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Dynamics of Kink and Multi-Kink Solitons in the Stochastic (2+1)-Dimensional Burgers Equation with Multiplicative White Noise

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Abstract: This paper investigates the stochastic (2+1)-dimensional Burgers equation, a fundamental nonlinear partial differential equation (PDE) widely used to model shock formation, soliton propagation, and complex spatiotemporal dynamics. The stochastic formulation incorporates multiplicative noise, allowing for the analysis of systems subject to random fluctuations and environmental disturbances. First, the equation is reformulated using a wave transformation technique to establish a unified analytical framework. Explicit solutions of the deterministic counterpart are then derived through two complementary approaches: the Singular manifold method and the multiple-kink wave solution technique. These solutions form the foundation for studying the impact of noise and parameter variations on soliton stability and morphology. Comprehensive numerical simulations are performed to evaluate the evolution of kink and multi-kink solitons under varying noise intensities and initial conditions. The results demonstrate that while solitons preserve their core structural features under weak stochastic perturbations, increased noise intensity leads to significant amplitude distortions, phase fluctuations, and, in some cases, loss of coherence. These findings highlight the dual role of multiplicative noise, which can induce minor modulations at low levels but severely disrupt nonlinear balance at higher intensities. This work fills a critical research gap by systematically examining the stochastic (2+1)-dimensional Burgers equation and provides insights relevant to practical applications in physics and engineering, particularly in areas where soliton stability in noisy environments is essential.

Keywords: Soliton dynamics, Additive white noise, Rössler system, Chaotic processes, Wave transformation, Stochastic partial differential equations, Burgers equation

1 Introduction

The Burgers equation is one of the most fundamental nonlinear partial differential equations, playing a central role in modeling a variety of physical phenomena such as fluid dynamics, gas dynamics, acoustics, and traffic flow [1,2,3]. Due to its inherent nonlinearity, it has served as a prototype for studying complex behaviors such as shock formation, wave interactions, and turbulence-like dynamics. Over the decades, numerous analytical and numerical techniques have been developed for solving different forms of the Burgers equation, including the Adomian decomposition method, variational iteration method, and homotopy perturbation approaches [4]-[7]. Exact solutions have also been obtained for special cases using techniques such as the power index method, singular manifold method, and other direct methods [8]-[15].

Extending the equation to higher dimensions, particularly the (2+1)-dimensional form, has enabled the exploration of richer nonlinear dynamics and wave structures [14,15]. Recent studies have shown that incorporating stochastic effects into nonlinear wave equations is crucial for accurately describing real-world systems subject to random fluctuations [16]-[19]. Stochastic partial differential equations (SPDEs) have been widely used in diverse applications ranging from climate modeling and fluid turbulence to nonlinear optics and quantum field theory[19,20,21,22]. Among the various stochastic forcing types, multiplicative noise has received considerable attention due to its ability to significantly modify soliton structures, induce phase shifts, and alter the stability of solutions [16,17,18], [25]. Similar studies on related nonlinear stochastic models have revealed that even weak noise can lead to long-term

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structural changes and energy redistribution [17, 18, 25, 26]

In this context, the stochastic (2+1)-dimensional Burgers equation provides a robust framework for investigating the combined effects of nonlinearity, dispersion, and stochasticity. Its solutions, including kink and multi-kink solitons, are particularly relevant for understanding topological structures and energy transport in noisy environments [9]-[15]. Building on classical deterministic results [1]-[15], this work focuses on analyzing the impact of multiplicative noise on kink and multi-kink wave solutions using advanced numerical simulations. Such investigations are essential for characterizing noise-induced instabilities and for designing noise-resilient applications in physics and engineering, where topological soliton-like structures play a pivotal role [19]-[26].

1.1 Study Focus, Objectives, and Research Gap

This study investigates the stochastic (2+1)-dimensional Burgers equation, a fundamental nonlinear partial differential equation (PDE) capable of modeling complex spatiotemporal phenomena such as shock formation, soliton interactions, and turbulence-like behavior. Unlike its deterministic counterpart, the stochastic formulation accounts for environmental noise and random perturbations, making it a powerful framework for analyzing real-world systems subject to uncertainty. A central focus is placed on examining the impact of multiplicative noise on kink and multi-kink soliton solutions, which are essential for understanding topological wave structures and energy transport in noisy media. The work aims to reformulate the stochastic (2+1)-dimensional Burgers equation using a wave transformation technique, derive explicit solutions for the deterministic case through the Singular Manifold Method and the Multiple-Kink Wave Solution approach, and analyze, via high-precision numerical simulations, how variations in noise intensity and other parameters affect soliton stability and evolution.

Although the deterministic Burgers equation and its higher-dimensional extensions have been studied extensively using various analytical and numerical methods [1]-[15], the stochastic (2+1)-dimensional form has received far less attention. Previous investigations on stochastic nonlinear models [16, 17, 18, 19], [25, 26] have shown that noise can significantly alter soliton dynamics, yet systematic studies addressing the influence of multiplicative noise on kink and multi-kink structures in the stochastic (2+1)-dimensional Burgers equation remain limited. This research addresses this gap by combining rigorous mathematical techniques with detailed numerical simulations, providing new insights into noise-induced instabilities and solution morphology. The outcomes not only advance the theoretical understanding of stochastic nonlinear wave equations but also offer practical implications for applications in condensed matter physics, nonlinear optics, and spintronics, where stable soliton-like structures are critical.

Structure of the Paper: Section 2 outlines the mathematical formulation of the governing equation and employs the wave transformation technique to reformulate the stochastic (2+1)-dimensional Burgers equation for further analysis. Section 3 introduces the Rossler system and examines its role within the stochastic modeling framework for spin dynamics. In Section 4, explicit solutions to the deterministic (2+1)-dimensional Burgers model are derived using two complementary approaches: the Singular Manifold Method and the Multiple-Kink Wave Solution technique. Section 5 investigates the effects of key parameters, including noise intensity and chaotic perturbations, on soliton dynamics, supported by comprehensive numerical simulations and graphical analyses. Finally, Section 6 presents the concluding remarks and suggests prospective directions for future research in stochastic nonlinear wave dynamics.

2 Stochastic (2+1)-Burgers equation

The (2+1)-Burgers equation is a nonlinear partial differential equation (PDE) that is very important for studying fluid dynamics, especially when it comes to things like turbulence, shock wave formation, and dissipative processes. This equation builds on the classical one-dimensional Burgers equation by adding two spatial dimensions and one temporal dimension. This helps us better explain how the velocity field changes in two-dimensional viscous flows. When you add viscosity, the equation loses energy. This makes it great for modeling situations where both nonlinear convection and diffusion are happening. One of the remarkable properties of the (2+1)-dimensional Burgers equation is its ability to capture the development and interaction of shock waves as well as the transition to turbulent flow structures. Its nonlinear and dissipative nature makes it a suitable model for studying energy dissipation mechanisms in complex fluid systems. Moreover, the equation exhibits solutions that can develop steep gradients, closely resembling discontinuities, which are often interpreted as shock-like structures.

The (2+1) Burgers equation is as follows

$$M_t = M_{xx} + 2NM_x, \qquad M_x = N_y, \tag{1}$$

where $M(x,y,t)$ and $N(x,y,t)$ are the components of the velocity field, t denotes time, and subscripts represent partial derivatives. The (2+1)-Burgers equation has found applications across various branches of science and engineering. In fluid dynamics, it serves as a simplified model for understanding the fundamental mechanisms underlying turbulence and shock dynamics, offering insights that are also relevant to the study of the Navier-Stokes equations. In addition, it is widely used in applied mathematics and computational physics as a benchmark problem for testing numerical algorithms designed to solve nonlinear PDEs. Beyond fluid mechanics, the equation is applied in acoustics to model nonlinear sound wave propagation and in traffic flow theory to describe vehicle density and velocity distributions over time and space. In plasma physics, it has been employed to study nonlinear wave phenomena and energy dissipation processes in magnetized plasmas. Furthermore, due to its analytical tractability in certain cases, the (2+1)-dimensional Burgers equation is often utilized in theoretical studies involving integrability, soliton dynamics, and perturbation analysis.

$$M_t = M_{xx} + 2NM_x + \sigma M W_t, \qquad M_x = N_y, \qquad (2)$$

The parameter σ represents the noise intensity, $W(t)$ denoted the white noise and $\sigma M W_t$ indicates multiplicative noise in the Ito´ sense, ensuring a rigorous mathematical treatment of the stochastic perturbations. This formulation generalizes the deterministic Burgers equation by introducing a multiplicative noise component, where the random fluctuations scale with the amplitude of M. This stochastic formulation is widely used for studying turbulence, nonlinear wave propagation, surface growth phenomena, traffic flow modeling, and other systems where randomness plays a critical role. Its close relationship with the Kardar-Parisi-Zhang (KPZ) equation makes it a valuable tool in interface dynamics and pattern formation studies. Moreover, it serves as an important benchmark for developing numerical methods for stochastic partial differential equations (SPDEs), offering insights into the interplay between nonlinear convection, diffusion, and random external influences.

