Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/200107

Graded (L,M)-Fuzzy α-Continuous and (L,M)-Fuzzy α-Irresolute Functions

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Received: 7 Jun. 2025, Revised: 21 Sep. 2025, Accepted: 1 Nov. 2025
Published online: 1 Jan. 2026

Abstract: The objective of this paper is to investigate α-openness, α-continuity, and α-irresoluteness for functions within the framework of (L,M)-fuzzy pretopology, using the implication operation and the (L,M)-fuzzy α-open operator as introduced by Li [1]. We further expand on the characteristics of these concepts so that they apply to the (L,M)-fuzzy pretopological environment, which is based on graded principles. Moreover, we analyze their connections to the concepts of α-compactness, α-connectedness, and the axioms associated with α-separation.

Keywords: (L,M)-fuzzy pretopology; (L,M)-fuzzy α open operator; (L,M)-fuzzy α continuous function; (L,M)-fuzzy α irresolute function, (L,M)-fuzzy α compactness function.

1 Introduction

After Chang introduced fuzzy set theory to topology, the principles of general topology have been effectively extended into the fuzzy realm using precise methods. In Chang's I-topology for a set X, each open set is considered fuzzy; however, the collection of these open sets forms a crisp subset of the I-powerset I^X, where I represents the unit interval $[0,1]$. Additionally, the topology itself is a crisp subset of all fuzzy subsets present in X.

On another note, the very concept of a fuzzy topology was initially articulated by Höhle in 1980. This idea was later independently expanded upon by Kubiak [4] and Šostak [5] in 1985 was also rediscovered by Ying in the original context proposed by Höhle. In 1991, Ying [7] put forth a logical perspective on fuzzifying topology, which shares similarities with Höhle's concept of fuzzy topology.

The emergence of fuzzy topology brought with it various topological properties, such as connectedness, separability, compactness, and filter convergence, each characterized by their own degrees. In 1988, Šostak introduced the concept of connectedness degree through level L-topological spaces [8,9]. Yue and Fang [10] later defined connectivity for the entire L-fuzzy topological space. Pang defined the degrees of openness, closeness, and continuity for functions in L-fuzzifying topology in [11], and he also generalized several general topology results to the framework of L-fuzzifying topology. More recently, Liang and Shi [12] explained the degrees of functions being open, closed, or continuous using implication operations, broadening many fundamental properties from regular topological spaces to (L,M)-fuzzy topological spaces using graded concepts. In 2009, Shi [13] presented degrees of separatedness and connectedness for L-subsets based on the L-fuzzy closure operator. Shi also introduced L-fuzzy semiopen and L-fuzzy preopen operators in L-fuzzy pretopological spaces, which respectively evaluate the degrees of semiopenness and preopenness of an L-subset [14]. Additionally, the idea of semi-compactness was defined and explored using Shi's operator in [15,16]. Furthermore, the degree of preconnectedness proposed by Ghareeb [17] was established with the help of the L-fuzzy preopen operator.

Wadei and Gareeb [18,19,31] recently introduced and examined various concepts such as semiopenness, semicontinuity, preopenness, precontinuity, irresoluteness, and preirresoluteness regarding functions in (L,M)-fuzzy topological spaces, with a focus on

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implication operations as well as several separation properties related to (L,M)-fuzzy topology and neighborhoods. Their foundational topological properties will be broadened to encompass the context of (L,M)-fuzzy topology through the application of graded concepts. The concepts of semi-preopenness, semi-precontinuity, and semi-preirresoluteness for functions within (L,M)-fuzzy pretopology, as introduced by Gareeb and Wadei [20], will be articulated using implication operations alongside the (L,M)-fuzzy semi-preopen operator.

This paper seeks to describe (L,M)-fuzzy pretopological spaces through the lens of sub-varieties representing new degrees of functions. We introduced novel degrees of weak forms of functions associated with (L,M)-fuzzy pretopological spaces, employing implication operations and Shi's operators in our analysis.

The organization of this paper is as follows. Section 2 outlines fundamental concepts in general topology and revisits (L,M)-fuzzy topology. In Section 3, we define the terms necessary to elucidate the concepts of α-openness, α-continuity, and α-irresoluteness pertaining to functions in (L,M)-fuzzy pretopological spaces. Additionally, we characterize these definitions and analyze various properties linked to them. Section 4 investigates the connections among α-compactness, α-connectedness, and the degrees of certain α-separation axioms in the realm of (L,M)-fuzzy pretopological spaces. We also address theoretical applications related to the degrees associated with (L,M)-fuzzy pretopology.

In a recent study, Wadei et al. [20,29,30] revealed novel degrees of weak function forms in (L,M)-fuzzy pretopological spaces through the implementation of implication operations and Ghareeb's operators. They investigated various properties including semi-preopenness, semi-precontinuity, and semi-preirresoluteness degree of functions in (L,M)-fuzzy pretopology. Moreover, Wadei and Gareeb [18] provided insights into the relationships involving the degrees of semi(pre)-compactness, semi(pre)-connectedness, as well as Semi-T_1, Pre-T_1, Semi-T_2, and Pre-T_2 properties of functions in the realm of (L,M)-fuzzy topological spaces, framed through implication operations. In 2016, Li and Jia et al. [32] introduced an innovative operator based on the L-fuzzy open operator to quantify the degree of α-openness in an L-subset. They also discussed the notions of α-compactness and α-separatedness for an L-subset in L-fuzzy topological spaces, driven by the L-fuzzy α-open operator, along with the relationship between L-fuzzy α-compactness and fuzzy α-compactness within L-topological spaces [32].

The primary objective of this study is to elucidate the notions of α-openness, α-continuity, and α-irresoluteness in relation to functions operating within (L,M)-fuzzy pretopological spaces, based on the (L,M)-fuzzy α-open operator [17] and the associated implication operation. We will delve into the distinctive properties associated with these new definitions from a graded conceptual perspective. Moreover, the paper will explore the interrelations among α-compactness, α-connectedness, and certain degrees of α-separation axioms within the context of (L,M)-fuzzy pretopological spaces.

2 Preliminaries

In this document, unless otherwise indicated, symbols L and M refer to completely distributive De Morgan algebras, and X is identified as a nonempty set. The minimum and maximum elements within L and M are represented by 0_L, 1_L, and 0_M, 1_M, respectively. For any elements a and b in L, we say that a is wedge below b [21], denoted as $a \leq b$, if every subset $\mathscr{A} \subseteq L$ satisfying $\mathscr{A} \geq b$ leads to the conclusion that some $c \in \mathscr{A}$ satisfies $c \geq a$. The complete lattice L is deemed a completely distributive lattice if and only if for every $b \in L$, it holds that $b = \bigvee_{a \leq b} a$. An element $b \in L$ is considered co-prime if the condition $a \leq b \vee c$ necessitates that $a \leq b$ or $a \leq c$. The set of all non-zero co-prime elements in L is designated as $P(L)$. We utilize $L^{\mathscr{X}}$ to refer to the collection of L-subsets formed on X. The least and greatest L-subsets in $L^{\mathscr{X}}$ are represented as $\underline{0}_{L^{\mathscr{X}}}$ and $\underline{1}_{L^{\mathscr{X}}}$, respectively.

Moreover, 2 denotes the set of all finite sub-families of contained in $L^{\mathscr{X}}$. It is apparent that $L^{\mathscr{X}}$ functions as a completely distributive De Morgan algebra. Additionally, the collection $\{x_a\}_{a \in P(L)}$ consists of non-zero co-prime elements found in $L^{\mathscr{X}}$. The implication operation $\mapsto$: $M \times M \to M$ is defined within any completely distributive De Morgan algebra M, expressed as $a \mapsto b = \bigvee_{a \wedge c \leq b} c$. The operation "$\leftrightarrow$" is also defined on M, based on the operation "$\mapsto$," as

$$a \leftrightarrow b = (a \mapsto b) \wedge (b \mapsto a).$$

An (L,M)-fuzzy inclusion [22,23] pertaining to the set X is specified by a mapping $\widetilde{\subset} : L^{\mathscr{X}} \times L^{\mathscr{X}} \to M$, characterized by the equation

$$\widetilde{\subset}(G_1, G_2) = \bigwedge_{x \in X} \left\{ G_1'(\mathscr{X}) \vee G_2(\mathscr{X}) \right\}.$$

It is customary to denote the (L,M)-fuzzy inclusion $\widetilde{\subset}(G_1, G_2)$ by the symbol $[G_1 \widetilde{\subset} G_2]$.

Lemma 21[24] *Given a completely distributive lattice $(M, \bigvee, \bigwedge)$ and an implication operation $\mapsto$ defined on the set M, we can assert the following:*

(0) $(x \mapsto y) \geq z \Longleftrightarrow x \wedge z \leq y$. $x \leq y \Leftrightarrow x \mapsto y = 1_M$. $x \mapsto (y \mapsto z) = (x \wedge y) \mapsto z$. $(z \mapsto x) \wedge (x \mapsto y) \leq z \mapsto y$. $z \mapsto x \leq (x \mapsto y) \mapsto (z \mapsto y)$. $x \mapsto \bigwedge_{i\in I} x_i = \bigwedge_{i\in I}(x \mapsto x_i)$, henze $x \mapsto y \leq x \mapsto z$ whenever $y \leq z$. $\bigvee_{i\in I} x_i \mapsto y = \bigwedge_{i\in I}(x_i \mapsto y)$, henze $x \mapsto z \geq y \mapsto z$ whenever $x \leq y$,

for each $x,y,z \in M$, $\{x_i\}_{i\in I}$, *and* $\{y_i\}_{i\in I} \subseteq M$.

5.Lemma 22[25] *Let* $g : \mathscr{Z} \to \mathscr{R}$ *be a mapping. Then for each* $\mathscr{H} \subseteq L^{\mathscr{R}}$, *we have*

$$\bigwedge_{y\in\mathscr{R}} \left\{ g_L^{\rightarrow}(G)'(y) \vee \bigvee_{H\in\mathscr{H}} H(y) \right\} = \bigwedge_{x\in\mathscr{Z}} \left\{ G'(\mathscr{Z}) \vee \bigvee_{H\in\mathscr{H}} g_L^{\leftarrow}(H)(\mathscr{Z}) \right\}.$$

Definition 23[26] *Let* $(\mathscr{Z},\Gamma)$ *be an L-fts and* $S \in L^{\mathscr{Z}}$. *Sn L-fuzzy* α*-open operator* $\mathscr{T}_\alpha : L^{\mathscr{Z}} \to L$ *is given by*

$$\mathscr{T}_\alpha(S) = \bigvee_{B\leq S} \left\{ \mathscr{T}(B) \wedge \bigwedge_{x_\lambda \prec S} \bigvee_{x_\lambda \prec C} \left(\mathscr{T}(C) \wedge \bigwedge_{y_\mu \prec C} \bigwedge_{y_\mu \not\leq D \geqslant B} (\mathscr{T}(D'))', \right.\right.$$

at this time, $\mathscr{T}_\alpha$ *is a* $\mathscr{T}$*-induced L-fuzzy* α*-open operation, and* $\mathscr{T}_\alpha(S)$ *says that S is a* α*-open set of degrees.*

Definition 24*For any non-empty set* $\mathscr{Z}$, *a function* $\mathscr{T} : L^{\mathscr{Z}} \longrightarrow M$ *which satisfies the following statements:*

($\mathscr{P}$*0)* $\mathscr{T}(\underline{1}_L) = \mathscr{T}(\underline{0}_L) = 1_M$, $\mathscr{T}\left(\bigvee_{i\in I} G_i\right) \geq \bigwedge_{i\in I} \mathscr{T}(G_i)$ *for all* $\{G_i\}_{i\in I} \subseteq L^{\mathscr{Z}}$,

Sn (L,M)*-fuzzy pretopology on* $\mathscr{Z}$ *is denoted as such* [27]*. The structure formed by the pair* $(\mathscr{Z},\Gamma)$ *is referred to as an* (L,M)*-fuzzy pretopological space, abbreviated as* (L,M)*-fpts. When an* (L,M)*-fuzzy pretopology* $\mathscr{T}$ *fulfills specific criteria, it is recognized as an* (L,M)*-fuzzy topology, commonly referred to as* (L,M)*-ft, on* $\mathscr{Z}$ [24,4,28,5]*. If it meets the following additional condition:*

($\mathscr{O}$*)* $\mathscr{T}(G_1 \wedge G_2) \geq \mathscr{T}(G_1) \wedge \mathscr{T}(G_2)$ *for all* $G_1, G_2 \in L^{\mathscr{Z}}$.

