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Singular Value Inequalities related to PPT

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Abstract: In this article, we use positive partial transpose blocks to establish a new set of singular value and unitarily invariant norm inequalities. Some of these inequalities improve, generalize, and even interpolate various well-known inequalities in the literature.

Keywords: Positive semidefinite matrices; positive partial transpose matrices; singular value inequalities; norm inequalities.

Let $\mathbb{M}_n$ be the algebra of all $n\times n$ complex matrices. For $X\in\mathbb{M}_n$, the singular values of X are the eigenvalues of the positive semidefinite matrix $|X|=(X^*X)^{1/2}$. They are denoted by $s_j(X), j=1,2,...,n$ and are arranged so that $s_1(X)\geq s_2(X)\geq ...\geq s_n(X)$.

Recall that a matrix $X\in\mathbb{M}_n$ is called positive semidefinite, denoted by $X\geq 0$, if $\langle Xx,x\rangle\geq 0$ for all $x\in\mathbb{C}^n$. Similarly, X is called positive definite, denoted by $X>0$, if $\langle Xx,x\rangle>0$ for all $x\in\mathbb{C}^n$. For $A,B,X\in\mathbb{M}_n$, let H be the 2×2 block matrix defined as follows

$$H=\begin{pmatrix} A & X\\ X^* & B\end{pmatrix}.$$

It is well known that $H\geq 0$ if and only if

$$B-X^*A^{-1}X\geq 0, \quad (1)$$

provided that A is strictly positive. See [11].

Notice that the notation $X\geq 0$ (resp. $X>0$) means that X is positive semidefinite (resp. positive definite). For two Hermitian $X,Y\in\mathbb{M}_n$, $X\leq Y$ means $Y-X\geq 0$.

We remark that the 2×2 block matrices are fundamental in the study of matrices in general, and they play a particularly significant role in the analysis of sectorial matrices. For instance, see [2], [3], and [4]. See also [5] and [6] for a broader discussion of matrix inequalities.

The block H is said to be positive partial transpose, or PPT for short, if both H and $\begin{pmatrix} A & X^*\\ X & B\end{pmatrix}$ are positive semidefinite.

The class of PPT matrices has been thoroughly studied in literature. For example see [1,12,13,18,19,21] and the references therein.

In this article, our main goal is to utilize the PPT blocks to present a new collection of singular value and unitarily invariant norm inequalities. Some of these inequalities extend, generalize and improve various well known singular value and norm inequalities that exist in the literature. Moreover, our proof methodology simplifies certain proofs found in the literature. As an illustrative example, refer to Theorem 2, Theorem 3, Theorem 4, and Theorem 5 presented below. In the first section, we establish singular value inequalities and unitarily invariant norm inequalities that connect the main and the off-diagonal of the PPT Block. Subsequently, in the second section, we provide a new set of inequalities.

1 Positive Partial Transpose Blocks

In this section, we present a set of inequalities that establish connections between the main diagonal and the off-diagonal elements of a PPT Block. While the proofs of some of these results are routine, we provide them here for the sake of completeness.

Before stating the results of this section, let us recall some important facts about geometric and weighted-geometric mean of two positive matrices. For positive definite $X,Y\in\mathbb{M}_n$ and $t\in[0,1]$, the weighted geometric mean of X and Y is defined as follows

$$X\#_tY=X^{1/2}(X^{-1/2}YX^{-1/2})^tX^{1/2}.$$

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When $t=\frac{1}{2}$, we drop t from the above definition, and we simply write $X\#Y$ and call it the geometric mean of X and Y. It is well known that

$$X\#_tY \le (1-t)X+tY. \quad (2)$$

See [11, Chapter 4].

We have for $r>0$

$$\|(X\#_tY)^r\| \le \left\|\left(e^{(1-t)\log X+t\log Y}\right)^r\right\| \le \left\|(Y^{rt/2}X^{(1-t)r}Y^{rt/2})\right\|. \quad (3)$$

See [10].

Before proceeding, we state the following lemma. Its proof can be found in [1] and [14]; however, for the sake of completeness, we include the proof here.