2.1 Wave transformation of the stochastic (2+1)-dimensional Burgers equation

The wave equation corresponding to the stochastic (2+1)-dimensional Burgers equation (2) is obtained by employing the following wave transformation.

$$\begin{aligned} M(x,y,t) &= m(x,y,t)\, e^{\sigma W(t) - \sigma^2 t/2}, \\ N(x,y,t) &= n(x,y,t)\, e^{\sigma W(t) - \sigma^2 t/2}, \end{aligned} \qquad (3)$$

where the functions $m(x,y,t)$ and $n(x,y,t)$ are considered deterministic, we obtain

$$\begin{aligned} M_t &= m_t + m(\sigma W_t + \sigma^2/2 - \sigma^2/2)\, e^{\sigma W(t) - \sigma^2 t/2}, \\ M_x &= m_x\, e^{\sigma W(t) - \sigma^2 t/2}, M_{xx} = m_{xx}\, e^{\sigma W(t) - \sigma^2 t/2}, \\ N_y &= n_y\, e^{\sigma W(t) - \sigma^2 t/2}, \end{aligned} \qquad (4)$$

By inserting Equation (3) into Equations (2) using equation (4), we have

$$m_t = m_{xx} + 2nm_x\, e^{\sigma W(t) - \sigma^2 t/2}, \qquad m_x = n_y, \qquad (5)$$

Given that $W(t)$ represents the Brownian white noise function, taking the expectation of both sides of equation (5) yields

$$m_t = m_{xx} + 2nm_x E(e^{\sigma W(t)}) e^{-\sigma^2 t/2}, \qquad m_x = n_y, \qquad (6)$$

As $W(t)$ is a Gaussian process $E(e^{\sigma W(t)}) = e^{\sigma^2 t/2}$, Equation (6) can therefore be expressed as

$$m_t = m_{xx} + 2nm_x, \qquad m_x = n_y, \qquad (7)$$

Which represents the deterministic (2+1)-dimensional Burgers equation. It should be noted that Equation (7) can be reformulated by applying the following transformation

$$m = n_y. \qquad (8)$$

By substituting the potential Equation (8) into Equation (7), the latter can be transformed into

$$n_{ty} = n_{xxy} + 2n_y n_{xy}. \qquad (9)$$

3 Rössler System

The Rössler system, first proposed by Otto Rössler in 1976, is a fundamental nonlinear dynamical model frequently employed in the study of chaotic phenomena. Comprising three coupled first-order ordinary differential equations, it serves as one of the canonical examples of chaotic systems and is often analyzed in parallel with the Lorenz system. The governing equations of the Rössler system are given by

$$\frac{df}{dt} = -(g+h), \quad \frac{dg}{dt} = f + ag, \quad \frac{dh}{dt} = b + h(f-c),$$

here $f(t), g(t)$ and $h(t)$ denote the state variables associated with the X, Y, and Z directions, respectively, while a, b, and c are real-valued parameters of the system. Although these variables are abstract in nature, they may be interpreted as components of a generalized nonlinear oscillator. A typical chaotic attractor of the Rössler system, obtained using the specified parameters and initial conditions, is presented in Figure 1.

$$a = 0.2,\, b = 0.2,\, c = 5.7,\, f(0) = 0.1,\, g(0) = 0.2,\, h(0) = 0.3.$$

The Rössler system demonstrates a broad spectrum of dynamical behaviors, ranging from periodic oscillations and quasi-periodicity to period-doubling bifurcations and fully developed chaos. These dynamics are commonly analyzed through tools such as Lyapunov exponents, bifurcation diagrams, and phase-space reconstructions. As with other chaotic systems, the Rössler model exhibits pronounced sensitivity to initial conditions, rendering it a pivotal framework for exploring deterministic chaos. Various control and synchronization strategies (including adaptive control and feedback linearization) have been formulated to regulate its chaotic trajectories. Numerical integration methods, such as the Runge-Kutta or Adams-Bashforth-Moulton schemes, are frequently employed to simulate the system's time evolution. Bifurcation analyses, achieved by systematically varying system parameters, reveal transitions between periodic and chaotic regimes.

In this study, simulations were conducted using MATLAB's ode45 solver to investigate the temporal behavior of the state variables $f(t), g(t)$, and $h(t)$. Results include time series plots, two-dimensional phase portraits in the $x-y, x-z$, and $y-z$ planes, as well as a three-dimensional trajectory in $x-y-z$ space. These simulations highlight characteristic features of chaotic systems, such as non-repeating oscillations and acute sensitivity to small perturbations in initial conditions (where even minute changes lead to markedly divergent trajectories). The resulting three-dimensional attractor displays the hallmark âœtwisted ribbonâ geometry associated with the Rössler system, illustrating how nonlinear coupling shapes chaotic dynamics. Owing to its structural simplicity, rich dynamical repertoire, and analytical accessibility, the Rössler system serves as a valuable model in studies of deterministic chaos, noise-driven systems, secure communication protocols, and nonlinear control design.

3.1 Stochastic Modeling of Spin Dynamics via the (2+1)-dimensional Burgers Equation

The stochastic (2+1)-dimensional Burgers equation serves as a robust analytical framework for examining nonlinear dynamical systems influenced by random fluctuations, such as those observed in the Rössler system. In these systems, spin dynamics exhibit extreme sensitivity to perturbations and environmental noise, which can trigger transitions from regular oscillations to fully chaotic regimes. Incorporating stochastic terms into the (2+1)-dimensional Burgers equation facilitates a systematic analysis of how diverse noise sources (thermal, quantum, or environmental) affect stability, coherence, and long-term evolution. This formulation is particularly relevant to the Rössler system, where irregular spin behavior may degrade energy efficiency, disrupt phase coherence, and destabilize quantum states, posing significant challenges for emerging technologies in spintronics and quantum information science [27,28, 29]. From a theoretical perspective, the stochastic (2+1)-dimensional Burgers equation provides a means to investigate critical nonlinear phenomena such as bifurcations, the emergence of chaos, and noise-induced structural transitions in wave propagation. Moreover, this approach enables the statistical characterization of system behavior over extended timescales, offering insight into the dual role of noise as both a destabilizing factor and a driver of self-organization in complex systems. When applied to the Rössler model, the integration of time-dependent magnetic interactions within this stochastic framework yields a systematic representation of chaotic spin phenomena. This comprehensive perspective not only improves the predictive accuracy of mathematical models but also enhances their applicability in controlling or exploiting chaos in advanced quantum and spin-based technologies.

4 Explicit solutions to the deterministic (2+1)-dimensional Burgers model

In this section, we derive the general solution of the deterministic (2+1)-dimensional Burgers equation using the singular manifold method and leading-order analysis. This approach yields a solution expressed in terms of three arbitrary functions, providing a broad class of exact solutions that capture the complex dynamics of the equation. In addition to the general solution, we also construct multiple-kink wave solutions by employing the exponential method in conjunction with the Cole-Hopf transformation. These solutions are of particular significance because they describe nonlinear wave interactions that can lead to steep gradients and localized structures resembling shock fronts.

The singular manifold method offers a systematic means to identify exact solutions by exploiting the integrability properties of the Burgers equation, while the Cole-Hopf transformation reduces the nonlinear partial differential equation to a linear diffusion-type equation, greatly simplifying the solution process. The multiple-kink wave solutions obtained through this methodology provide deeper insight into the equation's ability to model dissipative structures in two-dimensional flows. From a physical perspective, such solutions are valuable for understanding phenomena such as shock merging, wavefront interactions, and pattern formation in viscous fluids. Moreover, they serve as benchmark solutions for testing the accuracy of numerical schemes designed to solve nonlinear partial differential equations. Overall, the combination of the singular manifold procedure and the exponential-Cole-Hopf framework enriches the analytical understanding of the (2+1)-dimensional Burgers equation, highlighting its versatility in describing a variety of nonlinear dynamical behaviors.

4.1 Singular Manifold Techniques

Using the singular manifold method and leading-order analysis, the Painleve expansion of Equation (7) can be truncated as follows:

$$m=\phi^{-1}m_0+m_1, \quad n=\phi^{-1}n_0+n_1. \tag{10}$$

Here, ϕ denotes the fractional singular manifold, while m_1, n_1 represent arbitrary solutions of Equation (7).

