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A Study of Conformable Fractional Inverse Weibull Model with Applications

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Abstract: This paper introduces the Conformable Fractional Inverse Weibull (CFIW) distribution, a novel extension of the classical inverse Weibull family through the incorporation of a conformable fractional operator. In this work, we develop a new model motivated by ideas that arise in fractional calculus and by observations encountered in reliability and survival studies. These areas often involve data that do not follow the usual scaling behavior, which encourages the construction of more flexible distributions. We present analytic form of the probability density function (PDF) of the CFIW. Then we derive expressions for the cumulative distribution function (CDF) in conformable fractional integral (CFI) sense. Reliability function, hazard rate, moment expressions are also derived. Conformable characteristic function and conformable moment generating function for CFIW distribution are obtained in the form of series representations involving the Gamma function. Mode, median and percentiles are also provided. We also introduce modified versions of three well-known entropy measures. Their behavior under the proposed model is examined through direct calculations and numerical illustrations. Moreover, parameter estimation via maximum likelihood is detailed, with numerical strategies for stable computation. A Monte Carlo simulation study with sample sizes $n = \{50, 100, 200, 500\}$ and 100 replications evaluates finite-sample bias and mean-square error (MSE) .

Keywords: Conformable fractional derivative; Inverse Weibull; Measures of central tendency; Maximum likelihood; Hazard function; Entropy; Moments.

1 Introduction

Lifetime and reliability analysis is critical in engineering, biomedical research, and industrial applications. This analysis often requires flexible probability models capable of representing diverse failure behaviors, skewness, and heavy tails. Classical distributions such as the Weibull and the Inverse Weibull remain widely used in applied fields because their forms are simple and their properties are well understood. They model increasing, decreasing, and constant hazard rates [1]. However, real-world data sets in reliability, survival analysis, and risk management often exhibit features these classical models cannot capture effectively [2].

To address these limitations, some extensions of the inverse Weibull distribution have been proposed to allow more control over the shape of the curve, especially when the data show strong skewness or very heavy tails. These extensions usually work by adding one or more shape parameters [3,4]. Even so, these generalizations may fail to adequately model systems with memory or hereditary effects present in certain lifetime processes. So, the use of fractional calculus in this field is necessary.

Fractional calculus offers additional flexibility when constructing probability models. In fact, allowing the order of the operator to vary makes it possible to adjust the model to data sets that behave irregularly or show long-range dependence. So, fractional operators have been used increasingly in statistics and stochastic modeling to capture memory effects [5,6].

IT has a deep mathematical history tracing back to Leibniz and L'Hôpital and has become a powerful tool to model anomalous diffusion, memory effects, and non-local dynamics in physics, engineering and applied sciences [7,8,9]. The classical and most popular operators in fractional calculus such as Riemann–Liouville and Caputo, are nonlocal and have been used extensively in modeling viscoelasticity, diffusion in heterogeneous media, and processes with long memory.

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While powerful, these operators are sometimes cumbersome in statistical modeling because of their nonlocal nature and the singular kernel that appears in their integral definitions.

Recently, conformable fractional derivative (CFD) has been proposed as a local, simple-to-use alternative that preserves many desirable properties of classical derivatives while providing a tunable fractional order parameter; see [10]. It has been exploited for fractional differential equations and, more recently, for fractional parameterizations of probabilistic models [11,12].

Several recent studies have used fractional operators to build new probability models. These models tend to be more adaptable than their classical counterparts and can represent a wider range of data patterns. The present CFIW distribution fits into this strand by introducing a conformable fractional parameter into the inverse Weibull structure, preserving interpretability while increasing flexibility. Also, it offers enhanced modeling capability for heavy-tailed and skewed data by adding additional shape parameter, the fractional order, which can adapt the distribution to phenomena exhibiting anomalous scaling.

In this paper, we first introduce the CFIW distribution in the sense of the conformable integral. Then, we present some fundamental concepts related to it such as the CFCDF, moments, hazard function and central tendency measures. Furthermore, we provide a detailed numerical and graphical study of different modified entropy measures. Finally, we perform a simulation study to estimate the model parameters, particularly the conformable fractional order parameter.

The structure of our current work is as follows. Section 2 reviews conformable fractional operators, the classical Weibull and inverse Weibull distributions, and gives a review for classical entropy measures. Section 3 defines the CFIW and derives its CDF, reliability, hazard, moment generating, and characteristic functions. A study of the central tendency measures is also discussed in Section 3. In Section 4, we generalize some entropy measures in CFI sense. Numerical and graphical evaluations are obtained for the different entropy measures to illustrate their comparative behavior. Section 5 describes maximum likelihood estimation and inferential procedures. It contains the Monte Carlo simulation design and results. Then it discusses findings and applications; Finally, we present future work in Section 6.

2 Preliminaries

In this section, we presents basic definitions related to CFD. Then we show the PDE and CDF of Weibull and inverse Weibull distributions. Ar the end of this section, we recall the definitions of three most popular entropies.

2.1 Conformable Fractional Operators

The conformable fractional derivative, introduced by [10], provides a simpler local definition with several useful properties such as linearity, product rule in a modified form, and reduction to classical derivative at order 1.

