

Modeling Unemployment Dynamics in Ghana: Comparative Results of Caputo and Atangana-Baleanu Caputo Fractional Derivatives

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Abstract: In this study, we investigate unemployment dynamics which is a serious macroeconomic problem faced by all economies. For many decades, Ghana has experienced unemployment problems due to population growth, skill mismatch, etc. This research presents two fractional derivative-based unemployment models in Ghana. However, this work adopted an existing unemployment model for Ghana in the Caputo fractional derivative sense and developed it in the Atangana-Baleanu Caputo fractional sense to compare the superiority of the two models. The next generation matrix was used to obtain the unemployment-free equilibrium. The model had two fixed points. The existence and uniqueness of the ABC model were determined and proven to exist and unique. The generalized Adams–Bashford numerical method was used to develop a numerical scheme to simulate the ABC model, whereas an existing numerical scheme built on the Adams–Bashforth–Molton Predictor Corrector scheme was used to simulate the Caputo model. The results show that unemployment in Ghana can be controlled if policymakers invest in skill training with the necessary logistics, and further provide startup capital. The study further showed that government should create 3% vacancies in the public sector, and cut down taxes for various private institutions and individuals. Finally, the comparative results of the models show that the Atangana Baleanu Caputo model is superior to the Caputo model in terms of real-life applicability. This makes the application of the ABC in studying and controlling unemployment dynamics very suitable in Ghana’s labor market.

Keywords: Atangana-Baleanu Caputo, Caputo fractional derivative, Skill training, Startup capital, Unemployment.

1. Introduction

The study of the implications of the scarcity of resources, economic growth of production, production of goods and services, and the well-being of people over time has been a national and research concern throughout the globe. One of the major economic and social challenges experienced by countries worldwide is unemployment. This is because it is one of the critical economic indicators that reflects the welfare of people and the health of an economy [1].

Unemployment is caused by many factors. However, unemployment and labor force dynamics are closely related and are both influenced by population growth. Also, unemployment becomes complex if there is an unbalanced inflow of the labor force due to higher population growth compared to the rate of employment in the labor market [2-3]. Similarly, permanent structural changes and negative exogeneous shocks cause unemployment in every economy [4]. Although unemployment is a major macroeconomic problem in every economy, notwithstanding, the causes and averages differ for every economy and continent [5].

Using the modeled International Labor Organization data from 2012–2022 to comparing the unemployment rate of Arab world, European Union, Sub-Saharan Africa, Organization for Economic Co-operation and Development (OECD) member states, East Asia, the Pacific, and the world at large, Figure 1 shows that for the past decade, the world’s unemployment rate reached its highest peak of 6.9% in 2020, with its lowest peak of 5.54% in 2019. However, it was steady from 2012–2018, which consistently decreased after 2020. It is further seen that over the past ten years, unemployment rates in the Pacific and East Asia declined compared to those in the rest of the globe with the year 2020 (4.29%) and the year 2017 (3.87%) being the highest and lowest unemployment rates respectively. Again, Sub-Saharan Africa had an unemployment rate below the world’s rate until after 2017 when it exceeded world’s unemployment rate. Additionally, Sub-Saharan Africa had its lowest and highest unemployment rates in 2021 and 2013 as 6.8% and 5.52% respectively. Interestingly, unemployment in the European Union and the OECD sharply decreased from 2013–2019 and 2012–2019 respectively. However, both rose in 2020 at a variance of 0.38% and 1.72% respectively but declined after 2020. Figure 1 further reveals that the rate of unemployment in the Arab world has always been very high compared with that in other countries and the world at large.

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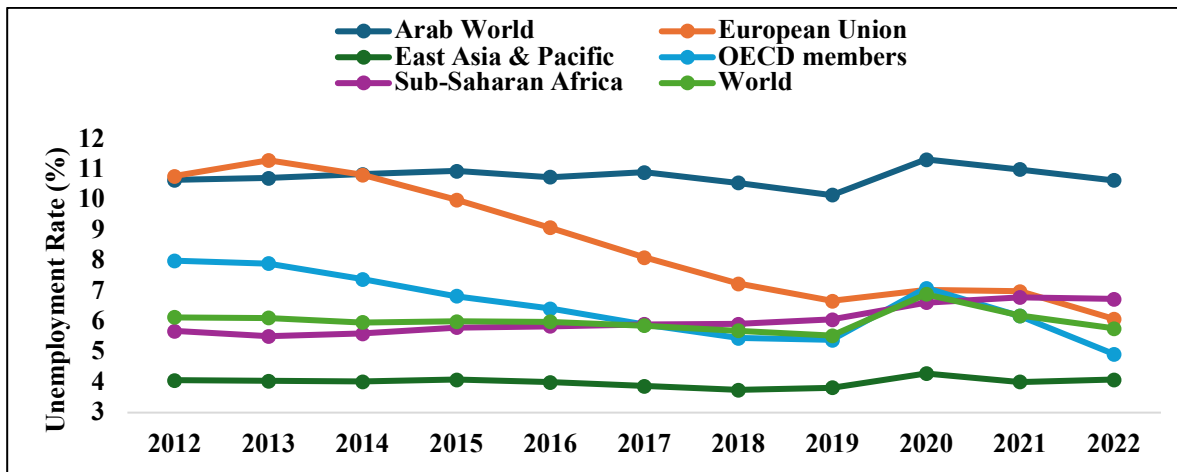


Fig. 1: Unemployment Rates for Different Parts of the World

It is further seen in Figure 1 that unemployment surged in in all the categories in 2020. [5-7] posited that the novel COVID-19 indeed affected economies throughout the world by causing various economies to contract.

[7] reported that the unemployment rate in the European Union was 6.2% in 2019 whereas it was 7.4% in the Euro region. The work further revealed that during the same period, Czechia had the lowest unemployment rate of 2.0%, followed by both the Netherlands and Germany with 3.2% each. Conversely, Greece on the other hand had the highest unemployment rate of 16.6% in October 2019, with Spain experiencing an unemployment rate of 13.7%, while Romania also experienced a 4.0% unemployment rate in December 2019.

As shown in Figure 2, their work also compared their 2018 and 2019 findings and reported that 21 European Union member states experienced a decrease in the unemployment rate in 2018. Moreover, between 2018 and 2019 the unemployment rate of countries such as Greece decreased from 18.6% (2018) to 16.6% (2019), that of Croatia also decreased from 7.3% (2018) to 6.4% (2019), and that of Bulgaria also decreased from 4.7% (2018) to 3.7% (2019). In contrast, other member states within the same period registered an increase in the unemployment rate. Typical examples include Cyprus, where the unemployment rate increased from 7.4% in 2018 to 7.6% in 2019, Portugal also increased from 6.6% (2018) to 6.9% (2019). Additionally, Sweden increased from 6.5% in 2018 to 6.9% in 2019. Lithuania also increased from 5.8% in 2018 to 6.1% in 2019, Luxembourg increased from 5.2% in 2018 to 5.6% in 2019, and Slovenia increased from 4.4% in 2018 to 4.6% in 2019.

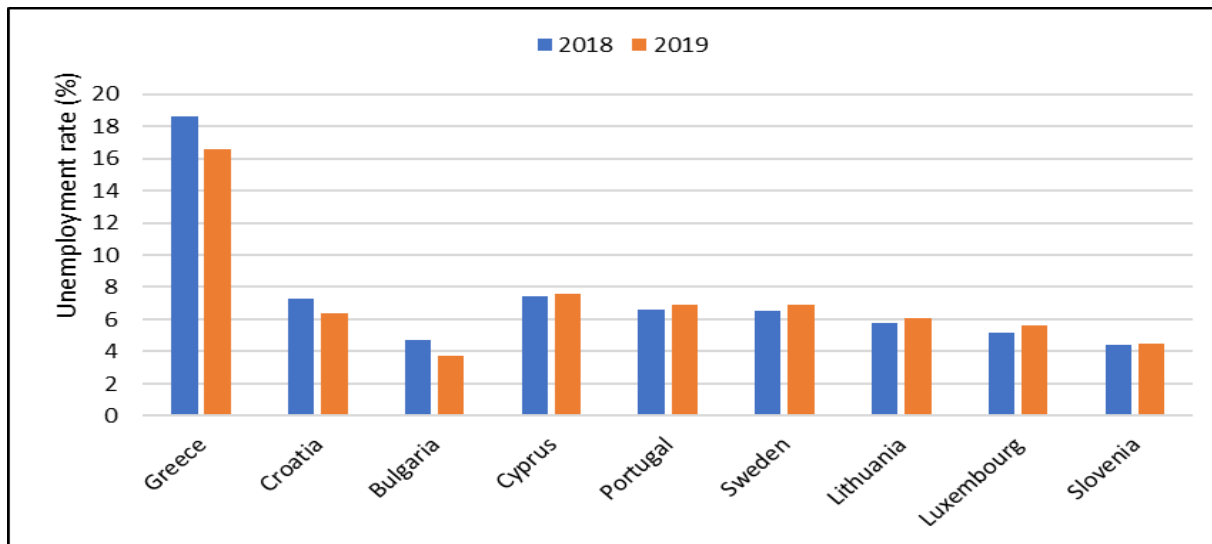


Fig. 2: Variation in the Unemployment Rates of the European Membership States

[8] posited that bridging the gap between unemployment involves entrepreneurial education at all levels of the educational ladder to provide students with skills, problem solving abilities, and risk taking to help rejuvenate their entrepreneurial mindset.

Unemployment in Africa continues to derail the gains made in the various African countries, especially in Sub-Saharan Africa. Among the Sub-Saharan African countries, unemployment has always been on the rise due to skill mismatch, population growth, political and economic crisis, and the difficulty of unemployed youth gaining access to the formal sector due to the small nature of the formal labor market [9-11]. The role of the labor force who participates within African countries cannot be underestimated in the sense that labor productivity is an ingredient necessary for every vibrant economy. Many studies have been conducted to understand the factors causing the rise in unemployment in Sub-Saharan Africa by considering and examining both the demand and supply factors of unemployment; however, this canker still persists [6, 12].

One of the most important problems faced by developing nations is unemployment. The work of [6, 12] posited that the unemployment problem faced by developing countries is due to the creation of inadequate job vacancies to match the proportion of the population that enter the labor market at a time [6, 12]. Ghana, the beacon of Africa for its first role in gaining independence has consistently suffered from unemployment. The population of Ghana has increased drastically. This has resulted in a consequential increase in its labor force. However, the country's unemployment problems stem from the fact that new work prospects have not expanded proportionately. Additionally, factors such as a lack of job-oriented educational systems, urban migration, skill mismatch, and economic contraction have greatly contributed to the high unemployment rate in Ghana.

Using entrepreneurship or skill training strategy to reduce unemployment is not instantaneous. Notwithstanding, the long-term effect has always been very significant and evident. [13] noted that using skill training strategy benefits industries such as construction, financial activities, utilities and transportation, professional and business, and services in the long run. The work further deduced that the underlying recipes for skill training in halting unemployment lies in the provision of funds, credit facilities, tax cuts, training, economic activities, etc.

Many models have been developed in the area of unemployment. [14 -15] developed a four-state model for unemployment on the basis of an existing housing allocation model by [16]. Also, [17] proposed a mathematical model based on a motivation from the work of [15]. Their work incorporated self-employment into the model to understand how self-employment as an intervention could help control unemployment. However, their model did not consider the provision of logistics to skill training institutions. To address this issue in poor countries with few resources, a three-state mathematical model was developed by [18]. The work of [18] also failed to capture skill training, logistics provisions for skill training institutions, and startup capital for high-skill graduates.

Many methods have been used to examine the dynamics of unemployment models. However, most of these models adopt traditional econometric and integer models. Although valuable, such models fail to capture the memory effect and the nuance nature of unemployment dynamics [1]. However, current studies on unemployment have applied fractional derivatives [19-23] to understand the dynamics of this social canker.

There are many fractional order operators with singular and local kernels or nonsingular and nonlocal kernels. The nonsingular and nonlocal kernels such as the Caputo, Riemann–Liouville, etc., have the advantages of allowing for more realistic mathematical modeling than integer order. They better describe the memory and hereditary properties, and better captures the behavior of complex systems than the integer order. In [24], the authors noted that the advantage of the Caputo derivative over the Riemann–Liouville derivative is that the Caputo fractional derivative generalizes the Laplace transform which is an integer-order derivative, whereas the Riemann–Liouville fractional derivative does not. Another benefit of the Caputo fractional derivative stems from its boundedness, meaning that even while it allows for the consideration of conditions that are easily and realistically interpreted, the derivative of a constant is equal to zero. However, the Atangana-Baleanu Caputo has both nonsingular and nonlocal kernels, and has the advantages of time relaxation or delay, and a crossover effect compared with those systems with local and singular kernels. These advantages make the ABC operator to be more realistic, applicable and interpretable in the real world.

[1] developed a “compartmental model for unemployment dynamics in Ghana using the Caputo fractional derivative.” In [1], the authors developed a four-state unemployment model in which skill training, provision of logistics by the government to training institutions, provision of startup capital to high-skilled trainees, and tax cuts were all incorporated into the model to assess their impact on unemployment in Ghana. Although valuable, the Caputo fractional derivative does not capture the crossover effect because of its local and singular kernel characteristics. This work adopts the model developed by [1]. The adopted model is developed in the Atangana-Baleanu Caputo sense, and the superiority of the two models is further compared.

2. Preliminary Information

The details of the Atangana-Baleanu Caputo operator, and the Caputo operator needed for this study is presented in this section.