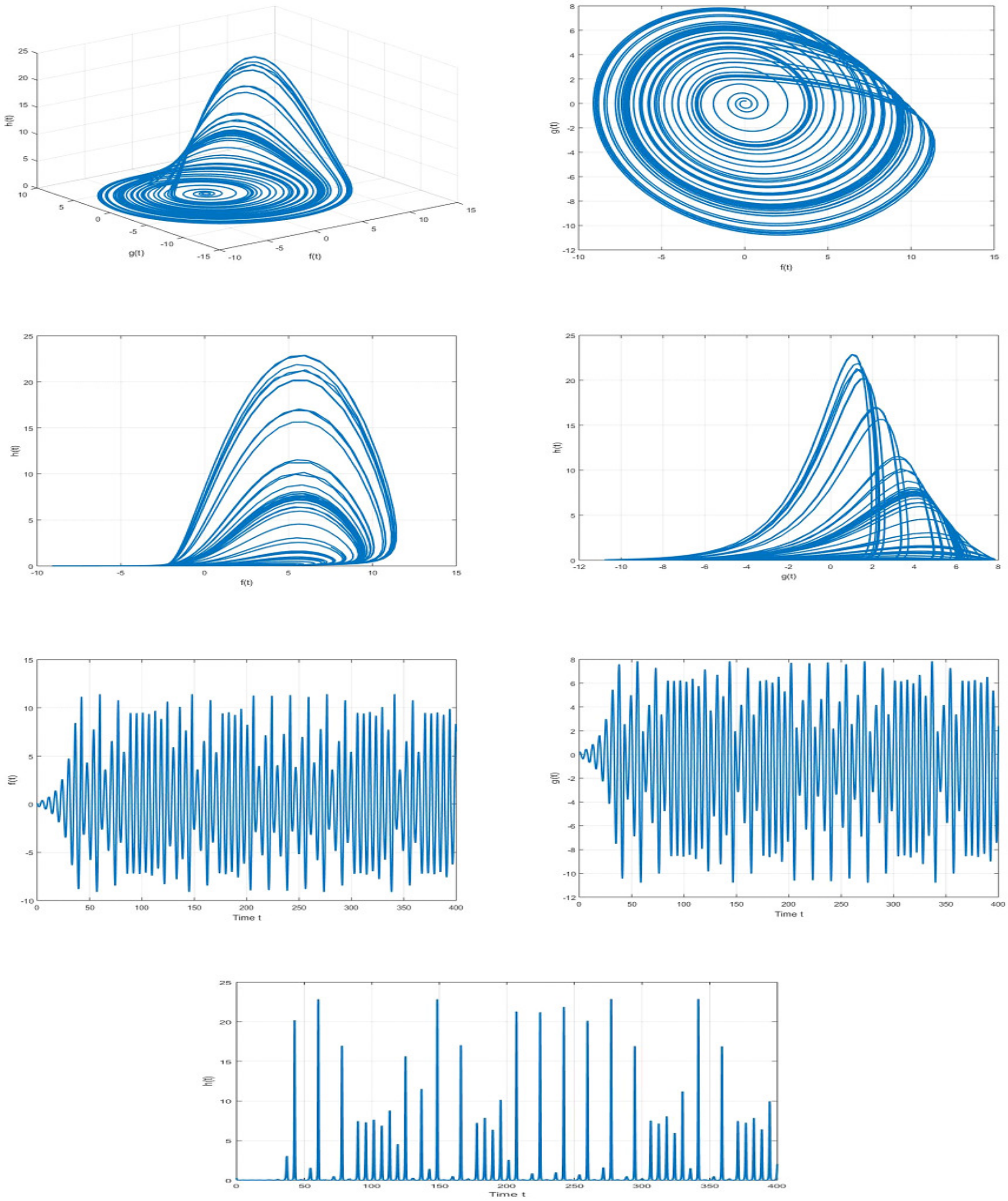

Fig. 1: Chaotic behavior of the Rössler system. Numerical simulations were performed using MATLAB's ode45 solver to examine the stochastic Rössler system. The figure illustrates the temporal evolution of the state variables $f(t), g(t)$, and $h(t)$; two-dimensional phase portraits in the $f-g, f-h$, and $g-h$ planes; and the complete three-dimensional trajectory in $f-g-h$ space. Simulations employed standard parameter values $a=0.2, b=0.2$, and $c=5.7$, with initial conditions $f_0=0.1, g_0=0.1$, and $h_0=0.1$. (a) the three-dimensional attractor in $f-g-h$ space; (b) phase portrait in the $f-g$ plane; (c) projection in the $f-h$ plane; (d) projection in the $g-h$ plane; (e) time series of $f(t)$; (f) time series of $f(t)$; and (g) time series of $h(t)$. These results demonstrate hallmark features of chaotic systems, such as aperiodic oscillations and strong sensitivity to initial conditions.

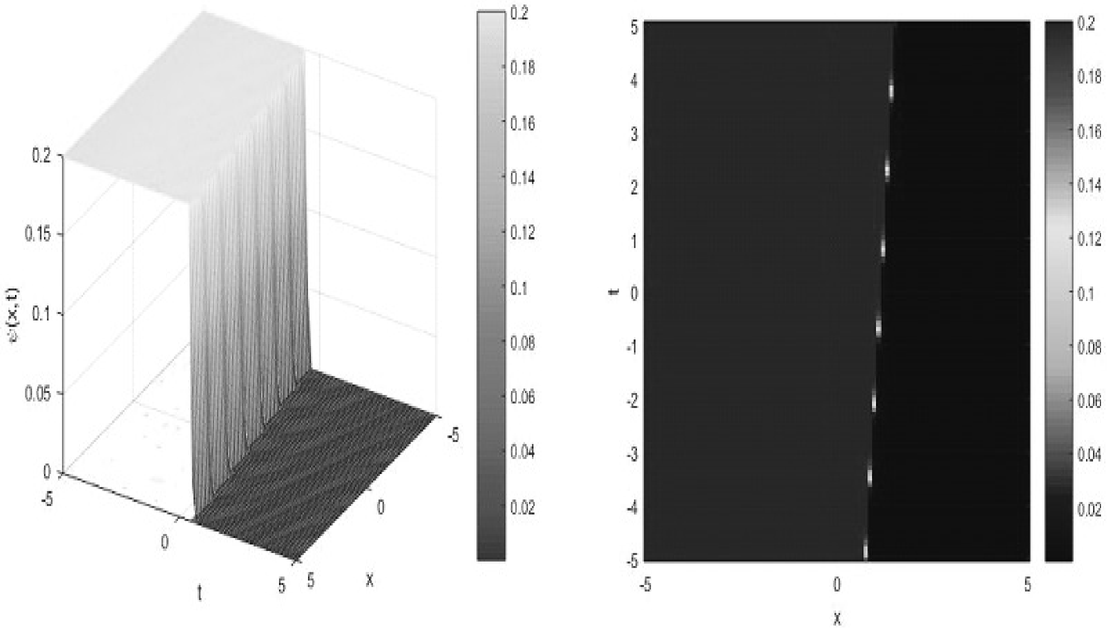

Fig. 2: illustrates the evolution of the kink-shaped soliton solution described by Equation (37). (a) provides a three-dimensional representation, whereas (b) shows the corresponding density plot. Both visualizations were obtained using the parameter values $\sigma = 0$, $k_1 = 0.3$, and $r_1 = 0.2$. These plots clearly capture the characteristic spatial structure and propagation dynamics of the kink-shaped soliton.

Substituting Equation (10) into Equation (7) and equating the coefficients of like powers of ϕ yields:

$$m_0 = \phi_y, \quad n_0 = \phi_x. \tag{11}$$

given that ϕ satisfies the relation

$$\phi_t = \phi_{xx} + 2m_1\phi_x. \tag{12}$$

Equation (12) is referred to as the singular manifold equation, while Equations (10)-(12) together define an auto-Böcklund transformation for Equation (7). If we now consider, $m_1 = \phi, n_{1y} = \phi_x$ then

$$m = \frac{\phi_x}{\phi} + \phi. \tag{13}$$

where ϕ satisfies

$$\phi_t = \phi_{xx} + 2\psi\phi_x, \qquad \psi_y = \phi_x. \tag{14}$$

Equations (13) and (14) represent an alternative form of an auto-Böcklund transformation for Equation (7). If $m_1 = 0, n_1 = 0$ is satisfied, a Cole-Hop-type transformation, also referred to as a hetero-Böcklund transformation, can be obtained.

$$m = \frac{\phi_x}{\phi}. \tag{15}$$

where ϕ satisfies

$$\phi_t = \phi_{xx}. \tag{16}$$

is applied to the deterministic (2+1)-dimensional Burgers equation, Equation (7). If we assume a particular solution of the form

$$m_1 = 0, \qquad n_1 = n_1(x,t), \tag{17}$$

where $n_1(x,t)$ is an arbitrary function of indicated variables. Then we can obtain systematically that Equation (13) has the following nonlinear separation function solution

$$\phi = F(x,t)G(y) + H(y), \tag{18}$$

where $F(x,t), G(y)$ and $H(y)$ are arbitrary functions in indicated variables, if

$$m_1 = \frac{F_t - F_{xx}}{2F_x} \tag{19}$$

Combining Equations (13), (14), (18), and (19) leads to a general functional separation solution of the (2+1)-dimensional Burgers equation (Equation (7)).