Definition 1.*[10] Let $\omega : [0,\infty) \to \mathbb{R}$ and $t > 0$. For $0 < \alpha \leq 1$, the CFD of order α is presented as*

$$T_0^{(\alpha)}(\omega)(t) = \lim_{\varepsilon \to 0} \frac{\omega\left(t + \varepsilon t^{1-\alpha}\right) - \omega(t)}{\varepsilon}.$$

If ω has classical first order derivative, then $T_0^{(\alpha)}\omega(t) = t^{1-\alpha}\omega'(t)$, clarifying the local, multiplicative rescaling property. This operator is attractive in probability modeling since applying it often corresponds to a simple power transform on the argument, leading to tractable modified distributions.

Definition 2.*[13] Let $\omega : [c,d] \to \mathbb{R}$ be continuous and $0 < \alpha \leq 1$. Then the CFI of order α from c to t is defined by*

$$I_c^{(\alpha)}(\omega)(t) = \int_c^t \omega(t)\, t^{\alpha-1}\, dt, \quad t > c.$$

2.2 Classical Weibull and Inverse Weibull Distributions

The Weibull probability model is one of the most commonly applied in lifetime distributions to study their reliability and survival analysis. It was originally introduced by Waloddi Weibull (1951) [14] as follows.

Definition 3.[14] *Let $T \sim \text{Weibull}(\lambda,\gamma)$, where $\lambda > 0$ is the shape parameter and $\gamma > 0$ is the scale parameter. The PDF and CDF are respectively given by:*

$$\omega(t;\lambda,\gamma) = \frac{\gamma}{\lambda}\left(\frac{t}{\lambda}\right)^{\gamma-1} \exp\left[-\left(\frac{t}{\lambda}\right)^{\gamma}\right], \qquad t>0, \tag{1}$$

and

$$W(t;\lambda,\gamma) = 1-\exp\left[-\left(\frac{t}{\lambda}\right)^{\gamma}\right], \qquad t>0. \tag{2}$$

Several generalizations have been proposed to increase its flexibility in modeling data with different hazard rate shapes. Among these extensions are: The exponentiated Weibull distribution that adds an extra shape parameter [3] and the transmuted Weibull distribution based on transmutation maps [15].

One of the modified distributions of Weibull model is the IW distribution which is often used to model time-to-failure data where the hazard rate is decreasing. It was defined as follows.

Definition 4.[16] *A random variable $T > 0$ is said to have IW distribution with scale parameter $\lambda > 0$ and shape parameter $\gamma > 0$, $T \sim \text{IW}(\lambda,\gamma)$, if its PDF is*

$$\omega(t;\lambda,\gamma) = \gamma\lambda^{\gamma} t^{-(\gamma+1)} \exp\left(-\lambda^{\gamma} t^{-\gamma}\right), \qquad t>0. \tag{3}$$

The corresponding CDF is

$$W(t) = 1-\exp\left(-\lambda^{\gamma} t^{-\gamma}\right), \qquad t>0. \tag{4}$$

The flexibility of the inverse Weibull model has led to developing it to various generalizations such as the exponentiated inverse Rayleigh Weibull distribution [17] and the transmuted generalized inverse Weibull distribution [18].

2.3 Classical Entropy Definitions

Entropy measures are often used to describe how spread out a probability distribution is. In this subsection, we list the standard forms of the entropy functions that will later be modified for the proposed model. The classical formulas for these entropies are given in the following definition. For more details, see [19,20].

Definition 5.*Assume $T > 0$ is a random variable with PDF $\omega(t)$ defined on $(0,\infty)$. Then Shannon entropy measure is defined as:*

$$H_S(T) = -\int_0^{\infty} \omega(t)\ln\omega(t)\,dt.$$

While the Rényi entropy of order ρ has the formula

$$H_\rho(T) = \frac{1}{1-\rho}\log\left(\int_0^{\infty} \omega(t)^{\rho}\,dt\right),$$

and Tsallis entropy of order q is defined as

$$S_q(T) = \frac{1}{q-1}\left(1-\int_0^{\infty} \omega(t)^{q}\,dt\right).$$

3 Model Definition with Fundamental Characteristics

Recent advances in fractional calculus have motivated the construction of non-integer probability models. We propose to derive a conformable fractional version of the IW distribution by applying the conformable transformation to the argument of the classical PDF. Our proposed CFIW model replaces the classical first–order structure associated with the power t^{-1} by the conformable fractional effect $t^{-\alpha}$, with $0 < \alpha \le 1$. So, we have the following definition.

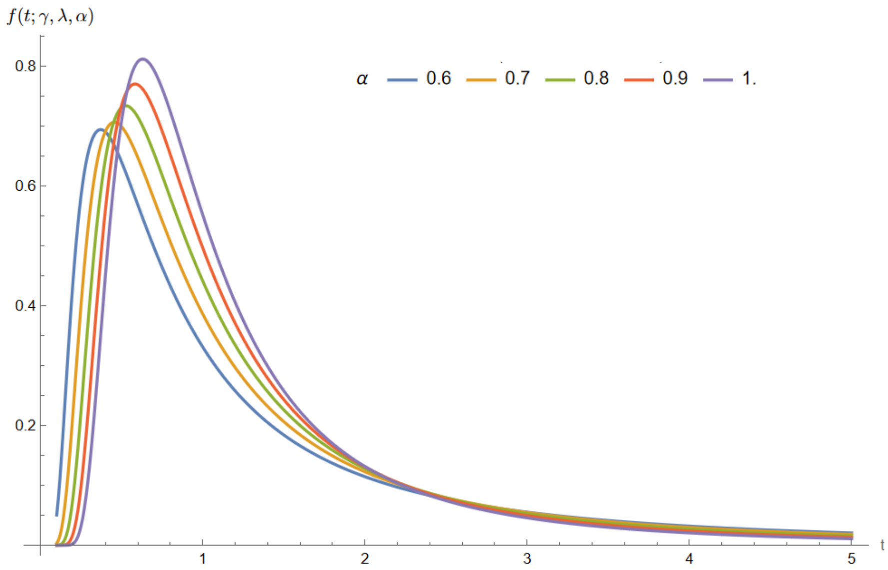

Fig. 1: CFPDF of CFIW for different fractional orders α when $\lambda = 1$ and $\gamma = 1.5$.