Definition 1. Suppose that $f \in H^1(0, b)$, $b > 0$, $\alpha \in [0, 1]$; then, the ABC operator can be written as:

$${}^{ABC}_0 D_t^\alpha (f(t)) = \frac{B(\alpha)}{1-\alpha} \int_0^t f'(z) E_\alpha \left[-\alpha \frac{(t-z)^\alpha}{1-\alpha} \right] dz$$

where $B(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$ is a normalized function that has $B(0) = B(1) = 1$, where E_α is the Mittag-Leffler function given as:

$$E_\alpha(\Theta) = \sum_{q=0}^{\infty} \frac{\Theta^q}{\Gamma(\alpha q + 1)}, \alpha > 0$$

The associated integral of order α to the ABC derivative operator in the R-L sense is given by:

$${}^{ABC}_a I_t^\alpha (f(t)) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t f(z)(t-z)^{(\alpha-1)} dz, \alpha \in [0,1]$$

Definition 2. (See [1]) for $\alpha > 0$, when $n-1 < \alpha < n$, $n \in \mathbb{N}$, the Caputo fractional derivative operator is defined as:

$${}^C D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-z)^{(n-\alpha-1)} f^{(n)}(z) dz.$$

where $\Gamma(\cdot)$ is the gamma function, with the Riemann-Liouville integral in the Caputo sense defined as:

$$I_t^\alpha f(t) = \frac{1}{\alpha} \int_0^t (t-z)^{(\alpha-1)} f(z) dz.$$

3. Model Formulation

The adopted four-state unemployment model from [1] has U as unemployed compartment, E as employed compartment, A as skill trainees compartment, and V as new vacancies.

3.1 Assumptions of the Model

The assumptions underpinning the development of the proposed unemployment model are that, new entrants (unemployed) into the labor market have the requisite skills to perform any job, but their skills lose relevance over time. The employed lose their job due to resignation, incapacitation, firing by bosses due to loss of relevant skills. Unemployed individuals who want to be trained can enroll into their preferred skill training institution. Also, trained graduates are categorized into low and high skilled trainees with startup capital from corporate institutions and government given to only high skilled graduates to create their own vacancies. Additionally, new vacancies in E and V are created as a result of incapacitation, immigration to foreign countries, retirement, and death. The unemployed with the relevance skills can be employed into E or V . additionally, government implements tax-cut and tax holiday policies to increase investment by the private sector to help a proportion of persons in E and V to create additional vacancies. Again, government also creates contingency vacancies through the National Youth Authority to control unemployment, however, are been influenced by economic growth and government revenue.

Considering the assumptions of the evolutionary labor force dynamics in Ghana and its schematic diagram in the work of [1], the model is described with Equation (1) as the model's systems of differential equations:

$$\left. \begin{aligned} \frac{dU}{dt} &= \Lambda + \varepsilon E + \rho(1-r)A - \psi\omega UA - (\eta + \tau + \sigma + \kappa)U \\ \frac{dE}{dt} &= \eta U - (\varepsilon + \kappa)E \\ \frac{dA}{dt} &= \psi\omega UA - \rho(1-r)A - (\theta\phi + \kappa)A \\ \frac{dV}{dt} &= \theta\phi A + (\tau + \sigma)U - \kappa V \end{aligned} \right\} \quad (1)$$

Equation (1) has its initial conditions as follows:

$$U_0 = U(0), E_0 = E(0), A_0 = A(0), V_0 = V(0).$$

The total labor force in Ghana Q per the model is given by:

$$Q = U + E + A + V \quad (2)$$

The model's parameters are described in Table 1 below as:

Table 1: Model parameter descriptions

Description	Parameters
New entrants into the unemployed labor force	Λ
Rate of death, incapacitation, migration, and retirement	κ
Rate of effective government revenue	r
The rate of successful skill training	ϕ
Rate of effective government startup capital	θ
Rate of provision of logistics and infrastructure for training	ω
Rate of desire to be trained	ψ
Rate of employed becoming unemployed	ε
Rate of unemployed becoming employed	η
The rate at which government creates new/emergency vacancies	τ
Rate of dropout during training, and becoming unemployed after training	ρ
Rate of new vacancies by the private sector due to tax cuts	σ
The rate of diminution of vacancies available due to financial crisis	γ

Additionally, Equation (1) has an economically feasible region in the domain Ξ_1 , with Ξ_1 defined in a four-dimensional real space $\mathbb{R}^4_+$ as:

$$\Xi_1 = \left\{ (U, E, A, V) \in \mathbb{R}^4_+ : 0 \leq Q \leq \frac{\Lambda}{\kappa} \right\} \quad (3)$$

This means that the model is economically feasible and that all associated results hold for the system. Since all the variables in Equation (1) are associated with human beings, it is nonnegative for which the region shown by Ξ_1 is positive invariant for the model in Equation (1). Hence, the defined region shown in Equation (3) for the model in Equation (1) is very well posed, and the initial value solution belonging to Ξ_1 remains in Ξ_1 at any time $t \geq 0$.

3.2 The Atangana-Baleanu Caputo (ABC) Model

The ABC fractional derivative operator was applied to Equation (1) and finally described in the ABC sense as:

$$\left. \begin{aligned} {}_0^{ABC} D_t^\alpha \frac{dU}{dt} &= \Lambda + \varepsilon E + \rho(1-r)A - \psi\omega UA - (\eta + \tau + \sigma + \kappa)U \\ {}_0^{ABC} D_t^\alpha \frac{dE}{dt} &= \eta U - (\varepsilon + \kappa)E \\ {}_0^{ABC} D_t^\alpha \frac{dA}{dt} &= \psi\omega UA - \rho(1-r)A - (\theta\phi + \kappa)A \\ {}_0^{ABC} D_t^\alpha \frac{dV}{dt} &= \theta\phi A + (\tau + \sigma)U - \kappa V \end{aligned} \right\} \quad (4)$$

Considering some appropriate associated initial conditions for the model, Equation (4) is further studied to obtain its existence and uniqueness.

3.3 Positivity of the Model

Lemma 1. Let Ξ_1 be a close set such that it is positive invariant for the unemployment model in Equation (4), then, Equation (3) holds. Thus,

$$\Xi_1 = \left\{ (U, E, A, V) \in \mathbb{R}_+^4 : Q = \frac{\Lambda}{\kappa} \right\}.$$

Proof. For the unemployment model in Equation (4), the fractional derivative of the total labor force is given as:

$${}_0^{ABC} D_t^\alpha Q = \Lambda - \kappa Q(t) \quad (5)$$

We make use of the Laplace transform computed by [25, 26] for the ABC derivative in the form:

$$L\left\{{}_a^{ABC} D_t^\alpha f(t)\right\} = \frac{B(a)f(z)z^\alpha - z^{\alpha-1}f(0)}{(1-\alpha)\left(z^\alpha + \frac{\alpha}{(1-\alpha)}\right)} \quad (6)$$

The application of Equation (6) gives:

$$Q(t) = \left(\frac{B(\alpha)}{B(\alpha) + (1-\alpha)\kappa} Q(0) + \frac{(1-\alpha)\Lambda}{B(\alpha) + (1-\alpha)\kappa} E_{\alpha,1}(-\Upsilon t^\alpha) + \frac{\alpha\Lambda}{B(\alpha) + (1-\alpha)\kappa} E_{\alpha,\alpha+1}(-\Upsilon t^\alpha) \right) \quad (7)$$

where,

$$\Upsilon = \frac{\alpha\kappa}{B(\alpha) + (1-\alpha)\kappa} \text{ and the two parameter Mittag-Leffler function } E_{\alpha,\Upsilon} \text{ is given as:}$$

$$E_{\alpha,\Upsilon} \Theta = \sum_{q=0}^{\infty} \frac{\Theta^q}{\Gamma(\alpha q + \Upsilon)} \quad (8)$$

Considering the asymptotic characteristic of the Mittag-Leffler function in Equation (8) [25, 26], it is seen that $Q(t) = \frac{\Lambda}{\kappa}$ as $t \rightarrow \infty$. Hence, Ξ_1 has an invariant region with respect to the unemployment model.

4. Existence and Uniqueness of the Model

To address the existence and uniqueness of solution for this unemployment model, we apply the fixed-point theorem to enable us explore the behaviour of our unemployment model to better understand its properties and dynamics.

Suppose that $\mathcal{X} = J(b) \times J(b)$, where the Banach space $J(b)$ is a continuous real value function defined with the norm

$$\|PU, E, A, V\|_{\mathcal{X}} = \|PU\|_{\mathcal{X}} + \|E\|_{\mathcal{X}} + \|A\|_{\mathcal{X}} + \|V\|_{\mathcal{X}}$$

where,

$$\|PU\|_{\mathcal{X}} = \sup\{|U(t)| : t \in b\},$$

$$\|E\|_{\mathcal{X}} = \sup\{|E(t)| : t \in b\},$$

$$\|A\|_{\mathcal{X}} = \sup\{|A(t)| : t \in b\},$$

$$\|V\|_{\mathcal{X}} = \sup\{|V(t)| : t \in b\}.$$

The unemployment model in Equation (4) is transformed into the following equation by taking into consideration the ABC fractional integral

$$\left. \begin{aligned} U(t) - U(0) &= \frac{1-\alpha}{B(\alpha)} \{ \Lambda + \varepsilon E + \rho(1-r)A - \psi\omega UA - (\eta + \tau + \sigma + \kappa)U \} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} \{ \Lambda + \varepsilon E + \rho(1-r)A - \psi\omega UA - (\eta + \tau + \sigma + \kappa)U \} dz, \\ E(t) - E(0) &= \frac{1-\alpha}{B(\alpha)} \{ \eta U - (\varepsilon + \kappa)E \} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} \{ \eta U - (\varepsilon + \kappa)E \} dz, \\ A(t) - A(0) &= \frac{1-\alpha}{B(\alpha)} \{ \psi\omega UA - \rho(1-r)A - (\theta\phi + \kappa)A \} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} \{ \psi\omega UA - \rho(1-r)A - (\theta\phi + \kappa)A \} dz, \\ V(t) - V(0) &= \frac{1-\alpha}{B(\alpha)} \{ \theta\phi A + (\tau + \sigma)U - \kappa V \} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} \{ \theta\phi A + (\tau + \sigma)U - \kappa V \} dz. \end{aligned} \right\} \quad (9)$$

Simplifying the above gives:

$$J_1(t, U) = \Lambda + \varepsilon E + \rho(1-r)A - \psi\omega UA - (\eta + \tau + \sigma + \kappa)U,$$

$$J_2(t, E) = \eta U - (\varepsilon + \kappa)E,$$

$$J_3(t, A) = \psi\omega UA - \rho(1-r)A - (\theta\phi + \kappa)A,$$

$$J_4(t, V) = \theta\phi A + (\tau + \sigma)U - \kappa V.$$

Theorem 1. The kernels J_1, J_2, J_3, J_4 satisfy the Lipchitz condition and contraction if the following inequality holds:

$$0 \leq \mathcal{G}_1 \leq 1, 0 \leq \mathcal{G}_2 \leq 1, 0 \leq \mathcal{G}_3 \leq 1, 0 \leq \mathcal{G}_4 \leq 1.$$

Proof. By taking the kernel $j_1(t, U) = \Lambda + \varepsilon E + \rho(1-r)A - \psi\omega UA - (\eta + \tau + \sigma + \kappa)U$. we let U and U_1 be any two functions such that:

$$\begin{aligned} & P j_1(t, U(t)) - j_1(t, U_1(t)) P \\ &= P \Lambda + \varepsilon E + \rho(1-r)A - \psi\omega UA - (\eta + \tau + \sigma + \kappa)U - (\Lambda + \varepsilon E + \rho(1-r)A - \psi\omega U_1 A - (\eta + \tau + \sigma + \kappa)U_1) P \\ &\leq P - (\psi\omega A + \eta + \tau + \sigma + \kappa) P P U(t) - U_1(t) P \\ &\leq P (\psi\omega P A(t) P + \eta + \tau + \sigma + \kappa) P P U(t) - U_1(t) P \\ &\leq (\psi\omega g_3 + \eta + \tau + \sigma + \kappa) P U(t) - U_1(t) P \\ &\leq \mathcal{G}_1 P U(t) - U_1(t) P \\ &\mathcal{G}_1 = \psi\omega g_3 + \eta + \tau + \sigma + \kappa \end{aligned}$$

where $g_1 = \max_{t \in j} P U(t) P, g_2 = \max_{t \in j} P E(t) P, g_3 = \max_{t \in j} P A(t) P, g_4 = \max_{t \in j} P V(t) P$ are bounded functions. This gives:

$$P j_1(t, U(t)) - j_1(t, U_1(t)) P \leq \mathcal{G}_1 P U(t) - U_1(t) P$$

It can be seen at this point that J_1 satisfies the Lipchitz condition, and if $0 \leq \mathcal{G}_1 \leq 0$, then it becomes a contraction for J_1 .

