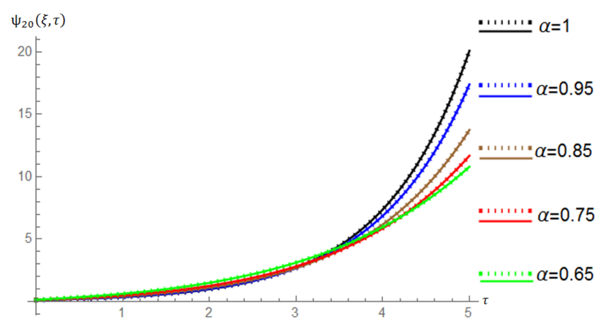

Fig. 1: Comparison between the exact solution (Dotted) and the 20$^{\text{th}}$ approximate solutions (solid) for the HGDE in Example 4.1.

Table 1: Absolute errors $|\psi_{20}(1,\tau)-\psi(1,\tau)|$ of approximation for Example 4.1

τ	$\alpha=1$	$\alpha=0.95$	$\alpha=0.85$	$\alpha=0.75$	$\alpha=0.65$
0.0	0.	0.	0.	0.	0.
0.5	0.	0.	1.1×10^{-16}	2.2×10^{-16}	0.
1.0	0.	0.	0.	0.	0.
1.5	2.2×10^{-16}	4.4×10^{-16}	2.2×10^{-16}	1.8×10^{-15}	1.6×10^{-14}
2.0	1.7×10^{-14}	2.4×10^{-14}	5.7×10^{-14}	1.9×10^{-13}	8.8×10^{-13}
2.5	1.8×10^{-12}	2.1×10^{-12}	3.1×10^{-12}	6.4×10^{-12}	1.9×10^{-11}
3.0	8.7×10^{-11}	8.1×10^{-11}	8.3×10^{-11}	1.1×10^{-10}	2.3×10^{-10}
3.5	2.3×10^{-9}	1.8×10^{-9}	1.3×10^{-9}	1.3×10^{-9}	1.9×10^{-9}
4.0	3.9×10^{-8}	2.6×10^{-8}	1.5×10^{-8}	1.1×10^{-8}	1.2×10^{-8}
4.5	4.7×10^{-7}	2.8×10^{-7}	1.2×10^{-7}	7.2×10^{-8}	6.2×10^{-8}
5.0	4.4×10^{-6}	2.4×10^{-7}	8.3×10^{-7}	3.9×10^{-7}	2.6×10^{-7}

On the other hand, it can be easily noticed that the 20$^{\text{th}}$ approximate solution can be rewritten as

$$\psi_{20}(\xi,\tau) = e^{-\xi}\left(1+\frac{\tau^{\alpha}}{\alpha}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^2}{2!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^3}{3!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^4}{4!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^5}{5!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^6}{6!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^7}{7!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^8}{8!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^9}{9!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^{10}}{10!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^{11}}{11!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^{12}}{12!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^{13}}{13!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^{14}}{14!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^{15}}{15!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^{16}}{16!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^{17}}{17!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^{18}}{18!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^{19}}{19!}+\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^{20}}{20!}\right) = e^{-\xi}\sum_{k=0}^{20}\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^k}{k!}.$$

So, we have

$$\psi(\xi,\tau) = \lim_{n\to\infty}\psi_n(\xi,\tau) = e^{-\xi}\sum_{k=0}^{\infty}\frac{\left(\frac{\tau^{\alpha}}{\alpha}\right)^k}{k!} = e^{-\xi+\frac{\tau^{\alpha}}{\alpha}},$$

which is identical to the exact solution in Eq (27).

4.2 *KdV-Burgers equation*

Another real-life application of CPDEs is the Korteweg-de Vries (KdV) Burgers equation which derived by Su and Gardner [31]. It is applicable to fluid dynamics, plasma physics, and other domains because it describes situations in which nonlinearity, dispersion, and dissipation coexist. In this subsection, we consider the conformable KdV equation (CKdVE) as given in [32].

Example 4.2. Consider the following CKdVE:

$$ {}^{C}_{\tau}T^{\alpha}_{0}\psi(\xi,\tau)+\varepsilon\psi(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau)-r\frac{d^2}{d\xi^2}\psi(\xi,\tau)+\mu\frac{d^3}{d\xi^3}\psi(\xi,\tau)=0, \quad 0<\alpha\le 1 \qquad (30)$$

with IC:

$$\psi(\xi,0) = -2\,\mathrm{sech}^2(\xi). \tag{31}$$

In order to solve the CKdVE in (30) and (31), we start by taking the CNT to both sides in (30) as follows:

$$\begin{aligned}
&{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{{}^{C}_{\tau}T^{\alpha}_{0}\psi(\xi,\tau)\right\} + {}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{\varepsilon\psi(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau)\right\}\\
&\quad - {}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{r\frac{d^2}{d\xi^2}\psi(\xi,\tau)\right\}\\
&\quad + {}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{\mu\frac{d^3}{d\xi^3}\psi(\xi,\tau)\right\} = 0.
\end{aligned}$$

Which gives

$$\begin{aligned}
&\frac{s\Psi_\alpha(\xi,s,u)}{u} - \frac{\psi(\xi,0)}{u} + {}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{\varepsilon\,\psi(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau)\right\}\\
&\quad - {}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{r\frac{d^2}{d\xi^2}\psi(\xi,\tau)\right\}\\
&\quad + {}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{\mu\frac{d^3}{d\xi^3}\psi(\xi,\tau)\right\} = 0
\end{aligned} \tag{32}$$

where $\Psi_\alpha(\xi,s,u) = {}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\{\psi(\xi,\tau)\}$.
Using the initial conditions in Eq (31) and rearranging the terms of Eq (32), we obtain

$$\begin{aligned}
\Psi(\xi,s,u) &= \frac{-2\,\mathrm{sech}^2(\xi)}{s}\\
&\quad + \frac{u}{s}\,{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{-\varepsilon\,\psi(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau)\right\}\\
&\quad + \frac{u}{s}\,{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{r\frac{d^2}{d\xi^2}\psi(\xi,\tau)\right\}\\
&\quad + \frac{u}{s}\,{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{\mu\frac{d^3}{d\xi^3}\psi(\xi,\tau)\right\}.
\end{aligned}$$

Now, we take the ICNT, so we have

$$\begin{aligned}
\psi(\xi,\tau) &= -2\,\mathrm{sech}^2(\xi)\\
&\quad + {}^{C}_{\tau}\mathcal{N}^{-1}_{\alpha}\left\{\frac{u}{s}\,\mathcal{N}^{+}_{\alpha}\left\{-\varepsilon\,\psi(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau)\right\}\right\}\\
&\quad + {}^{C}_{\tau}\mathcal{N}^{-1}_{\alpha}\left\{\frac{u}{s}\,{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{r\frac{d^2}{d\xi^2}\psi(\xi,\tau)\right\}\right\}\\
&\quad + {}^{C}_{\tau}\mathcal{N}^{-1}_{\alpha}\left\{\frac{u}{s}\,{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{\mu\frac{d^3}{d\xi^3}\psi(\xi,\tau)\right\}\right\}.
\end{aligned} \tag{33}$$

The CNDM presents the solution of Eq (30) and Eq (31) in the form of infinite series as

$$\psi(\xi,\tau) = \sum_{k=0}^{\infty} w_k(\xi,\tau) \tag{34}$$

while it expresses the nonlinear term $\psi(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau)$ in term of Adomian polynomials as follows:

$$\psi(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau) = \sum_{k=0}^{\infty}\mathscr{A}_k(\xi,\tau),$$

$$\mathscr{A}_k = \frac{1}{k!}\frac{d^k}{d\eta^k}\left[\left(\sum_{l=0}^{k}\eta^l w_l\right)\frac{d}{d\xi}\left(\sum_{l=0}^{k}\eta^l w_l\right)\right]_{\eta=0}, \quad k = 0,1,2,\ldots.$$

