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Path Integral Formulation of Fractional Systems with Multi-Parameters Fractional Derivatives

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Abstract: In this paper, which is based on the recent generalization of fractional variational formulations in terms of multi-parameter fractional derivatives introduced by Agrawal and Muslih, we develop a corresponding path-integral framework for mechanical systems incorporating such operators. We first construct the Lagrangian and Hamiltonian formalisms in the multi-parameter fractional setting, and then derive the associated evolution kernel in the path-integral representation. The resulting formulation offers a unified quantization scheme for systems that exhibit memory and non-local behavior induced by fractional dynamics. Representative examples illustrate the novel features introduced by the multi-parameter derivative structure.

Keywords: Fractional calculus; Multi-parameter fractional derivatives; Hilfer derivative; Path-integral quantization; Fractional Hamiltonian systems; Nonlocal dynamics .

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1 Introduction

The extension of conventional calculus, which relies on integer-order differentiation and integration, to fractional differential and integral operators has been developed through various mathematical formalisms [8, 11, 13, 15] . The growing significance of fractional calculus arises from its wide spectrum of applications in physics, engineering, and applied mathematics [1, 2, 8–10, 15] , where it effectively models systems with memory, non-locality, and complex dynamic behavior. Remarkable progress has been achieved in the formulation of fractional variational principles [1, 9, 10]. In this context, the pioneering studies of Klimek and Agrawal [9, 10] have inspired extensive research on fractional Lagrangian and Hamiltonian frameworks, making substantial contributions to the theoretical foundations of the field.

In the realm of constrained dynamical systems [2], both classical and quantum mechanical formulations play an essential role in theoretical physics. Following Dirac's quantization procedure for the electromagnetic field, major advancements have been achieved in the development of quantization methods, notably the canonical quantization and the path-integral approaches.

The objective of this work is to extend earlier results on fractional Lagrangian and Hamiltonian systems [1], where the dynamical variables in configuration and phase spaces depend on multi-parameter fractional derivatives. Within this generalized structure, we develop a fractional path integral formulation that integrates over extended canonical variables. This framework establishes a coherent pathway toward the quantization of fractional systems, offering deeper insight into the mathematical foundations and physical implications of nonlocal and memory-dependent dynamics.

2 Preliminaries

In the following discussion, we introduce the fundamental definitions and relationships involving fractional integrals and derivatives that will be employed throughout this work.

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2.1 Fractional Integrals

The *left (forward)* (LF) and *right (backward)* (RB) Riemann–Liouville fractional integrals of a function (r(t)) of order $(\gamma)((0 < \gamma < 1))$ with respect to (t) are defined as follows:

(LF) fractional integral:

$$(aI_t^{\alpha})r(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-t')^{\gamma-1}\, r(t')\, dt' \qquad (1)$$

(RB)fractional integral:

$$(tI_b^{\gamma})r(t) = \frac{1}{\Gamma(\gamma)} \int_t^b (t'-t)^{\gamma-1}\, r(t')\, dt' \qquad (2)$$

The Riesz potential, denoted as $(R_aI_b^{\alpha})r(t)$, is given by

$$\begin{aligned}(R_aI_b^{\gamma})r(t) &= \frac{1}{2\,\Gamma(\gamma)} \int_a^b |t'-t|^{\gamma-1}\, r(t')\, dt' \qquad (3)\\ &= \frac{1}{2}\left[(aI_t^{\gamma})r(t) + (tI_b^{\gamma})r(t)\right].\end{aligned}$$

In certain references, additional terms are sometimes included in the definition of the Riesz potential, which may produce singular contributions. To avoid such complications, the form in Eq. (3) is adopted here.

