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Interval-Valued Fuzzy Approach for Multi-Attribute Decision-Making in Water Quality Assessment

Atiqe Ur Rahman[1], *Diaa S. Metwally*[2], *M. M. Abd El-Raouf*[3,*], *M. A. El-Qurashi*[3], *Hamiden Abd El-Wahed Khalifa*[4], *and Heba Ghareb Goma*[5]

[1]Department of Mathematics, University of Management and Technology, Lahore, 54000, Pakistan

[2]Deanship of Scientific Research, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia

[3]Institute of Basic and Applied Science, College of Engineering and Technology, Arab Academy for Science, Technology and Maritime Transport, P.O. Box 1029, Abu Quir Campus, Alexandria, Egypt

[4]Department of Operations and Management Research,Faculty of Graduate Studies for Statistical Research, Cairo University, Giza 12613, Egypt

[5]Department of Mathematics and Statistics, Institute of Management Information System, Suiz, Egypt

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Abstract: Artificial intelligence and engineering are crucial for managing urban and industrial water resources, addressing declining drinking water quality, and analyzing water wellness, despite uncertainties. This article seeks to characterize an innovative mathematical framework called an interval-valued fuzzy parameterized interval-valued fuzzy soft set (IVFpIVFSS), better equipped to handle the uncertainties associated with multiple attribute decision-making (MADM) issues like drinking water quality assessment (DWQA). It features two interval-valued settings: the first is for interval-valued fuzzy parameterization, which is intended to assess the parameters' degree of uncertainty, and the second is for approximating the alternatives based on these parameters. Based on set-theoretic operations, appropriate parameters, and their respective interval-valued fuzzy parameterized grades, a decision-assisted system is established with the proposal of an algorithm for DWQA in certain areas. The flexibility of the study is assessed by its comparison with published literature based on structure and computations. The proposed IVFpIVFSS provides a robust approach to addressing the uncertainties in MADM challenges like DWQA, demonstrating improved capabilities for managing water quality in complex and uncertain environments.

Keywords: Computational intelligence; Decision making; Fuzzy parameterization; Interval-valued fuzzy set; Optimization; Water quality.

1 Introduction

Data-based uncertainties typically arise from incomplete, vague, or inconsistent information. Managing these uncertainties has been a major challenge for researchers across many important fields. As a result, the development of various mathematical models has increased. As a link in the same chain, Zadeh [34] proposed the idea of the fuzzy set (FS), which is capable of handling the uncertainty attached to any entity's membership in any given set by means of a membership function. This function assigns each element a specific real value from [0,1] called its membership grade. It is pertinent to understand, that in numerous practical scenarios, it is not possible to simply verify the level of inclusion in a fuzzy set. For describing the membership grade, interval-valued data makes more sense. Furthermore, Zadeh [35,36], Jahn [15] and Gorzałczany [11] jointly put out the idea of an interval-valued fuzzy set (IVFS) from this perspective. Molodtsov [20] initially put forth soft set theory (SOST) as a general mathematical tool for handling ambiguous, imprecise, and poorly specified things. The SOST differs from other

* Corresponding author e-mail: m_abdelraouf85@aast.edu

conventional methods of handling uncertainties in that it is not constrained by the limitations inherent in them regarding the parametrization techniques. He has successfully implemented his proposed notions in a number of areas, including probability, theory of measurement, game theory, operations research, and so forth. Approximate mapping is used in SOST to approximate the alternatives depending on suitable specifications. A subset of the initial universe is produced by this mapping for every parameter. The efforts of Saeed et al. [25] in investigating the basic concepts and operations of the SOSTs are commendable and unique. Fuzzy soft set (FSOS) [17] has been introduced in order to manage uncertainty and ambiguity in the most efficient manner through the integration of certain characteristics of SOS and FS. The researchers [32,16] made significant contributions to the field of SOST. To deal with interval-type data and parameterization collectively, the interval-valued fuzzy soft set (IVFSOS) [27,21] has been put forward. The idea of fuzzy parameterization (IFP) is used to regulate ambiguity in scenarios where there is uncertainty in the parameter selection process. To deal with the vagueness of each parameter, a specific fuzzy value is supplied to it. Çağman et al. [8] employed this idea to develop fuzzy parameterized soft set (FpSOS) and fuzzy parameterized fuzzy soft set (FpFSOS). They looked into a number of the fundamental features, operations, and results of FpFSOS. They also talked about the hiring procedure based on the FpFSOS aggregation operators. After that, Zhu and Zhan [38] presented t-norms and t-conorms products of FpFSOS and then discussed the selection of a suitable car based on AND-FpFSOS and OR-FpFSOS decision-making methods. Bashir and Salleh [5,12] integrated the IFP, SOS and expert set to put forward the generalized structures. Similarly, Riaz and Hashmi [24] discussed aggregation operators, AND-operation, comparison tables, OR-operation and reduct of FpFSOS. Alkhazaleh et al. [2] combined IFP and IVFSOS while developing fuzzy parameterized interval-valued fuzzy soft set (FpIVFSOS). Rahman et al. [23] discussed the market based problem using the idea of fuzzy parameterization with picture fuzzy soft set environment. Recently, Bataihah and Hazaymeh [6] extended the notions of FSOS by introducing the notions of time fuzzy parameterized fuzzy soft expert set which is meant to weigh time impact in the approximations. Memiş et al. [18] discussed the data classification using distance-based similarity measures of FpFSOS embedded with matrix manipulation. They [19] also discussed classification method in machine learning by proposing soft decision making strategy using matrix manipulation of FpFSOS. Thammajitr et al. [28] put forward the idea of fuzzy parameterized relative fuzzy soft sets as an extension of FSOS and FpFSOS. They also discussed its several rudiments as well. For more details see [29,30,31]

1.1 Relevant Literature and Motivation

Drinking water quality is crucial for people's health, public sanitation, and sustaining life. Poor water quality can lead to health issues like gastrointestinal disorders and waterborne illnesses. Maintaining high standards protects human health, the environment, and sustainable development. Ensuring clean drinking water is a basic human right. The DWQA is a significant concern nowadays since the wellness of society depends on the availability of clean drinking water. The ability to predict the quality of drinking water is a valuable tool for enhancing sanitation and combating pollution; artificial intelligence techniques have been shown to be effective in this field, as have practices and technologies for this purpose [26,3]. Since frequent inspections and assessments enable the deployment of appropriate waste water treatment and filtration methods to supply clean and safe drinking water for the general population, water quality evaluation is particularly crucial for regulating and guaranteeing the hygiene of drinking water [13,37]. To reduce water pollution, safeguard the environment, and lessen illnesses associated with water, water quality modeling and prediction have become crucial. Building sustainable and livable ecosystems depends on the assessment of water quality; modern technologies that analyze and regulate water quality in real-time improve living circumstances for people and enable societies to react quickly to calamities [7,1]. Barzegar et al. [4] handled DWQA while controlling uncertainties by putting forth a multilevel fuzzy design based on an actual data set with nine parameters. Hu et al. [14] employed an integrated probabilistic-fuzzy synthetic evaluation approach to evaluate the visual appeal and health hazards associated with DWQA in remote and rural areas. Based on eleven water-related factors, Gorai et al. [10] proposed a fuzzy aggregation technique to determine the DWQA of an extract to verify its appropriateness for consumption. Patil et al. [22] explored DWQA using a unique weight-integrated health hazard index in conjunction with soft computing approaches and artificial neural networks.
The literature reviewed above reveals that these are insufficient to manage the following challenges:

1. Decision-makers frequently encounter uncertainty due to differing opinions on the significance of various parameters. Fuzzy membership grades can help address this uncertainty by acknowledging the different levels of importance or certainty associated with each parameter. This approach allows for a more flexible and nuanced decision-making process that takes into account diverse perspectives and uncertainties.
2. Interval-valued membership degrees address the challenge of selecting membership degrees for fuzzy set elements due to subjectivity and ambiguity. This approach provides a realistic representation of uncertainty and imprecision, allowing greater flexibility.

3.The SOSs are flexible, requiring no immediate recalibration for parameter changes, making them ideal for complex data situations, providing a reliable and comprehensible structure for real-world problem-solving.