Hence, the first few Adomian polynomials can be obtain from the formulas:

$$\begin{aligned}
\mathscr{A}_0 &= w_0 w_{0\xi},\\
\mathscr{A}_1 &= w_{0\xi}w_1 + w_0 w_{1\xi},\\
\mathscr{A}_2 &= w_{0\xi}w_2 + w_1 w_{1\xi} + w_{2\xi}w_0,\\
\mathscr{A}_3 &= w_{0\xi}w_3 + w_2 w_{1\xi} + w_{2\xi}w_1 + w_{3\xi}w_0,\\
&\vdots
\end{aligned}$$

Substituting into Eq (33) implies:

$$\begin{aligned}
\sum_{k=0}^{\infty} w_k(\xi,\tau) &= -2\,\mathrm{sech}^2(\xi)\\
&\quad + {}^{C}_{\tau}\mathcal{N}^{-1}_{\alpha}\left\{\frac{u}{s}\,{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{-\varepsilon\sum_{k=0}^{\infty}\mathscr{A}_k(\xi,\tau)\right\}\right\}\\
&\quad + {}^{C}_{\tau}\mathcal{N}^{-1}_{\alpha}\left\{\frac{u}{s}\,{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{\sum_{k=0}^{\infty}\left\{r\frac{d^2}{d\xi^2}w_k(\xi,\tau)\right.\right.\right.\\
&\qquad \left.\left.\left. -\mu\frac{d^3}{d\xi^3}w_k(\xi,\tau)\right\}\right\}\right\}
\end{aligned}$$

which produces the recurrence relations:

$$w_0(\xi,\tau) = -2\,\mathrm{sech}^2(\xi) = g_0(\xi)$$

$$\begin{aligned}
w_{k+1}(\xi,\tau) &= {}^{C}_{\tau}\mathcal{N}^{-1}_{\alpha}\left\{\frac{u}{s}\,{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{-\varepsilon\mathscr{A}_k(\xi,\tau)\right\}\right\}\\
&\quad + {}^{C}_{\tau}\mathcal{N}^{-1}_{\alpha}\left\{\frac{u}{s}\,{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{r\frac{d^2}{d\xi^2}w_k(\xi,\tau)\right.\right.\\
&\qquad \left.\left. -\mu\frac{d^3}{d\xi^3}w_k(\xi,\tau)\right\}\right\}.
\end{aligned}$$

Hence, for $k=0$, $\mathscr{A}_0(\xi,\tau) = -8\,\mathrm{sech}^4\,\xi\tanh\xi = b_0(\xi)$,

$$w_1(\xi,\tau) = {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{-\varepsilon b_0(\xi)\right\}\right\}$$

$$+{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{r\frac{d^2}{d\xi^2}g_0(\xi)-\mu\frac{d^3}{d\xi^3}g_0(\xi)\right\}\right\}$$

$$= {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{-\varepsilon b_0(\xi)+rg_0''(\xi)-\mu g_0'''(\xi)\right\}\right\}$$

$$= {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s^2}\left\{-\varepsilon b_0(\xi)+rg_0''(\xi)-\mu g_0'''(\xi)\right\}\right\}$$

$$= \frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\left\{-\varepsilon b_0(\xi)+rg_0''(\xi)-\mu g_0'''(\xi)\right\}$$

$$= \frac{\tau^{\alpha}}{\Gamma(\alpha+1)}g_1(\xi),$$

where

$$g_1(\xi) = -2\,\mathrm{sech}^5(\xi)\Big(-3r\cosh(\xi)+r\cosh(3\xi) - 4\sinh(\xi)\left(-\mu\cosh(2\xi)+5\mu+\varepsilon\right)\Big).$$

To compute $w_2(\xi,\tau)$, let $k=1$ and proceed as follows:

$$\mathscr{A}_1(\xi,\tau) = \frac{-t^{\alpha}}{\alpha}b_1(\xi),$$

where

$$b_1(\xi) = 8\,\mathrm{sech}^8(\xi)\Big(44\mu+2\mu\cosh(4\xi)-7v\sinh(2\xi) + v\sinh(4\xi)-2\cosh(2\xi)(19\mu+3\varepsilon)+8\varepsilon\Big)$$

$$w_2(\xi,\tau) = {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\frac{\varepsilon\tau^{\alpha}}{\Gamma(\alpha+1)}b_1(\xi)\right\}\right\}$$

$$+{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\left\{r\frac{d^2g_1}{d\xi^2}(\xi) -\mu\frac{d^3g_1}{d\xi^3}(\xi)\right\}\right\}\right\}$$

$$= {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u^2}{s^3}\left\{\varepsilon b_1(\xi)+r\frac{d^2g_1}{d\xi^2}-\mu\frac{d^3g_1}{d\xi^3}\right\}\right\}$$

$$= \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\left(\varepsilon b_1(\xi)+r\frac{d^2g_1}{d\xi^2}-\mu\frac{d^3g_1}{d\xi^3}\right).$$

So, $w_2(\xi,\tau) = \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}g_2(\xi)$, where $g_2(\xi) = \varepsilon b_1(\xi)+r\frac{d^2g_1}{d\xi^2}-\mu\frac{d^3g_1}{d\xi^3}$.

Using Mathematica software, we compute $w_k(\xi,\tau), k=1,2,\ldots,10$ in order to get the tenth approximate solution Eq (30) as the truncated series $\psi_{10}(\xi,\tau) = \sum_{k=0}^{10}w_k(\xi,\tau)$. However, for $r=0, \varepsilon=-6, \mu=1$, the exact solution is $\psi(\xi,\tau) = -2\,\mathrm{sech}^2\left(\xi-\frac{4\tau^{\alpha}}{\alpha}\right)$, and the 10$^{\text{th}}$ approximate solution is

$$\psi_{10}(\xi,\tau) = -\frac{1}{120\alpha^5}\,\mathrm{sech}^7(\xi)\Big(15\alpha^5\cosh(5\xi)+240\alpha^4\tau^{\alpha}\sinh(\xi) +360\alpha^4\tau^{\alpha}\sinh(3\xi)+120\alpha^4\tau^{\alpha}\sinh(5\xi) +480\alpha^3\tau^{2\alpha}\cosh(5\xi)-12800\alpha^2\tau^{3\alpha}\sinh(\xi) -11520\alpha^2\tau^{3\alpha}\sinh(3\xi)+1280\alpha^2\tau^{3\alpha}\sinh(5\xi) +10\alpha\left(15\alpha^4-384\alpha^2\tau^{2\alpha}+10240\tau^{4\alpha}\right)\cosh(\xi) +5\alpha\left(15\alpha^4-96\alpha^2\tau^{2\alpha}-12800\tau^{4\alpha}\right)\cosh(3\xi) +1236992\tau^{5\alpha}\sinh(\xi)-233472\tau^{5\alpha}\sinh(3\xi) +4096\tau^{5\alpha}\sinh(5\xi)+2560\alpha\tau^{4\alpha}\cosh(5\xi)\Big).$$

Therefore we can check the accuracy of our methodology by computing the absolute errors for our approximations for different conformable orders. In Table 2, some values of the absolute errors $|\psi(\xi,\tau)-\psi_{10}(\xi,\tau)|$ for $\alpha\in\{0.95,0.85,0.75,0.65\}$ are presented. In Table 3, we compare the absolute errors resulting from our technique with those resulting by the variational iteration method (VIM) as given in [31] for classical integer order $\alpha=1$. The 3D plots for the exact and approximate solutions are presented in Figure 2 for $\alpha=0.95$ and $\alpha=0.75$.