$$n = \frac{FG_y + H_y}{FG + H}, \tag{20}$$

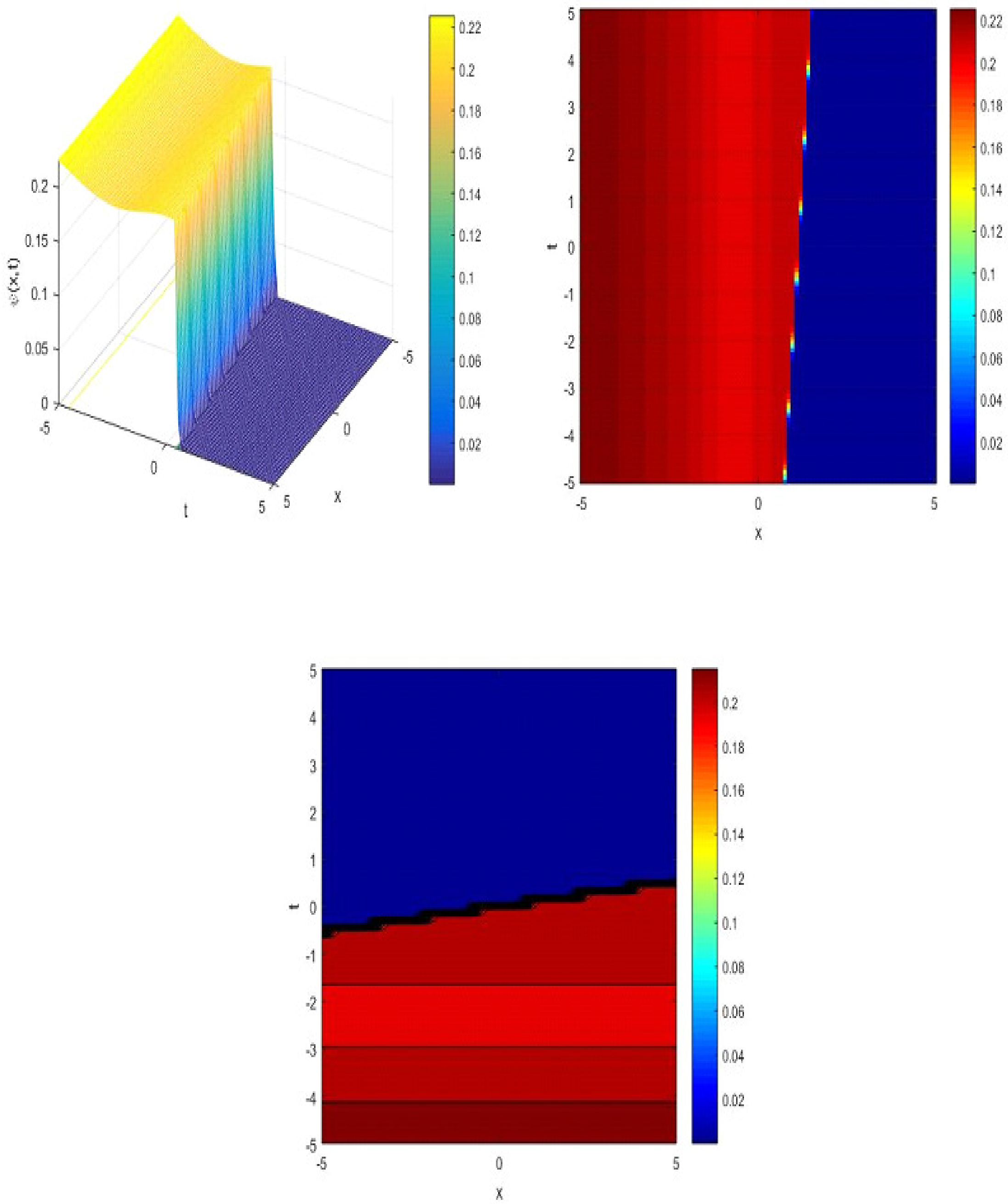

Fig. 3: presents the non-flat kink-shaped soliton solution derived from Equation (37). (a) shows the three-dimensional surface plot, (b) depicts the corresponding density plot, and (c) displays the heatmap. All visualizations were generated using the parameters $\sigma = 0.2$, $k_1 = 0.3$, and $r_1 = 0.2$. These results demonstrate the influence of a small, nonzero noise parameter (σ) on the structure and temporal evolution of the kink-shaped soliton.

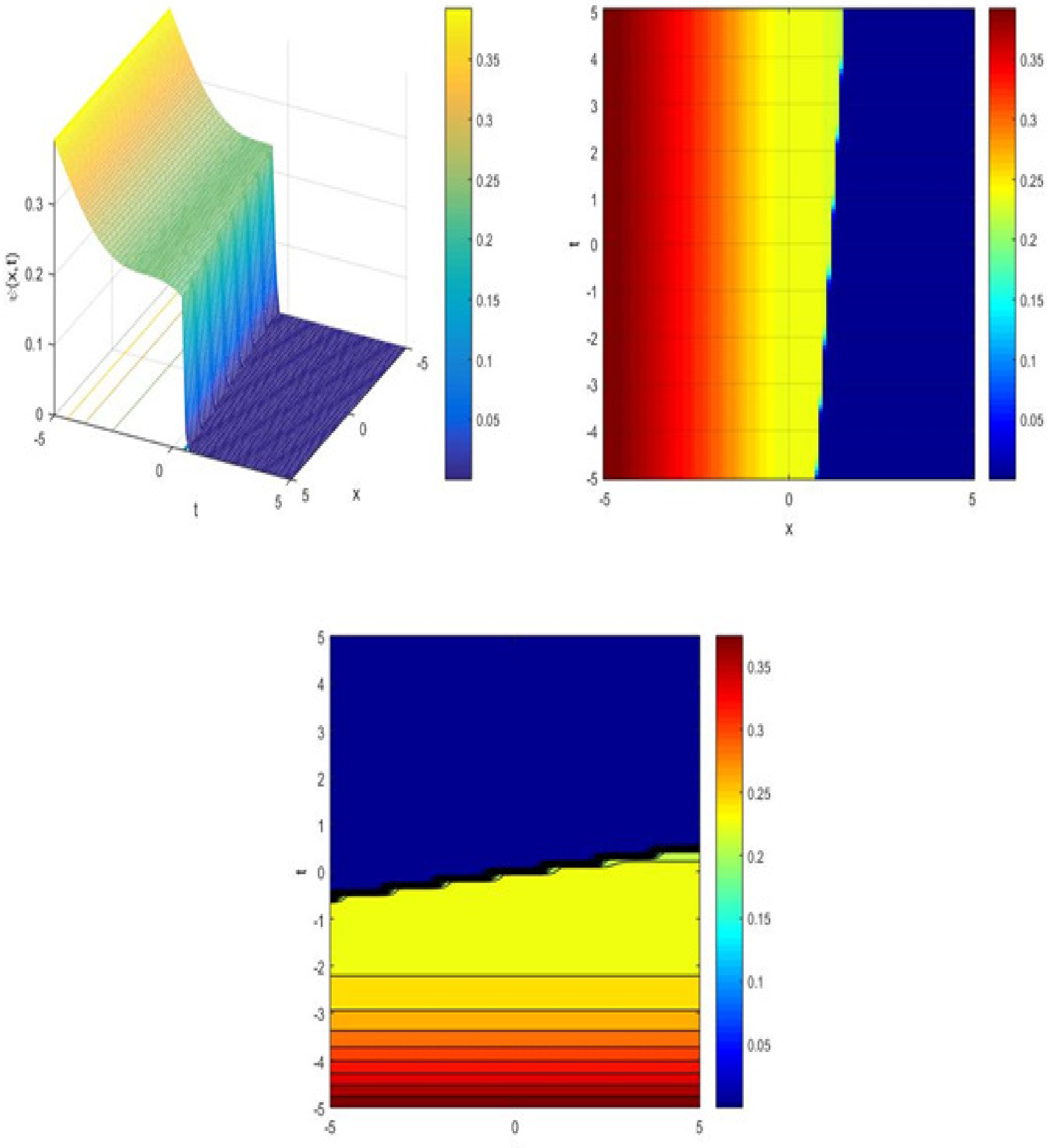

Fig. 4: presents the non-flat kink-shaped soliton solution derived from Equation (37). (a) shows the three-dimensional surface representation, (b) illustrates the corresponding density plot, and (c) depicts the heatmap. All visualizations were generated using the parameters $\sigma = 0.5$, $k_1 = 0.3$, and $r_1 = 0.2$. The results emphasize the increased structural deformation of the soliton as the noise intensity (σ) rises.

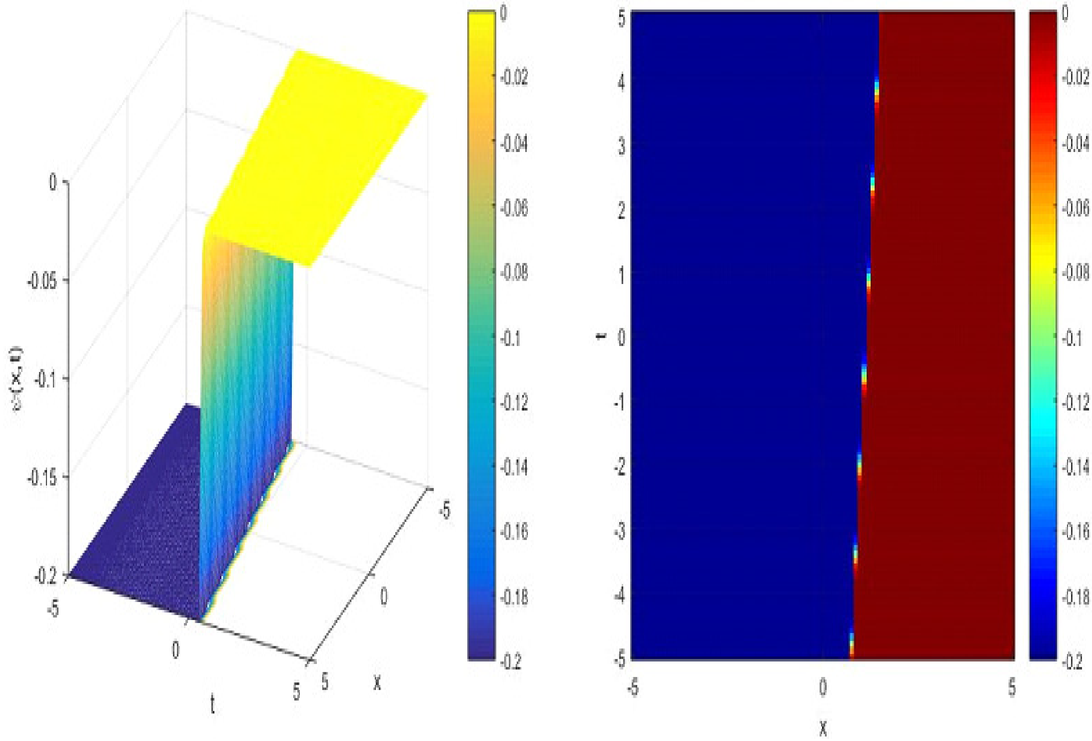

Fig. 5: illustrates the evolution of the anti–kink-shaped soliton solution described by Equation (37). (a) provides a three-dimensional representation, (b) shows the corresponding density plot. Both visualizations were generated using the parameter values $\sigma = 0$, $k_1 = 0.3$, and $r_1 = -0.2$. These plots clearly depict the characteristic spatial configuration and propagation dynamics of the anti–kink-shaped soliton.