Lemma 1. *If $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is PPT, then for every $t\in[0,1]$, the block $\begin{pmatrix} A\#_tB & X \\ X^* & A\#_{1-t}B \end{pmatrix}$ is PPT.*

Proof. Since $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is PPT, it is clear that both

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & X \\ X^* & A \end{pmatrix}$$

are positive definite. Without loss of generality we may assume they are positive definite, otherwise we use the well known continuous argument. Therefore,

$$X^*A^{-1}X \le B \quad \text{and} \quad X^*B^{-1}X \le A.$$

Observe,

$$\begin{aligned} X^*(A\#_tB)^{-1})X &= X^*(A^{-1}\#_tB^{-1})X \\ &= (X^*A^{-1}X)\#_t(X^*B^{-1})X \\ &\le B\#_tA \quad \text{(by the increasing property of means)}. \\ &= A\#_{1-t}B. \end{aligned} \quad (4)$$

Hence, $A\#_{1-t}B \ge X^*(A\#_tB)^{-1})X$. This implies that $\begin{pmatrix} A\#_tB & X \\ X^* & A\#_{1-t}B \end{pmatrix}$ is positive semidefinite. Similarly, it can be proved that $\begin{pmatrix} A\#_tB & X^* \\ X & A\#_{1-t}B \end{pmatrix}$ is also positive semidefinite. This completes the proof.

We remark that the proof method used above is similar to the one used in [7, Lemma 3.1] for establishing the special case when $t=1/2$.

Now, we state the following log majorization inequalities which governs the off-diagonal and the main diagonal of a PPT Block.

Theorem 1. *If $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is PPT, then for $k=1,2,\ldots,n$ and for $t\in[0,1]$.*

$$\prod_{j=1}^{k} s_j^{2r}(X) \le \prod_{j=1}^{k} s_j^{r}(A\#_tB)s_j^{r}(A\#_{1-t}B).$$

Proof. Since $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is PPT, Lemma 1 implies that the block $\begin{pmatrix} A\#_tB & X \\ X^* & A\#_{1-t}B \end{pmatrix}$ is positive semidefinite. Therefore, by [11, page 13],

$$X=(A\#_tB)^{1/2}K(A\#_{1-t}B)^{1/2} \quad \text{for some contraction} \quad K.$$

Then,

$$\begin{aligned} \prod_{j=1}^{k} s_j(X) &= \prod_{j=1}^{k} s_j\left((A\#_tB)^{1/2}K(A\#_{1-t}B)^{1/2}\right) \\ &\le \prod_{j=1}^{k} s_j\left((A\#_tB)^{1/2}\right)s_j(K)\,s_j\left((A\#_{1-t}B)^{1/2}\right) \\ &\le \prod_{j=1}^{k} s_j^{1/2}((A\#_tB))\,s_j^{1/2}((A\#_{1-t}B)). \end{aligned}$$

This completes the proof.

Recall that a norm $\|\cdot\|$ on $\mathbb{M}_n$ is called unitarily invariant norm if $\|UXV\|=\|X\|$ for all $X\in\mathbb{M}_n$ and all unitary elements $U,V\in\mathbb{M}_n$.

Let $\mathbb{R}^m_{+\downarrow}$ denote all the vectors $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)$ in $\mathbb{R}^n$ with $\gamma_1\ge\gamma_2\ge\ldots\ge\gamma_n\ge 0$. For each $\gamma\in\mathbb{R}^n_{+\downarrow}$ let $||\cdot||_\gamma$ be the norm defined on $\mathbb{M}_n$ as follows

$$||X||_\gamma=\sum_{j=1}^{n}\gamma_j s_j(X).$$

Let $\|\cdot\|$ be a unitarily invariant norm on $\mathbb{M}_n$. Then there is a compact set $K_{\|\cdot\|}\subset\mathbb{R}^n_{+\downarrow}$ such that

$$\|X\|=\max\{\|X\|_\gamma:\gamma\in K_{\|\cdot\|}\} \quad \text{for all} \quad X\in\mathbb{M}_n. \quad (5)$$

See [16].

Now, Theorem 1 implies the following inequality concerning unitarily invariant norms.