To address the challenges of uncertainty and complexity in decision-making, this study introduces a novel theoretical hybrid structure called the interval-valued fuzzy parameterized interval-valued fuzzy soft set (IVFpIVFSS). The advantages of fuzzy parameterization, interval-valued fuzzy settings, and soft sets are combined in this novel framework. The IVFpIVFSS provides a more thorough method of handling and modeling uncertainty by combining these components. It is so dependable and trustworthy that it can both address and improve upon the drawbacks of the previous literature. The contributing features of the study are:

1.By combining the three well-known concepts of IVFP, IVFS, and SOS, a novel fuzzy hybrid context known as IVFpIVFSS is presented. By providing a reliable technique for modelling and analyzing factors with varying degrees of uncertainty, this integration addresses the shortcomings in the literature and enhances the accuracy and applicability of decision-making processes.
2.To evaluate the level of ambiguity, specific interval-valued fuzzy parameterized ratings are obtained from experts and decision-makers for each parameter. By depicting a range of potential values, these grades address the ambiguity and variability in the parameters' relevance. By using interval-valued fuzzy parameterized grades, the method allows for different perspectives and degrees of certainty, leading to a more flexible and nuanced evaluation.
3.A robust algorithm has been developed to evaluate the quality of drinking water in educational institutions using interval-valued fuzzy parameters, choice values, scoring values, and IVFpIVFSS, ensuring it meets health and safety regulations and protects students' health.

The remaining part of the study is structured so that Section 2 serves as an overview of foundational information from published research. In Section 3, an algorithmic decision framework, the validation of the suggested algorithm, the exploration of IVFpIVFSS and its operational results are presented. The study's structural comparisons with a few existing works are shown in this section, and the investigation is ultimately concluded in Section 4.

2 Fundamental Knowledge

In order to help the readers grasp the concepts, some definitions that are required are provided in this section.

The symbols $\hat{\mho}$, $\mathbb{CI}([0,1])$ and $2^{\hat{\mho}}$ stand for universal set, family of closed unit subintervals and power set of $\hat{\mho}$ throughout the remaining paper.

Definition 1.[11] *An IVFS $\hat{\mathfrak{A}}$ over $\hat{\mho}$ is a mapping $\hat{\varpi}$ that maps $\hat{\mho}$ into $\mathbb{CI}([0,1])$. It can be stated as*

$$\hat{\mathfrak{A}} = \left\{ \begin{array}{l} (\hat{\hbar}, \hat{\varpi}(\hat{\hbar}) = [\hat{\varpi}_L(\hat{\hbar}), \hat{\varpi}_U(\hat{\hbar})]) : \\ \hat{\hbar} \in \hat{\mho}, \hat{\varpi}(\hat{\hbar}) \subseteq \mathbb{CI}([0,1]) \end{array} \right\}$$

where $\hat{\varpi}_L(\hat{\hbar}) \in [0,1]$ and $\hat{\varpi}_U(\hat{\hbar}) \in [0,1]$ are respectively the lower and upper belonging grades of $\hat{\hbar} \in \hat{\mho}$ such that $0 \leq \hat{\varpi}_L(\hat{\hbar}) \leq \hat{\varpi}_U(\hat{\hbar}) \leq 1$. The family of IVFSs over $\hat{\mho}$ is denoted by $IVFS(\hat{\mho})$. For convenience, an IVFS is represented as $\hat{\mathfrak{A}} = [\hat{\varpi}_L(\hat{\hbar}), \hat{\varpi}_U(\hat{\hbar})]$.

*Example 1.*Let $\hat{\mho} = \{\hat{\hbar}_1, \hat{\hbar}_2, \hat{\hbar}_3, \hat{\hbar}_4, \hat{\hbar}_5\}$ be the initial space of objects. The interval-valued membership grades of $\hat{\hbar}_i, i = 1,2,...,5$ are $\hat{\varpi}(\hat{\hbar}_1) = [0.21, 0.57]$, $\hat{\varpi}(\hat{\hbar}_2) = [0.34, 0.88]$, $\hat{\varpi}(\hat{\hbar}_3) = [0.18, 0.37]$, $\hat{\varpi}(\hat{\hbar}_4) = [0.51, 0.81]$ and $\hat{\varpi}(\hat{\hbar}_5) = [0.47, 0.79]$. Then, an IVFS $\hat{\mathfrak{A}}$ over $\hat{\mho}$ is constructed as
$\hat{\mathfrak{A}} = \left\{ \begin{array}{l} (\hat{\hbar}_1, \hat{\varpi}(\hat{\hbar}_1)), (\hat{\hbar}_2, \hat{\varpi}(\hat{\hbar}_2)), (\hat{\hbar}_3, \hat{\varpi}(\hat{\hbar}_3)), \\ (\hat{\hbar}_4, \hat{\varpi}(\hat{\hbar}_4)), (\hat{\hbar}_5, \hat{\varpi}(\hat{\hbar}_5)) \end{array} \right\}$ or $\hat{\mathfrak{A}} =$
$\left\{ \begin{array}{l} (\hat{\hbar}_1, [0.21, 0.57]), (\hat{\hbar}_2, [0.34, 0.88]), \\ (\hat{\hbar}_3, [0.18, 0.37]), (\hat{\hbar}_4, [0.51, 0.81]), (\hat{\hbar}_5, [0.47, 0.79]) \end{array} \right\}$.

Definition 2.[11] *Let $\hat{\mathfrak{A}}_1 = [\hat{\varpi}_{L_1}(\hat{\hbar}), \hat{\varpi}_{U_1}(\hat{\hbar})]$ and $\hat{\mathfrak{A}}_2 = [\hat{\varpi}_{L_2}(\hat{\hbar}), \hat{\varpi}_{U_2}(\hat{\hbar})]$ be IVFSs over $\hat{\mho}$ then*
1. $\hat{\mathfrak{A}}^c = [1 - \hat{\varpi}_U(\hat{\hbar}), 1 - \hat{\varpi}_L(\hat{\hbar})]$.
2.
$\hat{\mathfrak{A}}_1 \cup \hat{\mathfrak{A}}_2 = [sup\{\hat{\varpi}_{L_1}(\hat{\hbar}), \hat{\varpi}_{L_2}(\hat{\hbar})\}, sup\{\hat{\varpi}_{U_1}(\hat{\hbar}), \hat{\varpi}_{U_2}(\hat{\hbar})\}]$.
3.
$\hat{\mathfrak{A}}_1 \cap \hat{\mathfrak{A}}_2 = [inf\{\hat{\varpi}_{L_1}(\hat{\hbar}), \hat{\varpi}_{L_2}(\hat{\hbar})\}, inf\{\hat{\varpi}_{U_1}(\hat{\hbar}), \hat{\varpi}_{U_2}(\hat{\hbar})\}]$.

Definition 3.[20] *Let $\hat{\mathbb{Z}}$ be the collection of parameters and $\hat{\Psi}_{\hat{\mathbb{Z}}} : \hat{\mathbb{Z}} \to 2^{\hat{\mho}}$ be an approximate mapping then SOS $\hat{\mathbb{Y}}$ over $\hat{\mho}$ is stated as*

$$\hat{\mathbb{Y}} = \left\{ (\hat{z}, \hat{\Psi}_{\hat{\mathbb{Z}}}(\hat{z})) : \hat{z} \in \hat{\mathbb{Z}}, \hat{\Psi}_{\hat{\mathbb{Z}}}(\hat{z}) \subseteq 2^{\hat{\mho}} \right\}.$$