Table 2: Absolute errors $|\psi(\xi,\tau)-\psi_{10}(\xi,\tau)|$ of approximation for the CKdVE in Example 4.2

τ	ξ	$\alpha=0.95$	$\alpha=0.85$	$\alpha=0.75$	$\alpha=0.65$
0.01	-7.5	4.8×10^{-15}	1.5×10^{-13}	4.8×10^{-12}	1.7×10^{-10}
	-2.5	2.5×10^{-11}	8.0×10^{-10}	2.8×10^{-8}	1.1×10^{-6}
	2.5	2.4×10^{-11}	7.2×10^{-10}	2.3×10^{-8}	7.4×10^{-7}
	7.5	4.9×10^{-15}	1.5×10^{-13}	5.3×10^{-12}	2.1×10^{-10}
0.02	-7.5	2.4×10^{-13}	4.9×10^{-12}	1.0×10^{-10}	2.5×10^{-9}
	-2.5	1.3×10^{-9}	2.8×10^{-8}	6.5×10^{-7}	1.7×10^{-5}
	2.5	1.2×10^{-9}	2.3×10^{-8}	4.6×10^{-7}	8.3×10^{-6}
	7.5	2.6×10^{-13}	5.4×10^{-12}	1.2×10^{-10}	3.2×10^{-9}
0.03	-7.5	2.4×10^{-12}	3.8×10^{-11}	6.3×10^{-10}	1.1×10^{-8}
	-2.5	1.4×10^{-8}	2.3×10^{-7}	4.1×10^{-6}	8.3×10^{-5}
	2.5	1.2×10^{-8}	1.7×10^{-7}	2.5×10^{-6}	2.6×10^{-5}
	7.5	2.6×10^{-12}	4.3×10^{-11}	$7.\times10^{-10}$	1.6×10^{-8}
0.04	-7.5	1.2×10^{-11}	1.6×10^{-10}	2.3×10^{-9}	3.4×10^{-8}
	-2.5	7.3×10^{-8}	1.0×10^{-6}	1.5×10^{-5}	2.6×10^{-5}
	2.5	5.8×10^{-8}	7.0×10^{-7}	7.8×10^{-6}	3.3×10^{-4}
	7.5	1.4×10^{-11}	1.9×10^{-10}	3.0×10^{-9}	5.3×10^{-8}
0.05	-7.5	4.3×10^{-11}	5.0×10^{-10}	6.1×10^{-9}	7.9×10^{-8}
	-2.5	2.6×10^{-7}	3.2×10^{-6}	4.2×10^{-5}	6.2×10^{-5}
	2.5	2.0×10^{-7}	2.0×10^{-6}	1.7×10^{-5}	4.5×10^{-6}
	7.5	5.0×10^{-11}	6.1×10^{-10}	8.3×10^{-9}	1.3×10^{-14}

Table 3: Comparison between the absolute errors using CNDM and VIM when $\alpha = 1$ for the CKdVE in Example 4.2

τ	ξ	CNDM	VIM[32]
0.01	-7.5	8.81×10^{-16}	6.70×10^{-14}
	-2.5	4.65×10^{-12}	9.51×10^{-9}
	2.5	4.45×10^{-12}	9.79×10^{-9}
	7.5	9.01×10^{-16}	6.70×10^{-14}
0.02	-7.5	5.57×10^{-14}	2.08×10^{-12}
	-2.5	3.03×10^{-10}	3.00×10^{-7}
	2.5	2.78×10^{-10}	3.20×10^{-7}
	7.5	5.84×10^{-14}	2.20×10^{-12}
0.03	-7.5	6.28×10^{-13}	1.56×10^{-11}
	-2.5	3.52×10^{-9}	2.25×10^{-6}
	2.5	3.07×10^{-9}	2.47×10^{-6}
	7.5	6.72×10^{-13}	1.69×10^{-11}
0.04	-7.5	3.49×10^{-12}	6.50×10^{-11}
	-2.5	2.01×10^{-8}	9.33×10^{-6}
	2.5	1.67×10^{-8}	10.59×10^{-6}
	7.5	3.82×10^{-12}	7.22×10^{-11}
0.05	-7.5	1.32×10^{-11}	1.96×10^{-10}
	-2.5	7.77×10^{-8}	2.81×10^{-5}
	2.5	6.17×10^{-8}	3.29×10^{-5}
	7.5	1.48×10^{-11}	2.23×10^{-10}

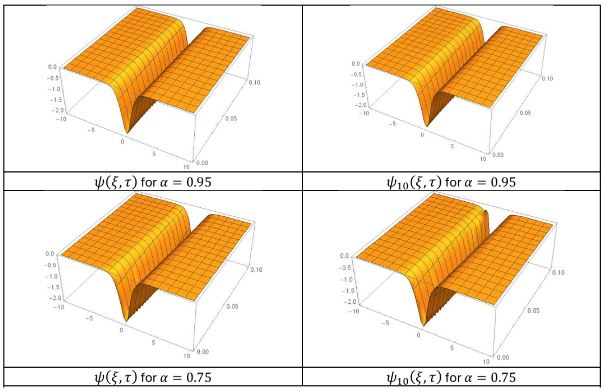

Fig. 2: The 3D plots for $\psi(\xi,\tau)$ and $\psi_{10}(\xi,\tau)$ for the CKdVE in Eq (30) for $\alpha = 0.95$ and $\alpha = 0.75$.

4.3 Burgers equation

Burgers' equation appears in fluid dynamics, traffic flow, and other applied mathematics fields. It is used to investigate shock waves, turbulence, and wave propagation and is frequently thought of as a simplified model for the Navier-Stokes equations. A numerical example of the modified conformable Burgers' equation (MCBE) is in the following example.

Example 4.3. [33] Consider the MCBE:

$$ {}_{\tau}^{C}T_{0}^{\alpha}\psi(\xi,\tau)+\psi^{2}(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau)-\mu\frac{d^{2}}{d\xi^{2}}\psi(\xi,\tau)=0, \quad 0<\alpha\le 1, \tag{35}$$

with IC:

$$\psi(\xi,0)=g_{0}(\xi). \tag{36}$$

In order to solve the MCBE in Eq (35) and Eq (36) by the CNDM, we start by taking the CNT to both sides of Eq (35) as

$$ {}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left\{{}_{\tau}^{C}T_{0}^{\alpha}\psi(\xi,\tau)\right\}+{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left\{\psi^{2}(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau)\right\} -{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left\{\mu\frac{d^{2}}{d\xi^{2}}\psi(\xi,\tau)\right\}=0.$$

Which gives

$$\frac{s\,\psi(\xi,s,u)}{u}-\frac{\psi(\xi,0)}{u}+{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left\{\psi^{2}(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau)\right\} -{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left\{\mu\frac{d^{2}}{d\xi^{2}}\psi(\xi,\tau)\right\}=0.$$

Using the IC in Eq (36) and rearranging the terms, we obtain

$$ {}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\{\psi(\xi,\tau)\}=\frac{\psi(\xi,0)}{s} -\frac{u}{s}{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left\{\psi^{2}(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau)\right\} +\frac{u}{s}{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left\{\mu\frac{d^{2}}{d\xi^{2}}\psi(\xi,\tau)\right\}=0.$$

Now, we take the inverse CNT, so we have

$$\psi(\xi,\tau)=\psi(\xi,0) -{}_{\tau}^{C}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left\{\psi^{2}(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau)\right\}\right\} +{}_{\tau}^{C}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}_{\tau}^{C}\mathscr{N}_{\alpha}^{+}\left\{\mu\frac{d^{2}}{d\xi^{2}}\psi(\xi,\tau)\right\}\right\}. \tag{37}$$