The pair $(\mathscr{Z},\Gamma)$ *is called* (L,M)*-fuzzy topological space ((L,M)-fts, for short). The gradation of openness and closeness of an L-subset G is given by* $\mathscr{T}(G)$ *and* $\mathscr{T}^*(G)$ *respectively, where* $\mathscr{T}^*(G) = \mathscr{T}(G')$*. S mapping* $g : (\mathscr{Z},\Gamma_1) \longrightarrow (\mathscr{R},\Gamma_2)$ *is called an* (L,M)*-fuzzy continuous iff* $\mathscr{T}_1(g_L^{\leftarrow}(H)) \geq \mathscr{T}_2(H)$ *for all* $H \in L^{\mathscr{R}}$.

2.Definition 25*Consider* $(\mathscr{Z},\Gamma)$ *as an* (L,M)*-pfts where S belongs to* $L^{\mathscr{Z}}$*. The* (L,M)*-fuzzy* α*-open operator, labeled as* $\mathscr{T}_\alpha : L^{\mathscr{Z}} \to M$*, is characterized by the formula* $\mathscr{T}_\alpha(S) = \bigvee_{B\leq S}\left\{\mathscr{T}(B) \wedge \bigwedge_{x_\lambda \prec S} \bigvee_{x_\lambda \prec C}\left(\mathscr{T}(C) \wedge \bigwedge_{y_\mu \prec C}\right.\right.$ $\bigwedge_{y_\mu leqD \geqslant B} (\mathscr{T}(D'))'$*. Here,* $\mathscr{T}_\alpha(S)$ *and* $\mathscr{T}_\alpha(S')$ *indicate the degrees of* α*-openness and* α*-closeness related to the L-subset S, respectively.*

Lemma 26*For any* (L,M)*-pfts* $(\mathscr{Z},\Gamma)$*, the* (L,M)*-fuzzy* α*-open operator* $\mathscr{T}_\alpha$ *fulfills the following assertions:*

(1) $\mathscr{T}_\alpha(\underline{0}_L) = \mathscr{T}_\alpha(\underline{1}_L) = 1_M$,
(2) $\mathscr{T}_\alpha(\bigvee_{i\in I} G_i) \geq \bigwedge_{i\in I} \mathscr{T}_\alpha(G_i)$ *for any* $\{G_i\}_{i\in I} \subseteq L^{\mathscr{Z}}$.

*Proof.*Trivial.

Corollary 27*In the context of any* (L,M)*-pfts* $(\mathscr{Z},\Gamma)$*, it follows that if* $\mathscr{T}_\alpha$ *represents the associated* (L,M)*-fuzzy* α*-open operator, then for every* $S \in L^{\mathscr{Z}}$*, it holds that* $\mathscr{T}(S) \leq \mathscr{T}_\alpha(S)$.

Definition 28*Consider* $(\mathscr{Z},\Gamma_2)$ *and* $(\mathscr{R},\Gamma_2)$ *as two* (L,M)*-pfts. The mapping* $g : (\mathscr{Z},\Gamma_1) \to (\mathscr{R},\Gamma_2)$ *is characterized as:*

1.(L,M)*-fuzzy* α*-continuous mapping iff* $\mathscr{T}_2(B) \leq (\mathscr{T}_1)_\alpha(g_L^{\leftarrow}(B))$ *for any* $B \in L^{\mathscr{R}}$.
2.(L,M)*-fuzzy* α*-irresolute mapping iff* $(\mathscr{T}_2)_\alpha(B) \leq (\mathscr{T}_1)_\alpha(g_L^{\leftarrow}(B))$ *for any* $B \in L^{\mathscr{R}}$.

Corollary 29*Consider a function* $g : (\mathscr{Z},\Gamma_1) \longrightarrow (\mathscr{R},\Gamma_2)$ *that establishes a relation between two* (L,M)*-pfts, namely* $(\mathscr{Z},\Gamma_1)$ *and* $(\mathscr{R},\Gamma_2)$.

1.g is (L,M)*-fuzzy* α*-continuous iff* $\mathscr{T}_2^*(B) \leq (\mathscr{T}_\alpha)_1^*(g_L^{\leftarrow}(B))$ *for all* $B \in L^{\mathscr{R}}$.
2.g is (L,M)*-fuzzy* α*-irresolute iff* $(\mathscr{T}_\alpha)_2^*(B) \leq (\mathscr{T}_\alpha)_1^*(g_L^{\leftarrow}(B))$ *for all* $B \in L^{\mathscr{R}}$.

Theorem 210*Consider a function* $g : (\mathscr{Z},\Gamma_1) \longrightarrow (\mathscr{R},\Gamma_2)$ *that represents a mapping between the two* (L,M)*-pfts* $(\mathscr{Z},\Gamma_1)$ *and* $(\mathscr{R},\Gamma_2)$.

1.If g is (L,M)*-fuzzy continuous, then g is* (L,M)*-fuzzy* α*-continuous.*
2.If g is (L,M)*-fuzzy* α*-irresolute, then g is* (L,M)*-fuzzy* α*-continuous.*

Proof. 1.If g is (L,M)-fuzzy continuous mapping, then $\mathscr{T}_2(B) \leq \mathscr{T}_1(g_L^{\leftarrow}(B))$ for any $B \in L^{\mathscr{R}}$. By using Corollary 27 i.e ($\mathscr{T}(S) \leq \mathscr{T}_\alpha(S)$), we have

$$\mathscr{T}_2(B) \leq \mathscr{T}_1(g_L^{\leftarrow}(B)) \leq (\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(B)),$$

for any $B \in L^{\mathscr{R}}$. Hence g is an (L,M)-fuzzy α-continuous.

2.If g is (L,M)-fuzzy α-irresolute mapping, then $(\mathscr{T}_\alpha)_2(B) \leq (\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(B))$ for all $B \in L^{\mathscr{R}}$. Based on Corollary 27, we get

$$\mathscr{T}_2(B) \leq (\mathscr{T}_\alpha)_2(B) \leq (\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(B)),$$

for any $B \in L^{\mathscr{R}}$. Hence g is an (L,M)-fuzzy α-continuous.

Definition 211*A mapping* $g : (\mathscr{Z},\mathscr{T}) \to (\mathscr{R},\Gamma_2)$ *between two* (L,M)*-pfts, which are* $(\mathscr{Z},\Gamma_1)$ *and* $(\mathscr{R},\Gamma_2)$*, is defined as an* (L,M)*-fuzzy* α*-open mapping if, for all S in* $L^{\mathscr{Z}}$*, the condition* $\mathscr{T}_1(S) \leq (\mathscr{T}_\alpha)_2(g_L^{\rightarrow}(S))$ *is satisfied.*

Theorem 212[1] *If $\mathscr{T}_\alpha$ is an (L,M)-fuzzy α-open operator on $\mathscr{Z}$ and αCl is an (L,M)-fuzzy α-closure operator. Then, we have*

$$\alpha Cl(S)(\mathscr{Z}_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq C} (\mathscr{T}_\alpha(B'))',$$

for all $x_\lambda \in P(L^{\mathscr{Z}})$ and $S \in L^{\mathscr{Z}}$.

Definition 213*An (L,M)-fuzzy α-closure operator on $\mathscr{Z}$ is defined as a function $\alpha Cl : L^{\mathscr{Z}} \to M^{P(L^{\mathscr{Z}})}$ that meets the subsequent criteria:*

1. $\alpha Cl(S)(\mathscr{Z}_\lambda) = \bigwedge_{\mu\leq\lambda} \alpha Cl(S)(\mathscr{Z}_\mu)$ *for all* $x_\lambda \in P(L^{\mathscr{Z}})$.
2. $\alpha Cl(\underline{0}_L)(\mathscr{Z}_\lambda) = 0_M$ *for all* $x_\lambda \in P(L^{\mathscr{Z}})$.
3. $\alpha Cl(S)(\mathscr{Z}_\lambda) = 1_M$ *for all* $x_\lambda \leq S$.
4. *for any* $\mu \in M_0$, $(\alpha Cl(\bigvee(\alpha Cl(S))_{[\mu]}))_{[\mu]} \subset (\alpha Cl(S))_{[\mu]}$.

The term $\alpha Cl(S)(\mathscr{Z}_\lambda)$ reflects how much x_λ is a member of the α-closure associated with the L-subset S.

Theorem 214[1] *The L-fuzzy topological space is denoted by $\mathscr{T}_\alpha$. Sssuming $a \in M(L)$ and considering $\mathscr{T}[a]$, we can assert that for every element x_λ that is less than S, there exists a set C with x_λ as a member of C such that $\mathscr{T}(C) \geq a$. Moreover, for all y_μ that are less than or equal to C and which do not exceed D that is at least B, the condition $(\mathscr{T}_\alpha(B'))' \geq a$ is satisfied. This can be restated as follows: for any x_λ less than S, a set C exists containing x_λ where $\mathscr{T}(C) \geq a$, and for every y_μ that is a subset of C, ky_μ must be less than or equal to B^-.*

Definition 215*Sn (L,M)-fuzzy quasi α-neighborhood system on $\mathscr{Z}$ is a family $\alpha Q = \{\alpha Q_{x_\lambda} \mid x_\lambda \in P(L^{\mathscr{Z}})\}$ of mappings $\{\alpha Q_{x_\lambda} : L^{\mathscr{Z}} \longrightarrow M\}$ which satisfying the following axioms:*

1. $\alpha Q_{x_\lambda}(\underline{0}_{L^{\mathscr{Z}}}) = 0_M$.
2. $\alpha Q_{x_\lambda}(S) \neq 0_M$, *then* $x_\lambda \not\leq S'$.
3. $\alpha Q_{x_\lambda}(B \wedge C) \leq \alpha Q_{x_\lambda}(B) \wedge \alpha Q_{x_\lambda}(C)$.
4. $\alpha Q_{x_\lambda}(S) = \bigvee_{x_\lambda \not\leq B \geq S'} \bigwedge_{y_\mu \not\leq B'} \alpha Q_{y_\mu}(B)$.

From [26] and based on Lemma 26, we can easily prove Theorems 216 and 219.

Theorem 216*Consider the pair $(\mathscr{Z},\Gamma)$ as an (L,M)-pfts, where $\mathscr{T}_\alpha$ represents the corresponding (L,M)-fuzzy α-open operator. Define $\alpha Q^{\mathscr{T}_\alpha} = \{\alpha Q_{x_\lambda} \mid x_\lambda \in P(L^{\mathscr{Z}})\}$ to denote the (L,M)-fuzzy quasi α-neighborhood system generated by $\mathscr{T}_\alpha$. If we establish the mapping $\alpha C^{\mathscr{T}_\alpha} : L^{\mathscr{Z}} \longrightarrow M^{P(L^{\mathscr{Z}})}$ as expressed by*

$$\alpha C^{\mathscr{T}_\alpha}(S)(\mathscr{Z}_\lambda) = \left(\alpha Q_{x_\lambda}(G')\right)',$$

it follows that $\alpha C^{\mathscr{T}_\alpha}$ functions as an (L,M)-fuzzy α-closure operator on $\mathscr{Z}$.