Corollary 1. *If $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is PPT. Let $r>0$ and $t\in[0,1]$, then*

$$\||X|^r\|^2 \le \|(A\#_tB)^r\|\;\|(A\#_{1-t}B)^r\|.$$

Proof. Theorem 1 implies that

$$\prod_{j=1}^{k} s_j^r(X) \le \prod_{j=1}^{k} s_j^{r/2}(A\#_tB)s_j^{r/2}(A\#_{1-t}B).$$

Let $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)\in K_{||\cdot||}$. Then

$$\prod_{j=1}^{k}\gamma_j s_j^r(X) \le \prod_{j=1}^{k}\gamma_j^{1/2}s_j^{r/2}(A\#_tB)\gamma_j^{1/2}s_j^{r/2}(A\#_{1-t}B).$$

Using Cauchy-Schwarz inequality and the fact that log majorization implies weak majorization, we have for $k = 1,2,\ldots,n$ and for $t \in [0,1]$

$$\begin{aligned}\sum_{j=1}^{k} \gamma_j s_j^r(X) &\leq \sum_{j=1}^{k} \gamma_j^{1/2} s_j^{r/2}(A\#_t B)\gamma_j^{1/2} s_j^{r/2}(A\#_{1-t}B)\\ &\leq \left(\sum_{j=1}^{k} \gamma_j s_j(A\#_t B)^r\right)^{1/2}\left(\sum_{j=1}^{k} \gamma_j s_j(A\#_{1-t}B)^r\right)^{1/2}\\ &= \|(A\#_t B)^r\|_\gamma^{1/2}\|(A\#_{1-t}B)^r\|_\gamma^{1/2}\\ &\leq \|(A\#_t B)^r\|^{1/2}\|(A\#_{1-t}B)^r\|^{1/2}.\end{aligned}$$

Hence,

$$\||X|^r\|_\gamma \leq \|(A\#_t B)^r\|^{1/2}\|(A\#_{1-t}B)^r\|^{1/2}.$$

The result follows by taking the maximum over all $\gamma \in K_{\|\cdot\|}$. The other inequalities follow from (3).

It is worth mentioning that the special cases ($r = 1$ and $t = 1/2$) of Theorem 1 and Corollary 1 were previously discussed in [19] and [7], respectively.

2 Applications

In this section, we introduce various applications of the results discussed in the preceding section. However, before proceeding, it is essential to consider the following remarks.

Remark. Let $A, B \in \mathbb{M}_n$.

1. Then the block $\begin{pmatrix} A^*A & A^*B \\ AB^* & B^*B\end{pmatrix}$ is positive definite since

$$\begin{pmatrix} A^*A & A^*B \\ AB^* & B^*B\end{pmatrix} = \begin{pmatrix} A & B\end{pmatrix}^*\begin{pmatrix} A & B\end{pmatrix}.$$

2. If $\begin{pmatrix} A & X \\ X^* & B\end{pmatrix}$ is positive definite block and $X = U|X|$ is the polar decomposition of X, then $\begin{pmatrix} U^*AU & |X| \\ |X| & B\end{pmatrix}$ is PPT since

$$\begin{pmatrix} U^*AU & |X| \\ |X| & B\end{pmatrix} = \begin{pmatrix} U^* & 0 \\ 0 & I\end{pmatrix}\begin{pmatrix} A & X \\ X^* & B\end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & I\end{pmatrix}.$$

2.1 The Sum of Products of pair of matrices

For $j = 1,2,\ldots,m$, let $A_j, B_j \in \mathbb{M}_n$ be such that $A_j^*B_j = B_j^*A_j$. Then

$$\sum_{j=1}^{m}\begin{pmatrix} A_j^*A_j & A_j^*B_j \\ A_jB_j^* & B_j^*B_j\end{pmatrix} = \begin{pmatrix} \sum_{j=1}^m |A_j|^2 & \sum_{j=1}^m A_j^*B_j \\ \sum_{j=1}^m A_jB_j^* & \sum_{j=1}^m |B_j|^2\end{pmatrix}$$

is PPT. Using Corollary 1 we have

$$\begin{aligned}&\left\|\left|\sum_{j=1}^{m} A_j^*B_j\right|^r\right\|^2\\ &\leq \left\|\left(\sum_{j=1}^{m}|A_j|^2\#_t\sum_{j=1}^{m}|B_j|^2\right)^r\right\|\left\|\left(\sum_{j=1}^{m}|A_j|^2\#_{1-t}\sum_{j=1}^{m}|B_j|^2\right)^r\right\|.\end{aligned} \tag{6}$$

In the forthcoming theorem we consider a particular case where $A_j, B_j \in \mathbb{M}_n$ are positive semidefinite.