*Example 2.*Let $\hat{\mho} = \{\hat{\hbar}_1, \hat{\hbar}_2, \hat{\hbar}_3, \hat{\hbar}_4, \hat{\hbar}_5\}$ and $\hat{\mathbb{Z}} = \{\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4, \hat{z}_5\}$ be the initial space of objects and the collection of parameters, respectively. The approximations of $\hat{\hbar}_i, i = 1,2,...,5$ with respect to parameters $\hat{z}_j, j = 1,2,...,5$ are $\hat{\Psi}_{\hat{\mathbb{Z}}}(\hat{z}_1) = \{\hat{\hbar}_1, \hat{\hbar}_3, \hat{\hbar}_5\}$, $\hat{\Psi}_{\hat{\mathbb{Z}}}(\hat{z}_2) = \{\hat{\hbar}_1, \hat{\hbar}_2, \hat{\hbar}_5\}$, $\hat{\Psi}_{\hat{\mathbb{Z}}}(\hat{z}_3) = \{\hat{\hbar}_2, \hat{\hbar}_4\}$, $\hat{\Psi}_{\hat{\mathbb{Z}}}(\hat{z}_4) = \{\hat{\hbar}_1, \hat{\hbar}_3\}$ and $\hat{\Psi}_{\hat{\mathbb{Z}}}(\hat{z}_5) = \{\hat{\hbar}_1, \hat{\hbar}_5\}$. Then, the SOS $\hat{\mathbb{Y}}$ over $\hat{\mho}$ is constructed as

$$\hat{\mathbb{Y}} = \left\{ \begin{array}{l} (\hat{z}_1, \hat{\Psi}_{\hat{\mathbb{Z}}}(\hat{z}_1)), (\hat{z}_2, \hat{\Psi}_{\hat{\mathbb{Z}}}(\hat{z}_2)), (\hat{z}_3, \hat{\Psi}_{\hat{\mathbb{Z}}}(\hat{z}_3)), \\ (\hat{z}_4, \hat{\Psi}_{\hat{\mathbb{Z}}}(\hat{z}_4)), (\hat{z}_5, \hat{\Psi}_{\hat{\mathbb{Z}}}(\hat{z}_5)) \end{array} \right\}$$

or

$$\hat{\mathbb{Y}} = \left\{ \begin{array}{l} (\hat{z}_1, \{\hat{\hbar}_1, \hat{\hbar}_3, \hat{\hbar}_5\}), (\hat{z}_2, \{\hat{\hbar}_1, \hat{\hbar}_2, \hat{\hbar}_5\}), (\hat{z}_3, \{\hat{\hbar}_2, \hat{\hbar}_4\}), \\ (\hat{z}_4, \{\hat{\hbar}_1, \hat{\hbar}_3\}), (\hat{z}_5, \{\hat{\hbar}_1, \hat{\hbar}_5\}) \end{array} \right\}.$$

Definition 4.[27,21] *Let $\hat{\mathbb{Z}}$ be the collection of parameters and $\hat{\Phi}_{\hat{\mathbb{Z}}}:\hat{\mathbb{Z}}\rightarrow IVFS(\hat{\mho})$ be an approximate mapping then IVFSOS $\hat{\mathbb{T}}$ over $\hat{\mho}$ is stated as*

$$\hat{\mathbb{T}}=\left\{\left(\hat{z},\hat{\Phi}_{\hat{\mathbb{Z}}}(\hat{z})\right):\hat{z}\in\hat{\mathbb{Z}},\hat{\Phi}_{\hat{\mathbb{Z}}}(\hat{z})\subseteq IVFS(\hat{\mho})\right\}.$$

3 Materials and Methods

In this portion, the main components of the suggested scheme are outlined. Two sessions are involved. In the first session, the proposed mathematical framework and related operational properties are presented. This session also includes a description of the selected parameters and their purposes. In the second session, a decision support framework for DWQA based on the algorithm is provided.

The IVFpIVFSS introduces a dual-level interval-valued fuzzy parameterization, which significantly enhances the capacity to model complex, uncertain, and parameter-dependent decision environments. Unlike conventional neutrosophic sets, this framework offers a more structured way to handle multi-parameter and sub-parameter uncertainty while preserving computational tractability.

3.1 Theoretical Development of IVFpIVFSS

Definition 5.*Let $P^{\hat{\mho}}$ and $\hat{\mathbb{Z}}$ be the sets consisting of sub collections of $\hat{\mho}$ and parameters respectively. Let $\hat{\mathfrak{A}}_{\hat{\mathbb{Z}}}=\{\frac{\hat{z}}{[\tilde{\zeta}_L(\hat{z}),\tilde{\zeta}_U(\hat{z})]}:\hat{z}\in\hat{\mathbb{Z}};\tilde{\zeta}_L(\hat{z}),\tilde{\zeta}_U(\hat{z})\in[0,1]\}$ be an IVFS over $\hat{\mathbb{Z}}$ consisting of interval-valued fuzzy parameterized grades (IVFPGs) for attributes $\hat{z}$ then IVFpIVFSS $\tilde{\Omega}$ is characterized by an approximate mapping $\tilde{\alpha}:\hat{\mathfrak{A}}_{\hat{\mathbb{Z}}}\rightarrow IVFS(\hat{\mho})$ and defined as*

$$\tilde{\Omega}=\left\{\begin{array}{c}\left(\frac{\hat{z}}{[\tilde{\zeta}_L(\hat{z}),\tilde{\zeta}_U(\hat{z})]},\tilde{\alpha}(\frac{\hat{z}}{[\tilde{\zeta}_L(\hat{z}),\tilde{\zeta}_U(\hat{z})]})\right):\\ \frac{\hat{z}}{[\tilde{\zeta}_L(\hat{z}),\tilde{\zeta}_U(\hat{z})]}\in\hat{\mathfrak{A}}_{\hat{\mathbb{Z}}},\tilde{\alpha}(\frac{\hat{z}}{[\tilde{\zeta}_L(\hat{z}),\tilde{\zeta}_U(\hat{z})]})\subseteq IVFS(\hat{\mho})\end{array}\right\}$$

such that $\tilde{\alpha}(\frac{\hat{z}}{[\tilde{\zeta}_L(\hat{z}),\tilde{\zeta}_U(\hat{z})]})=\varnothing$ for all $\frac{\hat{z}}{[\tilde{\zeta}_L(\hat{z}),\tilde{\zeta}_U(\hat{z})]}\notin\hat{\mathfrak{A}}_{\hat{\mathbb{Z}}}$ where

$$\tilde{\alpha}(\frac{\hat{z}}{[\tilde{\zeta}_L(\hat{z}),\tilde{\zeta}_U(\hat{z})]})=\left\{\begin{array}{c}\frac{\hat{\hbar}}{[\hat{\varpi}_L(\hat{\hbar}),\hat{\varpi}_U(\hat{\hbar})]}:\\ \hat{\hbar}\in\hat{\mho};\hat{\varpi}_L(\hat{\hbar}),\hat{\varpi}_U(\hat{\hbar})\in[0,1]\end{array}\right\}.$$

For convenience, the IVFpIVFSS $\tilde{\Omega}$ is represented by $(\tilde{\alpha},\hat{\mathfrak{A}}_{\hat{\mathbb{Z}}})$.

*Example 3.*Consider $\hat{\mho}=\{\hat{\hbar}_1,\hat{\hbar}_2,\hat{\hbar}_3,\hat{\hbar}_4\}$ and $\hat{\mathbb{Z}}=\{\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{z}_4\}$ be the sets consisting of options and parameters respectively. The IVFPGs of parameters $\hat{z}_1,\hat{z}_2,\hat{z}_3,$ and $\hat{z}_4$ are $\tilde{\zeta}(\hat{z}_1)=[0.21,0.32]$, $\tilde{\zeta}(\hat{z}_2)=[0.51,0.62]$, $\tilde{\zeta}(\hat{z}_3)=[0.31,0.42]$, and $\tilde{\zeta}(\hat{z}_4)=[0.61,0.72]$ respectively. Thus, by treating $\hat{\mathbb{Z}}$ and IVFPGs collectively, an IVFS

$$\hat{\mathfrak{A}}_{\hat{\mathbb{Z}}}=\left\{\frac{\hat{z}_1}{[0.21,0.32]},\frac{\hat{z}_2}{[0.51,0.62]},\frac{\hat{z}_3}{[0.31,0.42]},\frac{\hat{z}_4}{[0.61,0.72]}\right\}$$

over $\hat{\mathbb{Z}}$ is constructed. The interval-valued fuzzy parameterized graded-parameters are approximated as:

$$\tilde{\alpha}(\frac{\hat{z}_1}{[0.21,0.32]})=\left\{\begin{array}{c}\frac{\hat{\hbar}_1}{[0.21,0.33]},\frac{\hat{\hbar}_2}{[0.23,0.35]},\\ \frac{\hat{\hbar}_3}{[0.25,0.37]},\frac{\hat{\hbar}_4}{[0.27,0.39]}\end{array}\right\},$$

$$\tilde{\alpha}(\frac{\hat{z}_2}{[0.51,0.62]})=\left\{\begin{array}{c}\frac{\hat{\hbar}_1}{[0.11,0.21]},\frac{\hat{\hbar}_2}{[0.23,0.33]},\\ \frac{\hat{\hbar}_3}{[0.26,0.35]},\frac{\hat{\hbar}_4}{[0.37,0.57]}\end{array}\right\},$$

$$\tilde{\alpha}(\frac{\hat{z}_3}{[0.31,0.42]})=\left\{\begin{array}{c}\frac{\hat{\hbar}_1}{[0.11,0.41]},\frac{\hat{\hbar}_2}{[0.13,0.35]},\\ \frac{\hat{\hbar}_3}{[0.35,0.47]},\frac{\hat{\hbar}_4}{[0.17,0.37]}\end{array}\right\},$$

$$\tilde{\alpha}(\frac{\hat{z}_4}{[0.61,0.72]})=\left\{\begin{array}{c}\frac{\hat{\hbar}_1}{[0.41,0.63]},\frac{\hat{\hbar}_2}{[0.43,0.73]},\\ \frac{\hat{\hbar}_3}{[0.34,0.76]},\frac{\hat{\hbar}_4}{[0.45,0.83]}\end{array}\right\}.$$