where $F = F(x,t), G = G(y)$ and $H = H(y)$ denote arbitrary functions of the indicated variables. By employing the singular manifold method in conjunction with the separation of variables technique, a solution is obtained that depends on three independent functions of both temporal and spatial variables. This framework allows for the investigation of a wide variety of solution structures for Equation (7) of the (2+1)-dimensional Burgers equation, which incorporates fractional derivatives with respect to both space and time. Through appropriate selection of these arbitrary functions, it is possible to analyze an extensive range of solution behaviors and patterns intrinsic to the system. It is worth noting that Equations (10)-(20) correspond to Equations (5)-(14) in the work of Peng and Yamba [15].

4.2 Multiple-Kink Wave Solution

We now turn our attention to the multiple-kink wave solution. By selecting

$$n = e^{k_i x + r_i y - c_i t}, \tag{21}$$

Substituting into the linear term of Equation (9) yields the dispersion relation

$$c_i = -k_i^2, \tag{22}$$

We then obtain

$$\theta_i = k_i x + r_i y + k_i^2 t, \tag{23}$$

Applying the Coleâ“Hopf fractional transformation yields the multiple-kink wave solution of Equation (9) as

$$n = R\ln(f), \tag{24}$$

and

$$m = R\frac{f_y}{f}, \tag{25}$$

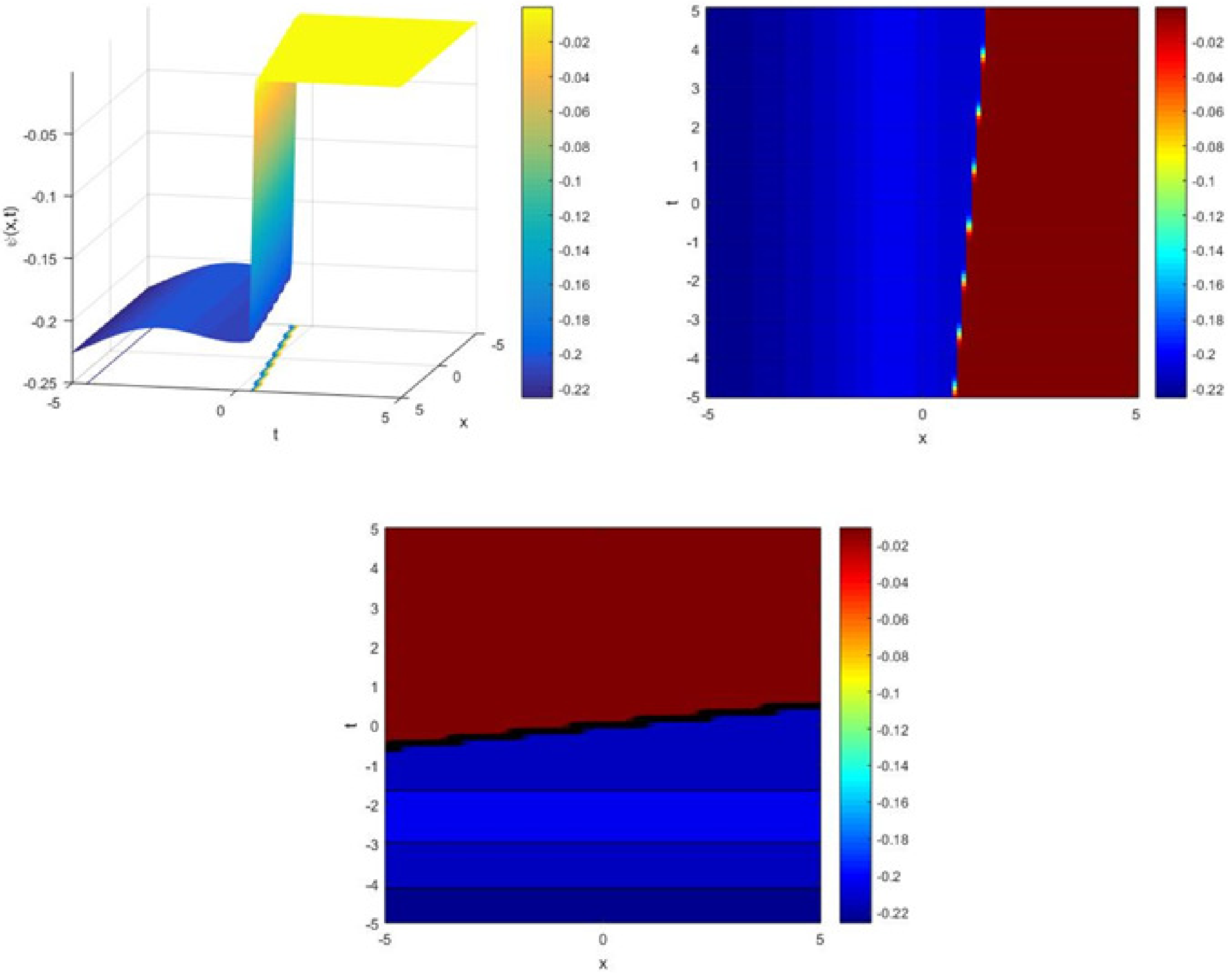

Fig. 6: presents the non-flat anti–kink-shaped soliton solution derived from Equation (37). (a) shows the three-dimensional surface plot, (b) depicts the corresponding density plot, and (c) displays the heatmap. All visualizations were generated using the parameters $\sigma = 0.2$, $k_1 = 0.3$, and $r_1 = -0.2$. These results demonstrate the impact of a small, nonzero noise parameter (σ) on the structure and temporal evolution of the anti–kink-shaped soliton.

For the case of the one-kink wave solution

$$f = 1 + e^{k_i x + r_i y + k_i^2 t}, \tag{26}$$

By substituting Equation (24) into Equation (9) and subsequently solving for R , we obtain

$$R = 1.$$

Therefore

$$n(x,y,t) = \ln(1 + e^{k_1 x + r_1 y + k_1^2 t}), \tag{27}$$

The one-kink wave solution is therefore given by

$$m(x,y,t) = \frac{r_1\, e^{k_1 x + r_1 y + k_1^2 t}}{1 + e^{k_1 x + r_1 y + k_1^2 t}}. \tag{28}$$

Accordingly, the two-kink wave solution is obtained in the form

$$f = 1 + e^{\theta_1} + + e^{\theta_2} + a_{12} + e^{\theta_1 + \theta_2}, \tag{29}$$

By substituting Equation (29) into Equation (24) and subsequently inserting the resulting expression into Equation (9), we obtain $R = 1$, with no phase shifts observed

$$a_{12} = 0,$$

therefore, we obtain

$$a_{ij} = 0, \qquad 1 \leqslant i < j \leqslant 3.$$

Accordingly, we have

$$n(x,y,t) = \ln(1 + e^{k_1 x + r_1 y + k_1^2 t} + e^{k_2 x + r_2 y + k_2^2 t}), \tag{30}$$