We also define a *reflection operator* P as

$$(Pr)(t) = r(a+b-t). \qquad (4)$$

This operator satisfies the following useful properties:

$$P(aI_t^{\gamma}) = (tI_b^{\gamma})P, \quad P(tI_b^{\gamma}) = (aI_t^{\gamma})P, \quad P(R_aI_b^{\gamma}) = (R_aI_b^{\gamma})P. \qquad (5)$$

2.2 Fractional Derivatives

Let n be an integer such that $n-1 < \gamma < n$. The *(LF)* and *(RB)* Riemann–Liouville fractional derivatives (RLFDs) are defined respectively by:

(LF) RL fractional derivative:

$$\begin{aligned}(aD_t^{\gamma})r(t) &= \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dt}\right)^n \int_a^t (t-t')^{n-\gamma-1}\, r(t')\, dt' \qquad (6)\\ &= D^n(aI_t^{n-\gamma})r(t).\end{aligned}$$

(RB) RL fractional derivative:

$$\begin{aligned}(tD_b^{\gamma})r(t) &= \frac{1}{\Gamma(n-\gamma)} \left(-\frac{d}{dt}\right)^n \int_t^b (t'-t)^{n-\gamma-1}\, r(t')\, dt' \qquad (7)\\ &= (-D)^n(tI_b^{n-\gamma})r(t).\end{aligned}$$

Here, $D = \frac{d}{dt}$ represents the ordinary differential operator. In contrast, the *Caputo fractional derivatives (CFDs)* are defined by interchanging the order of differentiation and integration:

(LF) Caputo derivative:

$$\begin{aligned}(C_aD_t^{\gamma})r(t) &= \frac{1}{\Gamma(n-\gamma)} \int_a^t (t-t')^{n-\gamma-1} \left(\frac{d^n r(t')}{dt'^n}\right) dt' \qquad (8)\\ &= (aI_t^{n-\gamma})D^n r(t).\end{aligned}$$

(RB) Caputo derivative:

$$\begin{aligned}(C_tD_b^{\gamma})r(t) &= \frac{1}{\Gamma(n-\gamma)} \int_t^b (t'-t)^{n-\gamma-1} \left(-\frac{d^n r(t')}{dt'^n}\right) dt' \qquad (9)\\ &= (tI_b^{n-\gamma})(-D)^n r(t).\end{aligned}$$

Thus, in Riemann–Liouville derivatives, the differentiation follows fractional integration, While in Caputo derivatives, the differentiation is applied first.

2.2 Riesz and Riesz–Caputo Derivatives

The *Riesz fractional derivative (RFD)* of order γ $(n-1 < \gamma < n)$ is defined as:

$$\begin{aligned}(R_aD_b^{\gamma})r(t) &= \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dt}\right)^n \int_a^b |t-t'|^{n-\gamma-1}\, r(t')\, dt' \qquad (10)\\ &= D^n(R_aI_b^{n-\gamma})r(t).\end{aligned}$$

The *Riesz–Caputo fractional derivative (RCFD)* is defined as:

$$\begin{aligned}(RC_aD_b^{\gamma})r(t) &= \frac{1}{\Gamma(n-\gamma)} \int_a^b |t-t'|^{n-\gamma-1} \left(\frac{d^n r(t')}{dt'^n}\right) dt' \qquad (11)\\ &= (R_aI_b^{n-\gamma})D^n r(t).\end{aligned}$$

These operators satisfy the following relationships:

$$(R_aD_b^{\gamma})r(t) = \frac{1}{2}\left[(aD_t^{\gamma})r(t) + (-1)^n(tD_b^{\gamma})r(t)\right], \qquad (12)$$

$$(RC_aD_b^{\gamma})r(t) = \frac{1}{2}\left[(C_aD_t^{\gamma})r(t) + (-1)^n(C_tD_b^{\gamma})r(t)\right]. \quad (13)$$

In the special case $0 < \gamma < 1$, these reduce to

$$(R_aD_b^{\gamma})r(t) = \frac{1}{2}\left[(aD_t^{\gamma})r(t) - (tD_b^{\gamma})r(t)\right], \quad (14)$$

$$(RC_aD_b^{\gamma})r(t) = \frac{1}{2}\left[(C_aD_t^{\gamma})x(t) - (C_tD_b^{\gamma})r(t)\right]. \quad (15)$$

Further details regarding the properties of Eqs. (12)–(15) and fractional integration by parts can be found in [1, 10].