The IVFpIVFSS $\tilde{\Omega}$ can be constructed as

$$\tilde{\Omega}=\left\{\begin{array}{c}\left(\frac{\hat{z}_1}{[0.21,0.32]},\left\{\frac{\hat{\hbar}_1}{[0.21,0.33]},\frac{\hat{\hbar}_2}{[0.23,0.35]},\frac{\hat{\hbar}_3}{[0.25,0.37]},\frac{\hat{\hbar}_4}{[0.27,0.39]}\right\}\right),\\ \left(\frac{\hat{z}_2}{[0.51,0.62]},\left\{\frac{\hat{\hbar}_1}{[0.11,0.21]},\frac{\hat{\hbar}_2}{[0.23,0.33]},\frac{\hat{\hbar}_3}{[0.26,0.35]},\frac{\hat{\hbar}_4}{[0.37,0.57]}\right\}\right),\\ \left(\frac{\hat{z}_3}{[0.31,0.42]},\left\{\frac{\hat{\hbar}_1}{[0.11,0.41]},\frac{\hat{\hbar}_2}{[0.13,0.35]},\frac{\hat{\hbar}_3}{[0.35,0.47]},\frac{\hat{\hbar}_4}{[0.17,0.37]}\right\}\right),\\ \left(\frac{\hat{z}_4}{[0.61,0.72]},\left\{\frac{\hat{\hbar}_1}{[0.41,0.63]},\frac{\hat{\hbar}_2}{[0.43,0.73]},\frac{\hat{\hbar}_3}{[0.34,0.76]},\frac{\hat{\hbar}_4}{[0.45,0.83]}\right\}\right)\end{array}\right\}.$$

Definition 6.*Let*

$$\tilde{\Omega}_1=\left\{\begin{array}{c}\left(\frac{\hat{z}}{[\tilde{\zeta}^1_L(\hat{z}),\tilde{\zeta}^1_U(\hat{z})]},\tilde{\alpha}^1_F\left(\frac{\hat{z}}{[\tilde{\zeta}^1_L(\hat{z}),\tilde{\zeta}^1_U(\hat{z})]}\right)\right):\\ \frac{\hat{z}}{[\tilde{\zeta}^1_L(\hat{z}),\tilde{\zeta}^1_U(\hat{z})]}\in\hat{\mathfrak{A}}^1_{\hat{\mathbb{Z}}}\end{array}\right\}$$

and

$$\tilde{\Omega}_2=\left\{\begin{array}{c}\left(\frac{\hat{z}}{[\tilde{\zeta}^2_L(\hat{z}),\tilde{\zeta}^2_U(\hat{z})]},\tilde{\alpha}^2_F\left(\frac{\hat{z}}{[\tilde{\zeta}^2_L(\hat{z}),\tilde{\zeta}^2_U(\hat{z})]}\right)\right):\\ \frac{\hat{z}}{[\tilde{\zeta}^2_L(\hat{z}),\tilde{\zeta}^2_U(\hat{z})]}\in\hat{\mathfrak{A}}^2_{\hat{\mathbb{Z}}}\end{array}\right\}$$

be IVFpIVFSSs then

1. The IVFpIVFSS $\tilde{\Omega}_3=\left\{\left(\frac{\hat{z}}{[\tilde{\zeta}^3_L(\hat{z}),\tilde{\zeta}^3_U(\hat{z})]},\tilde{\alpha}^3_F\left(\frac{\hat{z}}{[\tilde{\zeta}^3_L(\hat{z}),\tilde{\zeta}^3_U(\hat{z})]}\right)\right):\frac{\hat{z}}{[\tilde{\zeta}^3_L(\hat{z}),\tilde{\zeta}^3_U(\hat{z})]}\in\hat{\mathfrak{A}}^3_{\hat{\mathbb{Z}}}\right\}$ is their union such that $\tilde{\alpha}^3_F\left(\frac{\hat{z}}{[\tilde{\zeta}^3_L(\hat{z}),\tilde{\zeta}^3_U(\hat{z})]}\right)=$

$$\left\{\begin{array}{ll}\tilde{\alpha}^1_F\left(\frac{\hat{z}}{[\tilde{\zeta}^1_L(\hat{z}),\tilde{\zeta}^1_U(\hat{z})]}\right) & ;\frac{\hat{z}}{[\tilde{\zeta}^3_L(\hat{z}),\tilde{\zeta}^3_U(\hat{z})]}\in\hat{\mathfrak{A}}^1_{\hat{\mathbb{Z}}}\setminus\hat{\mathfrak{A}}^2_{\hat{\mathbb{Z}}}\\ \tilde{\alpha}^2_F\left(\frac{\hat{z}}{[\tilde{\zeta}^2_L(\hat{z}),\tilde{\zeta}^2_U(\hat{z})]}\right) & ;\frac{\hat{z}}{[\tilde{\zeta}^3_L(\hat{z}),\tilde{\zeta}^3_U(\hat{z})]}\in\hat{\mathfrak{A}}^2_{\hat{\mathbb{Z}}}\setminus\hat{\mathfrak{A}}^1_{\hat{\mathbb{Z}}}\\ \tilde{\alpha}^1_F\left(\frac{\hat{z}}{[\tilde{\zeta}^1_L(\hat{z}),\tilde{\zeta}^1_U(\hat{z})]}\right)\cup\tilde{\alpha}^2_F\left(\frac{\hat{z}}{[\tilde{\zeta}^2_L(\hat{z}),\tilde{\zeta}^2_U(\hat{z})]}\right) & ;\frac{\hat{z}}{[\tilde{\zeta}^3_L(\hat{z}),\tilde{\zeta}^3_U(\hat{z})]}\in\hat{\mathfrak{A}}^1_{\hat{\mathbb{Z}}}\cap\hat{\mathfrak{A}}^2_{\hat{\mathbb{Z}}}\end{array}\right.$$

where

$$\tilde{\alpha}_F^1\left(\frac{\hat{z}}{[\tilde{\zeta}_L^1(\hat{z}),\,\tilde{\zeta}_U^1(\hat{z})]}\right)\cup\tilde{\alpha}_F^2\left(\frac{\hat{z}}{[\tilde{\zeta}_L^2(\hat{z}),\,\tilde{\zeta}_U^2(\hat{z})]}\right)=$$

$$\left\{\frac{\hat{h}}{[sup\{\hat{\varpi}_L^1(\hat{h}),\hat{\varpi}_L^2(\hat{h})\},sup\{\hat{\varpi}_U^1(\hat{h}),\hat{\varpi}_U^2(\hat{h})\}]}:\hat{h}\in\hat{\mho}\right\}.$$