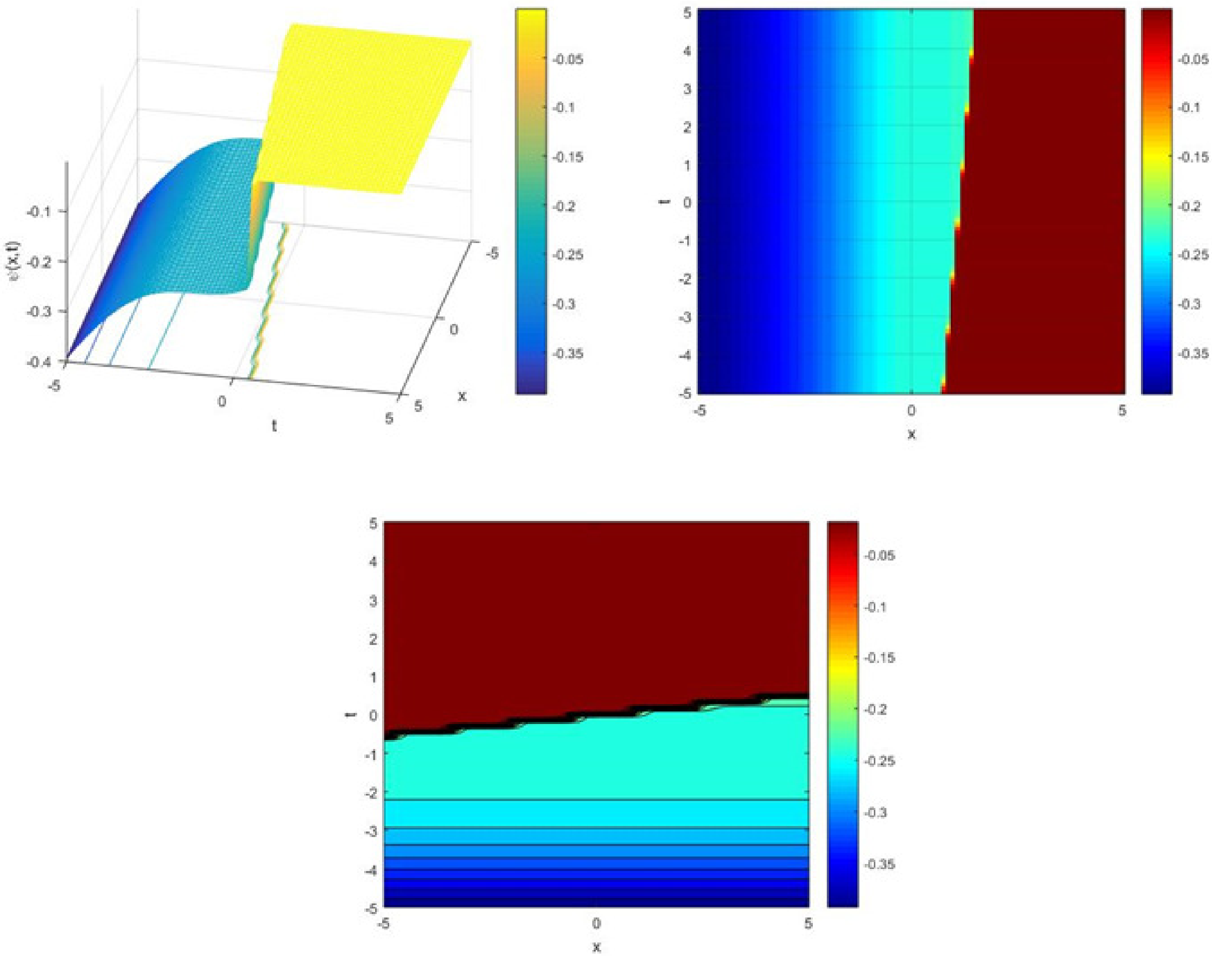

Fig. 7: presents the non-flat anti–kink-shaped soliton solution derived from Equation (37). (a) shows the three-dimensional surface representation, (b) illustrates the corresponding density plot, (c) depicts the heatmap. All visualizations were generated using the parameters $\sigma = 0.5$, $k_1 = 0.3$, and $r_1 = -0.2$. The results highlight the increased structural deformation of the anti–kink soliton as the noise intensity (σ) increases.

Accordingly, the two-kink wave solution takes the form

$$m(x,y,t) = \frac{r_1\, e^{k_1 x + r_1 y + k_1^2 t} + r_2\, e^{k_2 x + r_2 y + k_2^2 t}}{1 + e^{k_1 x + r_1 y + k_1^2 t} + e^{k_2 x + r_2 y + k_2^2 t}}. \tag{31}$$

To discuss the three-kink soliton solution, and following the same procedure as before, we obtain

$$n(x,y,t) = \ln(1 + e^{k_1 x + r_1 y + k_1^2 t} + e^{k_2 x + r_2 y + k_2^2 t} + e^{k_3 x + r_3 y + k_3^2 t}), \tag{32}$$

This yields the three-kink wave solution

$$m(x,y,t) = \frac{r_1\, e^{k_1 x + r_1 y + k_1^2 t} + r_2\, e^{k_2 x + r_2 y + k_2^2 t} + r_3 e^{k_3 x + r_3 y + k_3^2 t}}{1 + e^{k_1 x + r_1 y + k_1^2 t} + e^{k_2 x + r_2 y + k_2^2 t} + e^{k_3 x + r_3 y + k_3^2 t}}. \tag{33}$$

Proceeding in the same way as before, we have

$$n(x,y,t) = \ln(1 + e^{k_1 x + r_1 y + k_1^2 t} + e^{k_2 x + r_2 y + k_2^2 t} + e^{k_3 x + r_3 y + k_3^2 t} + e^{k_4 x + r_4 y + k_4^2 t}), \tag{34}$$

This yields the four-kink wave solution

$$m(x,y,t) = \frac{1}{\exp(n(x,y,t))}(r_1\, e^{k_1 x + r_1 y + k_1^2 t} + r_2\, e^{k_2 x + r_2 y + k_2^2 t} + r_3 e^{k_3 x + r_3 y + k_3^2 t} + r_4 e^{k_4 x + r_4 y + k_4^2 t}) \tag{35}$$

This is significant because the (2+1)-dimensional Burgers equation (9) admits N-kink solutions for any finite $N \geq 1$. These solutions are particularly important as they describe the interaction of multiple nonlinear wave fronts, which can coalesce, split, or evolve into complex patterns

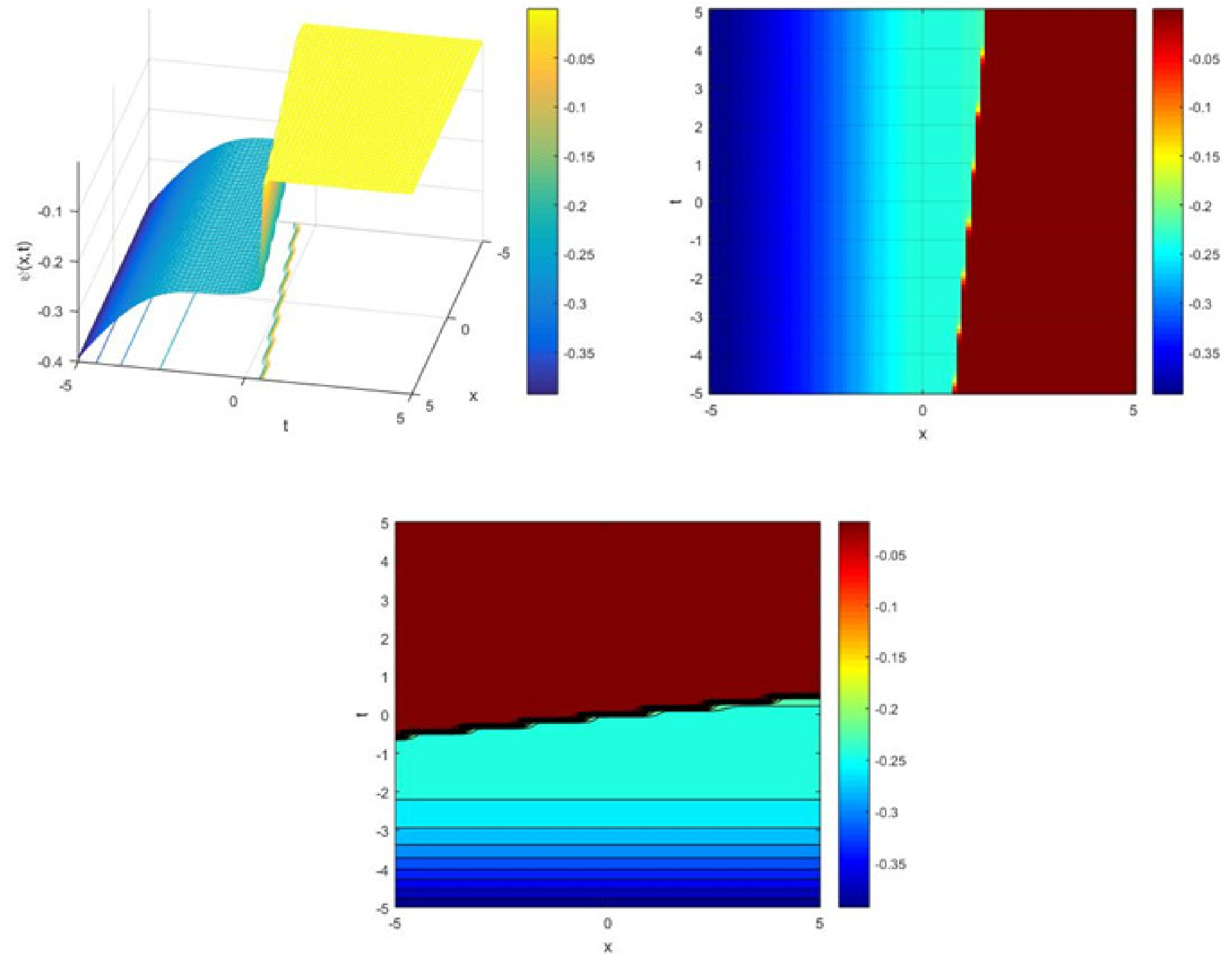

Fig. 8: illustrates the evolution of the two–kink-shaped soliton solution described by Equation (38). Subfigure (a) provides a three-dimensional representation, while subfigure (b) shows the corresponding density plot. Both visualizations were produced using the parameter values $\sigma = 0$, $k_1 = 2$, $k_2 = 2$, $r_1 = 2$ and $r_2 = 5$. These plots clearly depict the characteristic spatial structure and propagation dynamics of the two–kink-shaped soliton.