3 Two-Parameter Hilfer Fractional Derivatives

The two-parameter fractional derivative, first introduced by Hilfer, provides a flexible generalization that interpolates between the Riemann–Liouville and Caputo derivatives. In this section, we present the left (forward) and right (backward) Hilfer fractional derivatives [1, 15] and outline their principal properties.

Let $r(t)$ be a sufficiently smooth function, and let *gamma* and β be real parameters such that $n-1 < \gamma < n$ and $0 \le \beta \le 1$.

3.1 Definition

The (LF) and the (RB) two-parameter fractional derivatives (P2FDs) are defined as,

(LF) two-parameter fractional derivatives

$$({}_aD_t^{\gamma,\beta})r(t) = ({}_aI_t^{(1-\beta)(n-\gamma)})\, D^n\, ({}_aI_t^{\beta(n-\gamma)})r(t), \quad (16)$$

(RB) two-parameter fractional derivatives

$$({}_tD_b^{\gamma,\beta})r(t) = ({}_tI_b^{(1-\beta)(n-\gamma)})\, (-D)^n\, ({}_tI_b^{\beta(n-\gamma)})r(t). \quad (17)$$

Here γ, $n-1 < \gamma < n$,ourselves is the order of the fractional derivative, and β, $(0 \le \beta \le 1)$ is a parameter. Although γ could be any positive real number, we shall restrict ourselves to $0 < \gamma < 1$.

3.2 Properties

The Hilfer fractional derivative exhibits limiting behaviors that connect it with both the Caputo and Riemann–Liouville definitions:

For $\beta = 0$:

$$(aD_t^{\gamma,0})\, r(t) = (aI_t^{1-\gamma})\, D\, r(t) = (C_aD_t^{\gamma})\, r(t), \quad (18)$$

$$(tD_b^{\gamma,0})\, r(t) = (tI_b^{1-\gamma})\, (-D)\, r(t) = (C_tD_b^{\gamma})\, r(t). \quad (19)$$

For $\beta = 1$:

$$(aD_t^{\gamma,1})\, r(t) = D\, \left(aI_t^{1-\gamma} r(t)\right) = (aD_t^{\gamma})\, r(t), \quad (20)$$

$$(tD_b^{\gamma,1})\, r(t) = (-D)\, \left(tI_b^{1-\gamma} r(t)\right) = (tD_b^{r})\, r(t). \quad (21)$$

Hence, the Hilfer derivative introduces a two-parameter structure that generalizes traditional fractional derivatives, enabling a continuous transition between Riemann–Liouville and Caputo formulations. This property makes it especially suitable for describing dynamic processes governed by both local and non-local memory mechanisms.

4 Fractional Variational Formulations in Terms of Two-Parameter Hilfer and Multi-Parameter Fractional Derivatives

In this section, we extend the fractional variational principle to include both two-parameter Hilfer fractional derivatives and three-parameter generalized fractional derivatives (GFDs). These formulations provide a unified structure for describing dynamical systems that exhibit both non-locality and memory-dependent behavior.

4.1 Action Functional with Two-Parameter Hilfer Derivatives

Consider an action functional

$$S[r] = \int_a^b L\left(t,\, r,\, aD_t^{\gamma_1,\beta_1}\, r,\, tD_b^{\gamma_2,\beta_2}\, r\right) dt, \quad (22)$$

where L represents the fractional Lagrangian. The corresponding Euler–Lagrange equation and terminal conditions are expressed as:

$$\frac{\partial L}{\partial r} + tD_b^{\gamma_1,(1-\beta_1)}\left(\frac{\partial L}{\partial(aD_t^{\gamma_1,\beta_1}\, r)}\right) + aD_t^{\gamma_2,(1-\beta_2)}\left(\frac{\partial L}{\partial(tD_b^{\gamma_2,\beta_2}\, r)}\right) = 0, \quad (23)$$

with boundary (terminal) conditions:

$$\Big[tI_b^{(1-\beta_1)(1-\gamma_1)} \frac{\partial L}{\partial(aD_t^{\gamma_1,\beta_1}\, r)}\, aI_t^{\beta_1(1-\gamma_1)}\eta(t) - aI_t^{(1-\beta_2)(1-\gamma_2)} \frac{\partial L}{\partial(tD_b^{\gamma_2,\beta_2}\, r)}\, tI_b^{\beta_2(1-\gamma_2)}\eta(t)\Big]_{t=a} = 0, \quad (24)$$