Theorem 2. *For $j = 1,2,\ldots,m$, let $A_j, B_j \in \mathbb{M}_n$ be positive semidefinite such that, for each j, B_j commutes with A_j. Let $r \geq 1$, $X_r = \sum_{j=1}^m A_j^{2/r}$ and $Y_r = \sum_{j=1}^m B_j^{2/r}$. Then for all unitarily invariant norms*

$$\begin{aligned}&\left\|\sum_{j=1}^{m} A_jB_j\right\|^2\\ &\leq \left\|\left(\sum_{j=1}^{m} A_j^{1/r}B_j^{1/r}\right)^r\right\|\\ &\leq \|(X_r\#_t Y_r)^r\|\,\|(X_r\#_{1-t}Y_r)^r\|\\ &\leq \left\|\left(e^{(1-t)\log(X_r)+t\log(Y_r)}\right)^r\right\|\left\|\left(e^{t\log(X_r)+(1-t)\log(Y_r)}\right)^r\right\|\\ &\leq \left\|\left((Y_r)^{rt/2}(X_r)^{r(1-t)}(Y_r)^{rt/2}\right)\right\|\left\|\left((Y_r)^{r(1-t)/2}(X_r)^{rt}(Y_r)^{r(1-t)/2}\right)\right\|\\ &\leq \left\|(X_r)^{r(1-t)}(Y_r)^{rt}\right\|\left\|(X_r)^{rt}(Y_r)^{r(1-t)}\right\|.\end{aligned} \tag{7}$$

Proof. By replacing A_j and B_j with $A_j^{1/r}$ and $B_j^{1/r}$, respectively, in (6) and using (3), we obtain the following:

$$\begin{aligned}&\left\|\left(\sum_{j=1}^{m} A_j^{1/r}B_j^{1/r}\right)^r\right\|^2\\ &\leq \|(X_r\#_t Y_r)^r\|\,\|(X_r\#_{1-t}Y_r)^r\|\\ &\leq \left\|\left(e^{(1-t)\log(X_r)+t\log(Y_r)}\right)^r\right\|\left\|\left(e^{t\log(X_r)+(1-t)\log(Y_r)}\right)^r\right\|\\ &\leq \left\|\left((Y_r)^{rt/2}(X_r)^{r(1-t)}(Y_r)^{rt/2}\right)\right\|\left\|\left((Y_r)^{r(1-t)/2}(X_r)^{rt}(Y_r)^{r(1-t)/2}\right)\right\|\,\|\\ &\leq \left\|(X_r)^{r(1-t)}(Y_r)^{rt}\right\|\left\|(X_r)^{rt}(Y_r)^{r(1-t)}\right\|.\end{aligned}$$

Note that the last inequality above follows from the general facts that $\|Re(X)\| \leq \|X\|$ for all $X \in \mathbb{M}_n$, and if the product XY is Hermitian, then $\|XY\| \leq \|Re(YX)\|$.

Given that for each index j, B_j and A_j commute and are both positive definite, we can express the product A_jB_j as $(A_j^{1/r}B_j^{1/r})^r$. Moreover, for every nonnegative convex function f on $[0,\infty)$ such that $f(0) = 0$ we have: $\|\sum_{j=1}^m f(A_j)\| \leq \|f\left(\sum_{j=1}^m A_j\right)\|$. This result is presented in [17]. In particular, when we choose $f(x) = x^r$ with $r \geq 1$, we get:

$$\left\|\sum_{j=1}^{m} A_jB_j\right\| = \left\|\sum_{j=1}^{m}\left(A_j^{1/r}B_j^{1/r}\right)^r\right\| \leq \left\|\left(\sum_{j=1}^{m} A_j^{1/r}B_j^{1/r}\right)^r\right\|. \tag{8}$$

Combining the inequalities in (7) and (8) implies the result.