The CNDM solution has the infinite series form as

$$\psi(\xi,\tau)=\sum_{k=0}^{\infty}w_{k}(\xi,\tau), \tag{38}$$

and the nonlinear term $\psi^{2}(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau)$ is rewritten in term of Adomian polynomials as follows:

$$\psi^{2}(\xi,\tau)\frac{d}{d\xi}\psi(\xi,\tau)=\sum_{k=0}^{\infty}\mathscr{A}_{k}(\xi,\tau) \tag{39}$$

where the k-th Adomian polynomial for $k=0,1,2,\ldots$ is derived from the formula:

$$\mathscr{A}_k=\frac{1}{k!}\frac{d^k}{d\eta^k}\left[\left(\sum_{l=0}^{k}\eta^l w_l\right)^2\frac{d}{d\xi}\left(\sum_{l=0}^{k}\eta^l w_l\right)\right]_{\eta=0}. \quad (40)$$

So the first four Adomian polynomials for the MCBE in Eq (40) are:

$$\mathscr{A}_0=w_0^2w_{0\xi}$$

$$\mathscr{A}_1=w_0^2w_{1\xi}+2w_1w_0w_{0\xi}$$

$$\mathscr{A}_2=w_1^2w_{0\xi}+2w_0w_1w_{1\xi}+w_0\Big(2w_2w_{0\xi}+w_0(\xi)w_{2\xi}\Big)$$

$$\mathscr{A}_3=w_1^2w_{1\xi}+2w_1\Big(w_2w_{0\xi}+w_0w_{2\xi}\Big)+w_0\Big(2w_3w_{0\xi}+2w_2w_{1\xi}+w_0w_{3\xi}\Big)$$

Substituting Eq (40) and Eq (38) into Eq (37) implies:

$$\sum_{k=0}^{\infty}w_k(\xi,\tau)=\psi(\xi,0)-{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\sum_{k=0}^{\infty}\mathscr{A}_k(\xi,\tau)\right\}\right\}+{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\sum_{k=0}^{\infty}\mu\frac{d^2}{d\xi^2}w_k(\xi,\tau)\right\}\right\}$$

which produces the recurrence relations:

$$w_0(\xi,\tau)=g_0(\xi),$$
$$w_{k+1}(\xi,\tau)=-{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\mathscr{A}_k(\xi,\tau)\right\}\right\}+{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\mu\frac{d^2}{d\xi^2}w_k(\xi,\tau)\right\}\right\},\quad k=0,1,2,\ldots$$

For $k=0$,

$$\mathscr{A}_0(\xi,\tau)=g_0^2(\xi)g_0'(\xi)=b_0(\xi),$$

$$w_1(\xi,\tau)=-{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{b_0(\xi)\right\}\right\}+{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\mu\quad g_0''(\xi)\right\}\right\}$$
$$={}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s^2}\left\{-b_0(\xi)+\mu\quad g_0''(\xi)\right\}\right\}$$
$$=\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\left(-b_0(\xi)+\mu\quad g_0''\psi(\xi)\right).$$

That is, $w_1(\xi,\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}g_1(\xi)$, where $g_1(\xi)=-b_0(\xi)+\mu g_0''\psi(\xi)$.
Similarly, for $k=1$, $\mathscr{A}_1(\xi,\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}b_1(\xi)$, where $b_1(\xi)=b_0(\xi)\big(2g_1(\xi)b_0'(\xi)+b_0(\xi)g_1'(\xi)\big)$.

$$w_1(\xi,\tau)=-{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}b_1(\xi)\right\}\right\}+{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\mu\frac{d^2}{d\xi^2}\left(\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}g_1(\xi)\right)\right\}\right\}$$
$$={}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u^2}{s^3}\Big\{-b_1(\xi)+\mu g_1''(\xi)\Big\}\right\}$$
$$=\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\Big(-b_1(\xi)+\mu g_1''(\xi)\Big)$$
$$=\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}g_2(\xi),\quad\text{where } g_2(\xi)=-b_1(\xi)+\mu g_1''(\xi).$$

$$\mathscr{A}_2(\xi,\tau)=\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}b_2(\xi),\quad\text{where}$$
$$b_2(\xi)=\frac{1}{\Gamma(\alpha+1)^2}\Big(\Gamma(\alpha+1)^2b_0(\xi)(2g_2(\xi)b_0'(\xi)+b_0(\xi)g_2'(\xi))-\Gamma(2\alpha+1)g_1^2(\xi)b_0'(\xi)-2\Gamma(2\alpha+1)b_0(\xi)g_1(\xi)g_1'(\xi)\Big).$$

$$w_2(\xi,\tau)=-{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}b_2(\xi)\right\}\right\}+{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\mu\frac{d^2}{d\xi^2}\left(\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}g_2(\xi)\right)\right\}\right\}$$
$$={}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u^3}{s^4}\Big\{-b_2(\xi)+\mu g_2''(\xi)\Big\}\right\}$$
$$=\frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}\Big(-b_2(\xi)+\mu g_2''(\xi)\Big).$$

Using Mathematica software, with $g_0(\xi)=-\frac{\sqrt{3c}}{\sqrt{1-\cosh\left(\frac{2c(\xi+3r\mu)}{\mu}\right)-\sinh\left(\frac{2c(x+3r\mu)}{\mu}\right)}}$ where $c=0.1, r=0.01, \mu=1$, we compute the fourth approximate solution of (35) and compare it with the exact solution $\psi(\xi,\tau)=\frac{-\sqrt{3c}}{\sqrt{-\sinh\left(\frac{2c}{\mu}\left(-\frac{c\tau^{\alpha}}{\alpha}+\xi+3\mu r\right)\right)-\cosh\left(\frac{2c}{\mu}\left(-\frac{c\tau^{\alpha}}{\alpha}+\xi+3\mu r\right)\right)+1}}$ as presented in Table 4 in which the absolute errors for our approximations for different conformable orders are listed. The same MCBE was solved in [37] by an analytic approximate technique which is known as the conformable residual power series method (CRPSM), so we also tabulated the absolute errors that resulted in [37] in Table 4. The 3D plots for the exact and approximate solutions are presented in Figure 3 for $\alpha=0.5$ and $\alpha=0.75$. Finally, at $\xi=-1$, Figure 4 presents how the solution curves of the MCBE are affected by different conformable orders.

Table 4: Absolute errors $|\psi(-1,\tau)-\psi_4(-1,\tau)|$ of approximation for CMBE in Example 4.3

τ	$\alpha=0.95$		$\alpha=0.75$		$\alpha=0.5$	
	CNDM	CRPSM [37]	CNDM	CRPSM [37]	CNDM	CRPSM [37]
0.	0.	0.	0.	0.	0.	0.
0.1	8.1×10^{-11}	6.9×10^{-11}	2.6×10^{-9}	2.2×10^{-9}	3.4×10^{-7}	2.9×10^{-7}
0.2	2.1×10^{-9}	1.8×10^{-9}	3.4×10^{-8}	3.0×10^{-8}	1.9×10^{-6}	1.6×10^{-6}
0.3	1.5×10^{-8}	1.3×10^{-8}	1.6×10^{-7}	1.3×10^{-7}	5.1×10^{-6}	4.3×10^{-6}
0.4	5.7×10^{-8}	4.9×10^{-8}	4.5×10^{-7}	3.9×10^{-7}	1.0×10^{-5}	8.8×10^{-5}
0.5	1.6×10^{-7}	1.4×10^{-7}	1.0×10^{-6}	8.9×10^{-7}	1.8×10^{-5}	1.5×10^{-5}
0.6	3.8×10^{-7}	3.3×10^{-7}	2.0×10^{-6}	1.7×10^{-6}	2.8×10^{-5}	2.4×10^{-5}
0.7	7.9×10^{-7}	6.8×10^{-7}	3.6×10^{-6}	3.1×10^{-6}	4.0×10^{-5}	3.5×10^{-5}
0.8	1.5×10^{-6}	1.3×10^{-6}	5.8×10^{-6}	5.0×10^{-6}	5.6×10^{-5}	4.8×10^{-5}
0.9	2.5×10^{-6}	2.2×10^{-6}	9.0×10^{-6}	7.8×10^{-6}	7.4×10^{-5}	6.4×10^{-5}
1.	4.2×10^{-6}	3.6×10^{-6}	1.3×10^{-5}	1.1×10^{-5}	9.5×10^{-5}	8.2×10^{-5}