$$\Big[tI_b^{(1-\beta_1)(1-\gamma_1)} \frac{\partial L}{\partial(aD_t^{\gamma_1,\beta_1}\, r)}\, aI_t^{\beta_1(1-\gamma_1)}\eta(t) - aI_t^{(1-\beta_2)(1-\gamma_2)} \frac{\partial L}{\partial(tD_b^{\gamma_2,\beta_2}\, r)}\, tI_b^{\beta_2(1-\gamma_2)}\eta(t)\Big]_{t=b} = 0. \quad (25)$$

4.2 Three-Parameter Generalized Fractional Derivative (GFD)

We now introduce the three-parameter generalized fractional derivative (GFD), defined by:

$$D_t^{\gamma,\beta,\lambda} r(t) = \gamma \left(aD_t^{\gamma,\beta} r(t)\right) - (1-\lambda) \left(tD_b^{\gamma,\beta} r(t)\right), \quad (26)$$

where γ (with $n-1 \le \gamma \le n$) denotes the order of differentiation, while β and λ (with $0 \le \beta, \ \lambda \le 1$) are tunable parameters controlling the type and direction of the fractional derivative. For simplicity, we henceforth restrict to $0 \le \gamma \le 1$.

4.3 Special Cases

The GFD defined in Eq. (26) encompasses several well-known fractional derivatives as limiting cases:

$$D_t^{\gamma,1,1} r(t) = {}_0D_t^{\gamma,1} r(t) = {}_0D_t^{\gamma} r(t), \quad (27)$$

$$D_t^{\gamma,0,1} r(t) = {}_0D_t^{\gamma,0} r(t) = {}_0^CD_t^{\gamma} r(t), \quad (28)$$

$$D_t^{\gamma,1,0} r(t) = -{}_tD_b^{\gamma,1} r(t) = -{}_tD_b^{\gamma} r(t), \quad (29)$$

$$D_t^{\gamma,0,0} r(t) = -{}_tD_b^{\gamma,0} r(t) = -{}_t^CD_b^{\gamma} r(t), \quad (30)$$

$$\begin{aligned} D_t^{\gamma,1,1/2} r(t) &= \frac{1}{2}\left[{}_0D_t^{\gamma,1} r(t) - {}_tD_b^{\gamma,1} r(t)\right] \quad (31) \\ &= \frac{1}{2}\left[{}_0D_t^{\gamma} r(t) - {}_tD_b^{\gamma} r(t)\right] \\ &= {}^RD_t^{\gamma} r(t), \end{aligned}$$

and

$$\begin{aligned} D_t^{\gamma,0,1/2} r(t) &= \frac{1}{2}\left[{}_0D_t^{\gamma,0} r(t) - {}_tD_b^{\gamma,0} r(t)\right] \quad (32) \\ &= \frac{1}{2}\left[{}_0^CD_t^{\gamma} r(t) - {}_t^CD_b^{\gamma} r(t)\right] = {}^{RC}D_t^{\gamma} r(t). \end{aligned}$$

Hence, Riemann–Liouville, Caputo, Hilfer, Riesz, and Riesz–Caputo derivatives all emerge as special cases of the three-parameter GFD. In this sense, Eq. (26) can be viewed as a linear combination of left- and right-Hilfer derivatives, weighted by the parameter λ. When $\lambda = \frac{1}{2}$, the left and right contributions are equally weighted, while other values of λ represent asymmetric weighting.