Definition 217*The (L,M)-fuzzy α-neighborhood system situated on $\mathscr{Z}$ can be characterized by the set $\alpha N = \{\alpha N_{x_\lambda} \mid x_\lambda \in P(L^{\mathscr{Z}})\}$, comprising mappings $\{\alpha N_{x_\lambda} : L^{\mathscr{Z}} \longrightarrow M\}$ that fulfill the following criteria:*

1. $\alpha N_{x_\lambda}(\underline{0}_{L^{\mathscr{Z}}}) = 0_M$.
2. $\alpha N_{x_\lambda}(S) \neq 0_M$, *then* $x_\lambda \not\leq S$.
3. $\alpha N_{x_\lambda}(B \wedge C) \leq \alpha N_{x_\lambda}(B) \wedge \alpha N_{x_\lambda}(C)$.
4. $\alpha N_{x_\lambda}(S) = \bigvee_{x_\lambda \leq B \geq S} \bigwedge_{y_\mu \leq B} \alpha N_{y_\mu}(B)$.

Definition 218*The mapping known as the (L,M)-fuzzy α-interior operator on $\mathscr{Z}$, denoted $\alpha I : L^{\mathscr{Z}} \longrightarrow M^{P(L^{\mathscr{Z}})}$, is required to fulfill the following criteria:*

1. $\alpha I(S)(\mathscr{Z}_\lambda) = \bigwedge_{\mu\leq\lambda} \alpha I(S)(\mathscr{Z}_\mu)$ *for all* $x_\lambda \in P(L^{\mathscr{Z}})$.
2. $\alpha I(\underline{1}_{L^{\mathscr{Z}}})(\mathscr{Z}_\lambda) = 1_M$ *for all* $x_\lambda \in P(L^{\mathscr{Z}})$.
3. $\alpha I(S)(\mathscr{Z}_\lambda) = 0_M$ *for all* $x_\lambda \not\leq S$.
4. *For any* $\mu \in M_0$, $(\alpha I(S))^{(\mu)} \subseteq (\alpha I(\bigvee(\alpha I(S))^{(\mu)}))^{(\mu)}$.

The value $\alpha I(S)(\mathscr{Z}_\lambda)$ represent the degree to which x_λ belongs to the α-interior of S.

Theorem 219*Let $(\mathscr{Z},\Gamma)$ denote an (L,M)-pfts, and consider $\alpha N^{\mathscr{T}} = \{\alpha N^{\mathscr{T}}_{x_\lambda} \mid x_\lambda \in P(L^{\mathscr{Z}})\}$ as the corresponding (L,M)-fuzzy α-neighborhood system. If we define the function $\alpha I^{\mathscr{T}_\alpha} : L^{\mathscr{Z}} \to M^{P(L^{\mathscr{Z}})}$ through the equation $\alpha I^{\mathscr{T}_\alpha}(S)(\mathscr{Z}_\lambda) = \alpha N^{\mathscr{T}_\alpha}_{x_\lambda}(S)$, then it can be concluded that $\alpha I^{\mathscr{T}_\alpha}$ functions as an (L,M)-fuzzy α-interior operator on $\mathscr{Z}$.*

Definition 220*For an (L,M)-fpts $(\mathscr{Z},\Gamma)$ and $S \in L^{\mathscr{Z}}$. Let*

$$\alpha\mathbf{Con}(S) = \bigwedge_{\substack{B,C\in L^{\mathscr{Z}}\setminus\{\underline{0}\},\\ S=B\vee C}} \left\{ \bigvee_{x_\lambda\leq B} \alpha Cl(C)(\mathscr{Z}_\lambda) \vee \bigvee_{y_\mu\leq C} \alpha Cl(B)(y_\mu) \right\}.$$

Then $\alpha C(S)$ is called the degree of fuzzy α-connectedness of an L-subset S.

Theorem 221*Let $(\mathscr{Z},\Gamma)$ be an (L,M)-pfts and $S \in L^{\mathscr{Z}}$. Then*

$$\alpha\mathbf{Con}(S) = \bigwedge_{\substack{S\wedge B\neq\underline{0}_{L^{\mathscr{Z}}},\ S\wedge C\neq\underline{0}_{L^{\mathscr{Z}}},\\ S\wedge B\wedge C\neq\underline{0}_{L^{\mathscr{Z}}},\ S\leq B\vee C}} \left\{(\mathscr{T}_\alpha(B'))' \vee (\mathscr{T}_\alpha(C'))'\right\}.$$

In the work of Li [1], the idea of α-compactness was introduced in the setting of L-fuzzy topology, based on the (L,M)-fuzzy semi-preopen operator. The following definitions will clarify the concepts of degree α-compactness, as well as α–T_1 and α–T_2 in (L,M)-fuzzy pretopology by utilizing the implication operation.

Definition 222*Consider $(\mathscr{X},\Gamma)$ as an (L,M)-pfts. In the event that a subset $\mathscr{P}$ is present for $L^{\mathscr{X}}$, it follows that there is*

$$\alpha\mathbf{com}_{\mathscr{T}_\alpha}(S) = \bigwedge_{F\in\mathscr{P}} \mathscr{T}_\alpha(g) \wedge \bigwedge_{x\in\mathscr{X}} (S'(\mathscr{X}) \vee \bigwedge_{F\in\mathscr{P}} F(\mathscr{X})) \leq \bigvee_{\mathscr{Q}\in 2^{\mathscr{P}}} \bigwedge_{x\in\mathscr{X}} (S'(\mathscr{X}) \vee \bigwedge_{F\in\mathscr{Q}} F(\mathscr{X})),$$

Then say S is(L,M)-fuzzy-compact. Here $2^{\mathscr{P}}$ is a set of all finite subsets of $\mathscr{P}$

Definition 223*If $(\mathscr{X},\mathscr{T})$ be an (L,M)-pfts, then*

1.The degree α-$T_1(\mathscr{X},)$ to which $\mathscr{T}$ is α-T_1 is given by:

$$\alpha\text{-}T_1(\mathscr{X},\mathscr{T}) = \bigwedge_{v_1\not\leq v_2} \bigvee_{v_1\not\leq S\geq v_2} \mathscr{T}_\alpha(S').$$

2.The degree α-$T_2(\mathscr{X},\mathscr{T})$ to which $\mathscr{T}$ is α-T_2 is given by:

$$\alpha\text{-}T_2(\mathscr{X},\mathscr{T}) = \bigwedge_{v_1\not\leq v_2} \bigvee \Big\{ \mathscr{T}_\alpha(B') \wedge \mathscr{T}_\alpha(C) \mid v_1 \not\leq B \geq C \geq v_2 \Big\}.$$

3 α-openness, α-continuity and α-irresolutness degree of mappings based on (L,M)-fuzzy α-reopen operator

We present the concepts of α-openness, α-continuity, and the degree of α-irresolutness in the context of (L,M)-fuzzy pretopology. Furthermore, we define these concepts through the framework of (L,M)-fuzzy quasi α-neighborhood systems, (L,M)-fuzzy α-neighborhood systems, (L,M)-fuzzy α-interior operators, and (L,M)-fuzzy α-closure operators.

Definition 31*Suppose $g:(\mathscr{X},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ represents a function connecting two (L,M)-fpts $(\mathscr{X},\Gamma_1)$ and $(\mathscr{R},\Gamma_2)$; then:*

1.The degree $\alpha\mathbf{C}(g)$ to which g is α-continuous is given by

$$\alpha\mathbf{C}(g) = \bigwedge_{U\in L^{\mathscr{R}}} \Big\{ \mathscr{T}_2(U) \mapsto (\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U)) \Big\}.$$

2.The degree $\alpha\mathbf{OP}(g)$ to which g α-open is given by

$$\alpha\mathbf{OP}(g) = \bigwedge_{S\in L^{\mathscr{X}}} \Big\{ (\mathscr{T}_\alpha)_1(S) \mapsto (\mathscr{T}_\alpha)_2(g_L^{\rightarrow}(S)) \Big\}.$$

3.The degree $\alpha\mathbf{I}(g)$ to which g is α-irresolute is given by

$$\alpha\mathbf{I}(g) = \bigwedge_{U\in L^{\mathscr{R}}} \Big\{ (\mathscr{T}_\alpha)_2(U) \mapsto (\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U)) \Big\}.$$

Definition 32*When considering $g:(\mathscr{X},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ as a bijective function, the measure (g) of how g qualifies as an α-homomorphism is defined by*

$$(g) = \alpha\mathbf{I}(g) \wedge \alpha\mathbf{OP}(g).$$

and

$$(g) = \alpha\mathbf{C}(g) \wedge \alpha\mathbf{OP}(g).$$

Remark 33*According to Lemma 21 (2), when $\alpha\mathbf{C}(g)=1_M$, it follows that $(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(B)) \geq \mathscr{T}_2(B)$ for every B in $L^{\mathscr{R}}$. This represents the definition of an α-continuous function between two (L,M)-fpts. Furthermore, if we consider the identity function $id:(\mathscr{X},\Gamma_1)\to(\mathscr{X},\Gamma_1)$, it holds that $\alpha\mathbf{I}(id)=\alpha\mathbf{I}(id)=\alpha\text{-}\mathbf{Hom}(id)=1_M$.*

The ensuing corollaries are immediate implications of Definition 34 and Corollary 29.

Corollary 34*When $g:(\mathscr{X},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ represents a mapping from one (L,M)-fpts $(\mathscr{X},\Gamma_1)$ to another (L,M)-fpts $(\mathscr{R},\Gamma_2)$, we can state that:*

1.The degree $\alpha\mathbf{C}(g)$ to which g is α-continuous, is given by

$$\alpha\mathbf{C}(g) = \bigwedge_{U\in L^{\mathscr{R}}} \Big\{ \mathscr{T}_2^*(U) \mapsto (\mathscr{T}_\alpha)_1^*(g_L^{\leftarrow}(U)) \Big\}.$$

2.The degree $\alpha\mathbf{I}(g)$ to which g isα-irresolute, is given by

$$\alpha\mathbf{I}(g) = \bigwedge_{U\in L^{\mathscr{R}}} \Big\{ (\mathscr{T}_\alpha)_2^*(U) \mapsto (\mathscr{T}_\alpha)_1^*(g_L^{\leftarrow}(U)) \Big\}.$$

Definition 35*If we consider the mapping $g:(\mathscr{X},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ that connects the (L,M)-fpts's $(\mathscr{X},\Gamma_1)$ and $(\mathscr{R},\Gamma_2)$, then the degree $\alpha\mathbf{C}(g)$ is defined as the extent to which g is α-closed.*

$$\alpha\mathbf{C}(g) = \bigwedge_{B\in L^{\mathscr{R}}} \Big\{ \mathscr{T}_1^*(U) \mapsto (\mathscr{T}_\alpha)_2^*(g_L^{\leftarrow}(U)) \Big\}.$$

Theorem 36*Consider $(\mathscr{X},\Gamma_1)$, $(\mathscr{R},\Gamma_2)$, and $(Z,\mathscr{T}_3)$ as (L,M)-fpts, with mappings defined by $f:(\mathscr{X},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ and $f:(\mathscr{R},\Gamma_2)\to(Z,\mathscr{T}_3)$. The subsequent properties hold true:*

1.$\alpha\mathbf{I}(g)\wedge\alpha\mathbf{I}(g)\leq\alpha\mathbf{I}(f\circ g)$.
2.$\alpha\mathbf{OP}(g)\wedge\alpha\mathbf{OP}(g)\leq\alpha\mathbf{I}(f\circ g)$.
3.$\alpha\mathbf{Cl}(g)\wedge\alpha\mathbf{I}(g)\leq\alpha\mathbf{Cl}(f\circ g)$.