2. The IVFpIVFSS $\tilde{\Omega}_4=\left\{\left(\frac{\hat{z}}{[\tilde{\zeta}_L^4(\hat{z}),\tilde{\zeta}_U^4(\hat{z})]},\tilde{\alpha}_F^4\left(\frac{\hat{z}}{[\tilde{\zeta}_L^4(\hat{z}),\tilde{\zeta}_U^4(\hat{z})]}\right)\right):\frac{\hat{z}}{[\tilde{\zeta}_L^4(\hat{z}),\tilde{\zeta}_U^4(\hat{z})]}\in\hat{\mathfrak{A}}_{\hat{\mathbb{Z}}}^4\right\}$ *is their union such that* $\tilde{\alpha}_F^4\left(\frac{\hat{z}}{[\tilde{\zeta}_L^4(\hat{z}),\tilde{\zeta}_U^4(\hat{z})]}\right)=$

$$\tilde{\alpha}_F^1\left(\frac{\hat{z}}{[\tilde{\zeta}_L^1(\hat{z}),\,\tilde{\zeta}_U^1(\hat{z})]}\right)\cap\tilde{\alpha}_F^2\left(\frac{\hat{z}}{[\tilde{\zeta}_L^2(\hat{z}),\,\tilde{\zeta}_U^2(\hat{z})]}\right)$$

when $\frac{\hat{z}}{[\tilde{\zeta}_L^4(\hat{z}),\tilde{\zeta}_U^4(\hat{z})]}\in\hat{\mathfrak{A}}_{\hat{\mathbb{Z}}}^1\cap\hat{\mathfrak{A}}_{\hat{\mathbb{Z}}}^2$ *where* $\tilde{\alpha}_F^1\left(\frac{\hat{z}}{[\tilde{\zeta}_L^1(\hat{z}),\tilde{\zeta}_U^1(\hat{z})]}\right)\cap\tilde{\alpha}_F^2\left(\frac{\hat{z}}{[\tilde{\zeta}_L^2(\hat{z}),\tilde{\zeta}_U^2(\hat{z})]}\right)=$

$$\left\{\frac{\hat{h}}{[inf\{\hat{\varpi}_L^1(\hat{h}),\hat{\varpi}_L^2(\hat{h})\},inf\{\hat{\varpi}_U^1(\hat{h}),\hat{\varpi}_U^2(\hat{h})\}]}:\hat{h}\in\hat{\mho}\right\}.$$

*Example 4.*The IVFpIVFSSs $\tilde{\Omega}_1$, $\tilde{\Omega}_2$, $\tilde{\Omega}_3$, and $\tilde{\Omega}_4$ have been constructed after following Example 3. $\tilde{\Omega}_1=$

$$\left\{\begin{array}{l}\left(\frac{\hat{z}_1}{[0.21,0.32]},\left\{\frac{\hat{h}_1}{[0.21,0.33]},\frac{\hat{h}_2}{[0.23,0.35]},\frac{\hat{h}_3}{[0.25,0.37]},\frac{\hat{h}_4}{[0.27,0.39]}\right\}\right),\\ \left(\frac{\hat{z}_2}{[0.51,0.62]},\left\{\frac{\hat{h}_1}{[0.11,0.21]},\frac{\hat{h}_2}{[0.23,0.33]},\frac{\hat{h}_3}{[0.26,0.35]},\frac{\hat{h}_4}{[0.37,0.57]}\right\}\right),\\ \left(\frac{\hat{z}_3}{[0.31,0.42]},\left\{\frac{\hat{h}_1}{[0.11,0.41]},\frac{\hat{h}_2}{[0.13,0.35]},\frac{\hat{h}_3}{[0.35,0.47]},\frac{\hat{h}_4}{[0.17,0.37]}\right\}\right),\\ \left(\frac{\hat{z}_4}{[0.61,0.72]},\left\{\frac{\hat{h}_1}{[0.41,0.63]},\frac{\hat{h}_2}{[0.43,0.73]},\frac{\hat{h}_3}{[0.34,0.76]},\frac{\hat{h}_4}{[0.45,0.83]}\right\}\right)\end{array}\right\}.$$

$\tilde{\Omega}_2=$

$$\left\{\begin{array}{l}\left(\frac{\hat{z}_1}{[0.21,0.32]},\left\{\frac{\hat{h}_1}{[0.22,0.34]},\frac{\hat{h}_2}{[0.24,0.36]},\frac{\hat{h}_3}{[0.26,0.38]},\frac{\hat{h}_4}{[0.28,0.40]}\right\}\right),\\ \left(\frac{\hat{z}_2}{[0.52,0.63]},\left\{\frac{\hat{h}_1}{[0.12,0.22]},\frac{\hat{h}_2}{[0.24,0.35]},\frac{\hat{h}_3}{[0.27,0.37]},\frac{\hat{h}_4}{[0.38,0.59]}\right\}\right),\\ \left(\frac{\hat{z}_3}{[0.34,0.45]},\left\{\frac{\hat{h}_1}{[0.16,0.48]},\frac{\hat{h}_2}{[0.15,0.39]},\frac{\hat{h}_3}{[0.39,0.49]},\frac{\hat{h}_4}{[0.19,0.39]}\right\}\right),\\ \left(\frac{\hat{z}_4}{[0.64,0.79]},\left\{\frac{\hat{h}_1}{[0.45,0.68]},\frac{\hat{h}_2}{[0.46,0.77]},\frac{\hat{h}_3}{[0.36,0.79]},\frac{\hat{h}_4}{[0.47,0.89]}\right\}\right)\end{array}\right\}.$$

$\tilde{\Omega}_3=\tilde{\Omega}_1\cup\tilde{\Omega}_2=$

$$\left\{\begin{array}{l}\left(\frac{\hat{z}_1}{[0.21,0.32]},\left\{\frac{\hat{h}_1}{[0.22,0.34]},\frac{\hat{h}_2}{[0.24,0.36]},\frac{\hat{h}_3}{[0.26,0.38]},\frac{\hat{h}_4}{[0.28,0.40]}\right\}\right),\\ \left(\frac{\hat{z}_2}{[0.52,0.63]},\left\{\frac{\hat{h}_1}{[0.12,0.22]},\frac{\hat{h}_2}{[0.24,0.35]},\frac{\hat{h}_3}{[0.27,0.37]},\frac{\hat{h}_4}{[0.38,0.59]}\right\}\right),\\ \left(\frac{\hat{z}_3}{[0.34,0.45]},\left\{\frac{\hat{h}_1}{[0.16,0.48]},\frac{\hat{h}_2}{[0.15,0.39]},\frac{\hat{h}_3}{[0.39,0.49]},\frac{\hat{h}_4}{[0.19,0.39]}\right\}\right),\\ \left(\frac{\hat{z}_4}{[0.64,0.79]},\left\{\frac{\hat{h}_1}{[0.45,0.68]},\frac{\hat{h}_2}{[0.46,0.77]},\frac{\hat{h}_3}{[0.36,0.79]},\frac{\hat{h}_4}{[0.47,0.89]}\right\}\right)\end{array}\right\}.$$

$\tilde{\Omega}_4=\tilde{\Omega}_1\cap\tilde{\Omega}_2=$

$$\left\{\begin{array}{l}\left(\frac{\hat{z}_1}{[0.21,0.32]},\left\{\frac{\hat{h}_1}{[0.21,0.33]},\frac{\hat{h}_2}{[0.23,0.35]},\frac{\hat{h}_3}{[0.25,0.37]},\frac{\hat{h}_4}{[0.27,0.39]}\right\}\right),\\ \left(\frac{\hat{z}_2}{[0.51,0.62]},\left\{\frac{\hat{h}_1}{[0.11,0.21]},\frac{\hat{h}_2}{[0.23,0.33]},\frac{\hat{h}_3}{[0.26,0.35]},\frac{\hat{h}_4}{[0.37,0.57]}\right\}\right),\\ \left(\frac{\hat{z}_3}{[0.31,0.42]},\left\{\frac{\hat{h}_1}{[0.11,0.41]},\frac{\hat{h}_2}{[0.13,0.35]},\frac{\hat{h}_3}{[0.35,0.47]},\frac{\hat{h}_4}{[0.17,0.37]}\right\}\right),\\ \left(\frac{\hat{z}_4}{[0.61,0.72]},\left\{\frac{\hat{h}_1}{[0.41,0.63]},\frac{\hat{h}_2}{[0.43,0.73]},\frac{\hat{h}_3}{[0.34,0.76]},\frac{\hat{h}_4}{[0.45,0.83]}\right\}\right)\end{array}\right\}.$$

3.2 Decision Assisted Mechanism

In this part of the paper, a MADM system is developed for assessing the quality of drinking water.