In particular for $t = 1/2$, we have the following result.

Corollary 2. *For $j = 1,2,...,m$, let $A_j, B_j \in \mathbb{M}_n$ be positive semidefinite such that, for each j , B_j commutes with A_j. Then for all unitarily invariant norms*

$$\begin{aligned}\left\| \sum_{j=1}^{m} A_j B_j \right\| &\le \left\| \left(\sum_{j=1}^{m} A_j^{1/r} B_j^{1/r} \right)^r \right\| \\ &\le \left\| \left(\left(\sum_{j=1}^{m} A_j^{2/r} \right) \# \left(\sum_{j=1}^{m} B_j^{2/r} \right) \right)^r \right\| \\ &\le \left\| \left(e^{\frac{1}{2}\log\left(\sum_{j=1}^{m} A_j^{2/r}\right) + \frac{1}{2}\log\left(\sum_{j=1}^{m} B_j^{2/r}\right)} \right)^r \right\| \\ &\le \left\| \left(\sum_{j=1}^{m} B_j^{2/r} \right)^{r/4} \left(\sum_{j=1}^{m} A_j^{2/r} \right)^{r/2} \left(\sum_{j=1}^{m} B_j^{2/r} \right)^{r/4} \right\|. \\ &\le \left\| \left(\sum_{j=1}^{m} A_j \right)^{r/2} \left(\sum_{j=1}^{m} B_j \right)^{r/2} \right\| \quad \forall r \ge 1. \end{aligned} \tag{9}$$

Remark. It is evident that the above results refine and extend and interpolate Audenaert's inequality established in Theorem [8, Theorem 1]. They also provide further refinements of Audenaert's improvement presented in [15]. For an alternative proof of Audenaert's result, see [20].

A natural question arises: What are the consequences of relaxing the condition $A_j^* B_j = B_j^* A_j, j = 1,2,...,m$? In such cases, a weaker result can be obtained. To be more specific, let $A_j, B_j \in \mathbb{M}_n$ for $j = 1,2,...,m$. Then

$$\sum_{j=1}^{m} \begin{pmatrix} A_j^* A_j & A_j^* B_j \\ A_j B_j^* & B_j^* B_j \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{m} |A_j|^2 & \sum_{j=1}^{m} A_j^* B_j \\ \sum_{j=1}^{m} A_j B_j^* & \sum_{j=1}^{m} |B_j|^2 \end{pmatrix}$$

is positive semidefinite. Let $\sum_{j=1}^{m} A_j^* B_j = U \left| \sum_{j=1}^{m} A_j^* B_j \right|$ be the polar decomposition of $\sum_{j=1}^{m} A_j^* B_j$. By the second part of Remark 2, we conclude that the following block is PPT:

$$\begin{pmatrix} \sum_{j=1}^{m} U^* |A_j|^2 U & \left| \sum_{j=1}^{m} A_j^* B_j \right| \\ \left| \sum_{j=1}^{m} A_j^* B_j \right| & \sum_{j=1}^{m} |B_j|^2 \end{pmatrix}.$$

Therefore, Theorem 1, with $t = 1/2, r = 2$, implies the following result.

Theorem 3. *For $j = 1,2,...,m$, let $A_j, B_j \in \mathbb{M}_n$. Then for some unitary $U \in \mathbb{M}_n$ and for all unitarily invariant norms*

$$\begin{aligned}\left\| \left| \sum_{j=1}^{m} A_j^* B_j \right|^2 \right\| &\le \left\| \left(\left(\sum_{j=1}^{m} U^* |A_j|^2 U \right) \# \left(\sum_{j=1}^{m} |B_j|^2 \right) \right)^2 \right\| \\ &\le \left\| \left(\sum_{j=1}^{m} |B_j|^2 \right)^{1/2} U^* \left(\sum_{j=1}^{m} |A_j|^2 \right) U \left(\sum_{j=1}^{m} |B_j|^2 \right)^{1/2} \right\| \\ &\le \left\| \left(\sum_{j=1}^{m} |A_j|^2 \right) U \left(\sum_{j=1}^{m} |B_j|^2 \right) \right\|. \end{aligned}$$