4.4 Functional with Multiple Three-Parameter Derivatives

A more general fractional functional involving multiple three-parameter derivatives can be written as:

$$S[r] = \int_a^b L\left(t,\, r,\, D_t^{\gamma_1,\beta_1,\lambda_1} r, \ldots, D_t^{\gamma_n,\beta_n,\lambda_n} r\right) dt. \quad (33)$$

The corresponding fractional Euler–Lagrange equation reads:

$$\frac{\partial L}{\partial r} - \sum_{j=1}^{n} D_t^{\gamma_j,\,(1-\beta_j),\,(1-\lambda_j)} \left(\frac{\partial L}{\partial (D_t^{\gamma_j,\beta_j,\lambda_j} r)}\right) = 0, \quad (34)$$

with terminal conditions:

$$\begin{aligned} &\sum_{i=1}^{n} [\lambda_i \, tI_b^{(1-\beta_i)(1-\gamma_i)} \frac{\partial L}{\partial (D_t^{\gamma_i,\beta_i,\lambda_i} r)} \, aI_t^{\beta_i(1-\gamma_i)} \, \eta \quad (35) \\ &+ (1-\lambda_i) \, tI_b^{\beta_i(1-\gamma_i)} \, \eta \, aI_t^{(1-\beta_i)(1-\gamma_i)} \frac{\partial L}{\partial (D_t^{\gamma_i,\beta_i,\lambda_i} r)}]_{t=a,b} = 0. \end{aligned}$$

5 Fractional Hamiltonian Formulation

In this section, we extend the fractional variational principle to the Hamiltonian framework in order to derive the Hamiltonian equations of motion for systems described by multi-parameter fractional derivatives [1].

5.1 Generalized Hamiltonian Variables

Using the fractional variational formulation presented in Section 4, the generalized momenta corresponding to the fractional derivatives $aD_t^{\alpha_1,\beta_1} r$ and $tD_b^{\gamma_2,\beta_2} r$ are defined as

$$P_{\gamma_1,\beta_1} = \frac{\partial L}{\partial (aD_t^{\gamma_1,\beta_1} r)}, \quad (36)$$

$$P_{\gamma_2,\beta_2} = \frac{\partial L}{\partial (tD_b^{\gamma_2,\beta_2} r)}. \quad (37)$$

The corresponding Hamiltonian function is given by

$$H = -L + P_{\gamma_1,\beta_1} \left(aD_t^{\gamma_1,\beta_1} r\right) + P_{\gamma_2,\beta_2} \left(tD_b^{\gamma_2,\beta_2} r\right). \quad (38)$$

5.2 Action Functional and Canonical Equations

The action functional $S[r]$ in terms of these variables is written as

$$S[r] = \int_a^b \left(P_{\gamma_1,\beta_1} \, aD_t^{\gamma_1,\beta_1} r + P_{\gamma_2,\beta_2} \, tD_b^{\gamma_2,\beta_2} r - H\right) dt. \quad (39)$$

Applying the fractional variational principle to this functional yields the canonical Hamiltonian equations of motion:

$$aD_t^{\gamma_1,\beta_1} r = \frac{\partial H}{\partial P_{\gamma_1,\beta_1}}, \quad tD_b^{\gamma_2,\beta_2} r = \frac{\partial H}{\partial P_{\gamma_2,\beta_2}}, \tag{40}$$

$$tD_b^{\gamma_1,(1-\beta_1)} P_{\gamma_1,\beta_1} + aD_t^{\gamma_2,(1-\beta_2)} P_{\gamma_2,\beta_2} = \frac{\partial H}{\partial r}. \tag{41}$$

Furthermore, if the Hamiltonian explicitly depends on time, the following relation holds:

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{42}$$

5.2 Three-Parameter Fractional Systems

For systems characterized by the three-parameter generalized fractional derivative $D_t^{\gamma,\beta,\lambda} r$, the fractional momentum and Hamiltonian are defined respectively as:

$$P_{\gamma,\beta,\lambda} = \frac{\partial L}{\partial (D_t^{\gamma,\beta,\lambda} r)}, \tag{43}$$

$$H = -L + P_{\gamma,\beta,\lambda}\, D_t^{\gamma,\beta,\lambda} r. \tag{44}$$

The corresponding action functional becomes:

$$S[r] = \int_a^b \left(P_{\gamma,\beta,\gamma}\, D_t^{\gamma,\beta,\lambda} r - H\right) dt. \tag{45}$$

Applying the fractional variational principle once more, we obtain the fractional Hamiltonian equations of motion:

$$\frac{\partial H}{\partial r} = D_t^{\gamma,(1-\beta),(1-\lambda)} P_{\gamma,\beta,\lambda}, \tag{46}$$

$$\frac{\partial H}{\partial P_{\gamma,\beta,\lambda}} = D_t^{\gamma,\beta,\lambda} r, \tag{47}$$

and

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{48}$$

The formalism presented above generalizes the Hamiltonian structure of classical mechanics to systems governed by fractional dynamics. The equations (46)–(48) can also be extended to include multiple fractional derivatives with distinct parameters, as detailed in Ref.[1]. This framework establishes the foundation upon which the fractional path-integral formulation — the focus of the next section — is constructed.