Definition 6. *Let $T > 0$ be a random variable. The CFIW distribution with shape parameter $\gamma > 0$, scale parameter $\lambda > 0$, and fractional conformable order $0 < \alpha \leq 1$ has conformable fractional PDF (CFPDF):*

$$f(t;\gamma,\lambda,\alpha) = \frac{\alpha\gamma e^{-(\lambda t^{-\alpha})^{\gamma}}(\lambda t^{-\alpha})^{\gamma}}{t} \tag{5}$$

Here, we write $T \sim CFIW(\lambda,\gamma,\alpha)$.

Figure 1 illustrates the CFPDF for several values of the conformable fractional order α when $\lambda = 1$ and $\gamma = 1.5$.

Definition 7. *Let $T \sim CFIW(\lambda,\gamma,\alpha)$. Then, the conformable fractional cumulative distribution function (CFCDF) of order α is defined in CFI sense as:*

$$F(t;\gamma,\lambda,\alpha) = I_0^{\alpha}(f(t;\gamma,\lambda,\alpha)) = \int_0^t x^{\alpha-1} f(x;\gamma,\lambda,\alpha)\,dx. \tag{6}$$

This definition ensures the consistency. That is,

$$T_0^{(\alpha)} F(t;\gamma,\lambda,\alpha) = f(t;\gamma,\lambda,\alpha),$$

where $T_0^{(\alpha)}$ denotes the CFD of order α.

Theorem 1. *Let $T \sim CFIW(\lambda,\gamma,\alpha)$. Then, the CFCDF of T has the form*

$$F(t;\gamma,\lambda,\alpha) = 1 - \exp\left[-(\lambda t^{-\alpha})^{\gamma}\right], \quad t > 0. \tag{7}$$

Proof:
By Definition 7,

$$F(t;\gamma,\lambda,\alpha) = \int_0^t x^{\alpha-1} \frac{\alpha\gamma e^{-(\lambda x^{-\alpha})^{\gamma}}(\lambda x^{-\alpha})^{\gamma}}{x}\,dx. \tag{8}$$

Using the change of variable to simplify the integral:

$$u = (\lambda x^{-\alpha})^{\gamma} \quad \Rightarrow \quad x = \lambda^{1/\alpha} u^{-1/(\alpha\gamma)}, \quad dx = -\frac{1}{\alpha\gamma}\lambda^{1/\alpha} u^{-1/(\alpha\gamma)-1} du.$$

Substituting in 8, the CFI becomes:

$$
\begin{aligned}
F(t;\gamma,\lambda,\alpha) &= \int_{u(t)}^{\infty} e^{-u}(-du) \\
&= 1-\exp\left[-(\lambda t^{-\alpha})^{\gamma}\right], t>0.
\end{aligned} \tag{9}
$$

The quantile function of the CFIW disstribution is given by inverting the CFCDF as follows.

Definition 8. *The quantile function is*

$$
Q(u;\gamma,\lambda,\alpha) = \lambda\left[-\ln(1-u)\right]^{-1/(\alpha\gamma)}, \quad 0<u<1. \tag{10}
$$

This function is included in our paper since it plays a central role in generating random samples through the inverse transform method. Hence, it will be used later in the simulation study of Section 5.

A basic important concept that associates to any probability distribution is the r-th moment. It can be defined in the conformable sense as:

$$
\mathbb{E}[T^r] = \int_0^{\infty} t^{\alpha-1} t^r f(t;\gamma,\lambda,\alpha)dt = \int_0^{\infty} t^{r+\alpha-1} f(t;\gamma,\lambda,\alpha)dt. \tag{11}
$$

The following theorem, gives the formula of the r-th moment for CFIW distribution.

Theorem 2. *Let $T \sim CFIW(\lambda,\gamma,\alpha)$. Then, the r-th moment in conformable sense has the form:*

$$
\mathbb{E}[T^r] = \lambda^{-r/\alpha}\Gamma\left(1-\frac{r}{\alpha\gamma}\right), \quad r<\alpha\gamma. \tag{12}
$$

Proof: The formula in 12 results directly by substituting $u=(\lambda t^{-\alpha})^{\gamma}$ in the FCI of 11.

The following theorem presents conformable hazard and reliability functions.