Following the same procedure, the other kernels satisfy the Lipchitz condition as:

$$P j_2(t, E(t)) - j_2(t, E_1(t)) P \leq \mathcal{G}_2 P E(t) - E_1(t) P,$$

$$P j_3(t, A(t)) - j_3(t, A_1(t)) P \leq \mathcal{G}_3 P A(t) - A_1(t) P,$$

$$\mathbb{P} j_1(t, V(t)) - j_1(t, V_1(t)) \mathbb{P} \leq \mathcal{G}_4 \mathbb{P} V(t) - V_1(t) \mathbb{P}.$$

At this point, we consider the kernel of the model. We therefore write Equation (9) as follows:

$$\begin{aligned} U(t) &= U(0) + \frac{1-\alpha}{B(\alpha)} j_1(t, U) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} j_1(z, U) dz, \\ E(t) &= E(0) + \frac{1-\alpha}{B(\alpha)} j_2(t, E) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} j_2(z, E) dz, \\ A(t) &= A(0) + \frac{1-\alpha}{B(\alpha)} j_3(t, A) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} j_3(z, A) dz, \\ V(t) &= V(0) + \frac{1-\alpha}{B(\alpha)} j_4(t, V) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} j_4(z, V) dz. \end{aligned} \quad (10)$$

We represent Equation (10) recursively as:

$$\begin{aligned} U_a(t) &= \frac{1-\alpha}{B(\alpha)} j_1(t, U_{a-1}) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} j_1(z, U_{a-1}) dz, \\ E_a(t) &= \frac{1-\alpha}{B(\alpha)} j_2(t, E_{a-1}) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} j_2(z, E_{a-1}) dz, \\ A_a(t) &= \frac{1-\alpha}{B(\alpha)} j_3(t, A_{a-1}) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} j_3(z, A_{a-1}) dz, \\ V_a(t) &= \frac{1-\alpha}{B(\alpha)} j_4(t, V_{a-1}) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} j_4(z, V_{a-1}) dz. \end{aligned} \quad (11)$$

Equation (11) has initial conditions:

$$U_0(t) = U(0), E_0(t) = E(0), A_0(t) = A(0), V_0(t) = V(0).$$

We further used the initial conditions of the system, and the difference between the succeeding term to obtain Equation (12) as:

$$\begin{aligned} \nabla_a(t) &= U_a(t) - U_{a-1}(t) = \frac{1-\alpha}{B(\alpha)} j_1(t, U_{a-1}) - j_1(t, U_{a-2}) + \frac{1-\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} (j_1(t, U_{a-1}) - j_1(t, U_{a-2})) dz, \\ \Delta_a(t) &= E_a(t) - E_{a-1}(t) = \frac{1-\alpha}{B(\alpha)} j_2(t, E_{a-1}) - j_2(t, E_{a-2}) + \frac{1-\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} (j_2(t, E_{a-1}) - j_2(t, E_{a-2})) dz, \\ J_a(t) &= A_a(t) - A_{a-1}(t) = \frac{1-\alpha}{B(\alpha)} j_3(t, A_{a-1}) - j_3(t, A_{a-2}) + \frac{1-\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} (j_3(t, A_{a-1}) - j_3(t, A_{a-2})) dz, \\ h_a(t) &= V_a(t) - V_{a-1}(t) = \frac{1-\alpha}{B(\alpha)} j_4(t, V_{a-1}) - j_4(t, V_{a-2}) + \frac{1-\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} (j_4(t, V_{a-1}) - j_4(t, V_{a-2})) dz. \end{aligned} \quad (12)$$

where,

$$\begin{aligned} U_a(t) &= \sum_{i=1}^a \nabla_i(t), \\ E_a(t) &= \sum_{i=1}^a \Delta_i(t), \\ A_a(t) &= \sum_{i=1}^a J_i(t), \\ V_a(t) &= \sum_{i=1}^a h_i(t). \end{aligned} \quad (13)$$

At this point, we take Equation (12) and consider the triangular inequality, and also apply the norm on Equation (13) to transform the equation into Equation (14) as:

$$\begin{aligned} P \nabla_a(t) P &= P U_a(t) - U_{a-1}(t) P \\ &\leq \frac{1-\alpha}{B(\alpha)} P j_1(t, U_{a-1}) - j_1(t, U_{a-2}) P + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} P \int_0^t (t-z)^{\alpha-1} (j_1(t, U_{a-1}) - j_1(t, U_{a-2})) dz P. \end{aligned} \quad (14)$$

It can be seen that the Lipchitz condition is satisfied by the kernel. Hence, these equations hold:

$$P U_a(t) - U_{a-1}(t) P \leq \frac{1-\alpha}{B(\alpha)} \mathcal{G}_1 P U_{a-1}(t) - U_{a-2}(t) P + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \mathcal{G}_1 P \int_0^t (t-z)^{\alpha-1} P U_{a-1}(t) - U_{a-2}(t) P dz,$$

and,

$$P \nabla_a(t) P \leq \frac{1-\alpha}{B(\alpha)} \mathcal{G}_1 P \nabla_{a-1}(t) P + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \mathcal{G}_1 \int_0^t (t-z)^{\alpha-1} P \nabla_{a-1}(t) P dz \quad (15)$$

Following the same procedure, we achieve the following results:

$$\begin{aligned} P \Delta_a(t) P &\leq \frac{1-\alpha}{B(\alpha)} \mathcal{G}_2 P \Delta_{a-1}(t) P + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \mathcal{G}_2 \int_0^t (t-z)^{\alpha-1} P \Delta_{a-1}(t) P dz, \\ P J_a(t) P &\leq \frac{1-\alpha}{B(\alpha)} \mathcal{G}_3 P J_{a-1}(t) P + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \mathcal{G}_3 \int_0^t (t-z)^{\alpha-1} P J_{a-1}(t) P dz, \\ P h_a(t) P &\leq \frac{1-\alpha}{B(\alpha)} \mathcal{G}_4 P h_{a-1}(t) P + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \mathcal{G}_4 \int_0^t (t-z)^{\alpha-1} P h_{a-1}(t) P dz. \end{aligned}$$

Again, we further apply fixed-point theory, with the ABC model written in its compact form in Equation (16) as:

$$\begin{cases} {}_0^{ABC} D_t^\alpha a(t) = N(t, a(t)) \\ a(0) = a_0, 0 < t < T < \infty \end{cases} \quad (16)$$

The system in Equation (16) can be expressed in stable variable form if $a(t)$ is defined as $a(t) = (U, E, A, V)$, and N is also assumed to be a continuous vector function given as:

$$N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} \Lambda + \varepsilon E + \rho(1-r)A - \psi\omega UA - (\eta + \tau + \sigma + \kappa)U \\ \eta U - (\varepsilon + \kappa)E \\ \psi\omega UA - \rho(1-r)A - (\theta\phi + \kappa)A \\ \theta\phi A + (\tau + \sigma)U - \kappa V \end{bmatrix} \quad (17)$$

Suppose that the initial conditions for the state variables in vector form are defined as $a_0(t) = (U(0), E(0), A(0), V(0))$.

Then, the function N in Equation (17) above satisfies the Lipschitz continuity conditions and therefore can be denoted by:

$$\|N(t, a_1(t)) - N(t, a_2(t))\| \leq M \|a_1(t) - a_2(t)\| \quad (18)$$

Additionally showing the existence and solution of the Atangana-Baleanu Caputo model, the following theorem is provided, and the provision of the uniqueness of the model in Equation (4) is satisfied.

Theorem 1 For the existence and uniqueness of the model given by Equation (16), the model has a unique solution if these conditions below are satisfied.

$$\frac{1-\alpha}{ABC(\alpha)}M + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}T_{\max}^{\alpha}M < 1 \quad (19)$$

Proof: In proving Equation (19), the associated fractional integral of the ABC model in Equation (4) is applied to Equation (16) to yield the following nonlinear Volterra integral equations stated in Equation (20):

$$a(t) = a_0 + \frac{1-\alpha}{ABC(\alpha)}N(t, a(t)) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} N(z, a(z)) dz \quad (20)$$

Guaranteeing that $J = (0, T)$ and taking into consideration the operator $\varphi: \mathcal{F}(J, {}^\circ 4) \rightarrow \mathcal{F}(J, {}^\circ 4)$ defined by:

$$\varphi[a(t)] = a_0 + \frac{1-\alpha}{ABC(\alpha)}N(t, a(t)) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} N(z, a(z)) dz \quad (21)$$

Hence, Equation (20) becomes:

$$a(t) = \varphi[a(t)] \quad (22)$$

Additionally, the supremum norm on $J, \|\bullet\|_J$ is $\|a(t)\|_J = \sup_{t \in J} \|a(t)\|, a(t) \in \mathcal{F}$.

Additionally, $\mathcal{F}(J, {}^\circ 4)$ along the norm $\|\bullet\|_J$ represents a Banach space. Nevertheless, it is simply demonstrated with the following inequalities:

$$\left\| \int_0^t D(t, z) a(z) dz \right\| \leq T \|D(t, z)\|_J \|a(t)\|_J \quad (23)$$

$$\forall a(t) \in \mathcal{F}(J, {}^\circ 4), D(t, z) \in \mathcal{F}(J^2, {}^\circ),$$

such that,

$$\|D(t, z)\|_J = \sup_{t, z \in J} |D(t, z)| \quad (24)$$

Moreover, applying the definition of φ as stated in Equation (21), we derive Equation (24) as stated below:

$$\|\varphi[a_1(t)] - \varphi[a_2(t)]\|_J \leq \left\| \frac{1-\alpha}{ABC(\alpha)} N(t, a_1(t)) - N(t, a_2(t)) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} (N(z, a_1(z)) - N(z, a_2(z))) dz \right\| \quad (25)$$

Using the triangular inequality principle and the Lipschitz condition in Equation (18) combined with the outcome in Equation (23), we derive Equation (26):

$$\|\varphi[a_1(t)] - \varphi[a_2(t)]\|_J \leq \left(\frac{1-\alpha}{ABC(\alpha)}M + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}MT_{\max}^{\alpha} \right) \|a_1(t) - a_2(t)\|_J \quad (26)$$

Finally, upon simplification, we obtain

$$\|\varphi[a_1(t)] - \varphi[a_2(t)]\|_J \leq \Phi \|a_1(t) - a_2(t)\|_J \quad (27)$$

where

$$\Phi = \frac{1-\alpha}{ABC(\alpha)}M + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}MT_{\max}^{\alpha}$$

Hence, the operator φ therefore becomes a contraction as long as the condition stated in Equation (8) holds on $\mathcal{F}(J, {}^\circ 4)$. The system represented by Equation (5) has a unique solution as a result of the Banach fixed point theorem's conclusion.

4.1 Model Fixed Points

[1] deduced two fixed points for the model. These were the unemployment-free equilibrium and the risky unemployment equilibrium.

The unemployment-free equilibrium was obtained as follows:

$$U_f = (U^o, E^o, A^o, V^o) = \left(\frac{\Lambda(k+\varepsilon)}{d_0}, \frac{\Lambda\eta}{d_0}, 0, \frac{\Lambda(k+\varepsilon)(\tau+\sigma)}{\kappa d_0} \right) \quad (28)$$

where

$$d_0 = \kappa^2 + (\eta + \tau + \sigma + \varepsilon)\kappa + \varepsilon(\tau + \sigma).$$

The risky unemployment equilibrium is given by Equation (23) to Equation (24) in the work of [1].

4.2 Unemployment Reproduction Number

The unemployment reproduction number $\mathfrak{R}_0$ controls the unemployment labor force. When $\mathfrak{R}_0 < 1$, it becomes a situation where E, A, V is greater than the unemployed labor force, and this situation is closely monitored; otherwise, Ghana's economy will be seriously affected. [1] reported $\mathfrak{R}_0$ via the next generation matrix.

The next generation method (NGM) is used to determine the basic reproduction number of the unemployment model in

Equation (4). We let $x = (E, A, V)^T$, and decompose Equation (4) as $\frac{dx}{dt} = T - M$, with T being the transmission and M being the transition. This gives:

$$\frac{d}{dt} \begin{bmatrix} E \\ A \\ V \end{bmatrix} = \begin{bmatrix} \eta U \\ \psi \omega U A \\ \theta \phi A + d_8 U \end{bmatrix} - \begin{bmatrix} d_7 E \\ \rho(1-r)A + (\theta \phi + \kappa)A \\ \kappa V \end{bmatrix} \quad (29)$$

Solving the equation above gives:

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \psi \omega U^0 & 0 \\ 0 & \theta \phi & 0 \end{pmatrix}, \text{ and } M = \begin{pmatrix} d_7 & 0 & 0 \\ 0 & \rho(1-r) + \theta \phi + \kappa & 0 \\ 0 & 0 & \kappa \end{pmatrix}$$

The threshold for recruitment represents the spectral radius of the NGM which is mathematically written as:

$$\mathfrak{R}_0 = \rho(TM^{-1}) \quad (30)$$

The spectral radius of the dominant eigenvalue of the product of the transmission and the inverse of the transition matrix results as seen inequation (30) is:

$$\mathfrak{R}_0 = \frac{\psi \omega \Lambda(\kappa + \varepsilon)}{d_0(\rho(1-r) + \theta \phi + \kappa)} \quad (31)$$

From Equation (18), unemployment in Ghana will fail to exist if $\psi \omega \Lambda(\kappa + \varepsilon) < d_0(\rho(1-r) + \theta \phi + \kappa)$.

5. Sensitivity Analysis of the Unemployment Reproduction Number

In this section, we analyse the effect of the various model parameters to understand the comparative impact of the various parameters in the transmission dynamics of the unemployment model in Ghana. We do this by computing the partial

derivatives of the unemployment reproduction number with respect to the unemployment model's parameters. Using the normalized forward sensitivity index on the unemployment reproduction number, the sensitivity index is given by:

$$\Delta_{\zeta}^{\mathfrak{R}_0} = \frac{\partial \mathfrak{R}_0}{\partial \zeta} \times \frac{\zeta}{\mathfrak{R}_0} \quad (32)$$

where ζ is the parameter under consideration.

It is noted that when $\Delta_{\zeta_i}^{\mathfrak{R}_0} \leq 0$, $\forall i = 1, 2, \dots, n$ for the model's parameters, the unemployment basic reproduction number decreases with respect to $\zeta_1, \zeta_2, \dots, \zeta_n$, else it increases with respect to $\zeta_1, \zeta_2, \dots, \zeta_n$.