6 Path Integral Formulation of Fractional Mechanical Systems

Quantization of physical systems can be carried out by two primary approaches: (1) the canonical quantization method, originally developed by Dirac in 1926 [4], and (2) the path-integral formulation formulated by Feynman [7]. In the canonical approach, quantization is implemented via operator substitutions and commutation relations. Conversely, the path-integral method expresses quantum evolution as a sum over all possible trajectories connecting two states, each weighted by the phase factor $e^{iS/\hbar}$, where S is the classical action.

6.1 Feynman's Path Integral Framework

In the Schrödinger picture of quantum mechanics, the time-evolution of a quantum state $|\phi(t)\rangle$ can be written as

$$|\phi(t)\rangle = \sum_{n=0}^{\infty} e^{-iE_n t/\hbar} \langle n|\phi(t_0)\rangle\, |n\rangle. \tag{49}$$

Feynman proposed an equivalent representation in terms of the path integral:

$$|\phi(x,\acute{t})\rangle = \int_{-\infty}^{\infty} \langle \phi(\acute{x},\acute{t})|\phi(x,t)\rangle\, d\acute{x}\, |\phi(\acute{x},\acute{t})\rangle, \tag{50}$$

where the propagation kernel $\langle \phi(\acute{x},\acute{t})|\phi(x,t)\rangle$ is proportional to $e^{iS(x(t))/\hbar}$, and the action is defined by

$$S = \int L(q_i,\dot{q}_i,t)\, dt. \tag{51}$$

Here L is the Lagrangian of the system, and $(q_i,\dot{q}_i)$ denote generalized coordinates and velocities. The path-integral representation is then given by:

$$K = \int \prod dq_i\, \exp\left[i \int L\, dt\right]. \tag{52}$$

Because the Lagrangian $L(q_i,\dot{q}_i,t)$ and Hamiltonian $H(q_i,p_i,t)$ formulations are equivalent in classical mechanics, one may also express the kernel in the Hamiltonian form:

$$K = \int \prod dq_i\, dp_i\, \exp\left[i \int (p_i\, \dot{q}_i - H(q_i,p_i))\, dt\right]. \tag{53}$$

When $L(q_i,\dot{q}_i,t)$ is quadratic in $\dot{q}_i$, the integration over canonical momenta p_i yields:

$$K = \int \prod dq_i\, \exp\left[i \int L\, dt\right]. \tag{54}$$

6.2 Path Integral for Fractional Systems

The above formalism can be generalized to systems involving multi-parameter fractional derivatives.

(a) Two-Parameter Hilfer Fractional Systems For a system characterized by the two-parameter Hilfer fractional derivatives, the path-integral representation can be written as:

$$K = \int \prod dr\, dP_{\gamma_1,\beta_1}\, dP_{\gamma_2,\beta_2}$$
$$\exp\Big[i \int \big(P_{\gamma_1,\beta_1}\, aD_t^{\gamma_1,\beta_1}\, r + P_{\gamma_2,\beta_2}\, tD_b^{\gamma_2,\beta_2}\, r - H(t,r,P_{\gamma_1,\beta_1},P_{\gamma_2,\beta_2})\big)\, dt\Big]. \quad (55)$$

(b) Three-Parameter Generalized Fractional Systems For systems governed by the three-parameter generalized fractional derivative, the kernel becomes:

$$K = \int \prod dr\, dP_{\gamma,\beta,\lambda}$$
$$\exp\Big[i \int \big(P_{\gamma,\beta,\lambda}\, D_t^{\gamma,\beta,\lambda}\, r - H(t,r,P_{\gamma,\beta,\lambda})\big)\, dt\Big]. \quad (56)$$