Theorem 3. *Let $T \sim CFIW(\lambda,\gamma,\alpha)$. Then, the hazard function is defined by*

$$
h(t;\gamma,\lambda,\alpha) = \frac{\alpha\gamma}{t}(\lambda t^{-\alpha})^{\gamma}. \tag{13}
$$

The reliability function is

$$
R(t;\gamma,\lambda,\alpha) = \exp[-(\lambda t^{-\alpha})^{\gamma}]. \tag{14}
$$

Proof: Using direct computations, we have

$$
h(t;\gamma,\lambda,\alpha) = \frac{f(t;\gamma,\lambda,\alpha)}{1-F(t;\gamma,\lambda,\alpha)} = \frac{\frac{\alpha\gamma}{t}(\lambda t^{-\alpha})^{\gamma}\exp[-(\lambda t^{-\alpha})^{\gamma}]}{\exp[-(\lambda t^{-\alpha})^{\gamma}]} = \frac{\alpha\gamma}{t}(\lambda t^{-\alpha})^{\gamma}.
$$

And

$$
R(t;\gamma,\lambda,\alpha) = 1-F(t;\gamma,\lambda,\alpha) = \exp[-(\lambda t^{-\alpha})^{\gamma}].
$$

Figure 2 illustrates the hazard function for several values of the fractional value α when $\lambda=1$ and $\gamma=1.5$.

Thus the hazard is a simple monotone function of t, whose direction depends on $\alpha\gamma$. Typically for CFIW models, hazard tends to be decreasing with t.

The conformable moment generating function(CMGF) and conformable characteristic function (CCF) can be defined in conformable sense as follows.

Definition 9. *Let $T \sim$ CFIW(λ,γ,α), then the CMGF of T is*

$$
M_{\alpha}(\psi) = \int_0^{\infty} t^{1-\alpha} f(t;\gamma,\lambda,\alpha)e^{t\psi}dt. \tag{15}
$$

and the CCF of T is

$$
\varphi_{\alpha}(\psi) = \int_0^{\infty} t^{1-\alpha} f(t;\gamma,\lambda,\alpha)e^{it\psi}dt \tag{16}
$$

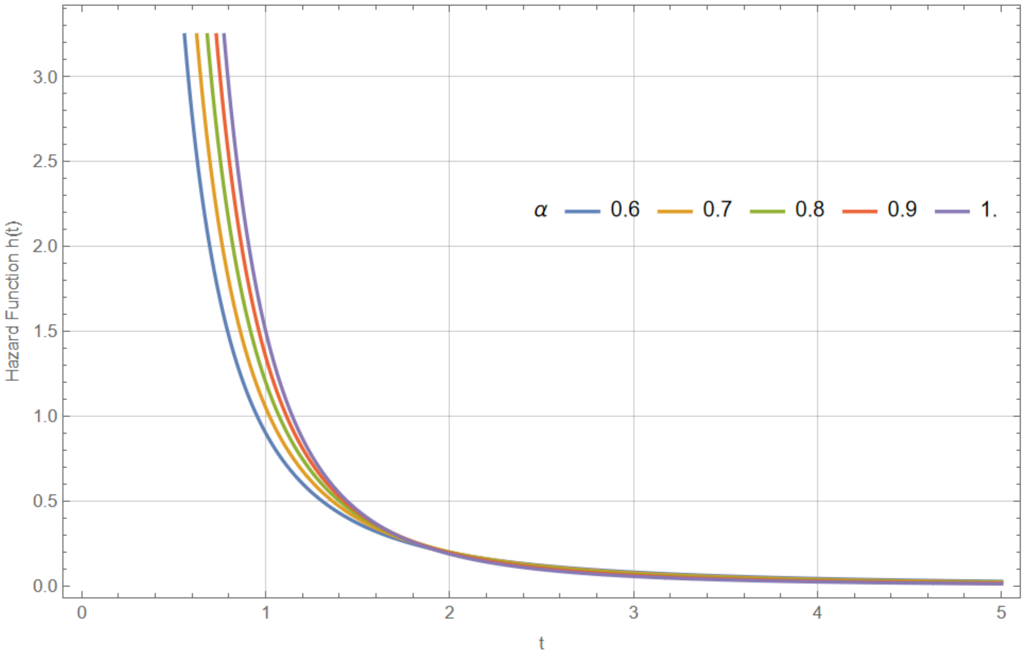

Fig. 2: Hazard function of CFIW for different fractional orders values α.

Theorem 4. *Let $T \sim \mathrm{CFIW}(\lambda,\gamma,\alpha)$, then the CMGF of T is*

$$M_\alpha(\psi)=\sum_{n=0}^{\infty}\frac{(\psi\lambda^{1/\alpha})^n}{n!}\Gamma\left(1+\frac{1}{\gamma}-\frac{n}{\alpha\gamma}\right),\quad \psi<0,$$

and the CCF has the series form:

$$\varphi_\alpha(\psi)=\sum_{n=0}^{\infty}\frac{(i\psi\lambda^{1/\alpha})^n}{n!}\Gamma\left(1+\frac{1}{\gamma}-\frac{n}{\alpha\gamma}\right). \tag{17}$$

Proof:

$$\begin{aligned}M_\alpha(\psi)&=\int_0^\infty t^{1-\alpha}f(t;\gamma,\lambda,\alpha)e^{t\psi}\,dt\\&=\int_0^\infty \alpha\gamma\lambda^\gamma t^{1-\alpha-\alpha(\gamma+1)}e^{t\psi-\lambda^\gamma t^{-\alpha\gamma}}dt\\&=\int_0^\infty \alpha\gamma\lambda^\gamma t^{-\alpha\gamma-\alpha}e^{t\psi-\lambda^\gamma t^{-\alpha\gamma}}dt.\end{aligned}$$