3.2.1 Criteria for Selection of Parameters

In many situations, attributes are key to the DMG process, from economic assessments to personal judgments. People evaluate the attributes of options, possibilities, or items when making decisions. Using attributes as the basis, information about numerous options can be gathered. When faced with a choice, individuals or groups typically gather data on various factors to assess and compare the options. Not all attributes carry the same weight in every decision. People assign different levels of importance to different aspects depending on their significance. The decision-maker compares the alternatives based on how well they fulfill each attribute to identify the best fit for their goals and interests. Tools like scorecards and decision matrices are often used for this purpose. In some cases, certain qualities may be more or less important. Awareness of context is essential for making informed decisions. Therefore, decision makers must be willing to sacrifice some rights and prioritize certain attributes over others when selecting the optimal option. Choosing attributes is both an art and a science. It combines judgments based on the specific decision scenario with data-driven analysis. The goal is to select relevant, meaningful attributes that align with the decisions objectives. The attributes for the suggested study are adopted through the analysis of Barzegar et al. [4] and Gorai et al. [10]. Two primary parameter groups (i.e., physical and chemical) are identified by referring to these references, and as a result, seven parameters are selected and shown in Table 1 and Figure 1. The attributes for the suggested study are adopted through the analysis of Barzegar et al. [4] and Gorai et al. [10]. Two primary parameter groups (i.e., physical and chemical) are identified by referring to these references, and as a result, seven parameters are selected and shown in Table 1 and Figure 1.

Table 1: Opted parameters with related classes

Sr. #	Category	Parameter
1	Physical parameters	Taste and odor
2	Physical parameters	Color
3	Chemical parameters	Dissolved metals like Ca, Mg, Fe, As, etc. [4, 10]
4	Chemical parameters	Dissolved salts like SO_4, NO_3, etc. [4, 10]
5	Chemical parameters	pH [4, 10]
6	Physical parameters	Hardness [4, 10]
7	Chemical parameters	Alkalinity [4, 10]

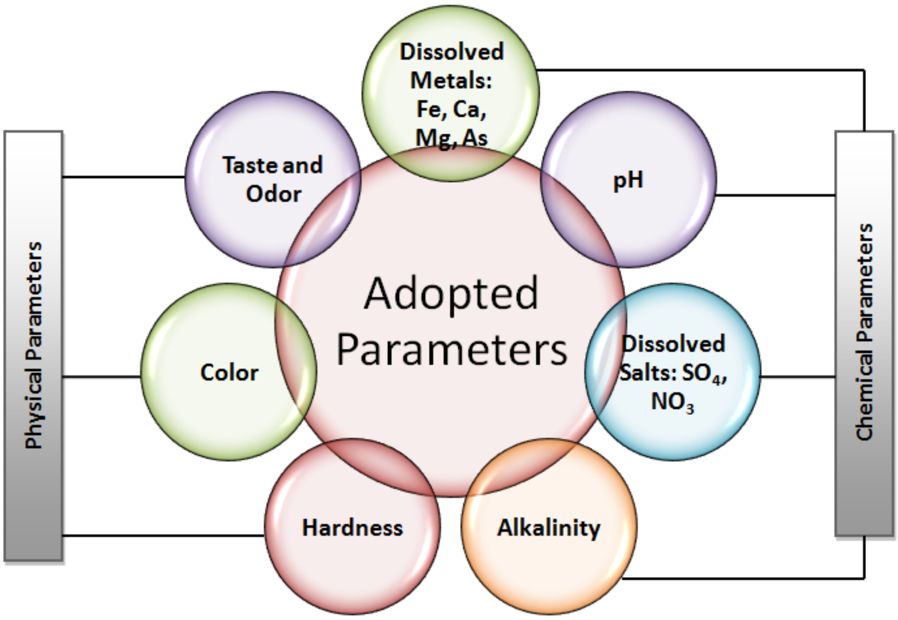

Fig. 1: Opted parameters

3.2.2 Problem Scenario

In Pakistan, only about 20% of the total population is provided with access to hygienic drinking water. Because there are limited sources of pure and nutritious drinking water, the other 80% of the population is compelled to utilize contaminated water. Fecal waste from sewers, which is frequently dumped into irrigation systems, is the main source of tainted The release of pesticides, fertilizers, and hazardous chemicals from manufacturing waste into water bodies is another cause of contamination. Waterborne illnesses account for around 80% of all diseases and 33% of mortality and are brought on by human-induced processes [9]. In the same way, Punjab, the most populous province in Pakistan, faces no different circumstances when it comes to the availability of safe water to drink. In the following lines, an algorithm is provided that is useful for DWQA in different schools located in different districts of Punjab. In this algorithm, some steps are partially followed from algorithm [33].

3.2.3 Proposed Algorithm

The following are the steps of the proposed algorithm:
==
1. **1st phase**
———1.1 Suppose the sets $\hat{\mho} = \{\hat{\hbar}_1, \hat{\hbar}_2, \hat{\hbar}_3, ..., \hat{\hbar}_m\}$,

Æ$= \{\tilde{\mathfrak{E}}_1, \tilde{\mathfrak{E}}_2, \tilde{\mathfrak{E}}_3, ..., \tilde{\mathfrak{E}}_p\}$, and
$\hat{\mathbb{Z}} = \{\hat{z}_1, \hat{z}_2, \hat{z}_3, ..., \hat{z}_n\}$ as set of options, set of experts, and set of
parameters respectively.
———1.2 From decision experts, obtain the IVFPGs $\rho_{i,j}, (i \in \hat{\mathbb{Z}}; j \in$ Æ$)$ for
each parameter.

2. **2nd phase**
———2.1 Using IVFPGs $\rho_{i,j}$ and opinions of decision experts for options based
on parameters, construct a IVFpIVFSS $(\tilde{\alpha}, \hat{\mathfrak{A}}_{\hat{\mathbb{Z}}})$ in tabulated form.
———2.2 Determine decisive intervals $[\hat{\theta}_L^{(i,k)}, \hat{\theta}_U^{(i,k)}]$ where

$$\hat{\theta}_L^{(i,k)} = \tilde{\zeta}_L^i(\hat{z}) \times \hat{\varpi}_L^k(\hat{h}) \quad (1)$$

and

$$\hat{\theta}_U^{(i,k)} = \tilde{\zeta}_U^i(\hat{z}) \times \hat{\varpi}_U^k(\hat{h}). \quad (2)$$

2.3 Determine choice values

$$\hat{\tau}_k = [\hat{\tau}_L^k, \hat{\tau}_U^k] = \left[\sum_{i,k=1}^{n,m} \hat{\theta}_L^{(i,k)}, \sum_{i,k=1}^{n,m} \hat{\theta}_U^{(i,k)} \right] \quad (3)$$

for each option $\hat{h}_k$ using $[\hat{\theta}_L^{(i,k)}, \hat{\theta}_U^{(i,k)}]$.
———2.4 Using choice values $\hat{\tau}_k$, calculate scores

$$\hat{\lambda}_q = \sum_{q,k=1}^{m} \left\{ \left(\hat{\tau}_L^q - \hat{\tau}_L^k \right) + \left(\hat{\tau}_U^q - \hat{\tau}_U^k \right) \right\}. \quad (4)$$

3. **3rd phase**
———3.1 The final decision is made on the highest score achieved.
==

Figure 2 is meant to depict the flow of Algorithm 3.2.3. Now Algorithm 3.2.3 is explained by the example given below.

*Example 5.*EDUVIEW is an organized network of schools distributed across all the districts in the province of Punjab. The central management of EDUVIEW decided to examine the drinking water quality in schools situated in some of the more disadvantaged areas out of concern for the well-being of their students and the standard of education in these areas. They committed to providing safe drinking water and a healthy learning environment for all children in these areas, so they chose four schools in four districts, one from each district, for the inspections to carry out this initiative. Furthermore, a committee comprising two senior executives and three physicians was established. The following items have been designated as crucial status in the committee's TORs:

1. Obtain specimens of drinking water from each of the chosen schools, then carry out relevant laboratory analyses.
2. Assign IVFPGs to each parameter to test its uncertainty.
3. Provide the approximated opinions based on the observations of particular schools under graded parameters using interval-valued fuzzy values.