The sensitivity of the unemployment reproduction number is very important because, it enables us to ascertain the robustness of the model's parameter values. Hence, it enables us to quantify the relative change in a state variable with respect to a change in parameter value.

The sensitivity indices derived from Equation (31) are given by:

$$\begin{aligned} \Delta_{\psi}^{\mathfrak{R}_0} &= 1 > 0, \Delta_{\omega}^{\mathfrak{R}_0} = 1 > 0, \Delta_{\Lambda}^{\mathfrak{R}_0} = 1 > 0, \Delta_{\kappa}^{\mathfrak{R}_0} = \kappa \left(\frac{1}{\kappa + \varepsilon} - \frac{q_1}{d_0} - \frac{1}{q_2} \right) < 0, \Delta_{\varepsilon}^{\mathfrak{R}_0} = \varepsilon \left(\frac{1}{\kappa + \varepsilon} - \frac{q_3}{d_0} \right) > 0, \\ \Delta_{\tau}^{\mathfrak{R}_0} &= -\frac{\tau(\kappa + \varepsilon)}{d_0} < 0, \Delta_{\eta}^{\mathfrak{R}_0} = -\frac{\eta\kappa}{d_0} < 0, \Delta_{\sigma}^{\mathfrak{R}_0} = -\frac{\sigma(\kappa + \varepsilon)}{d_0} < 0, \Delta_{\rho}^{\mathfrak{R}_0} = -\frac{\rho(1-r)}{q_2} < 0, \Delta_{\theta}^{\mathfrak{R}_0} = -\frac{\theta\phi}{q_2} < 0, \\ \Delta_{\phi}^{\mathfrak{R}_0} &= -\frac{\theta\phi}{q_2} < 0, \Delta_r^{\mathfrak{R}_0} = \frac{r\rho}{q_2} > 0 \end{aligned}$$

Where $q_1 = 2\kappa + \eta + \tau + \sigma + \varepsilon$, $q_2 = \rho(1-r) + \theta\phi + \kappa$, $q_3 = \kappa + \tau + \sigma$.

Table 2 shows the sensitivity indices, while Figure 3 shows the bar chart of the indices. These indices emphasize the main factors whose variation has a significant impact on the unemployment model. Figure 3 shows that the most sensitive parameters are ψ , ω , and Λ . Due to this, an increase or decrease in these sensitive parameters will either increase or decrease the unemployment reproduction number.

Table 2: Sensitivity Analysis of Parameters

Parameters	Sensitivity Value	Parameters	Sensitivity Value
Λ	1.000	ψ	1.0000
κ	-0.2802	ε	0.0866
r	0.0138	η	-0.7604
ϕ	-0.8165	τ	-0.0736
θ	-0.8165	ρ	-0.0089
ω	1.0000	σ	-0.1471

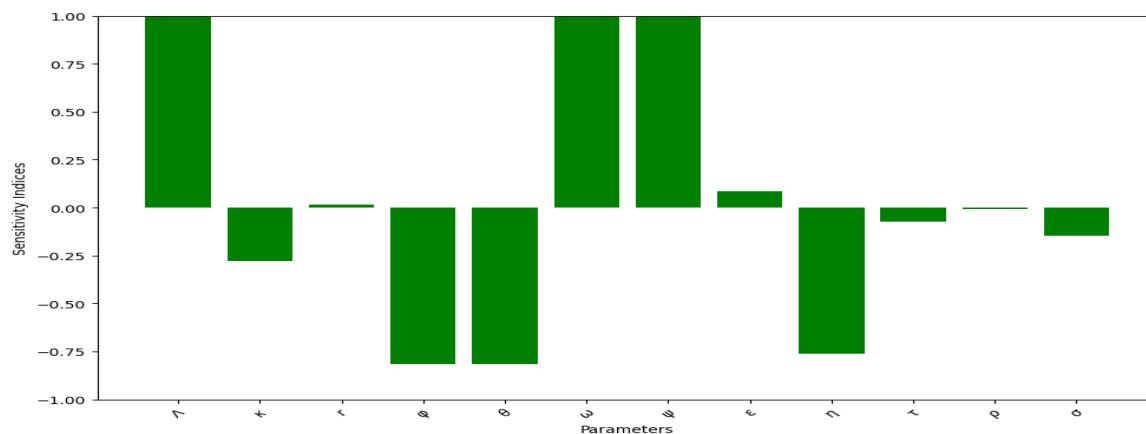


Fig. 3: Sensitivity Analysis of Parameters

6. Numerical simulations of the ABC model

To derive the numerical scheme of the ABC unemployment model from Equation (4), we used the generalized Adams–Bashford method as applied in the work of [27]. Applying the same procedure to Equation (4) gives:

$$\left. \begin{aligned} {}_0^{ABC} D_t^\alpha U &= N_1(t, U, E, A, V) \\ {}_0^{ABC} D_t^\alpha E &= N_2(t, U, E, A, V) \\ {}_0^{ABC} D_t^\alpha A &= N_3(t, U, E, A, V) \\ {}_0^{ABC} D_t^\alpha V &= N_4(t, U, E, A, V) \end{aligned} \right\} \quad (33)$$

We apply the fundamental theorem of fractional calculus to Equation (33) and convert it to a fractional to obtain the system of equations in Equation (34) as:

$$\left. \begin{aligned} U(t) - U(0) &= \frac{1-\alpha}{ABC(\alpha)} N_1(t, U) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} N_1(z, U) dz \\ E(t) - E(0) &= \frac{1-\alpha}{ABC(\alpha)} N_2(t, E) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} N_2(z, E) dz \\ A(t) - A(0) &= \frac{1-\alpha}{ABC(\alpha)} N_3(t, A) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} N_3(z, A) dz \\ V(t) - V(0) &= \frac{1-\alpha}{ABC(\alpha)} N_4(t, V) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} N_4(z, V) dz \end{aligned} \right\} \quad (34)$$

Simplifying further at $t = t_{n+1}, n = 0, 1, 2, 3, \dots$ yields Equation (35):

$$\left. \begin{aligned} U(t_{n+1}) - U(0) &= \frac{1-\alpha}{ABC(\alpha)} N_1(t_n, U) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - z)^{\alpha-1} N_1(z, U) dz \\ E(t_{n+1}) - E(0) &= \frac{1-\alpha}{ABC(\alpha)} N_2(t_n, E) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - z)^{\alpha-1} N_2(z, E) dz \\ A(t_{n+1}) - A(0) &= \frac{1-\alpha}{ABC(\alpha)} N_3(t_n, A) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - z)^{\alpha-1} N_3(z, A) dz \\ V(t_{n+1}) - V(0) &= \frac{1-\alpha}{ABC(\alpha)} N_4(t_n, V) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - z)^{\alpha-1} N_4(z, V) dz \end{aligned} \right\} \quad (35)$$

Through the two-point interpolation polynomial, the integrals in Equation (35) are approximated to give the iterative scheme for the unemployment model in Equation (4). The approximate solutions are finally obtained via Equation (36) after brief simplification.

$$\left. \begin{aligned} U(t_{n+1}) &= U(0) + \frac{1-\alpha}{ABC(\alpha)} N_1(t_n, U) + \frac{\alpha}{ABC(\alpha)} \sum_{j=0}^n \left(\frac{h^\alpha N_1(t_j, U)}{\Gamma(\alpha+2)} P_0 - \frac{h^\alpha N_1(t_{j-1}, U)}{\Gamma(\alpha+2)} P_1 \right) \\ E(t_{n+1}) &= E(0) + \frac{1-\alpha}{ABC(\alpha)} N_2(t_n, E) + \frac{\alpha}{ABC(\alpha)} \sum_{j=0}^n \left(\frac{h^\alpha N_2(t_j, E)}{\Gamma(\alpha+2)} P_0 - \frac{h^\alpha N_2(t_{j-1}, E)}{\Gamma(\alpha+2)} P_1 \right) \\ A(t_{n+1}) &= A(0) + \frac{1-\alpha}{ABC(\alpha)} N_3(t_n, A) + \frac{\alpha}{ABC(\alpha)} \sum_{j=0}^n \left(\frac{h^\alpha N_3(t_j, A)}{\Gamma(\alpha+2)} P_0 - \frac{h^\alpha N_3(t_{j-1}, A)}{\Gamma(\alpha+2)} P_1 \right) \\ V(t_{n+1}) &= V(0) + \frac{1-\alpha}{ABC(\alpha)} N_4(t_n, V) + \frac{\alpha}{ABC(\alpha)} \sum_{j=0}^n \left(\frac{h^\alpha N_4(t_j, V)}{\Gamma(\alpha+2)} P_0 - \frac{h^\alpha N_4(t_{j-1}, V)}{\Gamma(\alpha+2)} P_1 \right) \end{aligned} \right\} \quad (36)$$

where,

$$\left. \begin{aligned} P_0 &= (n+1-j)^\alpha (n-j+2+\alpha) - (n-j)^\alpha (n-j+2+2\alpha) \\ P_1 &= (n+1-j)^{\alpha+1} - (n-j)^\alpha (n-j+1+\alpha) \end{aligned} \right\}$$

6.1 Numerical simulations and results

The numerical simulations for the Atangana Baleanu Caputo model were carried out via the developed numerical scheme in Equation (23). The data for the variables and parameters for the simulation were sourced from Ghana Statistical Service (GSS), and the remaining data were assumed as indicated in Table 3 and Table 4. The simulation considered a 50-year duration. The fractional order or alpha values considered in simulating the model were $\alpha = [0.2, 0.4, 0.6, 0.8, 0.99]$. These alpha values represent the time scale of policy implementation, varied levels of historical influence on the various compartments, and distinct responses to policy interventions in the various compartments. Additionally, the initial condition value of new vacancies to be created was assumed to be zero. This is due to the fact that new vacancies are assumed to be created after the implemented of this model. This actually was to help us understand the dynamics of V from ground zero.

Table 3: Parameter Values

Parameter	Value	Source	Parameter	Value	Source
Λ	0.44	[28]	ψ	0.45	Assumed
κ	0.0077	[28]	ε	0.00099	Assumed
r	0.6096	[28]	η	0.35	Assumed
ϕ	0.6	Assumed	τ	0.03	Assumed
θ	0.06	Assumed	ρ	0.001	Assumed
ω	0.3	Assumed	σ	0.06	Assumed

Table 4: Initial Conditions Values

Parameter	Values (Million)	Source
U	1.77	[28]
E	11.22	[28]
A	0.472	[28]
V	0	Assumed

6.2 Results of the dynamics of the model's compartments

The dynamics of the various compartments were studied to understand how they behave over time. Figures 4(a) to 4(d) show the dynamics of the model at the unemployed, employed, skill trainees, and new vacancies respectively.

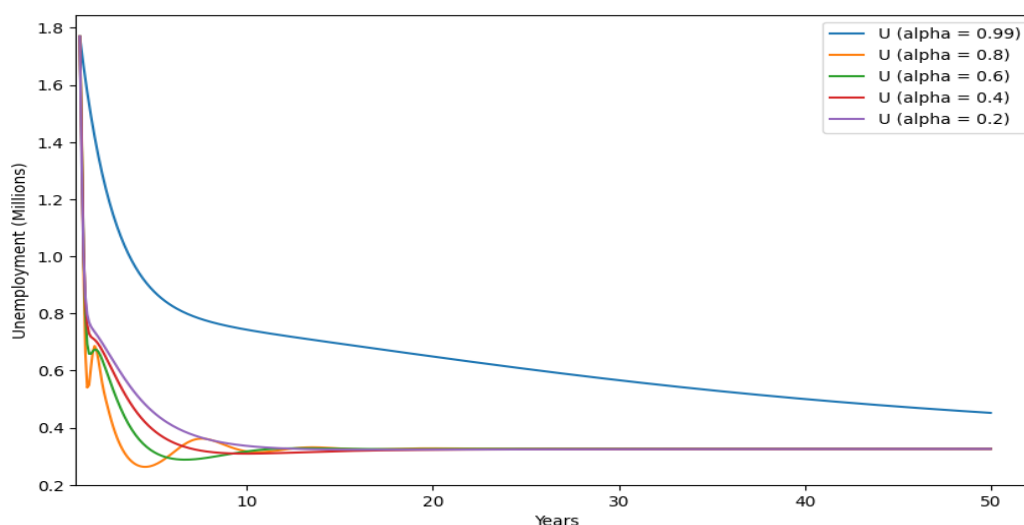


Fig. 4(a): Dynamics of Unemployment Compartment

Figure 4(a) shows the dynamics in the unemployment chamber. Unemployment decreases as the memory increases or α decreases. In terms of the integer order of ($\alpha \approx 1.0$), the results show a considerable influence of current unemployment rates and economic conditions on unemployment in Ghana, with current policy changes having extreme and swift impacts on the reduction in unemployment. This influence decreases unemployment significantly in the short term. Figure 4(a) further reveals that long-term memory effect ($\alpha = 0.8$) influenced by immediate past factors like flexible labor market adjustment which occurs within a year or two has positive effect on controlling unemployment. This causes drastic reduction in unemployment in the short run before converging to the constant rate. The unemployment rate decreased further as the memory effect increased (alpha decreased). This suggests that unemployment in Ghana is determined by the past state of the economy, with little impact from current state or short-term unemployment policies. The increase in memory additionally indicates that past unemployment rates and economic conditions or the influence of past economic factors, such as GDP and interest rates have stronger impact on unemployment in Ghana. As a result, the labor market is more sensitive to long-term economic variations. This indicates a very strong relationship between the past and present conditions of the labor market. This makes Ghana's labor market more rigid with strong unions, and high regulations compelling unemployment rate to adapt to very slow changes in the midst of economic conditions.