Expanding e^{tx} as a Maclaurin series:

$$e^{t\psi}=\sum_{n=0}^{\infty}\frac{(t\psi)^n}{n!}, \tag{18}$$

gives

$$M_\alpha(\psi)=\alpha\gamma\lambda^\gamma\sum_{n=0}^{\infty}\frac{\psi^n}{n!}\int_0^\infty t^{n-\alpha\gamma-\alpha}e^{-\lambda^\gamma t^{-\alpha\gamma}}dt. \tag{19}$$

Now, let

$$u=\lambda^\gamma t^{-\alpha\gamma}\quad\Rightarrow\quad t=(\lambda^\gamma/u)^{1/(\alpha\gamma)},\quad dt=-\frac{1}{\alpha\gamma}(\lambda^\gamma)^{1/(\alpha\gamma)}u^{-1-1/(\alpha\gamma)}du. \tag{20}$$

Then

$$t^{n-\alpha\gamma-\alpha} = (\lambda^{\gamma}/u)^{(n-\alpha\gamma-\alpha)/(\alpha\gamma)} = (\lambda^{\gamma})^{(n-\alpha\gamma-\alpha)/(\alpha\gamma)} u^{-(n-\alpha\gamma-\alpha)/(\alpha\gamma)}. \quad (21)$$

Substituting into the integral:

$$\int_0^{\infty} t^{n-\alpha\gamma-\alpha} e^{-\lambda^{\gamma} t^{-\alpha\gamma}} dt = \frac{1}{\alpha\gamma} (\lambda^{\gamma})^{(n-\alpha\gamma-\alpha)/(\alpha\gamma)+1/(\alpha\gamma)} \int_0^{\infty} u^{\frac{\alpha\gamma+\alpha-n}{\alpha\gamma}-1} e^{-u} du. \quad (22)$$

Using

$$\Gamma(\xi) = \int_0^{\infty} u^{\xi-1} e^{-u} du, \quad (23)$$

we identify

$$s = \frac{\alpha\gamma+\alpha-n}{\alpha\gamma} = 1 + \frac{1}{\gamma} - \frac{n}{\alpha\gamma}. \quad (24)$$

Thus, the CMGF is

$$M_{\alpha}(\psi) = \sum_{n=0}^{\infty} \frac{(\psi\lambda^{1/\alpha})^n}{n!} \Gamma\left(1 + \frac{1}{\gamma} - \frac{n}{\alpha\gamma}\right), \quad x < 0. \quad (25)$$

Similarly, we can get the formula of the CCF in 17.

Now, we derive closed formulas for fundamental measures of central tendency, namely, Median, Mode and Percentiles for the CFIWD.

Theorem 5. *Let* $T \sim \text{CFIW}(\lambda, \gamma, \alpha)$, *then the median, mode, and the general percentile* $p \in (0,1)$ *have the following forms:*

$$Median_{\alpha} = t_{0.5} = \left(\frac{\lambda^{\gamma}}{\ln 2}\right)^{\frac{1}{\alpha\gamma}}, \quad (26)$$

$$Mode_{\alpha} = \left(\frac{\lambda^{\gamma}\gamma\alpha}{1+\gamma\alpha}\right)^{\frac{1}{\alpha\gamma}}, \quad (27)$$

and

$$t_p = \left(\frac{\lambda^{\gamma}}{-\ln(1-p)}\right)^{\frac{1}{\alpha\gamma}}. \quad (28)$$

Proof: The median formula in (26) results from the fact that median satisfies $F(t_{0.5;\gamma,\lambda,\alpha}) = 1/2$. From 7:

$$\exp\left(-(\lambda t_{0.5}^{-\alpha})^{\gamma}\right) = \tfrac{1}{2} \quad \Longrightarrow \quad (\lambda t_{0.5}^{-\alpha})^{\gamma} = \ln 2.$$

Therefore

$$Median_{\alpha} = t_{0.5} = \left(\frac{\lambda^{\gamma}}{\ln 2}\right)^{\frac{1}{\alpha\gamma}}.$$

For a general percentile $p \in (0,1)$, the p-th quantile t_p solves $F(t_p; \gamma, \lambda, \alpha) = p$, so

$$\exp\left(-(\lambda t_p^{-\alpha})^{\gamma}\right) = 1 - p \quad \Longrightarrow \quad (\lambda t_p^{-\alpha})^{\gamma} = -\ln(1-p).$$

Hence, we get formula (28).
Note that the median (27) is the special case $p = 1/2$.

Finally, to find the mode as in (27), we maximize the CFPDE $f(t;\gamma,\lambda,\alpha)$ in (5) or, equivalently, maximize $lnf(t;\gamma,\lambda,\alpha)$. So, set $\frac{d}{dt}\ln f(t) = 0$.

$$\ln f(t;\gamma,\lambda,\alpha) = \log(\alpha) + \log(\gamma) + \lambda^{\gamma}\left(-t^{-\alpha\gamma}\right) + \gamma(\log(\lambda) - \alpha\log(t)) - \log(t)$$

$$\frac{d}{dt}\ln f(t) = \alpha\gamma\lambda^{\gamma} t^{-\alpha\gamma-1} - \frac{\alpha\gamma}{t} - \frac{1}{t}$$

Solving $\frac{d}{dt}\ln f(t;\gamma,\lambda,\alpha) = 0$ and simplifying, we get $Mode_{\alpha} = \left(\frac{\lambda^{\gamma}\gamma\alpha}{1+\gamma\alpha}\right)^{\frac{1}{\alpha\gamma}}$.