The chosen schools are enclosed in the set $\hat{\mho} = \{\hat{h}_1, \hat{h}_2, \hat{h}_3, \hat{h}_4\}$ and the parameters that are opted for their DWQA are enclosed in the set $\hat{\mathbb{Z}} = \{\hat{z}_1, \hat{z}_2, \hat{z}_3, ..., \hat{z}_7\}$. In set $\hat{\mathbb{Z}}$, $\hat{z}_1$ stands for "taste and odor", $\hat{z}_2$ for "color", $\hat{z}_3$ for "dissolved metals", $\hat{z}_4$ for "dissolved salts", $\hat{z}_5$ for "pH", $\hat{z}_6$ for "hardness", and $\hat{z}_7$ for "alkalinity". With mutual understanding and consensus, the decision experts Æ$= \{\tilde{\mathfrak{E}}_1, \tilde{\mathfrak{E}}_2, \tilde{\mathfrak{E}}_3, ..., \tilde{\mathfrak{E}}_5\}$ assign IVFPGs to all $\hat{z}_i, i = 1, 2, ..., 7$ and these grades are given in Table 2. After observing the fuzzy parameterized parameters, the

Table 2: The IVFPGs assigned to parameters $\hat{z}_i, i = 1, 2, ..., 7$

$\hat{z}_i$	Parameter	IVFPGs
$\hat{z}_1$	Taste and odor	$[0.32, 0.56]$
$\hat{z}_2$	Color	$[0.53, 0.76]$
$\hat{z}_3$	Dissolved metals	$[0.42, 0.74]$
$\hat{z}_4$	Dissolved salts	$[0.24, 0.57]$
$\hat{z}_5$	pH	$[0.19, 0.82]$
$\hat{z}_6$	Hardness	$[0.57, 0.83]$
$\hat{z}_7$	Alkalinity	$[0.29, 0.73]$

decision experts $\tilde{\mathfrak{E}}_j$ provide their expert opinions for the approximation of DWQA in chosen schools and the IVFpIVFSS $(\tilde{\alpha}, \hat{\mathfrak{A}}_{\hat{\mathbb{Z}}})$ is constructed below. $(\tilde{\alpha}, \hat{\mathfrak{A}}_{\hat{\mathbb{Z}}}) =$

$$\left\{ \begin{array}{l} \left(\frac{\hat{z}_1}{[0.32,0.56]}, \left\{ \frac{\hat{h}_1}{[0.24,0.37]}, \frac{\hat{h}_2}{[0.23,0.35]}, \frac{\hat{h}_3}{[0.25,0.77]}, \frac{\hat{h}_4}{[0.27,0.79]} \right\} \right), \\ \left(\frac{\hat{z}_2}{[0.53,0.76]}, \left\{ \frac{\hat{h}_1}{[0.13,0.28]}, \frac{\hat{h}_2}{[0.23,0.83]}, \frac{\hat{h}_3}{[0.26,0.65]}, \frac{\hat{h}_4}{[0.37,0.87]} \right\} \right), \\ \left(\frac{\hat{z}_3}{[0.42,0.74]}, \left\{ \frac{\hat{h}_1}{[0.17,0.49]}, \frac{\hat{h}_2}{[0.13,0.45]}, \frac{\hat{h}_3}{[0.35,0.57]}, \frac{\hat{h}_4}{[0.17,0.57]} \right\} \right), \\ \left(\frac{\hat{z}_4}{[0.24,0.57]}, \left\{ \frac{\hat{h}_1}{[0.42,0.68]}, \frac{\hat{h}_2}{[0.43,0.73]}, \frac{\hat{h}_3}{[0.34,0.76]}, \frac{\hat{h}_4}{[0.45,0.83]} \right\} \right), \\ \left(\frac{\hat{z}_5}{[0.19,0.82]}, \left\{ \frac{\hat{h}_1}{[0.27,0.85]}, \frac{\hat{h}_2}{[0.23,0.75]}, \frac{\hat{h}_3}{[0.25,0.57]}, \frac{\hat{h}_4}{[0.27,0.69]} \right\} \right), \\ \left(\frac{\hat{z}_6}{[0.57,0.83]}, \left\{ \frac{\hat{h}_1}{[0.18,0.63]}, \frac{\hat{h}_2}{[0.23,0.53]}, \frac{\hat{h}_3}{[0.26,0.65]}, \frac{\hat{h}_4}{[0.37,0.67]} \right\} \right), \\ \left(\frac{\hat{z}_4}{[0.29,0.73]}, \left\{ \frac{\hat{h}_1}{[0.21,0.62]}, \frac{\hat{h}_2}{[0.13,0.75]}, \frac{\hat{h}_3}{[0.35,0.97]}, \frac{\hat{h}_4}{[0.17,0.77]} \right\} \right) \end{array} \right\}.$$

The step 2.2 (tabulation of decisive intervals corresponding to opted parameters $\hat{z}_i, i = 1, 2, ..., 7$) of proposed algorithm is presented in matrix notation as given in Table 3. The step 2.3 (tabulation of choice values

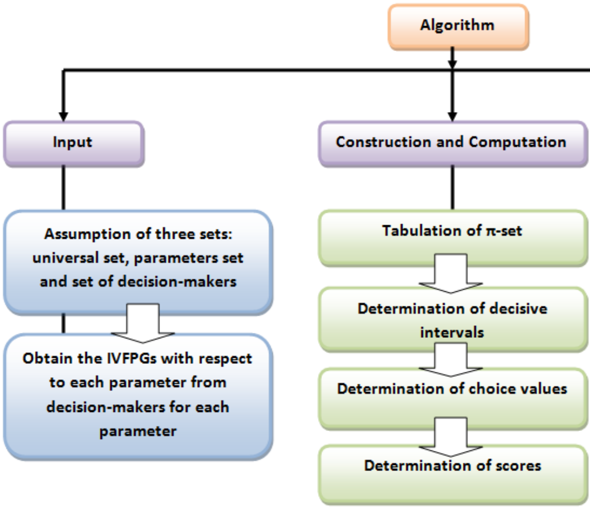

Fig. 2: Steps of Algorithm 3.2.3

Table 3: Tabulation of decisive intervals corresponding to opted parameters

$\hat{\mathbb{Z}} \downarrow \backslash \hat{\mho} \rightarrow$	$\hat{\hbar}_1$	$\hat{\hbar}_2$	$\hat{\hbar}_3$	$\hat{\hbar}_4$
$\hat{z}_1$	[0.0768, 0.2072]	[0.0736, 0.1960]	[0.0800, 0.4312]	[0.0864, 0.4424]
$\hat{z}_2$	[0.0689, 0.2128]	[0.1219, 0.6308]	[0.1378, 0.4940]	[0.1961, 0.6612]
$\hat{z}_3$	[0.0714, 0.3626]	[0.0546, 0.3330]	[0.1470, 0.4218]	[0.0714, 0.4218]
$\hat{z}_4$	[0.1008, 0.3876]	[0.1032, 0.4161]	[0.0816, 0.4332]	[0.1080, 0.4731]
$\hat{z}_5$	[0.0475, 0.6970]	[0.0437, 0.6150]	[0.0475, 0.4674]	[0.0513, 0.5658]
$\hat{z}_6$	[0.1026, 0.5229]	[0.1311, 0.4399]	[0.1482, 0.5395]	[0.2109, 0.5561]
$\hat{z}_7$	[0.0609, 0.4526]	[0.0377, 0.5475]	[0.1015, 0.7081]	[0.0493, 0.5621]

corresponding to alternatives $\hat{\hbar}_k, k = 1,2,...,4)$ of proposed algorithm is presented in matrix representation as given in Table 4. The step 2.4 (tabulation of scores corresponding to alternatives $\hat{\hbar}_k, k = 1,2,...,4)$ of proposed algorithm is presented in Table 5. According to above matrix, the options are ranked as $\hat{\hbar}_4 > \hat{\hbar}_3 > \hat{\hbar}_2 > \hat{\hbar}_1$ which means that the quality of drinking water in school $\hat{\hbar}_4$ is very poor and it needs special precautionary measures.

For the suggested algorithm, the computational complexity is determined as:
since the total chosen target educational institutions ($\hat{\hbar}$) is 4, and total parameters ($\hat{z}$) considered for evaluation is 7. As five experts have participated in the evaluation, then expected computational complexity is $O\big((\hat{\hbar}\times\hat{z})+\hat{\hbar}\log(\hat{\hbar})\big)$, i.e., $(4 \times 7) + 4\log 4 = 28 + 8\log 2 = 28 + 8(0.3010) = 28 + 2.4080 = 30.4080$ and $O(30.4080) = O(1)$ which means that the proposed algorithm consistently takes the same amount of time regardless of input size.