2.2 Some Singular Value Inequalities

Let $A, B \in \mathbb{M}_n$. Then

$$\begin{pmatrix} I + AA^* & A + B \\ (A+B)^* & I + B^* B \end{pmatrix} = \begin{pmatrix} I & A \\ B^* & I \end{pmatrix} \begin{pmatrix} I & B \\ A^* & I \end{pmatrix}$$

is positive semidefinite. Therefore, the second part of Remark 2 implies that $\begin{pmatrix} I + U^* |A|^2 U & |A+B| \\ (|A+B| & I + |B|^2 \end{pmatrix}$ is PPT, with $U = U_1 U_2$, where U_1 is the unitary in the polar decomposition of $(A+B)$ and U_2 is the unitary such that $AA^* = U_2^* A^* A U_2$. Then, by applying Theorem 1 with $t = 1/2$, and using the facts that $\prod_{j=1}^{k} s_j(X \# Y) \le \prod_{j=1}^{k} s_j^{1/2}(X) s_j^{1/2}(Y)$ and $\prod_{j=1}^{k} s_j(XY) \le \prod_{j=1}^{k} s_j(X) s_j(Y)$, we have the following inequalities for $r > 0$ and $k = 1,2,\ldots,n$:

$$\begin{aligned}\prod_{j=1}^{k} s_j(|A+B|^r) &\le \prod_{j=1}^{k} s_j^r \left((I + U^* |A|^2 U) \# (I + |B|^2) \right) \\ &\le \prod_{j=1}^{k} s_j^r \left((I + U^* |A|^2 U)^{1/2} \right) s_j^r \left((I + |B|^2)^{1/2} \right). \\ &= \prod_{j=1}^{k} s_j^{r/2} \left((I + |A|^2) \right) s_j^{r/2} \left((I + |B|^2) \right). \end{aligned}$$

Therefore,

$$\prod_{j=1}^{k} s_j(|A+B|^r) \le \prod_{j=1}^{k} s_j^{r/2} \left((I + |A|^2) \right) s_j^{r/2} \left((I + |B|^2) \right). \tag{10}$$

We remark that $(1 + x^2)^r \le (1 + x^r)^2$ for all positive real numbers x and for $1 \le r \le 2$. See [9, Lemma 2.7]. Then

$$s_j^r \left(I + |A|^2 \right) = \left(1 + s_j^2(A) \right)^r \le (1 + s_j^r(A))^2 = s_j^2(I + |A|^r).$$

Therefore,

$$s_j^{r/2} \left(I + |A|^2 \right) \le s_j(I + |A|^r). \tag{11}$$

By combining (10) and (11) we have the following result which was given in [9], where the proof provided was more complicated.

Theorem 4. *([9, Theorem 2.8]) Let $A, B \in \mathbb{M}_n$.*

$$\prod_{j=1}^{k} s_j(|A+B|^r) \le \prod_{j=1}^{k} s_j \left(I + |A|^r \right) s_j \left(I + |B|^r \right),$$

for all $1 \le r \le 2$.

An improvement of the above result can be given if A, B are Hermitian. In fact, if A, B are Hermitian, then $\begin{pmatrix} I + A^2 & A + B \\ A + B & I + B^2 \end{pmatrix}$ is PPT. Therefore, using Theorem 1 gives the following result.

Theorem 5. *Let $A,B \in \mathbb{M}_n$ be Hermitian. Then, for $k = 1,2,\ldots,n$*

$$\prod_{j=1}^{k} s_j(|A+B|^r) \le \prod_{j=1}^{k} s_j^r\left((I+A^2)\#(I+B^2)\right)$$
$$\le \prod_{j=1}^{k} s_j\left((I+B^2)^{r/4}(I+A^2)^{r/2}(I+B^2)^{r/4}\right), \quad \forall r \ge 0$$
$$\le \prod_{j=1}^{k} s_j\left((I+|A|^r)\right)s_j\left((I+|B|^r)\right), \quad (\text{for } 1 \le r \le 2).$$

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