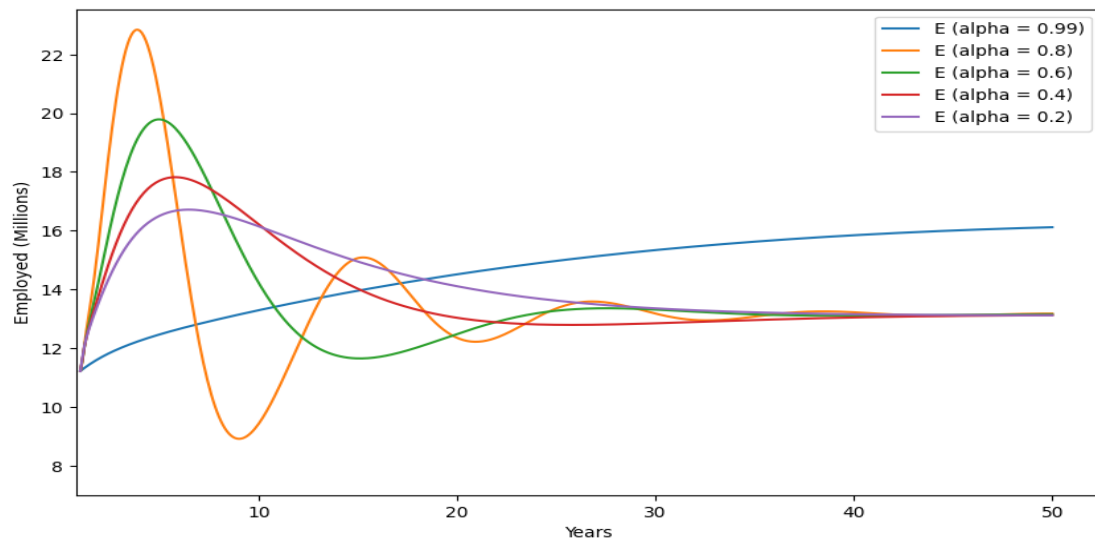


Fig. 4(b): Dynamics of Employed Compartment

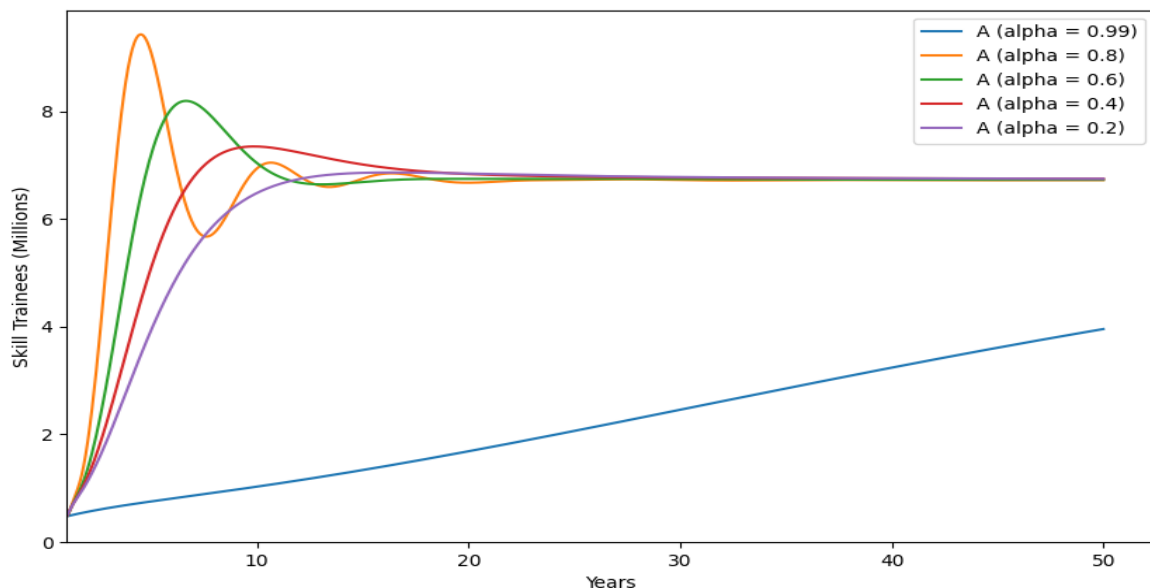


Fig. 4(c): Dynamics of Skill Training Compartment

Figure 4(b) shows the dynamics in the employed compartment. The figure highlights the memory effects ($\alpha = 0.2, 0.4, 0.6, 0.8, 1.0$) on the employed compartments to help understand dynamics of the employed compartments in Ghana. Figure 4(b) shows that a decrease in the alpha value results in a rise and fall in the employed compartment for a number of notable years before gradually converging in the long run, with its convergence level still greater than the initial number of employed persons. This explains why the entrance and retention rates of existing institutions are progressively higher than the exit rates of the current compartment. In general, a decline in the α value makes the labor force more sensitive to past economic conditions. This makes Ghana's labor market more elastic and adaptive and enables the occurrence of job transitions.

Moreover, Figure 4(c) shows the dynamics in the skill trainee compartment. The figure shows that at $\alpha \approx 1.0$, few people are engaged in skill training. However, as the memory increases (alpha decreases), more people acquire new skills to help mitigate the problem of job mismatch in Ghana. Moreover, Figure 4(c) shows that higher memory compels the labor force to learn new skills, that make them more responsive to the needs of the job market, thereby making them adaptable to Ghana's shifting skill requirements.

Figure 4(d) shows the dynamics at the new vacancies compartment. The graph examines how new vacancies are created with a variation in the memory effect. The Figure shows that few vacancies can be created over time at the integer order. This indicates that the current policy interventions do not influence the creation of more new vacancies. Figure 4(d) further shows that historical trends have a positive impact on skill training. This finding further shows that past training experiences are contributing factors for the unemployed to further indulge in skill training. In general, a decline in the α value generally depicts greater responsiveness to shifting economic conditions, heightened sensitivity to labor market conditions, influence of historical data, and improved capacity to adjust to new labor market demands.

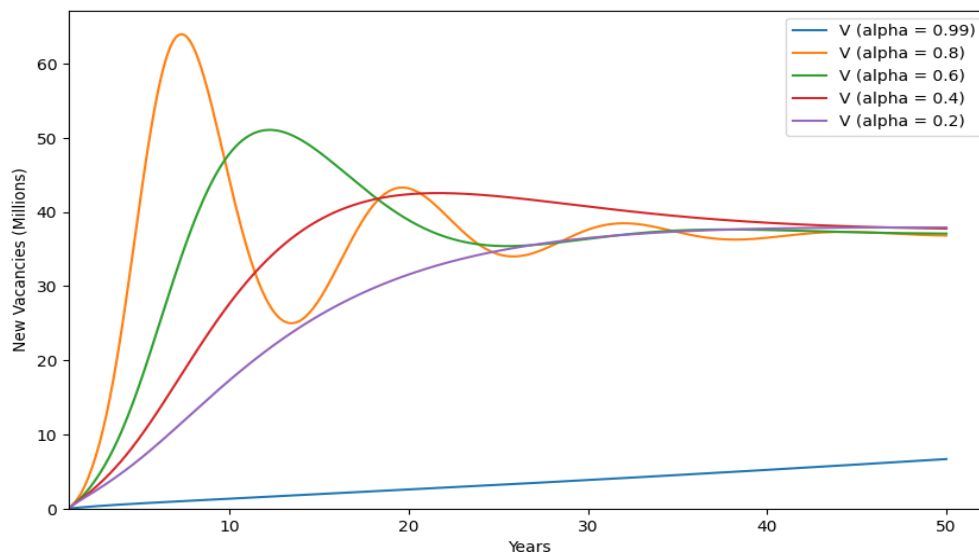


Fig. 4(d): Dynamics of New Vacancies Compartment

6.3 Impact of Startup Capital on the Model's Dynamics

Figures 5(a) to 5(c) show the impact of startup capital (θ) on unemployed, skill training, and newly created vacancies of the ABC model. The numerical simulations are conducted by varying the rate of startup capital (θ) to ascertain the dynamics of each compartment. Since more memory drives the system in a positive trend, a memory level of $\alpha = 0.2$ is chosen for this part of the analysis. Figures 5(a) to 5(c) show that a change in startup capital causes a change in the number of employed persons, skill trainees and newly created vacancies. However, Figure 5(a) shows that small startup capital given to support high-skilled trainees actually decreases unemployment compared with high startup capital, which might not be able to cover more people due to the country's budget constraints. Additionally, Figure 5(b) also shows that a small startup capital promotes more people to go into skill training. Moreover, Figure 5(c) adds that a small startup capital results in the creation of more new vacancies. This therefore indicates that to reduce unemployment in Ghana, startup capital needs to be given to a fraction of highly skilled trainees. This should be done at an optimal level to help achieve the intended purpose of curtailing unemployment problem in Ghana.

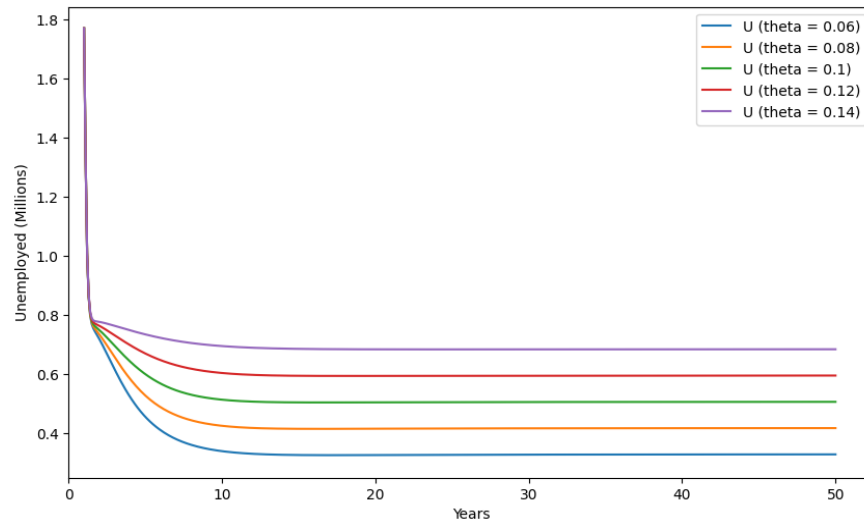


Fig. 5(a): Impact of startup capital on the unemployed compartment

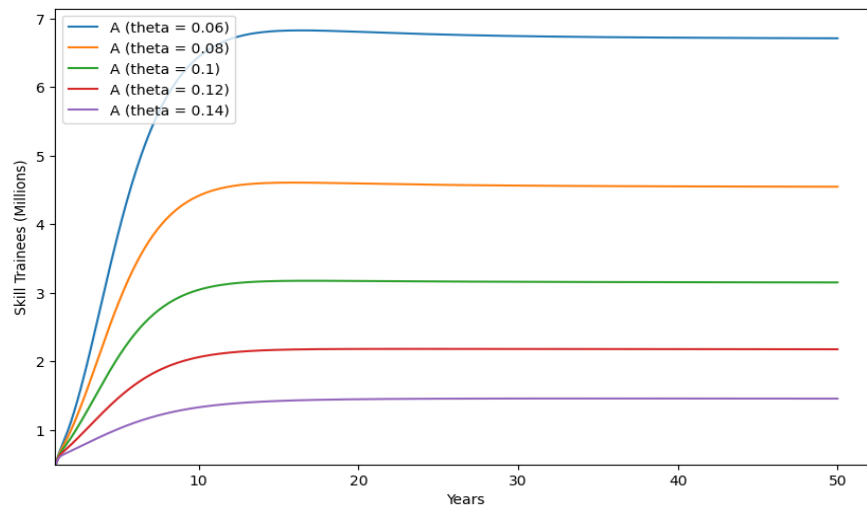


Fig. 5(b): Impact of Startup Capital on Skill Trainee

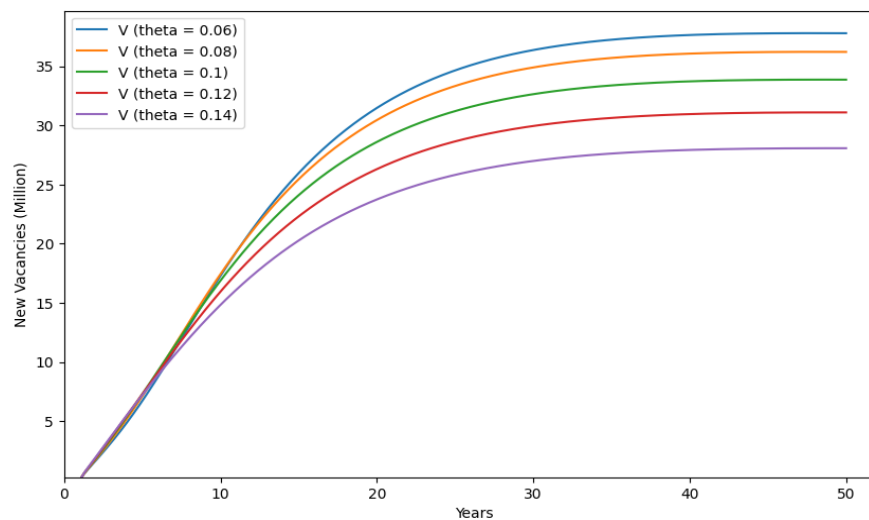


Fig. 5(c): Impact of Startup Capital on the Creation of New Vacancies

6.4 Impact of Government Revenue on Model Dynamics

Figures 6(a) to 6(d) show the impact of government revenue on the unemployed, employed, skill training, and newly created vacancies of the ABC model. The numerical simulations were conducted by varying the rate of government revenue to ascertain the dynamics of each compartment. Since more memory drives the system in a positive trend, a memory level of $\alpha = 0.2$ is also chosen for this part of the analysis.