Proof. Establishing (1) is sufficient, given that the other outcomes can be shown in a comparable way. From Lemma 21 and Definition 34, it follows that

$$\begin{aligned}\alpha\mathbf{I}(g)\wedge\alpha\mathbf{I}(g) &= \bigwedge_{U\in L^{\mathscr{R}}}\left\{(\mathscr{T}_\alpha)_2(U)\mapsto(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U))\right\}\wedge\\ &\quad\bigwedge_{V\in L^{Z}}\left\{(\mathscr{T}_\alpha)_3(V)\mapsto(\mathscr{T}_\alpha)_2(g_L^{\leftarrow}(V))\right\}\\ &\le\bigwedge_{V\in L^{Z}}\left\{\mathscr{T}_\alpha)_2(g_L^{\leftarrow}(V))\mapsto(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(g_L^{\leftarrow}(V)))\right\}\wedge\\ &\quad\bigwedge_{V\in L^{Z}}\left\{(\mathscr{T}_\alpha)_3(V)\mapsto(\mathscr{T}_\alpha)_2(g_L^{\leftarrow}(V))\right\}\\ &=\bigwedge_{V\in L^{Z}}\left\{\left(\mathscr{T}_\alpha)_2(g_L^{\leftarrow}(V))\mapsto(\mathscr{T}_\alpha)_1(g_L\circ g_L)^{\leftarrow}(V))\right)\wedge\right.\\ &\quad\left.\left((\mathscr{T}_\alpha)_3(V)\mapsto(\mathscr{T}_\alpha)_2(g_L^{\leftarrow}(V))\right)\right\}\\ &\le\bigwedge_{V\in L^{Z}}\left((\mathscr{T}_\alpha)_3(g_L^{\leftarrow}(V))\mapsto(\mathscr{T}_\alpha)_1(g_L\circ g_L)^{\leftarrow}(V))\right)\\ &=\alpha\mathbf{I}(g_L\circ g_L).\end{aligned}$$

Thus we completed the proof.

The subsequent corollaries arise directly from Definition 32 and Theorem 36.

Corollary 37 *Consider two functions $g:(\mathscr{X},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ and $g:(\mathscr{R},\Gamma_2)\to(Z,\mathscr{T}_3)$, which are (L,M)-fpts defined on the sets $\mathscr{X}$ and $\mathscr{R}$. Given that these mappings are bijective, the subsequent assertion is valid:*

$$(g)\wedge(g)\le(g\circ g).$$

Theorem 38 *Consider the mappings $g:(\mathscr{X},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ and $f:(\mathscr{R},\Gamma_2)\to(Z,\mathscr{T}_3)$, which are two (L,M)-fpts's defined on $\mathscr{X}$ and $\mathscr{R}$. If f is surjective, it follows that the subsequent assertions are true:*

1. $\alpha\mathbf{OP}(f\circ g)\wedge\alpha\mathbf{I}(g)\le\alpha\mathbf{OP}(f)$.
2. $\alpha\mathbf{Cl}(f\circ g)\wedge\alpha\mathbf{I}(g)\le\alpha\mathbf{Cl}(f)$.

Proof. Establishing (1) is sufficient, as the other assertions can be proved in like fashion. Since g functions as a surjective mapping, we derive that $(g_L\circ g_L)^{\rightarrow}(g_L^{\leftarrow}(U))=g_L^{\rightarrow}(U)$ for any element $U\in L^{\mathscr{R}}$.

Based on Lemma 21 (4), we know

$$\begin{aligned}\alpha\mathbf{OP}(g\circ f)\wedge\alpha\mathbf{I}(g) &=\\ &\bigwedge_{S\in L^{\mathscr{X}}}\left\{(\mathscr{T}_\alpha)_1(S)\mapsto(\mathscr{T}_\alpha)_3((g_L\circ g_L)^{\rightarrow}(S))\right\}\wedge\\ &\bigwedge_{U\in L^{\mathscr{R}}}\left\{(\mathscr{T}_\alpha)_2(U)\mapsto(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U))\right\}\\ \le\bigwedge_{U\in L^{\mathscr{R}}}&\left\{(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U))\mapsto(\mathscr{T}_\alpha)_3((g_L\circ g_L)^{\rightarrow}(g_L^{\leftarrow}(U)))\right\}\wedge\\ &\bigwedge_{U\in L^{\mathscr{R}}}\left\{(\mathscr{T}_\alpha)_2(U)\mapsto(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U))\right\}\\ &=\bigwedge_{U\in L^{\mathscr{R}}}\left\{\left((\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U))\mapsto(\mathscr{T}_\alpha)_3(g_L^{\rightarrow}(U))\right)\wedge\right.\\ &\quad\left.\left((\mathscr{T}_\alpha)_2(U)\mapsto(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U))\right)\right\}\\ &\le\bigwedge_{V\in L^{\mathscr{R}}}\left\{(\mathscr{T}_\alpha)_2(U)\mapsto(\mathscr{T}_\alpha)_3(g_L^{\rightarrow}(V))\right\}=\alpha\mathbf{OP}(g).\end{aligned}$$

It can be similarly confirmed that

Theorem 39 *Consider the functions $g:(\mathscr{X},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ and $g:(\mathscr{R},\Gamma_2)\to(Z,\mathscr{T}_3)$, which are both (L,M)-fpts defined on the sets $\mathscr{X}$, $\mathscr{R}$, and Z. If g is a surjective function, then the subsequent statements are valid:*

1. $\alpha\mathbf{OP}(g\circ f)\wedge\alpha\mathbf{I}(g)\le\alpha\mathbf{OP}(g)$.
2. $\alpha\mathbf{Cl}(g\circ f)\wedge\alpha\mathbf{I}(g)\le\alpha\mathbf{Cl}(g)$.

Theorem 310 *Let $g:(\mathscr{X},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ is a bijective mapping between (L,M)-fpts's on $\mathscr{X}$ and $\mathscr{R}$, then*

1. $\alpha\mathbf{I}(g)=\bigwedge_{S\in L^{\mathscr{X}}}\left\{(\mathscr{T}_\alpha)_2(g_L^{\rightarrow}(S))\mapsto(\mathscr{T}_\alpha)_1(S)\right\}$.
2. $\alpha\mathbf{OP}(g)=\bigwedge_{U\in L^{\mathscr{R}}}\left\{(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U)\mapsto(\mathscr{T}_\alpha)_2(U)\right\}$.
3. $\alpha\mathbf{I}(g^{-1})=\alpha\mathbf{OP}(g)=\alpha\mathbf{Cl}(g)$.

Proof. Demonstrating conditions (1) and (3) suffices, as condition (2) can be similarly established to condition (1)

(1) Since $g:\mathscr{X}\to\mathscr{R}$ is a bijective mapping, we have $g_L^{\leftarrow}(g_L^{\rightarrow}(S))=S$, $\forall S\in L^{\mathscr{X}}$, and $g_L^{\rightarrow}(g_L^{\leftarrow}(U))=U$, $\forall U\in L^{\mathscr{R}}$.

The following inequalities are obtained

$$\bigwedge_{S\in L^{\mathscr{Z}}}\left\{(\mathscr{T}\alpha)_2(g_L^{\leftarrow}(S))\mapsto(\mathscr{T}\alpha)_1(S)\right\}=\bigwedge_{S\in L^{\mathscr{Z}}}\left\{(\mathscr{T}\alpha)_2(g_L^{\leftarrow}(S))\mapsto(\mathscr{T}\alpha)_1(g_L^{\leftarrow}(g_L^{\rightarrow}(S)))\right\}$$
$$\geq\bigwedge_{U\in L^{\mathscr{R}}}\left\{(\mathscr{T}\alpha)_2(U)\mapsto(\mathscr{T}\alpha)_1(g_L^{\leftarrow}(U))\right\}$$
$$=\bigwedge_{U\in L^{\mathscr{R}}}\left\{(\mathscr{T}\alpha)_2(g_L^{\leftarrow}(g_L^{\rightarrow}(U)))\mapsto(\mathscr{T}\alpha)_1(g_L^{\leftarrow}(U))\right\}\geq\bigwedge_{S\in L^{\mathscr{Z}}}\left\{(\mathscr{T}\alpha)_2(g_L^{\rightarrow}(S))\mapsto(\mathscr{T}\alpha)_1(S)\right\}.$$

Hence

$$\alpha\mathbf{I}(g)=\bigwedge_{U\in L^{\mathscr{R}}}\left\{(\mathscr{T}\alpha)_2(g_L^{\leftarrow}(U))\mapsto(\mathscr{T}\alpha)_1(U)\right\}=\bigwedge_{S\in L^{\mathscr{Z}}}\left\{(\mathscr{T}\alpha)_2(g_L^{\leftarrow}(S))\mapsto(\mathscr{T}\alpha)_1(S)\right\}.$$

(3) Since $g:\mathscr{Z}\to\mathscr{R}$ is a bijective mapping, we have $(g^{-1})_L^{\leftarrow}(S)=g_L^{\rightarrow}(S)$ and $g_L^{\rightarrow}(S')=g_L^{\rightarrow}(S)'$ for any $S\in L^{\mathscr{Z}}$. Then

$$\alpha\mathbf{I}(g^{-1})=\bigwedge_{S\in L^{\mathscr{Z}}}\left\{(\mathscr{T}\alpha)_1(S)\mapsto(\mathscr{T}\alpha)_2((g^{-1})_L^{\leftarrow}(S))\right\}=\bigwedge_{S\in L^{\mathscr{Z}}}\left\{(\mathscr{T}\alpha)_1(S)\mapsto(\mathscr{T}\alpha)_2(g_L^{\rightarrow}(S))\right\}=\alpha\mathbf{OP}(g).$$

and

$$\alpha\mathbf{OP}(g^{-1})=\bigwedge_{S\in L^{\mathscr{Z}}}\left\{(\mathscr{T}\alpha)_1(S)\mapsto(\mathscr{T}\alpha)_2(g_L^{\rightarrow}(S))\right\}=\bigwedge_{S\in L^{\mathscr{Z}}}\left\{(\mathscr{T}\alpha)_1(S')\mapsto(\mathscr{T}\alpha)_2(g_L^{\rightarrow}(S'))\right\}$$
$$=\bigwedge_{S\in L^{\mathscr{Z}}}\left\{(\mathscr{T}\alpha)_1(S')\mapsto(\mathscr{T}\alpha)_2(g_L^{\rightarrow}(S)')\right\}=\alpha\mathbf{Cl}(g).$$

Corollary 311*Let $g:(\mathscr{Z},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ represent a bijective function connecting the (L,M)-fpts's $\mathscr{Z}$ and $\mathscr{R}$, such that*

1. $(g)=\alpha\mathbf{I}(g)\wedge\alpha\mathbf{I}(g^{-1})=\alpha\mathbf{I}(g)\wedge\alpha\mathbf{Cl}(g)$.
2. $(g)=\bigwedge_{S\in L^{\mathscr{Z}}}\left\{(\mathscr{T}\alpha)_2(g_L^{\rightarrow}(S))\mapsto(\mathscr{T}\alpha)_1(S)\right\}$.
3. $(g)=\bigwedge_{U\in L^{\mathscr{R}}}\left\{(\mathscr{T}\alpha)_1(g_L^{\leftarrow}(U))\mapsto(\mathscr{T}\alpha)_2(U)\right\}$.