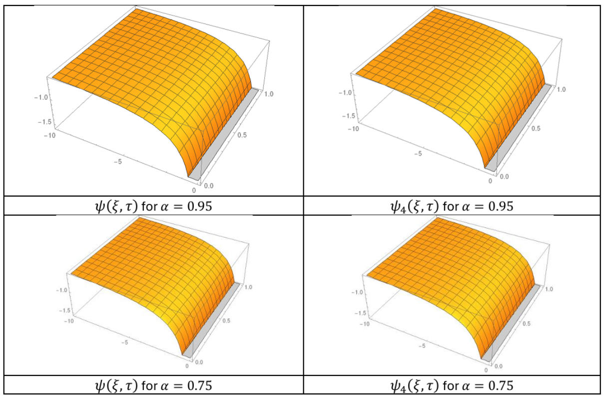

Fig. 3: The 3D plots for $\psi(\xi,\tau)$ and $\psi_4(\xi,\tau)$ for the KdV-Burgers equation in. Example (4.3) for $\alpha=0.95$ and $\alpha=0.75$.

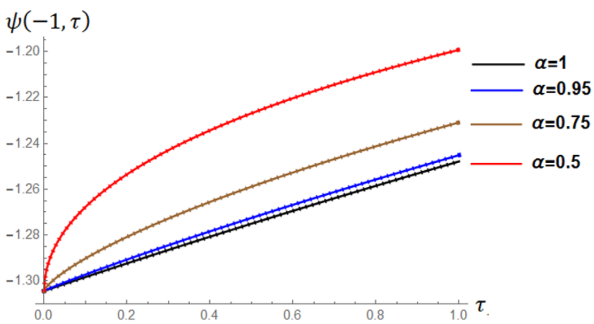

Fig. 4: Solution curves of the MCBE in. Example 4.3 for different conformable orders (Dashed: $\psi(-1,\tau)$; Solid: $\psi_4(-1,\tau)$).

4.4 Schrödinger equation

The conformable Schrödinger equation with a nonzero trapping potential is a fractional extension of classical quantum models by replacing integer order derivative by conformable derivative. This mathematical formulation enables accurate simulation of quantum systems with memory-dependent and non-local dynamics, particularly in the presence of dissipative forces or spatially constrained potentials. More generally, fractional Schrödinger equations are crucial to many engineering specialties, such as electronics, mechanics, materials science, and architecture, by increasing simulation accuracy and aiding in the development of innovative technologies such as nanodevices, smart materials, and responsive structural systems. An example of the conformable Schrödinger differential equation (CSDE) that is popular in quantum mechanics [37] is qiven below.

Example 4.4. Consider the following CSDE with nonzero trapping potential:

$$i\,{}^{C}_{\tau}T^{\alpha}_{0}Z(\xi,\tau)+\frac{d^2}{d\xi^2}Z(\xi,\tau)+2|Z(\xi,\tau)|^2Z(\xi,\tau)=0, \qquad (41)$$
$$\tau\geq 0,\ 0<\alpha\leq 1,$$

subject to the IC:

$$z(\xi,0)=e^{i\xi}. \qquad (42)$$

The exact solution for the CPDE in Eq (41) and its IC in Eq(42) is found to be

$$Z(\xi,\tau)=e^{i(\xi+\frac{\tau^{\alpha}}{\alpha})} \qquad (43)$$

To apply our methodology to the CSDE in Eq (41), we start by converting Eq (41) into a system of real-valued functions since $Z(\xi,\tau)$ can be decomposed as $Z(\xi,\tau)=\psi(\xi,\tau)+i\mu(\xi,\tau)$. As a result the CSDE in Eq (41) can be rewritten in the form of the following system

$$\begin{aligned}&{}^{C}_{\tau}T^{\alpha}_{0}\psi(\xi,\tau)+\frac{\partial^2}{\partial\xi^2}\mu(\xi,\tau)+2\psi^2(\xi,\tau)\mu(\xi,\tau)+2\mu^3(\xi,\tau)=0,\\&{}^{C}_{\tau}T^{\alpha}_{0}\mu(\xi,\tau)-\frac{\partial^2}{\partial\xi^2}\psi(\xi,\tau)-2\mu^2(\xi,\tau)\psi(\xi,\tau)-2\psi^3(\xi,\tau)=0,\\&\qquad 0<\alpha<1,\end{aligned} \qquad (44)$$

subject to initial conditions:

$$\begin{aligned}\psi(\xi,0)&=\cos(\xi),\\ \mu(\xi,0)&=\sin(\xi).\end{aligned} \qquad (45)$$

Applying the CNT for both sides of each in system (44) and using the IC's in system (45), we get

$$\begin{aligned}&{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{{}^{C}_{\tau}T^{\alpha}_{0}\psi(\xi,\tau)\right\}+{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{\frac{d^2}{d\xi^2}\mu(\xi,\tau)\right\}\\&+{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{2\psi^2(\xi,\tau)\mu(\xi,\tau)+2\mu^3(\xi,\tau)\right\}=0,\\&{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{{}^{C}_{\tau}T^{\alpha}_{0}\mu(\xi,\tau)\right\}-{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{\frac{d^2}{d\xi^2}\psi(\xi,\tau)\right\}\\&-{}^{C}_{\tau}\mathcal{N}^{+}_{\alpha}\left\{2\mu^2(\xi,\tau)\psi(\xi,\tau)+2\psi^3(\xi,\tau)\right\}=0\end{aligned} \qquad (46)$$

Which gives

$$\Psi_{N_\alpha}(\xi,s,u)+{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\frac{d^2}{d\xi^2}\mu(\xi,\tau)\right\} +{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{2\psi^2(\xi,\tau)\mu(\xi,\tau)+2\mu^3(\xi,\tau)\right\}=0,$$
$$\mu_{N_\alpha}(\xi,s,u)-{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\frac{d^2}{d\xi^2}\psi(\xi,\tau)\right\} -{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{2\mu^2(\xi,\tau)\psi(\xi,\tau)+2\psi^3(\xi,\tau)\right\}=0 \quad (47)$$

where $\Psi_{N_\alpha}(\xi,s,u) = {}^{C}_{\tau}\mathcal{N}_\alpha^{+}\{\psi(\xi,\tau)\}$ and $\mu_{N_\alpha}(\xi,s,u)={}^{C}_{\tau}\mathcal{N}_\alpha^{+}\{\mu(\xi,\tau)\}$.
Using the IC's in system (45) and rearranging the terms of system (44), we obtain

$$\Psi_{N_\alpha}(\xi,s,u)=\frac{\cos(\xi)}{s}-\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\frac{d^2}{d\xi^2}\mu(\xi,\tau)\right\} +\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{-2\psi^2(\xi,\tau)\mu(\xi,\tau)+\mu^3(\xi,\tau)\right\},$$
$$\mu_{N_\alpha}(\xi,s,u)=\frac{\sin(\xi)}{s}+\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\frac{d^2}{d\xi^2}\psi(\xi,\tau)\right\} +\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{2\mu^2(\xi,\tau)\psi(\xi,\tau)+\psi^3(\xi,\tau)\right\}.$$