Figure 6(a) shows that government revenue causes a drastic decline in unemployment. Also, Figure 6(b) shows that government revenue has positive impact on existing firms thereby causing a sharp expansionary effect on existing firms in the short run, but steadily converging declining to a stable point in the long run.

Again, Figures 6(c) and 6(d) show that government revenue increases skill training and the creation of new vacancies respectively.

Figures 6(a) to 6(d) finally show that regardless of the amount of government revenue mobilized, the government of Ghana can still reduce unemployment in the country.

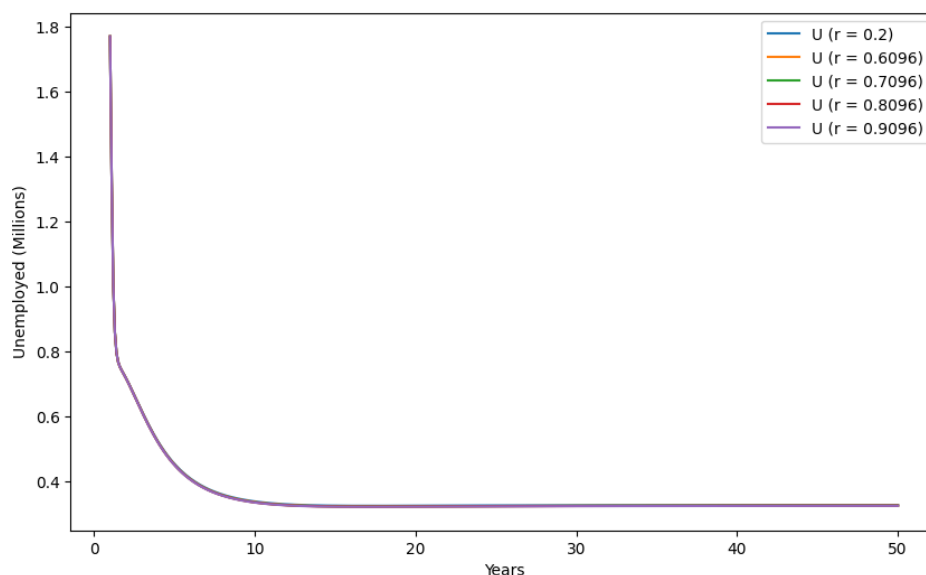


Fig. 6(a): Impact of Government Revenue on Unemployment

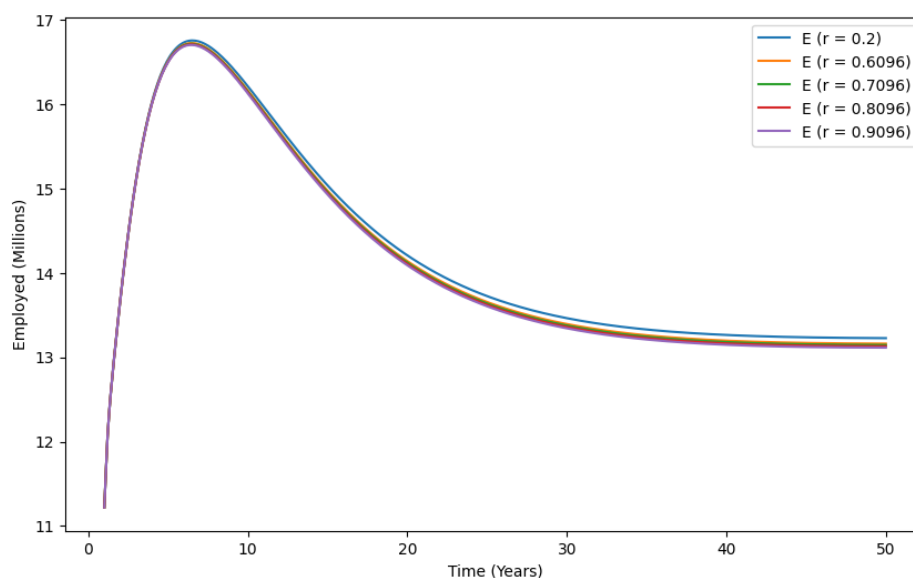


Fig. 6(b): Impact of Government Revenue on the Employed Compartment

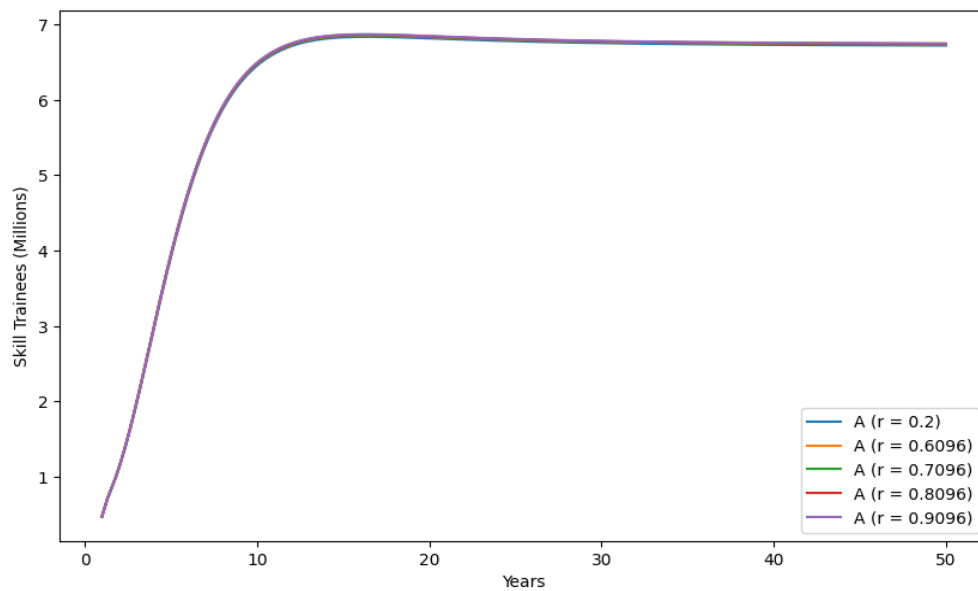


Fig. 6(c): Impact of Government Revenue on Skill Training

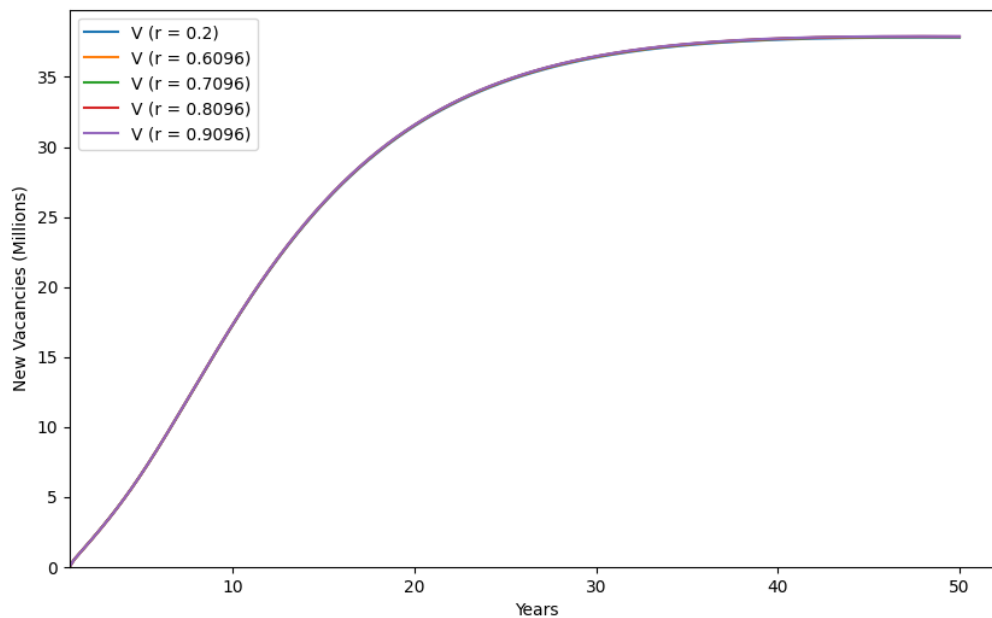


Fig. 6(d): Impact of Government Revenue on Newly Created Vacancy

6.5 Impact of Government and Private Partnership on Job Creation

Figures 7(a) to 7(c) show the combined impact of government and private partnership in creating more jobs to control unemployment in Ghana. The numerical simulations are conducted by varying the rate of the private sector (σ) while maintaining a 3% rate of job creation by the government ($\tau = 3\%$) to ascertain the dynamics of each compartment. Since more memory drives the system in a positive trend, a memory level of $\alpha = 0.2$ is also chosen for this part of the analysis.

Figure 7(a) shows that an increase in the combined effect of the government and the private sector with key emphasis on the private sector to drive the creation of more new vacancies causes a sharp decrease in unemployment in the short run and continues to decrease in the long run. Figure 7(b) also shows that as the private sector creates more vacancies; there will be variation in the number of skill trainees. However, a higher rate of job creation will cause lower number of skill trainees. Additionally, Figure 7(c) shows that the government (public) and private partnership create more new vacancies.

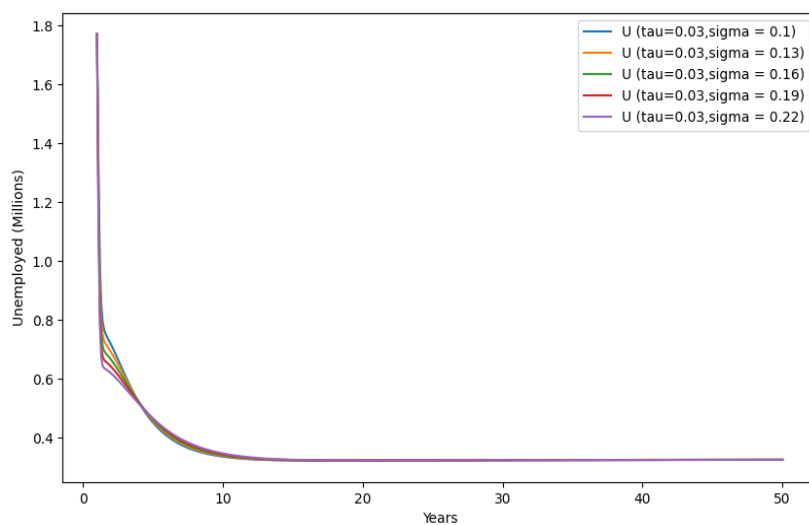


Fig. 7(a): Combined impact of job creation by the government and private sector on unemployment

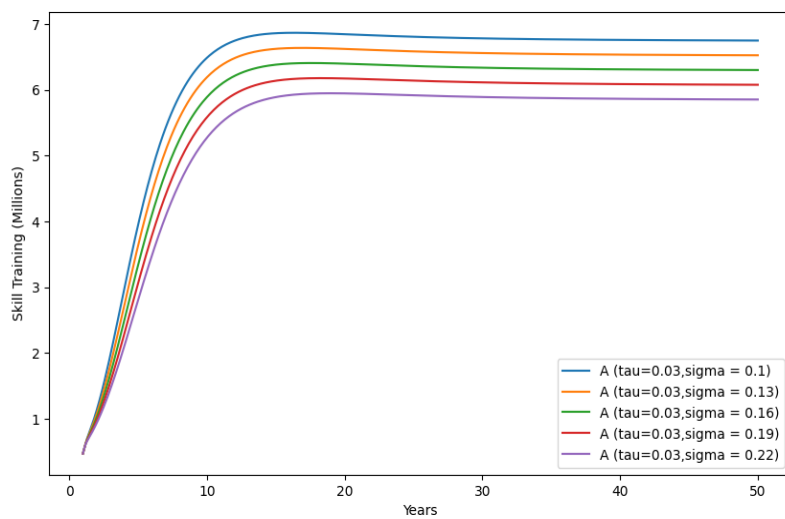


Fig. 7(b): Combined impact of job creation by the government and the private sector on skill training

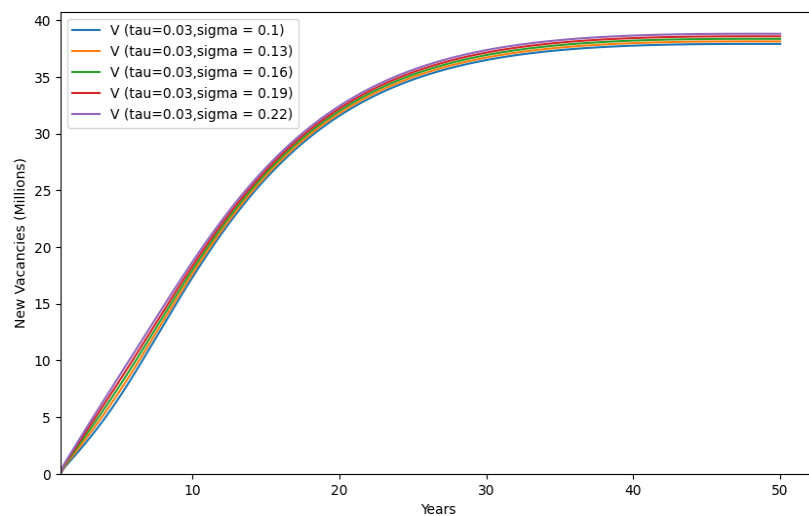


Fig. 7(c): Combined impact of job creation by the government and the private sector on new vacancies

6.6 Impact of Logistics Provision

Figures 8(a) to 8(c) show the impact of logistics provisions on skill training institutions. The numerical simulations are conducted by varying the rate of logistic provision to the various skill training institutions. Since more memory drives the system in a positive trend, a memory level of $\alpha = 0.2$ is chosen for this part of the analysis.