The subsequent corollaries and theorems define the concepts of degree of α-irresolutness and α-openness in relation to (L,M)-fuzzy quasi α-neighborhood systems, (L,M)-fuzzy α-closure operators, and (L,M)-fuzzy α-interior operators.

Corollary 312*Consider a mapping $g:(\mathscr{Z},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ that connects the (L,M)-fpts' $\mathscr{Z}$ with $\mathscr{R}$*

1. $\alpha\mathbf{I}(g)=\bigwedge_{U\in L^{\mathscr{R}}}\bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})}\left\{\alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}\alpha)_2}(U)\mapsto\alpha Q_{x_\lambda}^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(U))\right\}$.
2. $\alpha\mathbf{I}(g)=\bigwedge_{U\in L^{\mathscr{R}}}\bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})}\left\{\alpha N_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}\alpha)_2}(U)(g(\mathscr{Z})_\lambda)\mapsto\alpha N_{x_\lambda}^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(U))\right\}$.
3. $\alpha\mathbf{I}(g)=\bigwedge_{U\in L^{\mathscr{R}}}\bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})}\left\{\alpha I^{(\mathscr{T}\alpha)_2}(U)(g(\mathscr{Z})_\lambda)\mapsto\alpha I^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(U))(\mathscr{Z}_\lambda)\right\}$.
4. $\alpha\mathbf{I}(g)=\bigwedge_{U\in L^{\mathscr{R}}}\bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})}\left\{\alpha Cl^{(\mathscr{T}\alpha)_2}(U)(g(\mathscr{Z})_\lambda)'\mapsto\alpha Cl^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(U))(\mathscr{Z}_\lambda)'\right\}$.

Proof.(1) Since $\alpha Q_{x_\lambda}^{(\mathscr{T}_\alpha)_1}(S) = \bigvee_{x_\lambda \not\leq B' > S'} \mathscr{T}_\alpha(B)$ for all $S \in L^{\mathscr{Z}}$, and for any $x_\lambda \in P(L^{\mathscr{Z}})$ and $U_1, U \in L^{\mathscr{R}}$, we have $g(\mathscr{Z})_\lambda \not\leq {U_1}' \geq U' \Rightarrow x_\lambda \not\leq g_L^{\leftarrow}(U_1)' \geq g_L^{\leftarrow}(U)'$. Then

$$\bigwedge_{U\in L^{\mathscr{R}}} \bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})} \left\{\alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}_\alpha)_2}(U) \mapsto \alpha Q_{x_\lambda}^{(\mathscr{T}_\alpha)_1}(g_L^{\leftarrow}(U))\right\}$$

$$= \bigwedge_{U\in L^{\mathscr{R}}} \bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})} \left\{ \bigvee_{f(\mathscr{Z})_\lambda \not\leq {U_1}' \geq U'} (\mathscr{T}_\alpha)_2(U_1) \mapsto \bigvee_{x_\lambda \not\leq {U_1}' \geq g_L^{\leftarrow}(U)'} (\mathscr{T}_\alpha)_1(U_1)\right\}$$

$$\geq \bigwedge_{U\in L^{\mathscr{R}}} \bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})} \left\{ \bigvee_{f(\mathscr{Z})_\lambda \not\leq {U_1}' \geq U'} (\mathscr{T}_\alpha)_2(U_1) \mapsto \bigvee_{x_\lambda \not\leq g_L^{\leftarrow}(U_2)' \geq g^{\leftarrow}(U)'} (\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U_2))\right\}$$

$$\geq \bigwedge_{U\in L^{\mathscr{R}}} \bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})} \bigwedge_{f(\mathscr{Z})_\lambda \not\leq {U_1}' \geq U'} \left\{(\mathscr{T}_\alpha)_2(U_1) \mapsto (\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U_1))\right\}$$

$$\geq \bigwedge_{U\in L^{\mathscr{R}}} \left\{(\mathscr{T}_\alpha)_2(U) \mapsto (\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U))\right\} = \alpha\mathbf{I}(g).$$

Conversely, since

$$\mathscr{T}_{\alpha_1}(S) = \bigwedge_{x_\lambda \not\leq S'} \alpha Q_{x_\lambda}^{(\mathscr{T}_\alpha)_1}(S)$$

for each $S \in L^{\mathscr{Z}}$, and for each $x_\lambda \in P(L^{\mathscr{Z}})$, $U \in L^{\mathscr{R}}$, and $x_\lambda \not\leq g_L^{\leftarrow}(U)' \Rightarrow f(\mathscr{Z})_\lambda \not\leq U'$. The following is valid

$$\alpha\mathbf{I}(g) = \bigwedge_{U\in L^{\mathscr{R}}} \left\{(\mathscr{T}_\alpha)_2(U) \mapsto (\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U))\right\} = \bigwedge_{U\in L^{\mathscr{R}}} \left\{ \bigwedge_{y_\mu \not\leq U'} \alpha Q_{y_\mu}^{(\mathscr{T}_\alpha)_2}(U) \mapsto \bigwedge_{x_\lambda \not\leq g_L^{\leftarrow}(U)'} \alpha Q_{x_\lambda}^{(\mathscr{T}_\alpha)_1}(g_L^{\leftarrow}(U))\right\}$$

$$\geq \bigwedge_{U\in L^{\mathscr{R}}} \left\{ \bigwedge_{f(\mathscr{Z})_\lambda \not\leq U'} \alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}_\alpha)_2}(U) \mapsto \bigwedge_{x_\lambda \not\leq g_L^{\leftarrow}(U)'} \alpha Q_{x_\lambda}^{(\mathscr{T}_\alpha)_1}(g_L^{\leftarrow}(U))\right\}$$

$$\geq \bigwedge_{U\in L^{\mathscr{R}}} \bigwedge_{x_\lambda \not\leq g_L^{\leftarrow}(U)'} \left(\alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}_\alpha)_2}(U) \mapsto \alpha Q_{x_\lambda}^{(\mathscr{T}_\alpha)_1}(g_L^{\leftarrow}(U))\right\}$$

$$\geq \bigwedge_{U\in L^{\mathscr{R}}} \bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})} \left\{\alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}_\alpha)_2}(U) \mapsto \alpha Q_{x_\lambda}^{(\mathscr{T}_\alpha)_1}(g_L^{\leftarrow}(U))\right).$$

We have now concluded the proof of (1). Utilizing Theorem 216 and Theorem 219, we can establish the validity of (2), (3), and (4).

Theorem 313*Consider $g : (\mathscr{Z}, \Gamma_1) \to (\mathscr{R}, \Gamma_2)$ as a transformation linking the (L,M)-fpts's $\mathscr{Z}$ and $\mathscr{R}$, then*

1. $\alpha\mathbf{OP}(g) = \bigwedge_{S\in L^{\mathscr{Z}}} \bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})} \left\{\alpha Q_{x_\lambda}^{(\mathscr{T}_\alpha)_1}(S) \mapsto \alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}_\alpha)_2}(g_L^{\rightarrow}(S))\right\}.$
2. $\alpha\mathbf{OP}(g) = \bigwedge_{S\in L^{\mathscr{Z}}} \bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})} \left\{\alpha N_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}_\alpha)_1}(S)(\mathscr{Z}_\lambda) \mapsto \alpha N_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}_\alpha)_2}(g_L^{\rightarrow}(S))\right\}.$
3. $\alpha\mathbf{OP}(g) = \bigwedge_{S\in L^{\mathscr{Z}}} \bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})} \left\{\alpha I^{(\mathscr{T}_\alpha)_1}(S)(\mathscr{Z}_\lambda) \mapsto \alpha I^{(\mathscr{T}_\alpha)_2}(g_L^{\rightarrow}(S))(\mathscr{Z}_\lambda)\right\}.$
4. $\alpha\mathbf{OP}(g) = \bigwedge_{S\in L^{\mathscr{Z}}} \bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})} \left\{\alpha Cl^{(\mathscr{T}_\alpha)_1}(S')(\mathscr{Z}_\lambda)' \mapsto \alpha Cl^{(\mathscr{T}_\alpha)_2}(g_L^{\rightarrow}(S)')(g(\mathscr{Z})_\lambda)'\right\}.$
5. $\alpha\mathbf{OP}(g) = \bigwedge_{U\in L^{\mathscr{R}}} \bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})} \left\{\alpha Q_{x_\lambda}^{(\mathscr{T}_\alpha)_1}(g_L^{\leftarrow}(U)) \mapsto \alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}_\alpha)_2}(U)\right\}.$
6. $\alpha\mathbf{OP}(g) = \bigwedge_{U\in L^{\mathscr{R}}} \bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})} \left\{\alpha N_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}_\alpha)_1}(g_L^{\leftarrow}(U))(\mathscr{Z}_\lambda) \mapsto \alpha N_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}_\alpha)_2}(U)\right\}.$
7. $\alpha\mathbf{OP}(g) = \bigwedge_{U\in L^{\mathscr{R}}} \bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})} \left\{\alpha I^{(\mathscr{T}_\alpha)_1}(g_L^{\leftarrow}(U))(\mathscr{Z}_\lambda) \mapsto \alpha I^{(\mathscr{T}_\alpha)_2}(U)(\mathscr{Z}_\lambda)\right\}.$
8. $\alpha\mathbf{OP}(g) = \bigwedge_{U\in L^{\mathscr{R}}} \bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})} \left\{\alpha Cl^{(\mathscr{T}_\alpha)_1}(g_L^{\leftarrow}(U)')(\mathscr{Z}_\lambda)' \mapsto \alpha Cl^{(\mathscr{T}_\alpha)_2}(U')(g(\mathscr{Z})_\lambda)'\right\}.$

Proof. Demonstrating (5) suffices because the remaining conditions have been established in a similar manner.

$$\bigwedge_{U\in L^{\mathscr{R}}}\bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})}\left\{\alpha Q^{(\mathscr{T}_\alpha)_1}_{x_\lambda}(g^{\leftarrow}_L(U))\mapsto \alpha Q^{(\mathscr{T}_\alpha)_2}_{f(\mathscr{Z})_\lambda}(U)\right\}$$
$$=\bigwedge_{U\in L^{\mathscr{R}}}\bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})}\left\{\bigvee_{x_\lambda\not\leq U_1{}'\geq g^{\leftarrow}_L(U)'}(\mathscr{T}_\alpha)_1(U_1)\mapsto \bigvee_{f(\mathscr{Z})_\lambda\not\leq U_1{}'\geq (U)'}(\mathscr{T}_\alpha)_2(U_1)\right\}$$
$$\geq\bigwedge_{U\in L^{\mathscr{R}}}\bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})}\left\{\bigvee_{x_\lambda\not\leq U_1{}'\geq g^{\leftarrow}_L(U)'}(\mathscr{T}_\alpha)_1(U_1)\mapsto \bigvee_{f(\mathscr{Z})_\lambda\not\leq g^{\leftarrow}_L(U_2)'\geq U'}(\mathscr{T}_\alpha)_2(g^{\rightarrow}_L(U_2))\right\}$$
$$\geq\bigwedge_{U\in L^{\mathscr{R}}}\bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})}\left\{\bigvee_{x_\lambda\not\leq U_1{}'\geq g^{\leftarrow}_L(U)'}(\mathscr{T}_\alpha)_1(U_1)\mapsto \bigvee_{f(\mathscr{Z})_\lambda\not\leq g^{\leftarrow}_L(U_2)'\geq g^{\leftarrow}(U)'}(\mathscr{T}_\alpha)_2(g^{\rightarrow}_L(U_2))\right\}$$
$$\geq\bigwedge_{U\in L^{\mathscr{R}}}\bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})}\bigwedge_{x_\lambda\not\leq U_1{}'\geq g^{\leftarrow}_L(U)'}\left\{(\mathscr{T}_\alpha)_1(U_1)\mapsto \bigvee_{f(\mathscr{Z})_\lambda\not\leq g^{\leftarrow}_L(U_2)'\geq g^{\leftarrow}(U)'}(\mathscr{T}_\alpha)_2(g^{\rightarrow}_L(U_2))\right\}$$
$$\geq\bigwedge_{U\in L^{\mathscr{R}}}\bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})}\bigwedge_{x_\lambda\not\leq U_1{}'\geq g^{\leftarrow}_L(U)'}\left\{(\mathscr{T}_\alpha)_1(U_1)\mapsto (\mathscr{T}_\alpha)_2(g^{\rightarrow}_L(U_2))\right\}$$
$$\geq\bigwedge_{U\in L^{\mathscr{R}}}\left\{(\mathscr{T}_\alpha)_1(U)\mapsto (\mathscr{T}_\alpha)_2(g^{\rightarrow}_L(U))\right\}=SPo(g).$$