Now, we take the ICNT, so we have

$$\psi(\xi,\tau)=\cos(\xi)-{}^{C}_{\tau}\mathcal{N}_\alpha^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\frac{d^2}{d\xi^2}\mu(\xi,\tau)\right\}\right\} +{}^{C}_{\tau}\mathcal{N}_\alpha^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{-2\left(\psi^2(\xi,\tau)\mu(\xi,\tau)\right)+\mu^3(\xi,\tau)\right\}\right\},$$
$$\mu(\xi,\tau)=\sin(\xi)+{}^{C}_{\tau}\mathcal{N}_\alpha^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\frac{d^2}{d\xi^2}\psi(\xi,\tau)\right\}\right\} +{}^{C}_{\tau}\mathcal{N}_\alpha^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{2\left(\mu^2(\xi,\tau)\psi(\xi,\tau)\right)+\psi^3(\xi,\tau)\right\}\right\}. \quad (48)$$

Now, we replace $\psi(\xi,\tau)$ and $\mu(\xi,\tau)$ in the linear terms by the series forms:

$$\psi(\xi,\tau)=\sum_{k=0}^{\infty}w_k(\xi,\tau),$$
$$\mu(\xi,\tau)=\sum_{k=0}^{\infty}m_k(\xi,\tau), \quad (49)$$

and the nonlinear terms $-2\left(\psi^2(\xi,\tau)\mu(\xi,\tau)+\mu^3(\xi,\tau)\right)$ and $2\left(\mu^2(\xi,\tau)\psi(\xi,\tau)+\psi^3(\xi,\tau)\right)$ by the Adomian polynomials as follows:

$$-2\left(\psi^2(\xi,\tau)\mu(\xi,\tau)+\mu^3(\xi,\tau)\right)=\sum_{k=0}^{\infty}\mathcal{A}_k(\xi,\tau),$$
$$2\left(\mu^2(\xi,\tau)\psi(\xi,\tau)+\psi^3(\xi,\tau)\right)=\sum_{k=0}^{\infty}B_k(\xi,\tau), \quad (50)$$

where

$$\mathcal{A}_k=-\frac{2}{k!}\frac{d^k}{d\eta^k}\left[\left(\sum_{i=0}^{k}\eta^i w_i\right)^2\left(\sum_{i=0}^{k}\eta^i m_i\right)+\left(\sum_{i=0}^{k}\eta^i m_i\right)^3\right],$$
$$B_k=\frac{1}{k!}\frac{d^k}{d\eta^k}\left[\left(\sum_{i=0}^{k}\eta^i m_i\right)^2\left(\sum_{i=0}^{k}\eta^i w_i\right)+\left(\sum_{i=0}^{k}\eta^i w_i\right)^3\right],$$
$$k=0,1,2,\ldots. \quad (51)$$

Hence, the first few Adomian polynomials are

$$\mathcal{A}_0=-2\sin(\xi),$$
$$\mathcal{A}_1=2m_1(\xi)(\cos(2\xi)-2)-2w_1(\xi)\sin(2\xi),$$
$$\mathcal{A}_2=-2\Big(2m_1(\xi)w_1(\xi)\cos(\xi)+3m_1(\xi)^2\sin(\xi) -m_2(\xi)(\cos(2\xi)-2)+\sin(\xi)\left(w_1(\xi)^2+2w_2(\xi)\cos(\xi)\right)\Big),$$

$$B_0=2\cos(\xi),$$
$$B_1=2\left(m_1(\xi)\sin(2\xi)+w_1(\xi)(\cos(2\xi)+2)\right),$$
$$B_2=2\Big(2m_1(\xi)w_1(\xi)\sin(\xi)+m_2(\xi)\sin(2\xi)+m_1(\xi)^2\cos(\xi) +2w_2(\xi)+3w_1(\xi)^2\cos(\xi)+w_2(\xi)\cos(2\xi)\Big).$$

Substituting in system (49) and system (51) in system (48) implies:

$$\sum_{k=0}^{\infty}w_k(\xi,\tau)=\cos(\xi)-{}^{C}_{\tau}\mathcal{N}_\alpha^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\frac{d^2}{d\xi^2}\sum_{k=0}^{\infty}m_k(\xi,\tau)\right\}\right\} +{}^{C}_{\tau}\mathcal{N}_\alpha^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\sum_{k=0}^{\infty}\mathcal{A}_k(\xi,\tau)\right\}\right\},$$
$$\sum_{k=0}^{\infty}m_k(\xi,\tau)=\sin(\xi)+{}^{C}_{\tau}\mathcal{N}_\alpha^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\frac{d^2}{d\xi^2}\sum_{k=0}^{\infty}w_k(\xi,\tau)\right\}\right\} +{}^{C}_{\tau}\mathcal{N}_\alpha^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\sum_{k=0}^{\infty}B_k(\xi,\tau)\right\}\right\}. \quad (52)$$

This produces the recurrence relations:

$$w_0(\xi,\tau)=\cos(\xi),$$
$$m_0(\xi,\tau)=\sin(\xi)$$

$$w_{k+1}(\xi,\tau)=-{}^{C}_{\tau}\mathcal{N}_\alpha^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\frac{d^2}{d\xi^2}\sum_{k=0}^{\infty}m_k(\xi,\tau)\right\}\right\} +{}^{C}_{\tau}\mathcal{N}_\alpha^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\sum_{k=0}^{\infty}\mathcal{A}_k(\xi,\tau)\right\}\right\},$$
$$m_{k+1}(\xi,\tau)={}^{C}_{\tau}\mathcal{N}_\alpha^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\frac{d^2}{d\xi^2}\sum_{k=0}^{\infty}w_k(\xi,\tau)\right\}\right\} +{}^{C}_{\tau}\mathcal{N}_\alpha^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathcal{N}_\alpha^{+}\left\{\sum_{k=0}^{\infty}B_k(\xi,\tau)\right\}\right\},$$
$$k=0,1,2,\ldots \quad (53)$$

Putting $k=0$ in system (53), we have

$$w_1(\xi,\tau) = -{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\{-\sin(\xi)\}\right\} + {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\{-2\sin(\xi)\}\right\} = {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{-\frac{u\sin(\xi)}{s^2}\right\} = -\frac{\tau^{\alpha}}{\alpha}\sin(\xi).$$

$$m_1(\xi,\tau) = {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\{-\cos(\xi)\}\right\} + {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\{2\cos(\xi)\}\right\} = {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}\cos(\xi)\right\} = \frac{\tau^{\alpha}\cos(\xi)}{\alpha\Gamma(2)}.$$

Similarly, for $k=1$, the recurrence relation in system (53) yields.

$$w_2(\xi,\tau) = -{}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\frac{\partial^2}{\partial\xi^2}\left\{\frac{\tau^{\alpha}}{\alpha}\cos(\xi)\right\}\right\}\right\} + {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\frac{-2\cos(\xi)\tau^{\alpha}}{\alpha}\right\}\right\} = {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{-\frac{\cos(\xi)}{\alpha}\tau^{\alpha}\right\}\right\} = {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{-\frac{u}{s}\frac{\Gamma(2)u}{s^2}\cos(\xi)\right\} = {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{-\frac{u^2}{s^3}\cos(\xi)\right\} = -\frac{\tau^{2\alpha}}{2\alpha^2}\cos(\xi),$$

$$m_2(\xi,\tau) = {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{\frac{\partial^2}{\partial\xi^2}\left\{-\frac{\tau^{\alpha}}{\alpha}\sin(\xi)\right\}\right\}\right\} + {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{-2\sin(\xi)\frac{\tau^{\alpha}}{\alpha}\right\}\right\} = {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{\frac{u}{s}{}^{C}_{\tau}\mathscr{N}_{\alpha}^{+}\left\{-\sin(\xi)\frac{\tau^{\alpha}}{\alpha}\right\}\right\} = {}^{C}_{\tau}\mathscr{N}_{\alpha}^{-1}\left\{-\frac{u}{s}\sin(\xi)\frac{u}{s^2}\Gamma(2)\right\} = -\frac{\tau^{2\alpha}}{2\alpha^2}\sin(\xi)$$