6.3 Regular and Singular Fractional Systems

For a dynamical system with Lagrangian $L = L(q_i, \dot{q}_i, t)$, the regularity condition is determined by the Hessian matrix:

$$H_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i\, \partial \dot{q}_j}, \quad i,j = 1,\ldots,n. \quad (57)$$

If the rank of H_{ij} is n, the system is regular and $\det(H_{ij}) \neq 0$. If the rank is less than n and $\det(H_{ij}) = 0$, the system is classified as singular (i.e., constrained). A similar classification applies to systems with fractional derivatives [3].

In cases where the fractional Lagrangian is singular (i.e., not quadratic in generalized fractional velocities), the Dirac method can be employed to formulate the path integral in the extended phase space.

6.4 Constrained Fractional Systems

Consider a system with first-class constraints ϕ_α that vanish on the constraint surface (i.e., $\phi_\alpha = 0$) and satisfy the Poisson bracket relations [8]:

$$\{\phi_\alpha, \phi_\beta\} = f^{\gamma}_{\alpha\beta}\, \phi_\gamma, \quad (58)$$

where $f^{\gamma}_{\alpha\beta}$ are the structure functions.

To quantize such systems within the path-integral framework[3, 5–7, 14], one may utilize the Faddeev–Popov method[6], which transforms first-class constraints into second-class constraints by introducing gauge-fixing conditions $\chi_\beta = 0$ such that $\det\{\phi_\alpha, \chi_\beta\} \neq 0$.

Under these conditions, the path integral for a constrained fractional system takes the form:

$$K = \int \prod dq_i\, dp_i\, \delta(\phi_\alpha)\, \delta(\chi_\beta)\, \det\big|\{\phi_\alpha, \chi_\beta\}\big|$$
$$\exp\Big[i \int (p_i\, \dot{q}_i - H(t,q_i,p_i) + \lambda\, \phi_\alpha)\, dt\Big], \quad (59)$$

where λ denotes the Lagrange multipliers determined by the chosen gauge-fixing conditions [8].

7 Examples

To illustrate the fractional formulations derived in the preceding sections, we now examine several representative examples that demonstrate how the fractional path-integral method applies to physical systems with memory and non-local dynamics.

7.1 Example 1: Fractional Free Particle

Consider a free particle described by the fractional action functional

$$S[y] = \frac{1}{2} \int_0^1 \big[(aD_t^{\alpha,\beta}\, y)^2\big]\, dt. \quad (60)$$

The corresponding path-integral representation is expressed as

$$K = \int \prod dy\, dp_{\alpha,\beta}$$
$$\exp\Big[\, i \int \big(p_{\alpha,\beta}\, aD_t^{\alpha,\beta}\, y \;-\; \frac{1}{2}(p_{\alpha,\beta})^2\big)\, dt\Big]. \quad (61)$$

Performing the integration over the canonical momentum $p_{\alpha,\beta}$ yields

$$K = \int \prod dy\; \exp\Big[\, i \int \frac{1}{2}\big((aD_t^{\alpha,\beta}\, y)^2\big)\, dt\Big]. \quad (62)$$

Thus, the path-integral of the fractional free particle takes a form analogous to its classical counterpart, with the ordinary derivative replaced by the fractional Hilfer derivative.

7.2 Example 2: Fractional Harmonic Oscillator

Next, consider the fractional harmonic oscillator, defined by the action [12]

$$S[y] = \int_0^1 \left[\frac{1}{2} (aD_t^{\alpha,\beta} y)^2 + \frac{1}{2} k^2 y^2 \right] dt. \tag{63}$$

The corresponding path-integral is given by

$$K = \int \prod dy\, dp_{\alpha,\beta} \tag{64}$$
$$\exp\Big[i \int \Big(p_{\alpha,\beta}\, aD_t^{\alpha,\beta} y - \frac{1}{2} (p_{\alpha,\beta})^2 - \frac{1}{2} k^2 y^2 \Big) dt \Big].$$

After integrating over the momenta $p_{\alpha,\beta}$, the kernel becomes

$$K = \int \prod dy \ \exp\Big[i \int \Big(\frac{1}{2} (aD_t^{\alpha,\beta} y)^2 + \frac{1}{2} k^2 y^2 \Big) dt \Big]. \tag{65}$$

The resulting formulation clearly shows that the fractional harmonic oscillator retains the structure of the classical system, with the time derivative replaced by a fractional operator, thereby incorporating the effects of non-locality and memory.