It must be shown that for every $S\in L^{\mathscr{Z}}$, the following holds true

$$\bigwedge_{y_b\not\leq(g^{\rightarrow}_L(S))'}\alpha Q^{(\mathscr{T}_\alpha)_2}_{y_b}(g^{\rightarrow}_L(S))=\bigwedge_{f(\mathscr{Z})_\lambda\not\leq(g^{\rightarrow}_L(S))'}\alpha Q^{(\mathscr{T}_\alpha)_2}_{f(\mathscr{Z})_\lambda}(g^{\rightarrow}_L(S)).$$

It is obvious that

$$\bigwedge_{y_b\not\leq(g^{\rightarrow}_L(S))'}\alpha Q^{(\mathscr{T}_\alpha)_2}_{y_b}(g^{\rightarrow}_L(S))\leq\bigwedge_{f(\mathscr{Z})_\lambda\not\leq(g^{\rightarrow}_L(S))'}\alpha Q^{(\mathscr{T}_\alpha)_2}_{f(\mathscr{Z})_\lambda}(g^{\rightarrow}_L(S)).$$

We will show that

$$\bigwedge_{y_b\not\leq(g^{\rightarrow}_L(S))'}\alpha Q^{(\mathscr{T}_\alpha)_2}_{y_b}(g^{\rightarrow}_L(S))\geq\bigwedge_{f(\mathscr{Z})_\lambda\not\leq(g^{\rightarrow}_L(S))'}\alpha Q^{(\mathscr{T}_\alpha)_2}_{f(\mathscr{Z})_\lambda}(g^{\rightarrow}_L(S)).$$

For any $y_b\in P(L^{\mathscr{R}})$ with $y_b\not\leq g^{\rightarrow}_L(S)'$, we obtain $\mu\not\leq(g^{\rightarrow}_L(S)(y)'=\bigwedge_{f(\mathscr{Z})=y}S(\mathscr{Z})'$. Then there exists $x\in\mathscr{Z}$ such that $g(\mathscr{Z})=y$ and $b\leq S(\mathscr{Z})'$. This implies $b\not\leq\bigwedge_{f(\mathscr{Z})=f(z)}S(z)'=g^{\rightarrow}_L(S)(g(\mathscr{Z}))'$. Hence $g(\mathscr{Z})_\mu\leq g^{\rightarrow}_L(S)'$. On the other hand, since
$\bigwedge_{f(\mathscr{Z})_\lambda\not\leq(g^{\rightarrow}_L(S))'}\alpha Q^{(\mathscr{T}_\alpha)_2}_{f(\mathscr{Z})_\lambda}(g^{\rightarrow}_L(S))\leq\alpha Q^{(\mathscr{T}_\alpha)_2}_{f(\mathscr{Z})_\lambda}(g^{\rightarrow}_L(S))=\alpha Q^{(\mathscr{T}_\alpha)_2}_{y_b}(g^{\rightarrow}_L(S))$. Then

$$\bigwedge_{f(\mathscr{Z})_\lambda\not\leq(g^{\rightarrow}_L(S))'}\alpha Q^{(\mathscr{T}_\alpha)_2}_{f(\mathscr{Z})_\lambda}(g^{\rightarrow}_L(S))\leq\bigwedge_{y_b\not\leq(g^{\rightarrow}_L(S))'}\alpha Q^{(\mathscr{T}_\alpha)_2}_{y_b}(g^{\rightarrow}_L(S)).$$

Therefore

$$\bigwedge_{y_b\not\leq(g^{\rightarrow}_L(S))'}\alpha Q^{(\mathscr{T}_\alpha)_2}_{y_b}(g^{\rightarrow}_L(S))\leq\bigwedge_{f(\mathscr{Z})_\lambda\not\leq(g^{\rightarrow}_L(S))'}\alpha Q^{(\mathscr{T}_\alpha)_2}_{f(\mathscr{Z})_\lambda}(g^{\rightarrow}_L(S)).$$

(2) $\bigwedge_{S\in L^{\mathscr{Z}}}\left\{\bigwedge_{x_\lambda\not\leq S'}\alpha Q^{(\mathscr{T}_\alpha)_1}_{x_\lambda}(g^{\leftarrow}(g^{\rightarrow}(S)))\mapsto\bigwedge_{f(\mathscr{Z})_\lambda\not\leq g^{\rightarrow}(S)'}\right.$
$\left.\alpha Q^{(\mathscr{T}_\alpha)_2}_{f(\mathscr{Z})_\lambda}(g^{\rightarrow}(S))\right\}$

$$\geq\bigwedge_{U\in L^{\mathscr{R}}}\bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})}\left\{\alpha Q^{(\mathscr{T}_\alpha)_1}_{x_\lambda}(g^{\leftarrow}(U))\mapsto\alpha Q^{(\mathscr{T}_\alpha)_1}_{f(\mathscr{Z})_\lambda}(U)\right\}.$$

For each $\mathbf{c}\in M$ such that

$$\mathbf{c}\leq\bigwedge_{U\in L^{\mathscr{R}}}\bigwedge_{x_\lambda\in P(L^{\mathscr{Z}})}\left\{\alpha Q^{(\mathscr{T}_\alpha)_1}_{x_\lambda}(g^{\leftarrow}(U))\mapsto\alpha Q^{(\mathscr{T}_\alpha)_1}_{f(\mathscr{Z})_\lambda}(U)\right\}.$$

Then $\mathbf{c} \leq \alpha Q_{x_\lambda}^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(U)) \mapsto \alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}\alpha)_2}(U)$ for any $U \in L^{\mathscr{R}}$ and $x_\lambda \in P(L^{\mathscr{Z}})$. Uy Lemma 21(1), we have $\mathbf{c} \wedge \alpha Q_{x_\lambda}^{\mathscr{T}\alpha 1}(g_L^{\leftarrow}(U)) \leq \alpha Q_{f(\mathscr{Z})_\lambda}^{\mathscr{T}\alpha 2}(U)$. For all $S \in L^{\mathscr{Z}}$ and $g(\mathscr{Z})_\lambda \leq g_L^{\rightarrow}(S)'$, we obtain $\lambda \leq g_L^{\rightarrow}(S)(g(\mathscr{Z}))' = \bigwedge_{f(\mathscr{Z})=f(z)} S(z)'$. Then there exists $z \in \mathscr{Z}$ such that $g(z) = f(\mathscr{Z})$ and $\lambda \not\leq S(z)'$. This implies $\lambda \not\leq S(z)'$. On the other hand, since

$$\begin{aligned}&\mathbf{c} \wedge \bigwedge_{x_\lambda \not\leq S'} \alpha Q_{x_\lambda}^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(g_L^{\rightarrow}(S)))\\ \leq& \mathbf{c} \wedge \alpha Q_{z_\lambda}^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(g_L^{\rightarrow}(S)))\\ \leq& \alpha Q_{f(z)_\lambda}^{(\mathscr{T}\alpha)_2}(g_L^{\rightarrow}(S)) = \alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}\alpha)_2}(g_L^{\rightarrow}(S)).\end{aligned}$$

The subsequent statement is accurate.

$$\mathbf{c} \wedge \bigwedge_{x_\lambda \not\leq S'} \alpha Q_{x_\lambda}^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(g_L^{\rightarrow}(S))) \leq \bigwedge_{f(\mathscr{Z})_\lambda \not\leq g_L^{\rightarrow}(S)'} \alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}\alpha)_2}(g_L^{\rightarrow}(S)).$$

By Lemma 21(1), we have
$\mathbf{c} \leq \bigwedge_{x_\lambda \not\leq S'} \alpha Q_{x_\lambda}^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(g_L^{\rightarrow}(S))) \mapsto \bigwedge_{f(\mathscr{Z})_\lambda \not\leq g_L^{\rightarrow}(S)'}$
$\alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}\alpha)_2}(g_L^{\rightarrow}(S))$.
Therefore

$$\mathbf{c} \leq \bigwedge_{S \in L^{\mathscr{Z}}} \left\{ \bigwedge_{x_\lambda \not\leq S'} \alpha Q_{x_\lambda}^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(g_L^{\rightarrow}(S))) \leq \bigwedge_{f(\mathscr{Z})_\lambda \not\leq g_L^{\rightarrow}(S)'} \alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}\alpha)_2}(g_L^{\rightarrow}(S)) \right\}.$$

By the arbitrariness of **c**, we have

$$\bigwedge_{S \in L^{\mathscr{Z}}} \left\{ \bigwedge_{x_\lambda \not\leq S'} \alpha Q_{x_\lambda}^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(g_L^{\rightarrow}(S))) \leq \bigwedge_{f(\mathscr{Z})_\lambda \not\leq g_L^{\rightarrow}(S)'} \alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}\alpha)_2}(g_L^{\rightarrow}(S)) \right\}$$

$\geq \bigwedge_{U \in L^{\mathscr{R}}} \bigwedge_{x_\lambda \in P(L^{\mathscr{Z}})} \left\{ \bigwedge_{x_\lambda \not\leq S'} \alpha Q_{x_\lambda}^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(S)) \leq \right.$
$\left. \bigwedge_{f(\mathscr{Z})_\lambda \not\leq g_L^{\rightarrow}(S)'} \alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}\alpha)_2}(g_L^{\rightarrow}(S)) \right\}$. For each $S \in L^{\mathscr{Z}}$ and $S \leq g_L^{\leftarrow}(g_L^{\rightarrow}(S))$, can be verified that

$$\begin{aligned}&= \bigwedge_{S \in L^{\mathscr{Z}}} \left\{ (\mathscr{T}\alpha)_1(S) \mapsto (\mathscr{T}\alpha)_2(g_L^{\rightarrow}(S)) \right\}\\ &= \bigwedge_{S \in L^{\mathscr{Z}}} \left\{ \bigwedge_{x_\lambda \not\leq S'} \left\{ \alpha Q_{x_\lambda}^{(\mathscr{T}\alpha)_1}(S) \mapsto \bigwedge_{y_b \not\leq g^{\rightarrow}(S)'} \alpha Q_{y_b}^{(\mathscr{T}\alpha)_2}(g_L^{\rightarrow}(S)) \right\}\right.\\ &= \bigwedge_{S \in L^{\mathscr{Z}}} \left\{ \bigwedge_{x_\lambda \not\leq S'} \alpha Q_{x_\lambda}^{(\mathscr{T}\alpha)_1}(S) \mapsto \bigwedge_{f(\mathscr{Z})_\lambda \not\leq g_L^{\rightarrow}(S)'} \alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}\alpha)_2}(g_L^{\rightarrow}(S)) \right\}\\ &\geq \bigwedge_{S \in L^{\mathscr{Z}}} \left\{ \bigwedge_{x_\lambda \not\leq S'} \alpha Q_{x_\lambda}^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(g_L^{\rightarrow}(S))) \mapsto \bigwedge_{f(\mathscr{Z})_\lambda \not\leq g_L^{\rightarrow}(S)'} \alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}\alpha)_2}(g_L^{\rightarrow}(S)) \right\}\\ &\geq \bigwedge_{U \in L^{\mathscr{R}}} \left\{ \bigwedge_{x_\lambda \in P(L^{\mathscr{Z}})} \alpha Q_{x_\lambda}^{(\mathscr{T}\alpha)_1}(g_L^{\leftarrow}(U) \mapsto \alpha Q_{f(\mathscr{Z})_\lambda}^{(\mathscr{T}\alpha)_2}(U)) \right\}.\end{aligned}$$

The desired equality is obtained.