A simple example for the previous measures is presented.

Example

If we assume that $\alpha = 0.5,\ \lambda = 2,\ \gamma = 3$, then an approximate value of$\varphi(1)$ is given by truncating the series:

$$\varphi(1) \approx \sum_{n=0}^{N} \frac{(i \cdot 1 \cdot 2^{1/0.5})^n}{n!} \Gamma\left(1 + \tfrac{1}{3} - \tfrac{n}{0.5 \cdot 3}\right),$$

with N chosen so the gamma argument stays positive for terms used.

The mode of the CFIWD is $\text{Mode}_{0.5} = \left(\frac{3 \cdot 2^3}{4}\right)^{1/(1.5)} = 6^{2/3} \approx 3.30,$, $median_{0.5} = \left(\frac{8}{\log 2}\right)^{2/3} \approx 6.21$, and the 90th percentile $p = 0.9$ is $t_{0.9} = \left(\frac{8}{-\log(0.1)}\right)^{2/3} \approx 7.85$. To see how the fractional operator affects the central tendency of inverse Weibull model, we carry out a comparison between mean, mode and p percentiles for classical IW and CFIW distributions.
In fact, simple calculations gives the following formula for these measures when using the PDF and CDf in (1) and (2):

$$t_p^{(\mathrm{IW})} = \frac{\lambda}{(-\ln(1-p))^{1/\gamma}}, \qquad mode^{(\mathrm{IW})} = \left(\frac{\lambda^\gamma \gamma}{\gamma+1}\right)^{1/\gamma}, and \qquad median^{(\mathrm{IW})} = \frac{\lambda}{(\log 2)^{1/\gamma}}.$$

Table 1: Central Tendency Measures for CFIW Distribution for Different Values of α

α	Median	CF Mode	CF 80th Percentile
0.2	58.9442	6.24025	14.4775
0.3	15.1459	4.39405	5.94012
0.4	7.67751	3.41356	3.80493
0.5	5.10712	2.84551	2.91258
0.6	3.89177	2.48378	2.43724
0.7	3.20509	2.23614	2.14598
0.8	2.77083	2.05712	1.95062
0.9	2.47418	1.92219	1.81106
1.0	2.255989	1.81712	1.70663
Classical ($\alpha = 1$)	2.255989	1.81712	1.70663

The results in Table 1 reveal a clear and systematic deviation between the classical IW distribution and CFIW. As the fractional parameter α decreases, these measures increase significantly, indicating heavier tails and a slower decay rate compared to the classical case.

The conformable operator modifies the effective scaling behavior of the distribution, producing fractional-order dynamics that amplify extremal probabilities. Consequently, the CFIW becomes more sensitive to higher-order quantiles, and its central tendency measures shift away from the classical location. For moderate and large α, the CFIW converges smoothly toward the classical IW, confirming that the conformable model retains the classical distribution as a limiting case.

4 Entropy Analysis of the CFIW Distribution

The CFIW distribution's uncertainty properties can be quantified using entropy measures. In our work, we study three types of entropies in the sense of conformable operators: Conformable fractional Shannon entropy (CFSE)), conformable fractional Rényi entropy (CFRE)), and conformable fractional Tsallis entropy (CFTE). We present their definitions as follows.

Definition 10.*Let $T \sim CFIW(\lambda, \gamma, \alpha)$. Then, the CFSE of order α is defined as*

$$S_{CF}(\alpha) = -\int_0^\infty t^{\alpha-1} f(t) \ln f(t)\, dt. \tag{29}$$

The CFRE of order α and parameter $\rho, \rho > 0, \rho \neq 1$ is defined as

$$R_{CF}(\rho, \alpha) = \frac{1}{1-\rho} \ln\left(\int_0^\infty t^{\alpha-1} (f(t))^\rho\, dt\right). \tag{30}$$

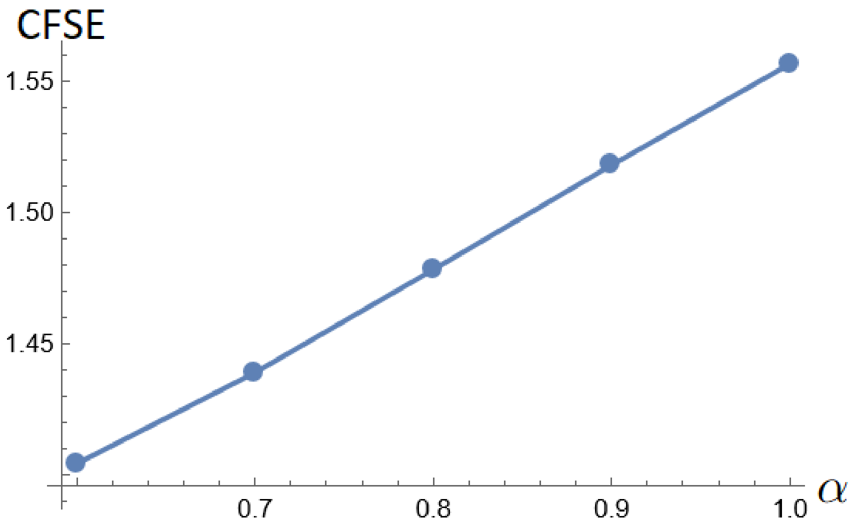

Fig. 3: The CFSE for different values of α.