The proposed framework demonstrates strong scalability characteristics, meaning it can efficiently manage larger datasets with higher dimensions and complexity. Its structure and computational design ensure that increases in data size do not lead to disproportionate rises in computation time or memory consumption. This adaptability allows the framework to maintain stable performance and accuracy, even when applied to extensive or complex datasets.

Table 4: Tabulation of choice values corresponding to alternatives

$\hat{\mathbb{Z}} \downarrow \backslash \hat{\mho} \rightarrow$	$\hat{\hbar}_1$	$\hat{\hbar}_2$	$\hat{\hbar}_3$	$\hat{\hbar}_4$
$\hat{z}_1$	[0.0768, 0.2072]	[0.0736, 0.1960]	[0.0800, 0.4312]	[0.0864, 0.4424]
$\hat{z}_2$	[0.0689, 0.2128]	[0.1219, 0.6308]	[0.1378, 0.4940]	[0.1961, 0.6612]
$\hat{z}_3$	[0.0714, 0.3626]	[0.0546, 0.3330]	[0.1470, 0.4218]	[0.0714, 0.4218]
$\hat{z}_4$	[0.1008, 0.3876]	[0.1032, 0.4161]	[0.0816, 0.4332]	[0.1080, 0.4731]
$\hat{z}_5$	[0.0475, 0.6970]	[0.0437, 0.6150]	[0.0475, 0.4674]	[0.0513, 0.5658]
$\hat{z}_6$	[0.1026, 0.5229]	[0.1311, 0.4399]	[0.1482, 0.5395]	[0.2109, 0.5561]
$\hat{z}_7$	[0.0609, 0.4526]	[0.0377, 0.5475]	[0.1015, 0.7081]	[0.0493, 0.5621]
$\hat{\tau}_k$	$\hat{\tau}_1$=[0.5289,2.8427]	$\hat{\tau}_2$=[0.5658,3.1783]	$\hat{\tau}_3$=[0.7436,3.4952]	$\hat{\tau}_4$=[0.7734,3.6825]

Table 5: Tabulation of scoring values corresponding to alternatives

	$\hat{\hbar}_1$	$\hat{\hbar}_2$	$\hat{\hbar}_3$	$\hat{\hbar}_4$
$\hat{\lambda}_q$	$\hat{\lambda}_1$ = -2.324	$\hat{\lambda}_2$ = -0.834	$\hat{\lambda}_3$ = 1.1448	$\hat{\lambda}_4$ = 2.0132

3.2.4 Discussion and Comparison Analysis

The previous section offers a systematic decision-making framework for evaluating drinking water quality in schools across four districts. After thorough investigation and review of the literature, it is evident that no one has yet addressed DWQA with uncertain quantification. Therefore, comparing this framework with any existing standards based on calculations is not feasible. However, to demonstrate its versatility, a comparison based on various indicators is shown in Table 6, clearly indicating that this framework possesses all the features of previous FSOS-like frameworks. In Table 6, the acronyms FPD, SoS, IVS, and DWQA are meant for fuzzy parameterized domain, soft settings, interval settings and drinking water quality assessment respectively.

4 Conclusion

The main objective of this research is to evaluate the quality of drinking water while taking into account interval-valued uncertainty and fuzzy parameterized features. The interval-valued fuzzy parameterization framework, accompanied by IVFS settings, is employed for using a quite flexible and prototype context. As a result, the structure IVFpIVFSS is initiated, and its union and intersection operations are explained by examples. The IVFPGs are given to parameters by decision experts as part of the suggested technique. The options are approximated using interval-valued fuzzy numbers based on fuzzy parameterized parameters, which allows the decision experts to evaluate the alternatives more easily by giving their unbiased opinions. An algorithm is proposed in the decision support framework part to evaluate the drinking water quality of four distinct area-based educational institutions using seven of the most important parameters. An example is provided to validate the algorithm that has been suggested. Based on the structure and mathematical computation, the suggested method is compared with published references. Despite its effectiveness, the study has certain limitations. The proposed framework, IVFpIVFSS, has been tested on a limited dataset comprising only four area-based institutions, which may restrict the generalizability of the results. The evaluation also relies heavily on expert opinions, which can introduce subjectivity despite the fuzzy-based settings. Moreover, while the model handles interval-valued uncertainty effectively, it does not consider dynamic environmental or temporal variations in water quality parameters, which could further enhance the robustness of the analysis. According to the study, the unique mathematical framework could be expanded by using IVFpIVFSSs in domains other than DWQA that deal with similar MADM issues, such as air quality control and environmental sustainability. The combination of machine learning (ML) and artificial intelligence (AI) approaches can be investigated further to improve the decision-assisted system's real-time adaptability and predictive powers. Collaborations across disciplines with social researchers and economists may also offer a more comprehensive understanding of the

Table 6: Flexibility of IVFpIVFSS

References	FPD	SoS	IVS	DWQA
Min [21]	×	✓	✓	×
Çağman et al. [8]	✓	✓	×	×
Zhu and Zhan [38]	✓	✓	×	×
Bashir and Salleh [5]	✓	✓	×	×
Hazaymeh et al. [12]	✓	✓	×	×
Riaz and Hashmi [24]	✓	✓	×	×
Alkhazaleh et al. [2]	✓	✓	✓	×
IVFpIVFSS	✓	✓	✓	✓

socioeconomic effects of managing water quality, which could help to improve the framework's robustness and refine its parameters. The practical effectiveness and scalability of the IVFpIVFSS-based approach in various environmental and socio-economic contexts will also require validation through field trials and case studies conducted in different geographical regions.

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Atique Ur Rahman received Ph.D. from University of Management and Technology, Lahore, Pakistan. He has published over 100 research articles in peer-reviewed international journals with over 1600 citations. He has published 5 book chapters too. Currently, he is attached with the Department of Mathematics, University of Management and Technology, Lahore, Pakistan, for research purposes. He is the member of advisory board for Journal of Fuzzy Extensions and Applications and member of editorial board for Journal of Soft Computing and Artificial Intelligence. He is the reviewer of more than 60 peer-reviewed journals. He

introduced over 40 new hybrid set structures in the Hypersoft set environment. His areas of interest include fuzzy set theory and extensions, soft set theory and extensions, neutrosophic set theory, hypersoft set theory and extensions, optimization, multi-attribute decision-making, mathematical inequalities, time scales calculus.

Diaa S. Metwally works as an assistant professor in the Department of Financial Accounting and Auditing, Faculty of Commerce, Zagazig University, Egypt. He is also currently working as an Assistant Professor in the Department of Accounting at the College of Business, Imam Muhammad bin Saud Islamic University in the Kingdom of Saudi Arabia. He completed his doctorate in Financial Accounting at Zagazig University, Egypt in 2015. He has published numerous research papers in the field of accounting, food economics, and Cost estimation in food.

M. M. Abd El-Raouf has been an associate professor at the Arab Academy for Science, Technology, and Maritime Transport (AASTMT) since 2024. He was assigned to head the International Research Projects Department at AASTMT in 2021. He received his Ph.D. in mathematical statistics in 2019. His research interests are probability distributions and statistics, reliability theory, statistical Modelling and inference, regression analysis, data analysis, data science, and machine learning.

Mohamed Abd El-Aal Mustafa El-Qurashi is a lecturer and researcher specializing in mathematical statistics at the Arab Academy for Science, Technology and Maritime Transport (AASTMT). He holds a PhD in Mathematical Statistics from the Faculty of Science, Tanta University (2024), where he also earned his master's, pre-master's, and bachelor's degrees in statistics and computer science.

Hamiden Abd El-Wahed Khalifa received the Ph.D. degree from the Faculty of Science, Tanta University, Tanta, Gharbia Governorate, Egypt. She have full professor in Operations Research (Mathematical Programming). She is recently attached with the Operations Research Department, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt, as Professor. She is also attached with the Department of Mathematics, College of Science and Arts, Qassim University, Buraydah, Saudi Arabia, for research projects. She has published more than 200 publications in SCI journals. Her research interests include Game Theory, Multi-objective Programming, Fuzzy Mathematics, Rough Sets, Decision Making

Heba Ghareb Goma received the Ph.D.degree from the faculty of Science, Tanta University, Tanta, Gharbia Governorate, Egypt. She is recently attached With the Mathematical Department,Suez Institute for Management of Information System, Suez, Egypt. She has published about 10 publications in SCI journals. Her research interests include Multi-objective Programming, Fuzzy Mathematics, Rough Sets, Decision Making.