4 α-compactness, α-connectedness, α-T_1, and α-T_2 degree in (L,M)-fuzzy pretopological spaces

Theorem 41 *Consider $(\mathscr{X},\Gamma_1)$ and $(\mathscr{R},\Gamma_2)$ as two (L,M)-fpts, with $g:\mathscr{X}\to\mathscr{R}$ designated as a mapping. Then, it holds that $\alpha\mathbf{com}_{(\mathscr{T}_\alpha)_1}(S)\wedge\alpha\mathbf{I}(g)\le\alpha\mathbf{com}_{(\mathscr{T}_\alpha)_2}(g_L^{\rightarrow}(S))$ for all $S\in L^{\mathscr{X}}$.*

Proof. Take any $\mathbf{c}\in M$ such that $\mathbf{c}\le\alpha\mathbf{com}_{(\mathscr{T}_\alpha)_1}(S)\wedge\alpha\mathbf{I}(g)$. Then

$$\mathbf{c}\le\alpha\mathbf{I}(g)=\bigwedge_{U\in L^{\mathscr{R}}}\Big((\mathscr{T}_\alpha)_2(U)\mapsto(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U))\Big),$$

and $\mathbf{c}\le\alpha\mathbf{com}_{(\mathscr{T}_\alpha)_1}(S)$

$$=\bigwedge_{\S\in L^{\mathscr{X}}}\Bigg\{\Bigg(\bigwedge_{B\in\S}(\mathscr{T}_\alpha)_1(B)\wedge\bigwedge_{x\in\mathscr{X}}\Big(S'\vee\bigvee_{B\in\S}B\Big)(\mathscr{X})\Bigg)$$

$$\mapsto\bigvee_{\in 2^{(\S)}}\bigwedge_{x\in\mathscr{X}}\Big(S'\vee\bigvee_{B\in}B\Big)(\mathscr{X})\Bigg\}$$

Then for all $U\in L^{\mathscr{R}}$ and $\S\subseteq L^{\mathscr{X}}$, by Lemma 22, we have $\mathbf{c}\le(\mathscr{T}_\alpha)_2(U)\mapsto(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U))$, and

$$\mathbf{c}\le\Bigg(\bigwedge_{S_\in}(\mathscr{T}_\alpha)_1(B)\wedge\bigwedge_{x\in\mathscr{X}}\Big(S'\vee\bigvee_{B\in}B\Big)(\mathscr{X})\Bigg)$$

$$\mapsto\bigvee_{\in 2^{(\S)}}\bigwedge_{x\in\mathscr{X}}\Big(S'\vee\bigvee_{B\in}B\Big)(\mathscr{X}).$$

Sccording to Lemma 21(1), it is possible to obtain

$$\mathbf{c}\wedge(\mathscr{T}_\alpha)_2(U)\le(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U)),\forall U\in L^{\mathscr{R}}$$

and

$$\mathbf{c}\wedge\bigwedge_{B\in\S}(\mathscr{T}_\alpha)_1(B)\wedge\bigwedge_{x\in\mathscr{X}}\Big(S'\vee\bigvee_{B\in\S}B\Big)(\mathscr{X})\le\bigvee_{H\in 2^{(\S)}}\bigwedge_{x\in\mathscr{X}}\Big(S'\vee\bigvee_{B\in\S}B\Big)(\mathscr{X}).$$

It is necessary to establish that for all sets B contained within $L^{\mathscr{R}}$,

$$\begin{aligned}\mathbf{c}\le&\alpha\mathbf{com}_{(\mathscr{T}_\alpha)_2}(g_L^{\rightarrow}(S))\\=&\bigwedge_{\in L^{\mathscr{R}}}\Bigg\{\Bigg(\bigwedge_{U_1\in}(\mathscr{T}_\alpha)_2(U_1)\wedge\bigwedge_{y\in\mathscr{R}}\Big(g_L^{\rightarrow}(S)'\vee\bigvee_{U_1\in}U_1\Big)(y)\Bigg)\\&\mapsto\bigvee_{\in 2^{()}}\bigwedge_{y\in\mathscr{R}}\Big(g_L^{\rightarrow}(S)'\vee\bigvee_{U_1\in}U_1\Big)(y)\Bigg\},\end{aligned}$$

Let $g_L^{\leftarrow}()=\{g_L^{\leftarrow}(U_1)\mid U_1\in\}\subseteq L^{\mathscr{X}}$. Then we have

$$\begin{aligned}&\mathbf{c}\wedge\bigwedge_{U_1\in}(\mathscr{T}_\alpha)_2(U_1)\wedge\bigwedge_{y\in\mathscr{R}}\Big(g_L^{\rightarrow}(S)'\vee\bigvee_{U_1\in}U_1\Big)(y)\\&\le\mathbf{c}\wedge\bigwedge_{U_1\in}(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U_1))\wedge\bigwedge_{y\in\mathscr{R}}\Big(g_L^{\rightarrow}(S)'\vee\bigvee_{U_1\in}U_1\Big)(y)\\&=\mathbf{c}\wedge\bigwedge_{U_1\in}(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U_1))\bigwedge_{x\in\mathscr{X}}\Big(S'\vee\bigvee_{U_1\in}g_L^{\leftarrow}(U_1)\Big)(\mathscr{X})\\&=\mathbf{c}\wedge\bigwedge_{B\in g_L^{\leftarrow}()}(\mathscr{T}_\alpha)_1(B)\wedge\bigwedge_{x\in\mathscr{X}}\Big(S'\vee\bigvee_{B\in g_L^{\leftarrow}()}(B)\Big)(\mathscr{X})\\&\le\bigvee_{\in 2^{(g_L^{\leftarrow}())}}\bigwedge_{x\in\mathscr{X}}\Big(S'\vee\bigvee_{B\in}(B)\Big)(\mathscr{X})\\&=\bigvee_{\in 2^{()}}\bigwedge_{x\in\mathscr{X}}\Big(S'\vee\bigvee_{U_1\in}g_L^{\leftarrow}(U_1)\Big)(\mathscr{X})\\&=\bigvee_{\in 2^{()}}\bigwedge_{y\in\mathscr{R}}\Big(g_L^{\rightarrow}(S)'\vee\bigvee_{U_1\in}U_1\Big)(y).\end{aligned}$$

By Lemma 21(1), we have

$$\mathbf{c}\le\Bigg(\bigwedge_{U_1\in}(\mathscr{T}_\alpha)_2(U_1)\wedge\bigwedge_{y\in\mathscr{R}}\Big(g_L^{\rightarrow}(S)'\vee\bigvee_{U_1\in}U_1\Big)(y)\Bigg)$$

$$\mapsto\bigvee_{\in 2^{()}}\bigwedge_{y\in\mathscr{R}}\Big(g_L^{\rightarrow}(S)'\vee\bigvee_{U_1\in}U_1\Big)(y).$$

Therefore

$$\begin{aligned}\mathbf{c}\le&\bigwedge_{\subseteq L^{\mathscr{R}}}\Bigg\{\Bigg(\bigwedge_{U_1\in}(\mathscr{T}_\alpha)_2(U_1)\wedge\bigwedge_{y\in\mathscr{R}}\Big(g_L^{\rightarrow}(S)'\vee\bigvee_{U_1\in}U_1\Big)(y)\Bigg)\\&\mapsto\bigvee_{\in 2^{()}}\bigwedge_{y\in\mathscr{R}}\Big(g_L^{\rightarrow}(S)'\vee\bigvee_{U_1\in}U_1\Big)(y)\Bigg\}=\alpha\mathbf{com}_{(\mathscr{T}_\alpha)_2}(g_L^{\rightarrow}(S)).\end{aligned}$$

By the arbitrariness of $\mathbf{c}$, we obtain that $\alpha\mathbf{com}_{(\mathscr{T}_\alpha)_1}(S)\wedge\alpha\mathbf{I}(g)\le\alpha\mathbf{com}_{(\mathscr{T}_\alpha)_2}(g_L^{\rightarrow}(S))$.

The corollaries that follow are immediate consequences of Theorem 41.

Corollary 42 *Consider a surjective mapping $g:(\mathscr{X},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ that connects the (L,M)-fpts's $\mathscr{X}$ with $\mathscr{R}$. In this context, it follows that $\alpha\mathbf{com}_{(\mathscr{T}_\alpha)_1}(1_{L^{\mathscr{X}}})\wedge\alpha\mathbf{I}(g)\le\alpha\mathbf{com}_{(\mathscr{T}_\alpha)_2}(0_{L^{\mathscr{R}}})$.*

In general topology, the principle states that if G is a connected space and g is continuous, then the image $g(G)$ is also connected. We will now broaden this idea to the framework of (L,M)-fpt.

Theorem 43 *Consider a mapping $g:(\mathscr{X},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ that relates the (L,M)-fpts's $\mathscr{X}$ and $\mathscr{R}$. It follows that $\alpha\mathbf{com}_{(\mathscr{T}_\alpha)_1}(G)\wedge\alpha\mathbf{I}(g)\le\alpha\mathbf{com}_{(\mathscr{T}_\alpha)_2}(g_L^{\rightarrow}(G))$ for every $G\in L^{\mathscr{X}}$.*

Theorem 44*Let $g:(\mathscr{Z},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ be a mapping between (L,M)-fpts's $\mathscr{Z}$ and $\mathscr{R}$, then $\alpha\mathbf{Cl}_{(\mathscr{T}_\alpha)_2}(g_L^{\to}(S))'\wedge\alpha\mathbf{I}(g)\leq\alpha\mathbf{C}_{(\mathscr{T}_\alpha)_1}(S)'$ for all $S\in L^{\mathscr{Z}}$.*

*Proof.*Take any $\mathbf{c}\in M$ such that $\mathbf{c}\leq\alpha\mathbf{C}_{(\mathscr{T}_\alpha)_2}(g_L^{\to}(S))'\wedge\mathbf{SPi}(g)$. By Theorems 221 and 34 (2), we obtain

$$\mathbf{c}\leq\alpha\mathbf{C}_{(\mathscr{T}_\alpha)_2}(g_L^{\to}(S))'$$

$$=\bigvee_{\substack{g_L^{\to}(S)\wedge U_1\neq 0_{L^{\mathscr{R}}},\ g_L^{\to}(S)\wedge U_2\neq 0_{L^{\mathscr{R}}},\\ g_L^{\to}(S)\wedge U_1\wedge U_2=0_{L^{\mathscr{R}}},\ g_L^{\to}(S)\leq U_1\vee U_2}}\left\{(\mathscr{T}_\alpha)_2(U_1')\vee(\mathscr{T}_\alpha)_2(U_2')\right\},$$

and

$$\mathbf{c}\leq\alpha\mathbf{I}(g)=\bigwedge_{U_3\in L^{\mathscr{R}}}\left\{(\mathscr{T}_\alpha)_2(U_3')\mapsto(\mathscr{T}_\alpha)_1(g_L^{\to}(U_3)')\right\}.$$