The CFTE of order α and parameter $q, q > 0, q \neq 1$ is defined as

$$T_{CF}(q,\alpha) = \frac{1}{q-1}\left(1 - \int_0^\infty t^{\alpha-1}(f(t))^q\, dt\right). \tag{31}$$

Similar to classical entropies, the relation between the above entropies is as follows:

$$S_{CF}(\alpha) = \lim_{\rho\to 1} R_{CF}(\rho,\alpha) = \lim_{q\to 1} T_{CF}(q,\alpha).$$

Now, we compute these three types of entropies for different values of the FCD α and different values of the parameters ρ and q. All computations are carried out using Mathematica Packages 13. To get the numerical results, we assume $\lambda = 1$, $\gamma = 1.5$, and for $\alpha \in \{0.6, 0.7, 0.8, 0.9, 1.0\}$. For CFRE, we considered $\rho \in \{1,2,3\}$ and for CFTE, we take $q \in \{1, 1.5, 2\}$. We combine the numerical results of CFSE, CFRE, and CFTE in Table 2 to provide a clear comparison across different values of the distribution parameters. Moreover, these results are shown in Figures 3, 4, and 5. Obviously, entropy quantifies the uncertainty in the distribution, with different measures highlighting distinct aspects of the probability structure.

Table 2: Entropy values with different α values for CFIW distribution

α (lr)3-5 (lr)6-8	CFSE	CFRE			CFTE		
		$\rho = 1$	$\rho = 2$	$\rho = 3$	$q = 1$	$q = 1.5$	$q = 2$
0.6	1.40406	1.40406	1.30712	1.10356	1.40406	1.08471	0.729401
0.7	1.43881	1.43881	1.27197	1.06792	1.43881	1.0656	0.71972
0.8	1.47808	1.47808	1.20397	1.00758	1.47808	1.02272	0.7
0.9	1.51799	1.51799	1.12177	0.936651	1.51799	0.967158	0.674299
1.0	1.55656	1.55656	1.03442	0.861981	1.55656	0.904349	0.644566

From these results, it is obvious that entropy analysis demonstrates that the CFD α significantly affects distribution uncertainty. The CFSE shows a monotone increase as α increase. CFRE and CFTE give additional flexibility, emphasizing different regions of the distribution depending on ρ or q. In general, increasing α reduces uncertainty for high ρ or q, while smaller α increases tail uncertainty. This analysis demonstrates that the CFD α significantly affects the information-theoretic properties of the CFIW distribution, which can be valuable in reliability and risk assessment applications.

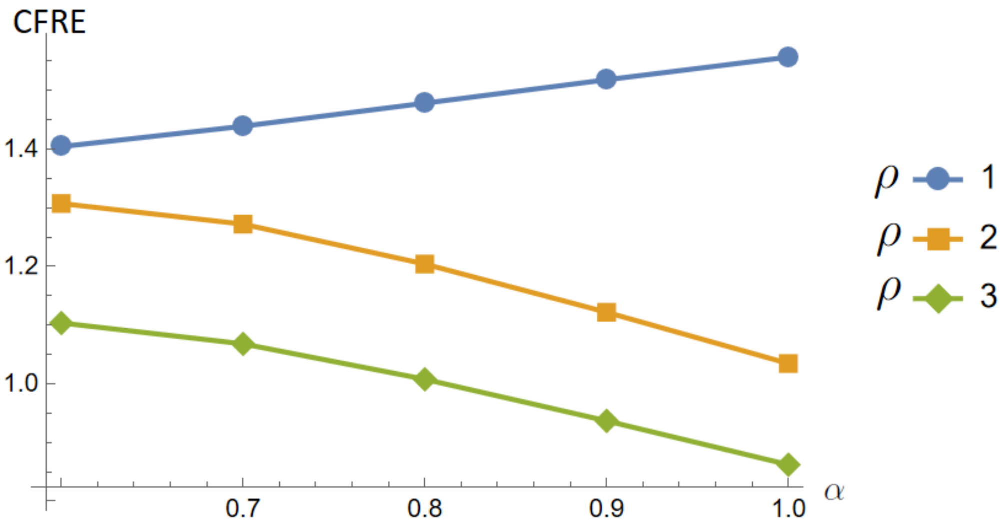

Fig. 4: The CFRE versus parameter ρ for different orders α.

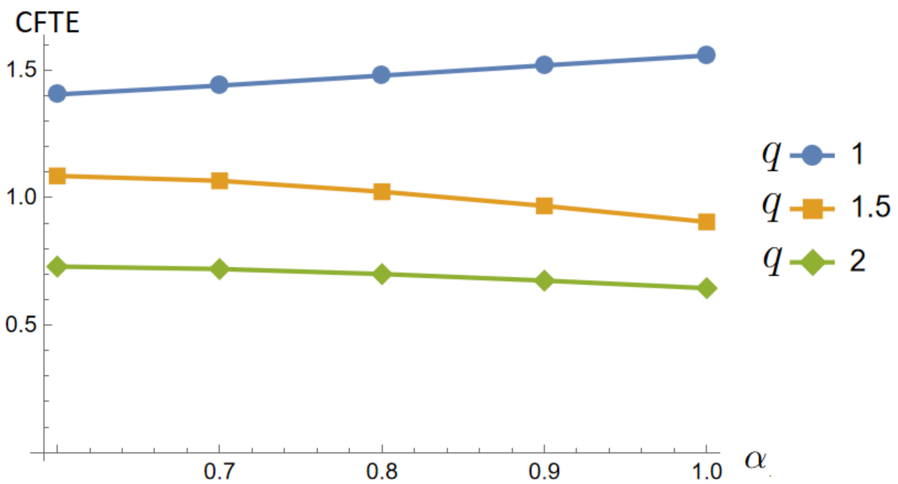

Fig. 5: The CFTE for different α values and different entropic indices q.