7.3 Example 3: Constrained Fractional System

Let us now examine a fractional system with constraints. Consider a model depending on two dynamical variables x and y, described by the Lagrangian

$$L = \frac{1}{2} (aD_t^{\alpha,\beta} x)^2. \tag{66}$$

Here, the Hessian matrix has rank one, indicating that the system possesses one true canonical variable and one redundant variable, which can be removed through a suitable gauge-fixing condition[12].

The generalized momenta are given by

$$p_x = \frac{\partial L}{\partial (aD_t^{\alpha,\beta} x)} = aD_t^{\alpha,\beta} x, \tag{67}$$

$$p_y = \frac{\partial L}{\partial (aD_t^{\alpha,\beta} y)} = 0. \tag{68}$$

Thus, the system exhibits one primary first-class constraint:

$$\phi \equiv p_y = 0. \tag{69}$$

The corresponding Hamiltonian reads

$$H = \frac{p_x^2}{2} + \lambda p_y, \tag{70}$$

where λ denotes the Lagrange multiplier enforcing the constraint.

The action functional in Hamiltonian form becomes

$$S[x] = \int \Big(p_x\, aD_t^{\alpha,\beta} x + p_y\, aD_t^{\alpha,\beta} y - \frac{1}{2} p_x^2 + \lambda p_y \Big) dt. \tag{71}$$

To remove redundancy, we impose the gauge-fixing condition

$$\chi = y = 0. \tag{72}$$

Accordingly, the path integral takes the form

$$K = \int \prod dx\, dy\, dp_x\, dp_y\ \delta(y)\, \delta(p_y)\ \det\big|\{\phi, \chi\}\big| \tag{73}$$
$$\exp \int \Big(p_x\, aD_t^{\alpha,\beta} x + p_y\, aD_t^{\alpha,\beta} y - \frac{1}{2} p_x^2 + \lambda p_y \Big) dt.$$

Integrating over p_x, p_y, and y gives the final result:

$$K = \int \prod dx \ \exp\Big[\Big] [\, i \int \Big(\frac{1}{2} (aD_t^{\alpha,\beta} x)^2 \Big) dt \Big]. \tag{74}$$

This demonstrates how the Dirac–Faddeev–Popov formalism can be naturally extended to fractional constrained systems, maintaining consistency with the principles of canonical quantization.

8 Conclusions

In this work, we have proposed a novel generalization of the fractional path integral formulation for dynamical systems described by Hilfer fractional derivatives, which provide a continuous interpolation between the Riemann–Liouville and Caputo operators. This framework has been further extended to functionals defined through a three-parameter fractional derivative, enabling a richer description of systems that exhibit both memory effects and nonlocal interactions.

By applying this approach to three illustrative cases—a free particle, a fractional harmonic oscillator, and a particle with first-class constraints—we derived explicit fractional path integral representations within the canonical phase-space formalism. The obtained results reveal how the presence of fractional derivatives modifies the kernel structure, the propagation amplitude, and ultimately the dynamical behavior of the system. These findings highlight the capacity of fractional calculus to capture intermediate regimes between classical and quantum dynamics and to provide new perspectives on anomalous diffusion and dissipative processes.

The formulation presented here offers a unified and flexible tool for analyzing fractional dynamical systems and can naturally be generalized to fractional field theories involving multi-parameter derivatives. Such extensions could open new avenues for modeling nonlocal field interactions, fractional quantum field dynamics, and complex systems with hierarchical memory. Future work will focus on developing the corresponding field-theoretic generalizations and exploring their implications for fractional statistical mechanics, quantum dissipation, and wave propagation in complex media.

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