This implies that there exist $U_1,\ U_2\in L^{\mathscr{R}}$ with $g_L^{\to}(S)\wedge U_1\neq 0_{L^{\mathscr{R}}}$, $g_L^{\to}(S)\wedge U_2\neq 0_{L^{\mathscr{R}}}$, $g_L^{\to}(S)\wedge U_1\wedge U_2=0_{L^{\mathscr{R}}}$, $g_L^{\to}(S)\leq U_1\vee U_2$ such that $\mathbf{c}\leq(\mathscr{T}_\alpha)_2(U_1')\wedge(\mathscr{T}_\alpha)_2(U_2')$, and

$\mathbf{c}\leq(\mathscr{T}_\alpha)_2(U_3')\mapsto(\mathscr{T}_\alpha)_1(g^{\leftarrow}(U_3)')$ for all $U_3\in L^{\mathscr{R}}$. This implies that there exist $U_1,\ U_2\in L^{\mathscr{R}}$ with $S\wedge g_L^{\leftarrow}(U_1)\neq 0_{L^{\mathscr{Z}}}$, $S\wedge g^{\leftarrow}(U_2)\neq 0_{L^{\mathscr{Z}}}$, $S\wedge g_L^{\leftarrow}(U_1)\wedge g_L^{\leftarrow}(U_2)=0_{L^{\mathscr{Z}}}$, $S\leq g_L^{\leftarrow}(U_1)\vee g_L^{\leftarrow}(U_2)$ where

$$\mathbf{c}\leq(\mathscr{T}_\alpha)_2(U_1')\wedge(\mathscr{T}_\alpha)_2(U_2'),\tag{4.1}$$

and

$$\mathbf{c}\wedge(\mathscr{T}_\alpha)_2(U_3')\leq(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U_3)'),\ \forall U_3\in L^{\mathscr{R}}.\tag{4.2}$$

Upon examination of Eqs. (4.1) and (4.2), the subsequent findings are obtained:

$$\mathbf{c}=\mathbf{c}\wedge(\mathscr{T}_\alpha)_2(U_1')\wedge(\mathscr{T}_\alpha)_2(U_2')rightarrow$$
$$leq(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U_1)')\wedge(\mathscr{T}_\alpha)_1(g_L^{\leftarrow}(U_1)')$$

$$\leq\bigvee_{\substack{S\wedge B\neq 0_{L^{\mathscr{Z}}},\ S\wedge C\neq 0_{L^{\mathscr{Z}}},\\ S\wedge B\wedge C\neq 0_{L^{\mathscr{Z}}},\ S\leq B\vee C}}\left\{\mathscr{T}_{\alpha_1}(B')\vee\mathscr{T}_{\alpha_1}(U_2')\right\}=\mathbf{SPc}_{(\mathscr{T}_\alpha)_1}(S)'.$$

Therefore, given that $\mathbf{c}$ is not fixed, we can assert that $\alpha\mathbf{C}_{(\mathscr{T}_\alpha)_2}(g_L^{\to}(S))'\wedge\alpha\mathbf{I}(g)\leq\alpha\mathbf{C}_{(\mathscr{T}_\alpha)_1}(S)'$

We broaden the understanding of αT_1 and αT_2 in the context of our discussion in this paper.

Lemma 45*Let $g:(\mathscr{Z},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ is a bijective mapping between (L,M)-fuzzy pretopological spaces $\mathscr{Z}$ and $\mathscr{R}$. Then*

1.$\alpha T_1(\mathscr{Z},\mathscr{T}_1)\wedge\alpha\mathbf{OP}(g)\leq\alpha T_1(\mathscr{R},\mathscr{T}_2)$.
2.$\alpha T_2(\mathscr{Z},\mathscr{T}_1)\wedge\alpha\mathbf{OP}(g)\leq\alpha T_2(\mathscr{R},\mathscr{T}_2)$.

Proof.(1) Take any $\mathbf{c}\in M$ such that

$$\mathbf{c}\leq\alpha T_1(\mathscr{Z},\mathscr{T}_1)\wedge\mathbf{SPo}(g)$$
$$=\bigwedge_{a_1\not\leq a_2}\bigvee_{a_1\not\leq S\geq q_1}(\mathscr{T}_\alpha)_1(S')$$
$$\wedge\bigwedge_{B\in L^{\mathscr{Z}}}\Big((\mathscr{T}_\alpha)_1(B)\mapsto(\mathscr{T}_\alpha)_2(g_L^{\to}(B))\Big).$$

Consider any a_1 and a_2 in $P(L^{\mathscr{Z}})$ where a_1 is not comparable to a_2. It follows that there exists an element S in $L^{\mathscr{Z}}$ such that it holds that a_1 is not less than or equal to S, and S is at least as large as a_2, while $\mathbf{c}$ remains less than or equal to $(\mathscr{T}_\alpha)_1(S')$. Sdditionally, for any S_1 found in $L^{\mathscr{Z}}$, if $\mathbf{c}\leq(\mathscr{T}_\alpha)_1(B)$, it leads to the conclusion that $(\mathscr{T}_\alpha)_2(g_L^{\to}(B))$ is valid. Based on Lemma 21 (1), we derive that $\mathbf{c}\wedge(\mathscr{T}_\alpha)_1(B)$ is indeed less than or equal to $(\mathscr{T}_\alpha)_2(g_L^{\to}(S_1))$. In order to demonstrate

$$\mathbf{c}\leq\alpha T_1(\mathscr{R},\mathscr{T}_2)=\bigwedge_{u_1\not\leq u_2}\bigvee_{u_1\not\leq U\geq u_2}(\mathscr{T}_\alpha)_2(U'),$$

let $u_1,\ u_2\in P(L^{\mathscr{R}})$ with $c\not\leq e$. Since g is a bijective mapping, there exist $a_1,\ a_2\in J(L^{\mathscr{Z}})$ with $a_1\not\leq a_2$, such that $u_1=g_L^{\to}(a_1)$ and $u_2=g_L^{\to}(a_2)$. From $a_1\not\leq a_2$, there exists $S\in L^{\mathscr{Z}}$, with $a_1\not\leq S\geq a_2$ such that $\mathbf{c}\leq(\mathscr{T}_\alpha)_1(S')$. Then $u_1=g_L^{\to}(a_1)\not\leq g_L^{\to}(S)\geq g_L^{\to}(a_2)=u_2$. Since g is a bijective mapping, we have

$$\mathbf{c}=\mathbf{c}\wedge(\mathscr{T}_\alpha)_1(S')\leq(\mathscr{T}_\alpha)_2(g_L^{\to}(S'))=(\mathscr{T}_\alpha)_2(g_L^{\to}(S)').$$

This implies

$$\mathbf{c}\leq\bigwedge_{u_1\not\leq u_2}\bigvee_{u_1\not\leq U\geq u_2}(\mathscr{T}_\alpha)_2(V')=\alpha T_1(\mathscr{R},\Gamma_2).$$

By the arbitrariness of $\mathbf{c}$, we obtain

$$\alpha T_1(\mathscr{Z},\mathscr{T}_1)\wedge\alpha OP(g)\leq T_1(\mathscr{R},\mathscr{T}_2)$$

. Other case are similarly proved.

Lemma 46*Let $g:(\mathscr{Z},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ is a bijective mapping between (L,M)-fuzzy pretopological spaces $\mathscr{Z}$ and $\mathscr{R}$. Then*

1.$\alpha T_1(\mathscr{Z},\mathscr{T}_2)\wedge\alpha\mathbf{I}(g)\leq\alpha T_1(\mathscr{R},\mathscr{T}_1)$.
2.$\alpha T_2(\mathscr{Z},\mathscr{T}_2)\wedge\alpha\mathbf{I}(g)\leq\alpha T_2(\mathscr{R},\mathscr{T}_1)$.

*Proof.*It is proved in the same way of Lemma 45

Integrating Lemmas 45, 46, and Definition 32(1) allows us to establish the following theorem.

Theorem 47*Let $g:(\mathscr{Z},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ be a bijective mapping between (L,M)-fpts's $\mathscr{Z}$ and $\mathscr{R}$. Then*

1.$\alpha T_1(\mathscr{Z},\mathscr{T}_1)\wedge(g)\leq\alpha T_1(\mathscr{R},\mathscr{T}_2)$, $\alpha T_1(\mathscr{Z},\mathscr{T}_2)\wedge(g)\leq\alpha T_1(\mathscr{R},\mathscr{T}_1)$.

2. $\alpha T_2(\mathscr{X},\mathscr{T}_1)\wedge(g)\leq\alpha T_2(\mathscr{R},\mathscr{T}_2)$, $\alpha T_2(\mathscr{X},\mathscr{T}_2)\wedge(g)\leq\alpha T_2(\mathscr{R},\mathscr{T}_1)$

The following corollary obtained from Theorem 48.

Corollary 48 *Let $g:(\mathscr{X},\Gamma_1)\to(\mathscr{R},\Gamma_2)$ be a bijective mapping between (L,M)-fpts's $\mathscr{X}$ and $\mathscr{R}$. Then*

1. $\alpha T_1(\mathscr{X},\mathscr{T}_1)\wedge(g)=\alpha T_1(\mathscr{R},\mathscr{T}_2)\wedge(g)$.
2. $\alpha T_2(\mathscr{X},\mathscr{T}_1)\wedge(g)=\alpha T_2(\mathscr{R},\mathscr{T}_2)\wedge(g)$

5 Conclusion

In our work, we have established the α-open operation within the scope of L-fuzzy topological spaces. Using this definition as a starting point, we examine the features of (L,M)-fuzzy pretopology. This specific form of (L,M)-fuzzy α-pretopology paves the way for further exploration of (L,M)-fuzzy extensions. By focusing on pretopological attributes like connectivity and separability, our aim is to investigate these notions more extensively in the (L,M)-fuzzy pretopological space. The present paper serves as a detailed survey of (L,M)-fuzzy pretopological spaces, concentrating on sub-categories of innovative function degrees. We have introduced new weaker variants of function degrees within the (L,M)-fuzzy pretopological spaces through the application of implication operations and Shi's operators. In addition, we analyze the characteristics of α-openness, α-continuity, and the extent of α-irresoluteness in functions related to (L,M)-fuzzy pretopology, confirming that a variety of functions can be regarded as α-open, α-continuous, and exhibit differing levels of α-irresoluteness. Moreover, we constructed and reviewed relationships involving α-compactness, α-connectedness, **Semi**-T_1, **Pre**-T_1, **Semi**-T_2, and **Pre**-T_2.

6 Conclusion

We have defined the α-open operation in the context of L-fuzzy topological spaces. Utilizing this definition, we investigate the properties of the (L,M)-fuzzy pretopology. This particular space of (L,M)-fuzzy α-pretopology facilitates future investigations into (L,M)-fuzzy extensions. By considering pretopological features like connectivity and separability, we aim to further explore these concepts within the (L,M)-fuzzy pretopological space. In this paper, we present a comprehensive overview of (L,M)-fuzzy pretopological spaces, focusing on sub-varieties of innovative function degrees. We have established new weak forms of function degrees within the (L,M)-fuzzy pretopological spaces through the use of implication operations and Shi's operators. Additionally, we explore the properties of α-openness, α-continuity, and the degree of α-irresoluteness in functions associated with (L,M)-fuzzy pretopology, demonstrating that various functions can be considered α-open, α-continuous, and α-irresolute to some degree. We also constructed and examined relationships concerning α-compactness, α-connectedness, **Semi**-T_1, **Pre**-T_1, **Semi**-T_2, and **Pre**-T_2.

7 Acknowledgment

The authors are grateful to the Deanship of Scientific Research at Jadara University for providing financial support for this publication.
The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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