Continuing this process for $k=2,3,\cdots,9$, we get

$$w_3(\xi,\tau) = \frac{\tau^{3\alpha}\sin(\xi)}{6\alpha^3}, \qquad m_3(\xi,\tau) = -\frac{\tau^{3\alpha}\cos(\xi)}{6\alpha^3},$$
$$w_4(\xi,\tau) = \frac{\tau^{4\alpha}\cos(\xi)}{24\alpha^4}, \qquad m_4(\xi,\tau) = \frac{\tau^{4\alpha}\sin(\xi)}{24\alpha^4},$$
$$w_5(\xi,\tau) = -\frac{\tau^{5\alpha}\sin(\xi)}{120\alpha^5}, \qquad m_5(\xi,\tau) = \frac{\tau^{5\alpha}\cos(\xi)}{120\alpha^5},$$
$$w_6(\xi,\tau) = -\frac{\tau^{6\alpha}\cos(\xi)}{720\alpha^6}, \qquad m_6(\xi,\tau) = -\frac{\tau^{6\alpha}\sin(\xi)}{720\alpha^6},$$
$$w_7(\xi,\tau) = \frac{\tau^{7\alpha}\sin(\xi)}{5040\alpha^7}, \qquad m_7(\xi,\tau) = -\frac{\tau^{7\alpha}\cos(\xi)}{5040\alpha^7},$$
$$w_8(\xi,\tau) = \frac{\tau^{8\alpha}\cos(\xi)}{40320\alpha^8}, \qquad m_8(\xi,\tau) = \frac{\tau^{8\alpha}\sin(\xi)}{40320\alpha^8},$$
$$w_9(\xi,\tau) = -\frac{\tau^{9\alpha}\sin(\xi)}{362880\alpha^9}, \qquad m_9(\xi,\tau) = \frac{\tau^{9\alpha}\cos(\xi)}{362880\alpha^9},$$
$$w_{10}(\xi,\tau) = -\frac{\tau^{10\alpha}\cos(\xi)}{3628800\alpha^{10}}, \qquad m_{10}(\xi,\tau) = -\frac{\tau^{10\alpha}\sin(\xi)}{3628800\alpha^{10}}$$

Consequently, the 10^{th} approximate sock for in system (44) is

$$\psi_{10}(\xi,\tau) = \sum_{k=0}^{10} w_k(\xi,\tau)$$
$$= \cos(\xi) - \sin(\xi)\frac{\tau^{\alpha}}{\alpha} - \cos(\xi)\frac{\tau^{2\alpha}}{2!\alpha^2} + \sin(\xi)\frac{\tau^{3\alpha}}{3!\alpha^3} + \cos(\xi)\frac{\tau^{4\alpha}}{4!\alpha^4} - \sin(\xi)\frac{\tau^{5\alpha}}{5!\alpha^5} - \cos(\xi)\frac{\tau^{6\alpha}}{6!\alpha^6} + \sin(\xi)\frac{\tau^{7\alpha}}{7!\alpha^7} + \cos(\xi)\frac{\tau^{8\alpha}}{8!\alpha^8} - \sin(\xi)\frac{\tau^{9\alpha}}{9!\alpha^9} - \cos(\xi)\frac{\tau^{10\alpha}}{10!\alpha^{10}}$$
$$= \cos(\xi)\left(1 - \frac{\tau^{2\alpha}}{2!\alpha^2} + \frac{\tau^{4\alpha}}{4!\alpha^4} - \frac{\tau^{6\alpha}}{6!\alpha^6} + \frac{\tau^{8\alpha}}{8!\alpha^8} - \frac{\tau^{10\alpha}}{10!\alpha^{10}}\right) - \sin(\xi)\left(\frac{\tau^{\alpha}}{\alpha} - \frac{\tau^{3\alpha}}{3!\alpha^3} + \frac{\tau^{5\alpha}}{5!\alpha^5} - \frac{\tau^{7\alpha}}{7!\alpha^7} + \frac{\tau^{9\alpha}}{9!\alpha^9}\right)$$
$$= \cos(\xi)\sum_{j=0}^{5}\frac{(-1)^j\left(\frac{\tau^{\alpha}}{\alpha}\right)^{2j}}{(2j)!} - \sin(\xi)\sum_{j=0}^{4}\frac{(-1)^j\left(\frac{\tau^{\alpha}}{\alpha}\right)^{2j+1}}{(2j+1)!}.$$

And

$$\mu_{10}(\xi,\tau) = \sum_{k=0}^{10} m_k(\xi,\tau)$$
$$= \sin(\xi) + \cos(\xi)\frac{\tau^{\alpha}}{1!\alpha} - \sin(\xi)\frac{\tau^{2\alpha}}{2!\alpha^2} - \cos(\xi)\frac{\tau^{3\alpha}}{3!\alpha^3} \sin(\xi)\cdot\frac{\tau^{4\alpha}}{4!\alpha^4} + \cos(\xi)\frac{\tau^{5\alpha}}{5!\alpha^5} - \sin(\xi)\cdot\frac{\tau^{6\alpha}}{6!\alpha^6} - \cos(\xi)\frac{\tau^{7\alpha}}{7!\alpha^7} + \sin(\xi)\cdot\frac{\tau^{8\alpha}}{8!\alpha^8} + \cos(\xi)\frac{\tau^{9\alpha}}{9!\alpha^9} - \sin(\xi)\frac{\tau^{10\alpha}}{10!\alpha^{10}}$$
$$= \sin(\xi)\left(1 - \frac{\tau^{2\alpha}}{2!\alpha^2} + \frac{\tau^{4\alpha}}{4!\alpha^4} - \frac{\tau^{6\alpha}}{6!\alpha^6} + \frac{\tau^{8\alpha}}{8!\alpha^8} - \frac{\tau^{10\alpha}}{10!\alpha^{10}}\right) + \cos(\xi)\left(\frac{\tau^{\alpha}}{1!\alpha} - \frac{\tau^{3\alpha}}{3!\alpha^3} + \frac{\tau^{5\alpha}}{5!\alpha^5} - \frac{\tau^{7\alpha}}{7!\alpha^7} + \frac{\tau^{9\alpha}}{9!\alpha^9}\right)$$
$$= \sin(\xi)\sum_{j=0}^{5}\frac{(-1)^j\left(\frac{\tau^{\alpha}}{\alpha}\right)^{2j}}{(2j)!} + \cos(\xi)\sum_{j=0}^{4}\frac{(-1)^j\left(\frac{\tau^{\alpha}}{\alpha}\right)^{2j+1}}{(2j+1)!}$$

The series form that has resulted in the 10^{th} approximate solution leads us to the form of the N^{th} approximate solution as

$$\psi_N(\xi,\tau)=\cos(\xi)\sum_{k=0}^{[\frac{N}{2}]}\frac{(-1)^j\left(\frac{\tau^\alpha}{\alpha}\right)^{2j}}{(2j)!}-\sin(\xi)\sum_{k=0}^{[\frac{N-1}{2}]}\frac{(-1)^j\left(\frac{\tau^\alpha}{\alpha}\right)^{2j+1}}{(2j)!},$$

and

$$\mu_N(\xi,\tau)=\sin(\xi)\sum_{k=0}^{[\frac{N}{2}]}\frac{(-1)^j\left(\frac{\tau^\alpha}{\alpha}\right)^{2j}}{(2j)!}+\cos(\xi)\sum_{k=0}^{[\frac{N-1}{2}]}\frac{(-1)^j\left(\frac{\tau^\alpha}{\alpha}\right)^{2j+1}}{(2j)!}.$$