Figures 8(a) to 8(c) show that a variation in the provision of logistics causes a variation in the dynamics in the unemployed, skill training and new vacancy compartments. Figure 8(a) shows that higher logistics provision drastically reduces unemployment. Also, Figure 8(b) shows that higher logistics provision to skill training institutions motivates many unemployed individuals to enroll in skill training than inadequate logistic provided to the various training institutions. Again, Figure 8(c) shows that as more logistics are provided to skill training institutions, more people acquire high skills to enable them transition into creating their own vacancies.

In summary, Figure 8(a) to 8(c) indicates that the provision of adequate logistics motivates many unemployed individuals to engage in skill training. This reduces the number of unemployed persons, increases the number of skill trainees, and invariably causes high skill acquisition to enable trained graduates to create their own vacancies.

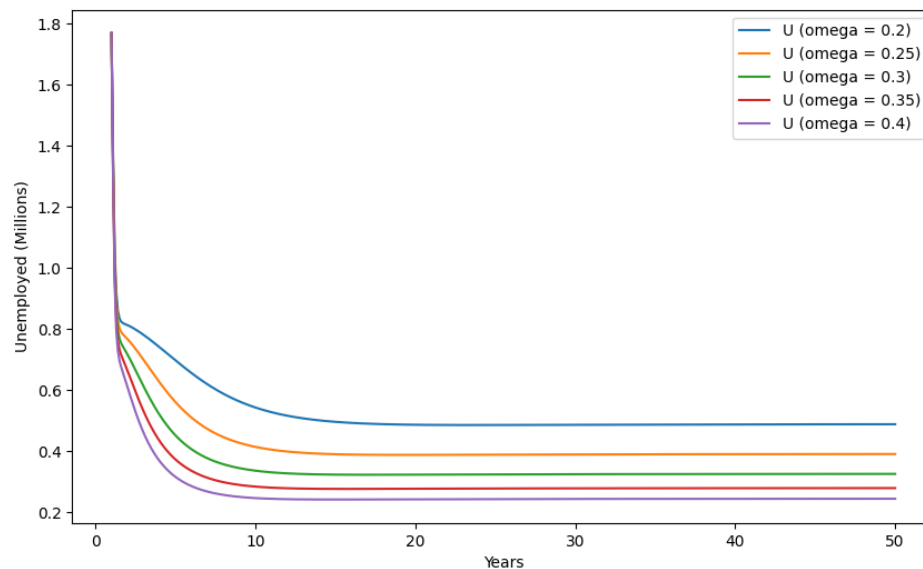


Fig. 8(a): Impact of Logistics Provision on Unemployment

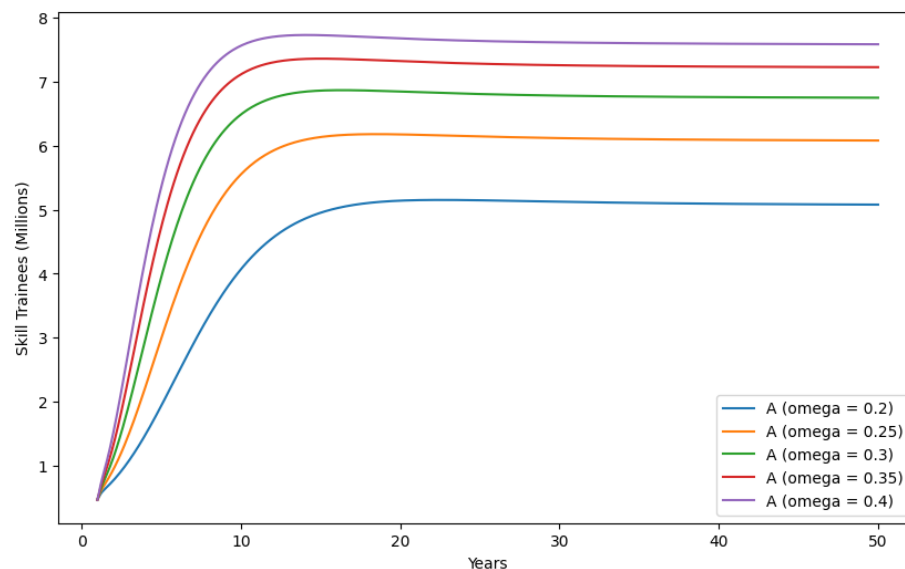


Fig. 8(b): Impact of Logistics Provision on Skill Training

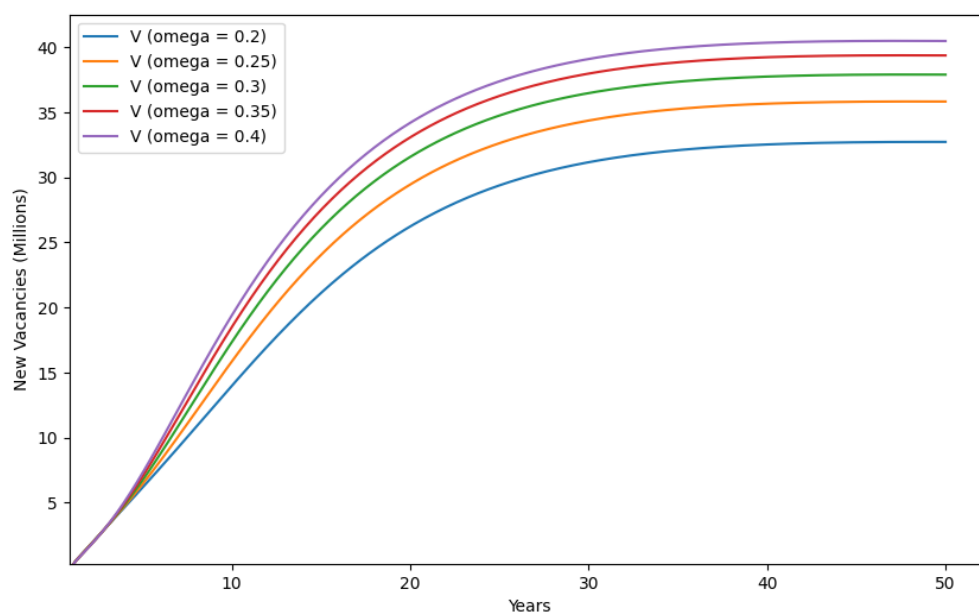


Fig. 8(c): Impact of Logistics Provision on New Vacancies

6.7 The General Dynamics of the Atangana Baleanu Caputo Model

In considering the generalized state of the model for its implementation, we considered a time duration of 50 years using moderate or intermediate memory ($\alpha = 0.5$). This memory indicates a fair balance from both distant past and recent rates of unemployment. The general dynamics of the model as indicated in Figure 9 shows that as the number of skill trainees increases in the short run and flattens after the 10th year, the employed rate also increases in the short run and stabilizes at a far higher nominal term than its initial point in the long run. This creates more new vacancies by the private sector, thereby reducing unemployment drastically. This means that the Atangana-Baleanu-Caputo model has the power to control unemployment over time in Ghana.

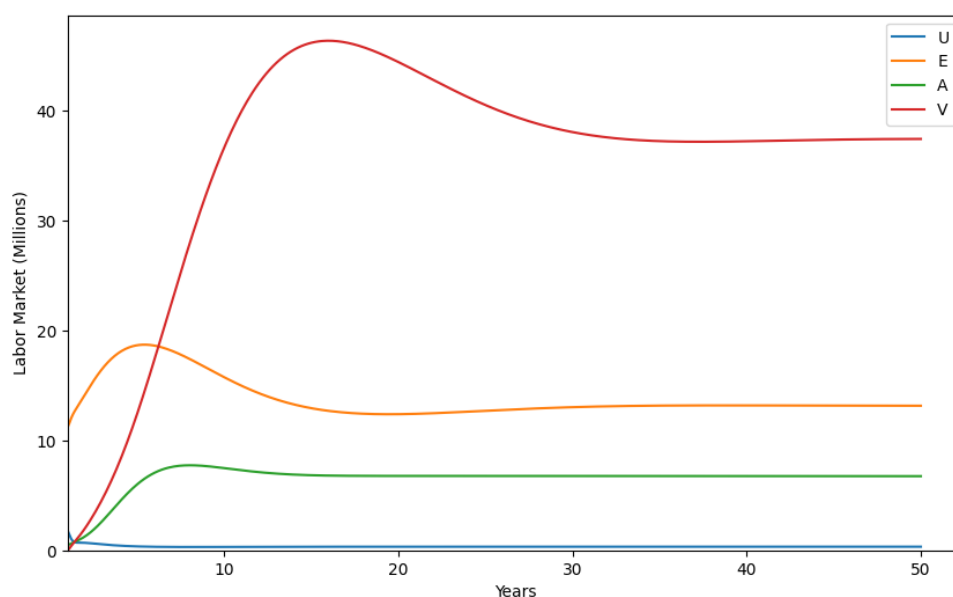


Fig. 9: General dynamics of the ABC model at alpha (0.5)

6.8 Comparison of the Caputo Fractional Model and the Atangana Baleanu Caputo Model

This subsection presents a graphical comparison of the Atangana Baleanu Caputo fractional order unemployment model, and the Caputo fractional order unemployment model developed by [1] with its numerical scheme by [29]. The simulations for

the Caputo model were also carried out with the parameter and variable values in Table 3 and Table 4. To compare the two models, several alpha values ($\alpha = 0.2, 0.5, 0.8$) were used, with the results displayed in Figure 10 to Figure 12.

With $\alpha = 0.2$, Figures 10(a) to 10(c) show that the ABC model outperforms the Caputo model. This is because in Figure 10(a), the ABC model decreases unemployment far better than the Caputo model does. Additionally, Figure 10(b) shows that in the quest to promote skill training, the ABC model has a greater number of skill trainees than the Caputo model. Finally, more vacancies are created with the Atangana Baleanu Caputo model than with the Caputo model, as shown in Figure 10(c). A similar trend is shown in Figure 11(a) to 11(c) at an alpha value of 0.5 and Figure 12(a) to 12(c) at an alpha value of 0.8.

This shows that for all memories (alpha values), the Atangana Baleanu Caputo unemployment model performs better than the Caputo unemployment model in Ghana.

Comparing the two models at alpha value of 0.2 gives Figures 10(a) to 10(c) as:

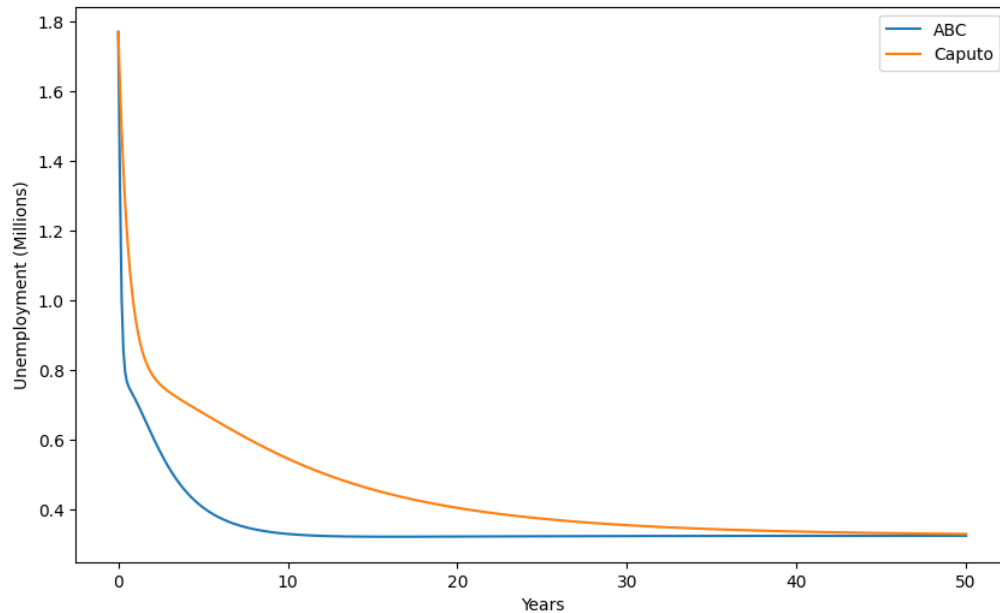


Fig. 10(a): Unemployment Dynamics of ABC and Caputo models at Alpha (0.2)

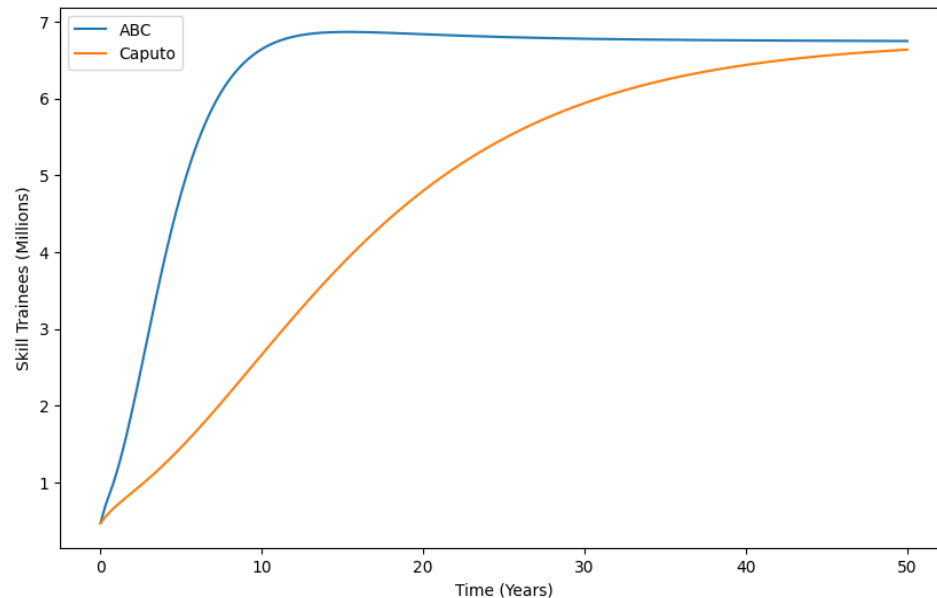


Fig. 10(b): Skill Training Dynamics of ABC and Caputo models at Alpha (0.2)