over time. Each kink represents a localized steep gradient or shock-like structure, and the ability of the system to support multiple kinks highlights its capacity to model intricate dynamical behaviors such as shock merging, multi-wave collisions, and pattern formation in dissipative media. In practical terms, N-kink solutions are valuable for understanding physical processes where multiple discontinuities or steep transitions occur simultaneously, such as in turbulent fluid flows, traffic dynamics, acoustic wave propagation, and interface growth phenomena. The superposition and interaction properties of these solutions provide insights into the mechanisms driving energy dissipation and the emergence of ordered structures in nonlinear systems. Furthermore, from a mathematical perspective, the existence of exact N-kink solutions demonstrates the richness of the solution space of the Burgers equation. These solutions can serve as benchmark cases for validating numerical algorithms and can be used as initial conditions to study stability and long-term dynamics. Based on the results derived, the general form of the kink solutions can be expressed as:

$$\begin{aligned} n(x,y,t) &= \ln\left(1+\sum_{i=1}^{N} e^{k_i x + r_i y + k_i^2 t}\right) \\ m(x,y,t) &= \frac{\sum_{i=1}^{N} e^{k_i x + r_i y + k_i^2 t}}{1+\sum_{i=1}^{N} e^{k_i x + r_i y + k_i^2 t}}. \end{aligned} \tag{36}$$

5 Simulation Results of Wave Pattern Behavior

This section presents and analyzes the simulation results of the stochastic (2+1)-dimensional Burgers equation,

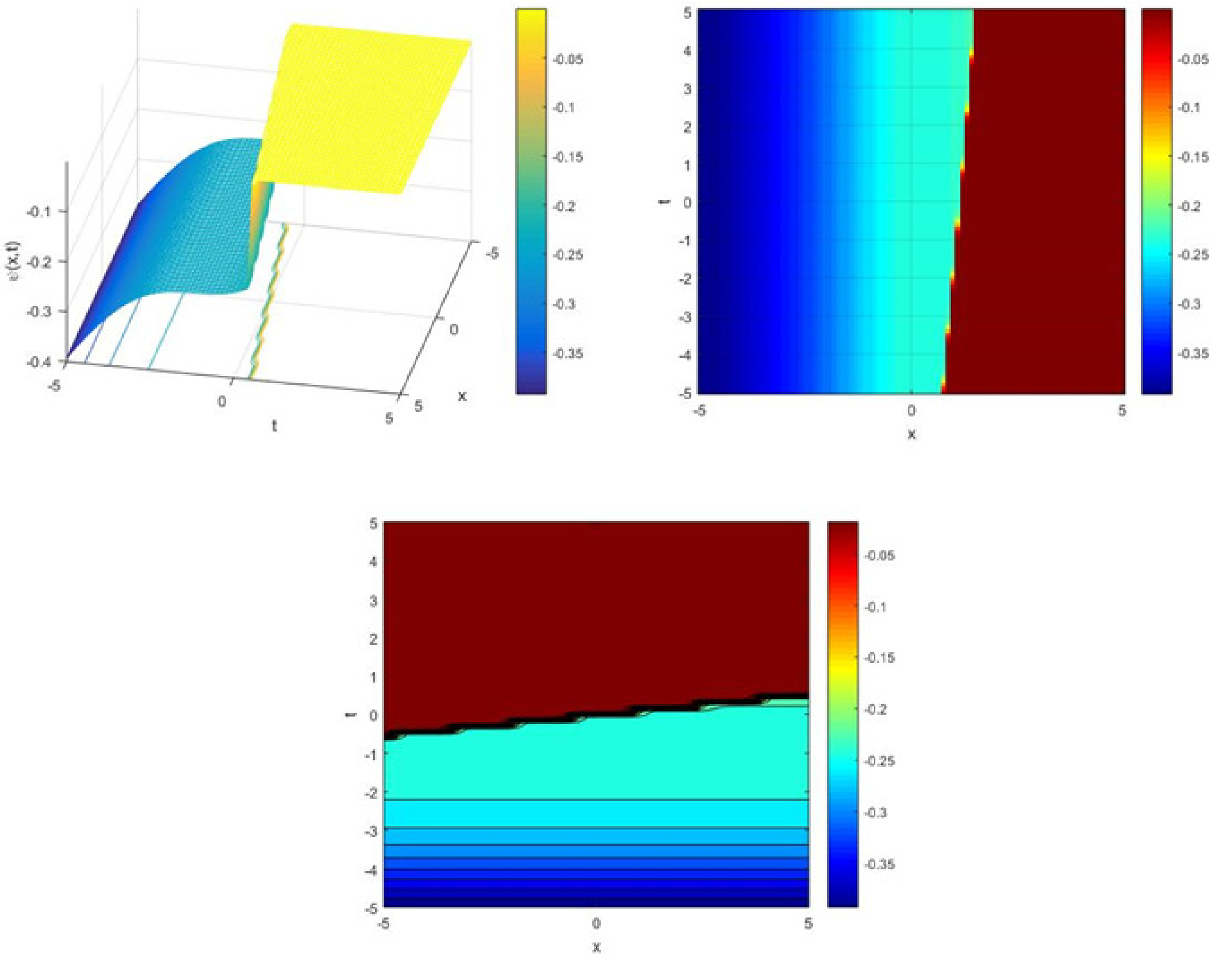

Fig. 9: presents the non-flat two–kink-shaped soliton solution derived from Equation (38). (a) shows the three-dimensional surface plot, (b) depicts the corresponding density plot, and subfigure (c) displays the heatmap. All visualizations were produced using the parameter values $\sigma = 0.2$, $k_1 = 2$, $k_2 = 2$, $r_1 = 2$ and $r_2 = 5$. These results illustrate the influence of a small, nonzero noise parameter (σ) on the structure and temporal evolution of the two–kink-shaped soliton.

validating the theoretical N-kink wave solutions derived in the preceding sections and illustrating the diverse dynamical behaviors exhibited by the system. By systematically varying initial conditions, parameters, and the number of interacting kinks, the simulations reveal distinct wave front structures, collisions, and complex spatiotemporal patterns. High-accuracy numerical integration schemes were used to ensure reliability, and the results are visualized through time-series plots, phase-space trajectories, and spatial-temporal contour maps. These visualizations highlight key features such as the persistence of localized wave fronts, shock-like structures, and intricate interference patterns, while demonstrating how variations in viscosity, noise intensity, and initial wave amplitude influence propagation speed, amplitude modulation, and eventual dissipation. The analysis also explores the impact of additive white noise on solitary and kink-type solutions, showing that low noise levels induce only minor perturbations in amplitude, phase, and propagation velocity, whereas higher noise intensities result in significant deformations and eventual destabilization of solitons.

Kink solitons, which are topological excitations characterized by step-like profiles, play a crucial role in nonlinear physical systems as they represent transitions between distinct asymptotic states. These structures are prevalent in condensed matter systems (e.g., magnetic domain walls, crystal dislocations, and Josephson junctions), spintronics (as models for magnetization reversal and spin-transfer torque dynamics), and nonlinear optics and field theory (to describe phase transitions and localized excitations). The present