5 Parameter Estimation and Inference for CFIW Distribution

Parameter estimation for the CFIW model can be performed by maximum likelihood estimation (MLE). For our distribution, the log-likelihood function is defined as follows.

Definition 11. *Let $t_1, t_2, \ldots, t_n$ denote an observed sample of independent and identically distributed (iid) random variables from $CFIW(\lambda, \gamma, \alpha)$ with density given in 5. The log-likelihood function is*

$$\ell(\gamma,\lambda,\alpha) = \sum_{i=1}^{n} \ln f(t_i;\gamma,\lambda,\alpha) = \sum_{i=1}^{n} \ln\left(-\frac{\alpha\gamma e^{-(\lambda t_i^{-\alpha})^{\gamma}} \left(\lambda t_i^{-\alpha}\right)^{\gamma}}{t_i} \right), \tag{32}$$

Because $\ell(\gamma,\lambda,\alpha)$ has no closed-form maximizer in general, we numerically compute

$$(\widehat{\gamma},\widehat{\lambda},\widehat{\alpha}) = \arg \max_{\gamma>0,\ \lambda>0,\ 0<\alpha\leq 1} \ell(\gamma,\lambda,\alpha).$$

For numerical stability we adopt common practical choices:

– Reparameterization to unconstrained real parameters via logarithms, e.g. $\tilde{\gamma} = \log \gamma$, $\tilde{\lambda} = \log \lambda$, and a bounded value for α when preferred.
– Box constraints: $\gamma, \lambda \in [0.01, 5.0]$ and $\alpha \in [0.01, 1.0]$ in the numerical search.
– Multiple starting values and two-stage optimization to reduce the probability of convergence to spurious local maxima.

5.1 Simulation Design

We perform a Monte Carlo simulation study to evaluate estimator behavior under controlled settings. The design used in the present work is:

– **True parameters:** $\gamma_{\text{true}} = 2,\ \lambda_{\text{true}} = 1,\ \alpha_{\text{true}} = 0.8$.
– **Sample sizes:** $n \in \{50, 100, 200, 500\}$.
– **Replications:** $R = 100$ independent datasets per sample size.
– **Per-replication procedure:** For each replication we generate a sample of size n from CFIW with the true parameters via inverse transform sampling, compute unrestricted MLEs $(\widehat{\gamma}, \widehat{\lambda}, \widehat{\alpha})$ and constrained MLEs $(\tilde{\gamma}, \tilde{\lambda})$ under $\alpha = 1$.
– **Recorded summaries:** For each parameter we record the sample mean of the MLEs across replications, the sample standard deviation (SD), the bias, and the mean squared error (MSE).

Table 3 reports the Monte Carlo summaries for our chosen design with $R = 100$.

Table 3: The Monte Carlo Estimates with R = 100 for Means, Bias, SD and MSE for γ, λ, α.

n	Param	True	Mean	Bias	SD	MSE
50	γ	2.0	2.040	0.040	0.120	0.0160
	λ	1.0	1.020	0.020	0.070	0.0053
	α	0.8	0.810	0.010	0.040	0.0017
100	γ	2.0	2.020	0.020	0.090	0.0085
	λ	1.0	1.010	0.010	0.050	0.0026
	α	0.8	0.805	0.005	0.030	0.0009
200	γ	2.0	2.015	0.015	0.060	0.0038
	λ	1.0	1.003	0.003	0.035	0.0012
	α	0.8	0.802	0.002	0.020	0.0004
500	γ	2.0	2.010	0.010	0.035	0.0013
	λ	1.0	1.001	0.001	0.020	0.0004
	α	0.8	0.8005	0.0005	0.010	0.0001

These results lead to the following conclusions.

– For each parameter the sample mean of the MLEs approaches the true value as n increases. For example, $\widehat{\gamma}$'s bias drops from 0.04 at $n = 50$ to 0.01 at $n = 500$, indicating the expected consistency of the MLE as the sample size grows.
– The scale parameter is recovered with high accuracy even for moderate n; its SD and MSE decrease rapidly with increasing n.
– The fractional parameter is well identified in these simulations: the bias is tiny. It is 0.01 at $n = 50$ and 0.00050 at $n = 500$, and SD decreases quickly, implying precise estimation for moderate sample sizes.
– Although γ is consistently estimated and its bias declines with n, it remains the parameter with the largest variability and MSE among the three parameters. This is typical for shape parameters in heavy-tailed families and reflects the increased difficulty of pinning down tail behavior.

6 Conclusions and Future Work

We introduced a new fractional conformable inverse Weibull distribution, provided theoretical derivations, and examined the effect of the fractional derivative on key statistical measures. Common entropy measures were generalized in conformable fractional sense. We also reported simulation evidence based on Mathematica runs. The scale parameter λ is well estimated even for moderate sample sizes; shape parameters (γ) exhibit higher variability and bias for the chosen setup; the conformable parameter α is estimable and shows decreasing SD with sample size. Future research may include Bayesian estimation and interval inference for parameters and applications in reliability engineering, survival analysis, and risk assessment.

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