As a result, the exact solution of system (4) is

$$\begin{aligned}\psi(\xi,\tau)&=\lim_{N\to\infty}\psi_N(\xi,\tau)\\&=\cos(\xi)\sum_{k=0}^{\infty}\frac{(-1)^j\left(\frac{\tau^\alpha}{\alpha}\right)^{2j}}{(2j)!}-\sin(\xi)\sum_{k=0}^{\infty}\frac{(-1)^j\left(\frac{\tau^\alpha}{\alpha}\right)^{2j+1}}{(2j)!}\\&=\cos(\xi)\cos\left(\frac{\tau^\alpha}{\alpha}\right)-\sin(\xi)\sin\left(\frac{\tau^\alpha}{\alpha}\right)\\&=\cos\left(\xi+\frac{\tau^\alpha}{\alpha}\right)\end{aligned}$$

and

$$\begin{aligned}\mu(\xi,\tau)&=\lim_{N\to\infty}\mu_N(\xi,\tau)\\&=\sin(\xi)\sum_{k=0}^{\infty}\frac{(-1)^j\left(\frac{\tau^\alpha}{\alpha}\right)^{2j}}{(2j)!}+\cos(\xi)\sum_{k=0}^{\infty}\frac{(-1)^j\left(\frac{\tau^\alpha}{\alpha}\right)^{2j+1}}{(2j)!}\\&=\sin(\xi)\cos\left(\frac{\tau^\alpha}{\alpha}\right)+\cos(\xi)\sin\left(\frac{\tau^\alpha}{\alpha}\right)\\&=\sin\left(\xi+\frac{\tau^\alpha}{\alpha}\right).\end{aligned}$$

In fact, $\psi(\xi,\tau)=\cos\left(\xi+\frac{\tau^\alpha}{\alpha}\right)$ and $\mu(\xi,\tau)=\sin\left(\xi+\frac{\tau^\alpha}{\alpha}\right)$ satisfy the system in (44 and 45) which ensures that they are the exact solution. Consequently, the solution of the CSDE in Eq (41) is

$$z(\xi,\tau)=\cos\left(\xi+\frac{\tau^\alpha}{\alpha}\right)+i\sin\left(\xi+\frac{\tau^\alpha}{\alpha}\right)=e^{i\left(\xi+\frac{\tau^\alpha}{\alpha}\right)}$$

However, to evaluate the accuracy of our methodology when using a few iterations only, Table 5 present the absolute errors for $\alpha\in\{1,0.9,0.8,0.7,0.6\}$ with limited number of iterations. Moreover, the effect of the conformable operator is clearly observed when comparing the $2D$ plots for various values of α and $\xi=0.1$, as shown in Figure 5, and the $3D$ plots as illustrated in Figure 6.

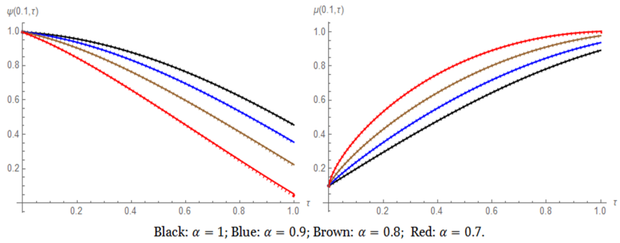

Fig. 5: Comparison between the exact (Dotted) and approximate (solid) curves of the solution of CSDE in (44).

Table 5: Absolute errors for the CSDE in system (44) for different values of α

		$\lvert\psi(0.1,\tau)-\psi_{10}(0.1,\tau)\rvert$			
τ	$\alpha=1$	$\alpha=0.9$	$\alpha=0.8$	$\alpha=0.7$	$\alpha=0.6$
0.0	0.	0.	0.	0.	0.
0.2	1.1×10^{-16}	1.1×10^{-15}	2.6×10^{-14}	7.3×10^{-13}	2.6×10^{-11}
0.4	1.4×10^{-13}	1.3×10^{-12}	1.4×10^{-11}	1.8×10^{-10}	2.9×10^{-9}
0.6	1.4×10^{-11}	8.0×10^{-11}	5.5×10^{-10}	4.5×10^{-9}	4.7×10^{-8}
0.8	3.6×10^{-10}	1.5×10^{-9}	7.6×10^{-9}	4.5×10^{-8}	3.5×10^{-7}
1.0	4.6×10^{-9}	1.5×10^{-8}	5.9×10^{-8}	2.7×10^{-7}	1.6×10^{-6}
		$\lvert\mu(0.1,\tau)-\mu_{10}(0.1,\tau)\rvert$			
τ	$\alpha=1$	$\alpha=0.9$	$\alpha=0.8$	$\alpha=0.7$	$\alpha=0.6$
0.0	0.	0.	0.	0.	0.
0.2	6.1×10^{-16}	9.5×10^{-15}	2.0×10^{-13}	5.2×10^{-12}	1.7×10^{-10}
0.4	1.0×10^{-12}	9.1×10^{-12}	9.1×10^{-11}	1.1×10^{-9}	1.6×10^{-8}
0.6	9.0×10^{-11}	5.0×10^{-10}	3.2×10^{-9}	2.4×10^{-8}	2.3×10^{-7}
0.8	2.1×10^{-9}	8.6×10^{-9}	4.0×10^{-8}	2.2×10^{-7}	1.5×10^{-6}
1.0	2.5×10^{-8}	7.8×10^{-8}	2.8×10^{-7}	1.2×10^{-6}	6.7×10^{-6}

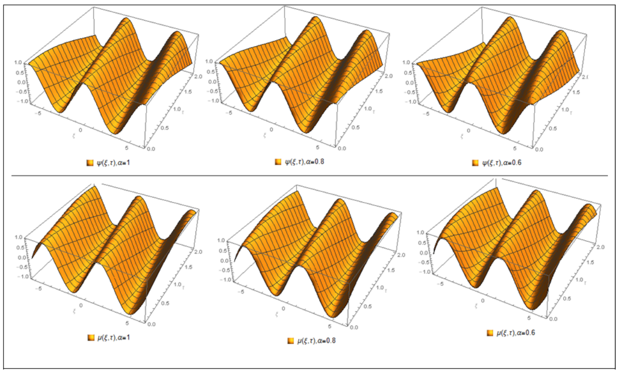

Fig. 6: The 3D plots for $\psi(\xi,\tau)$ and $\mu(\xi,\tau)$ for the (CSDE) in. Example (4.4) for $\alpha=0.1$,$\alpha=0.8$ and $\alpha=0.6$.

5 Conclusion

In this paper, an effective technique that depends on both the natural transform and the wellknown Adomian polynomials was employed to develop approximate and analytic solutions of nonlinear CPDEs. Dealing with conformable derivative reduced the complexities of other fractional derivatives due to its resemblance with the classical integer order derivative in most features. On the other hand, the CNDM proved its efficiency through its ability to solve different nonlinear real-life models with high accuracy.

We applied the proposed method to solve four nonlinear CPDEs that are related to real-life phenomena. We compared the approximate solution with the exact solution and with another methodology to check the accuracy of our proposed method. The results showed that a small number of iterations may achieve very small error. Comparing our results with the exact solutions, we noticed that the error tends to zero when increasing the number of iterations. Indeed, the approximate solutions rapidly converged to the exact solutions. The exact solution was obtained for some examples using the CNDM which is an advantage cannot be found in most approximate methodologies.

The effect of the CD to the solution of different models was obvious through plotting the solution curves for different conformable orders. Also, the results showed that the solutions of all examples and models approach the solution of classical integer order systems as the conformable order approaches the integer order. We did not face any difficulties or round off errors when using Mathematica especially when obtaining many terms in the approximate solution.

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