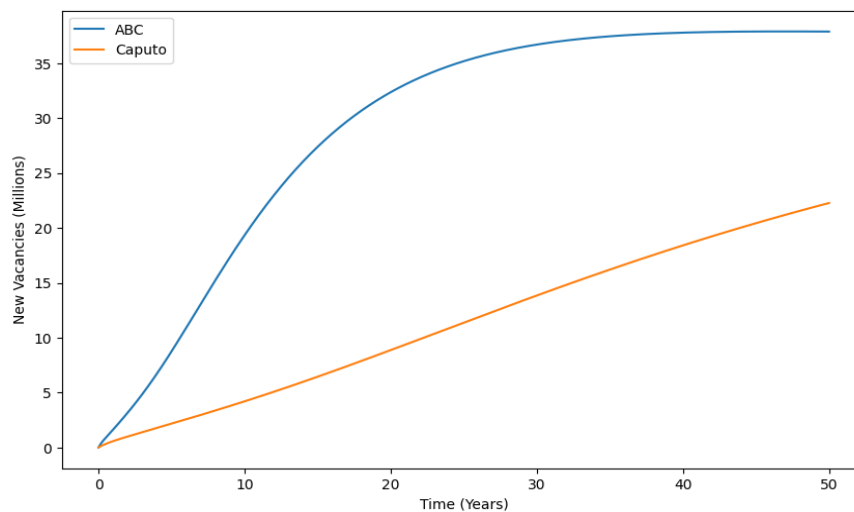


Fig. 10(c): New Vacancies Dynamics of ABC and Caputo models at Alpha (0.2)

Comparing the two models at alpha value of 0.5 gives Figures 11(a) to 11(c) as:

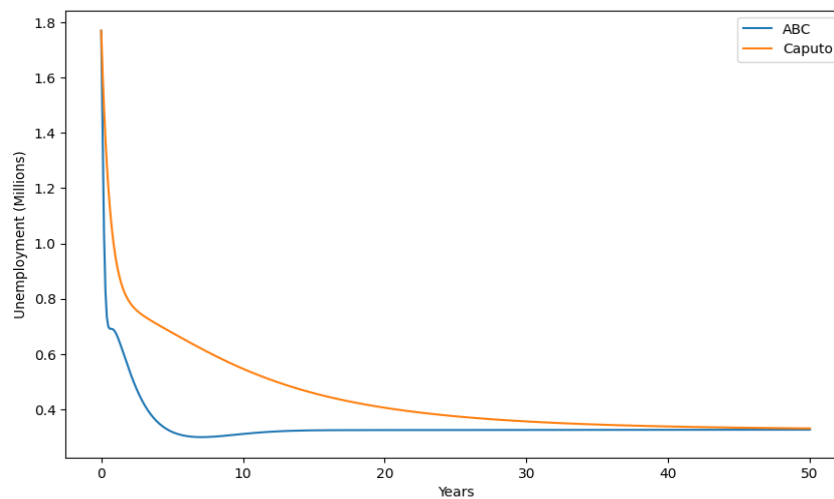


Fig. 11(a): Unemployment Dynamics of ABC and Caputo models at Alpha (0.5)

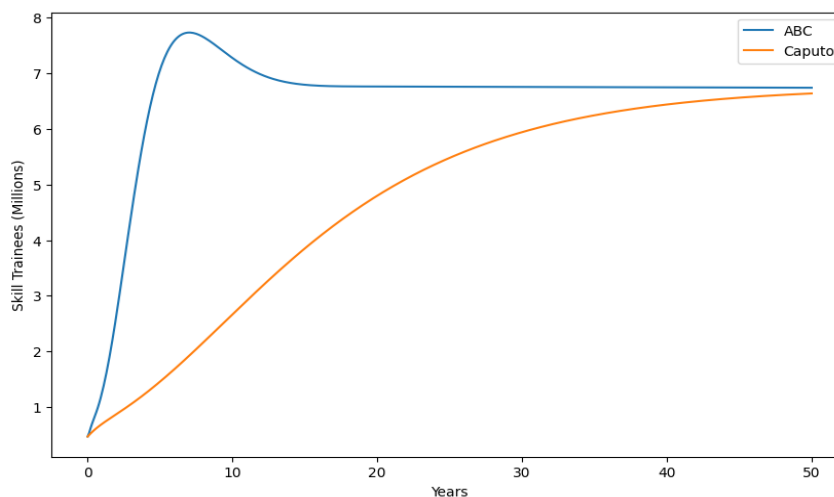


Fig. 11(b): Skill Training Dynamics of ABC and Caputo models at Alpha (0.5)

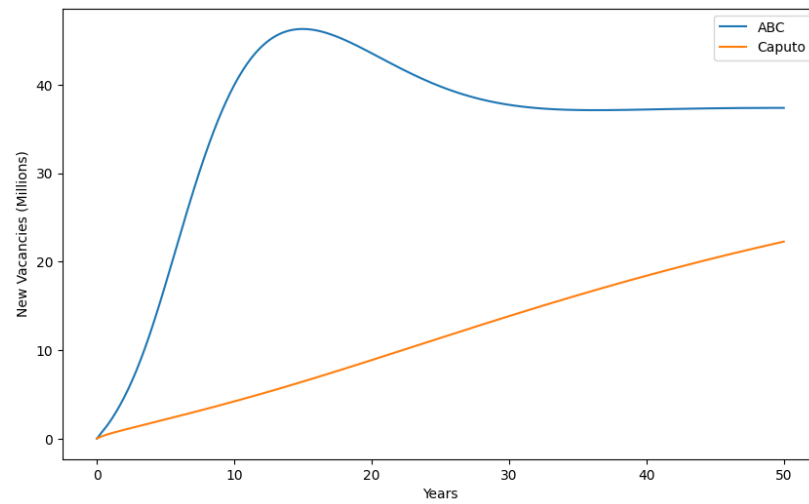


Fig. 11(c): New Vacancies Dynamics of ABC and Caputo models at Alpha (0.5)

Finally, we compare the two models at alpha value of 0.8. this gives Figures 12(a) to 12(c) as:

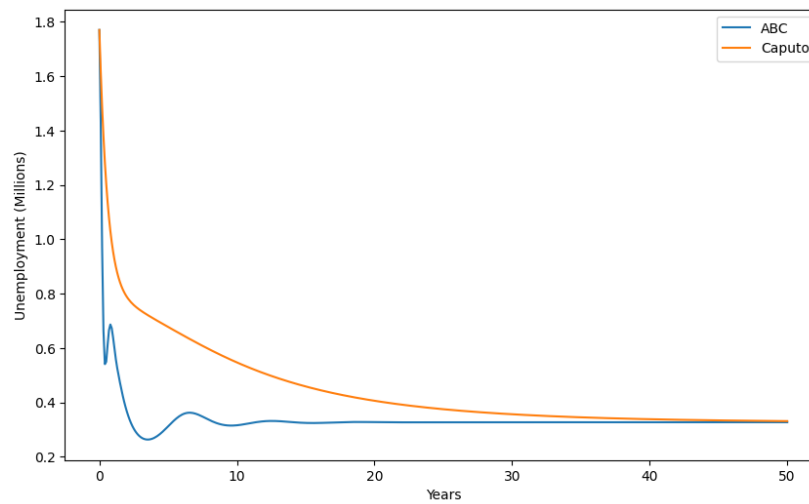


Fig. 12(a): Unemployment Dynamics of ABC and Caputo models at Alpha (0.8)

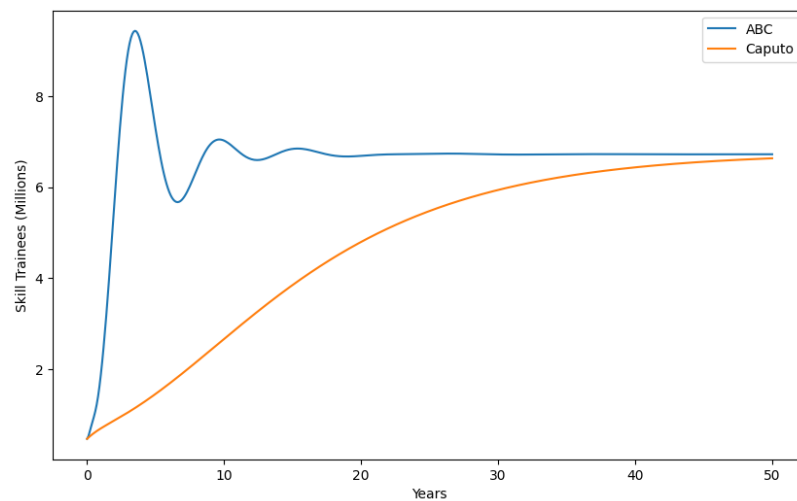


Fig. 12(b): Skill Training Dynamics of ABC and Caputo models at Alpha (0.8)

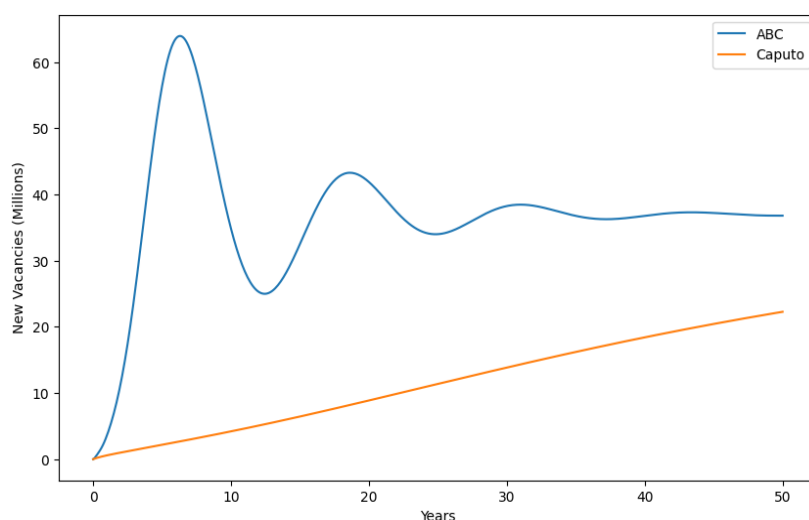


Fig. 12(c): New Vacancies Dynamics of ABC and Caputo models at Alpha (0.8)

7. Conclusion

In this study, we adopted a proposed Caputo fractional derivative unemployment model in Ghana, and extended the study in the Atangana Baleanu Caputo sense. We presented the existence and uniqueness of the ABC model, the unemployment fixed points, and the unemployment basic reproduction number. Again, we developed a numerical scheme for the ABC model via the generalized Adams–Bashford method. Finally, we conducted a comparative study of the two models' robustness to Ghana's labor market dynamics.

We identified that the ABC model is flexible in its applicability, it provides better results and outperforms the Caputo model in real-life situations in Ghana. The results indicated that long-distance unemployment rates have great significance for current unemployment, showing a very strong relationship between the past and present conditions of the labor market. This behaviour results in the labor market becoming more flexible, thereby allowing unemployment rate to adapt to changing economic conditions more quickly in Ghana. Again, we noted that a decline in the alpha values or memory yields a decline in unemployment. This shows a strengthening of the relationship between past economic conditions and unemployment. This further implies that past economic conditions, or the impact of past economic factors, such as GDP growth and interest rates, on unemployment in Ghana are stronger. This therefore causes the labor market to be more sensitive to Ghana's long-term economic variations, making the labor market more elastic and adaptive and enabling job transitions to occur steadily over time.

Additionally, an increase in startup capital increases the creation of new vacancies. Conversely, a decrease in startup capital to cover more beneficiaries also creates more vacancies in the long run to curtail unemployment canker. Thus, the most important thing is for high-skilled trainee graduates to have a little startup capital. Additionally, the provision of logistics to various training institutions increases the number of participants in skill acquisition, drastically reduces the number of unemployed persons and finally increases the number of new vacancies created.

The sensitivity analysis showed that whiles population growth, rate of employed becoming unemployed, etc increase unemployment in Ghana due to skill mismatch, however, startup capital, creation of contingency vacancies by the government, private sector participation, and successful skill training help to control unemployment in Ghana.

The simulation further proved that controlling unemployment should be a conscious effort. Thus, there must be provision of logistic to the various skill training institutions, while existing companies should be given tax incentives to enable them create more vacancies through investment.

Recommendations

From the results it is recommended that government should pass policies that promote public and private sector partnership to support the private sector to create more vacancies. Additionally, government should provide the necessary logistics to the various training institutions to promote industry skill acquisition. Also, government should pass tax reform policies to give tax incentives to the private sector to enable them invest in the economy to create additional vacancies. Finally, government should give moderate startup capital to high skilled graduate to enable them create vacancies for themselves and others.

Limitations

The limitation of this work is that the model is purposely designed to suit Ghana's labour market, except for other economies with the same conditions as enshrined in the assumptions. Additionally, a variation in any of the assumptions might result into a variation in the model's dynamics.

Authors Agreement

All the authors have agreed to the processing of this paper.

Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this work.

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