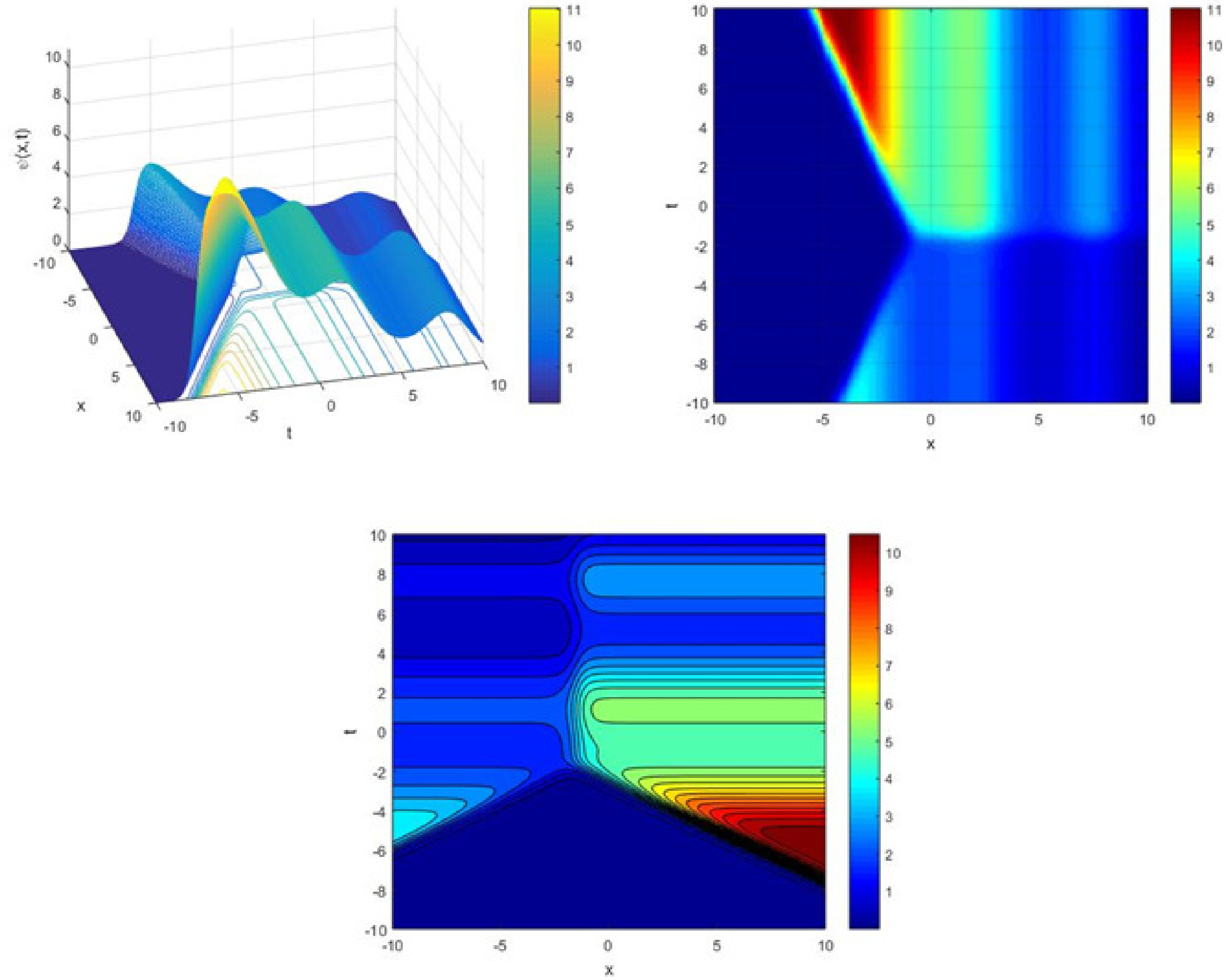

Fig. 10: presents the non-flat two–kink-shaped soliton solution derived from Equation (38). (a) shows the three-dimensional surface representation, (b) illustrates the corresponding density plot, (c) depicts the heatmap. All visualizations were produced using the parameter values $\sigma = 0.5$, $k_1 = -2$, $k_2 = 2$, $r_1 = 2$ and $r_2 = 5$. The results highlight the enhanced structural deformation of the two–kink soliton as the noise intensity (σ) increases.

simulations highlight the sensitivity of kink solitons in the stochastic (2+1)-dimensional Burgers equation to random perturbations, which can impair their structural stability and practical applicability. Although low-intensity noise allows solitons to maintain coherence, strong stochastic forcing cause's significant amplitude and phase disturbances, and beyond a threshold, complete destabilization may occur. The resilience of solitons is further influenced by key parameters such as ki and ri ; appropriate tuning of these values can enhance robustness. These results deepen the understanding of noise nonlinearity interactions and provide critical insights for the development of noise-tolerant soliton-based technologies in applied physics and engineering. Future research should extend this work to include colored or multiplicative noise models and broader parameter-space explorations to better characterize noise-driven transitions and soliton persistence.

Based on the results derived in equation (28) and applying the relationship established in equation (3) the corresponding expressions for

$$m(x,y,t) = \frac{r_1\, e^{k_1 x + r_1 y + k_1^2 t}}{1 + e^{k_1 x + r_1 y + k_1^2 t}} e^{\sigma W(t) - \sigma^2 t/2}. \qquad (37)$$

The influence of multiplicative noise on the one-kink soliton solution of the stochastic (2+1)-dimensional

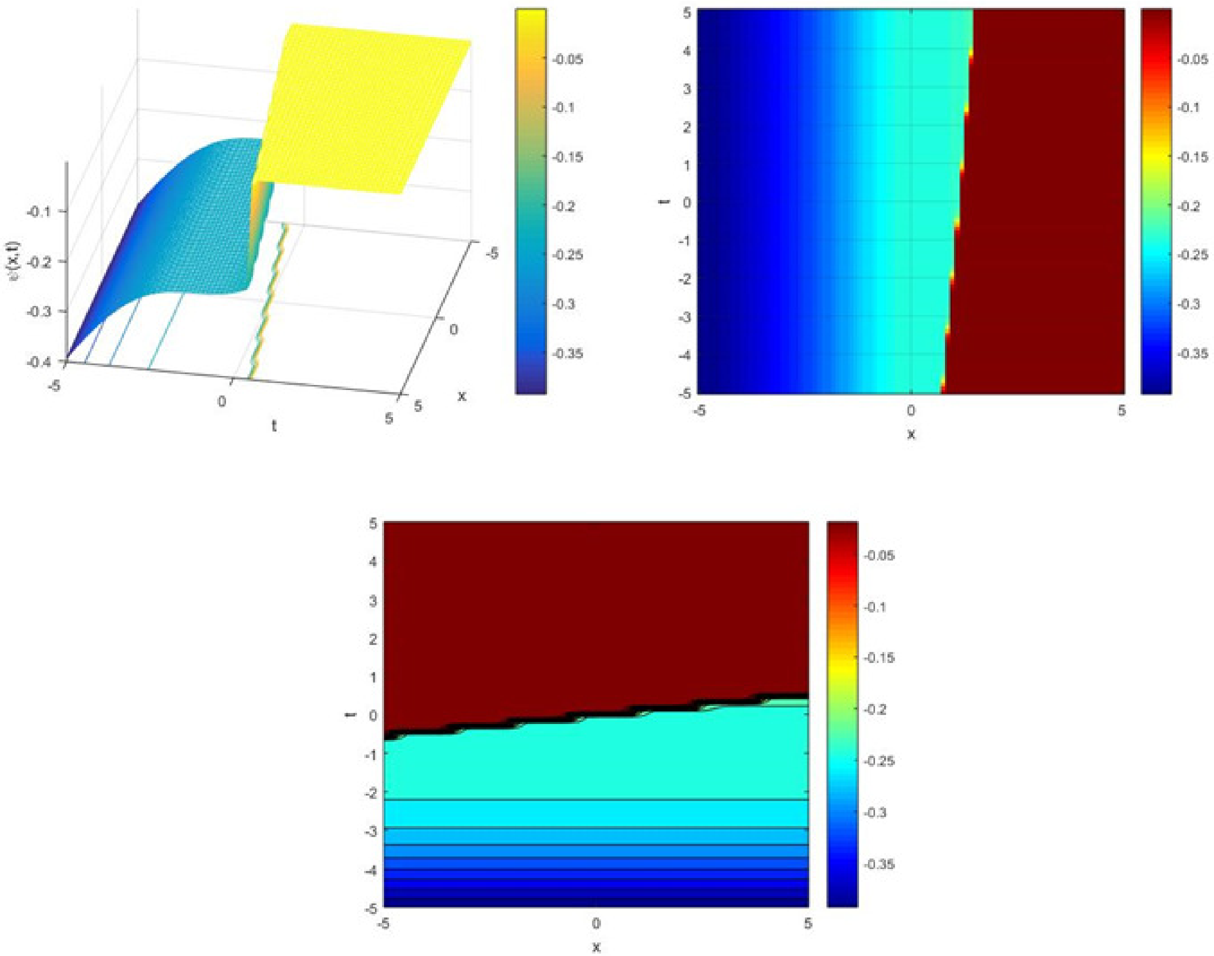

Fig. 11: illustrates the evolution of the anti-two–kink-shaped soliton solution described by Equation (38). Subfigure (a) provides a three-dimensional representation, while subfigure (b) shows the corresponding density plot. Both visualizations were generated using the parameter values $\sigma = 0$, $k_1 = -2$, $k_2 = 2$, $r_1 = -2$ and $r_2 = -5$. These plots clearly depict the characteristic spatial configuration and propagation dynamics of the two–kink-shaped soliton.

Burgers equation was examined through extensive numerical simulations. Solution profiles were analyzed under varying noise intensities to evaluate soliton robustness. Figure 2 shows the deterministic one-kink soliton profiles derived from Equation (37) in the absence of noise ($\sigma = 0$), which serve as the baseline for stability assessment. Figures 3 and 4 illustrate the system's behavior under low ($\sigma = 0.2$) and intermediate ($\sigma = 0.5$) noise levels, respectively. Under weak stochastic perturbations, the soliton largely maintains its structural integrity, exhibiting only minor amplitude and phase fluctuations. As the noise intensity increases, however, more substantial distortions and nonlinear interactions become evident, highlighting the solution's heightened sensitivity to external fluctuations.

Figures 5, 6, and 7 present the mirrored soliton structures obtained by applying the negative sign of r_1, demonstrating the same noise-dependent behavior. All simulations were performed using fixed parameter values of $k_1 = 0.3$ and $r_1 = 0.2$. The results show that weak noise introduces small perturbations without disrupting coherence, while stronger noise significantly affects soliton morphology and, in some cases, leads to partial loss of coherence. This behavior underscores the dual role of multiplicative noise in nonlinear systems: low-level fluctuations can be tolerated, but higher noise intensities disrupt the critical balance between nonlinearity and dispersion required for soliton stability. These findings have practical implications for systems dependent on kink-like structures, such as domain walls in condensed matter physics, spin-transfer torque devices, and signal

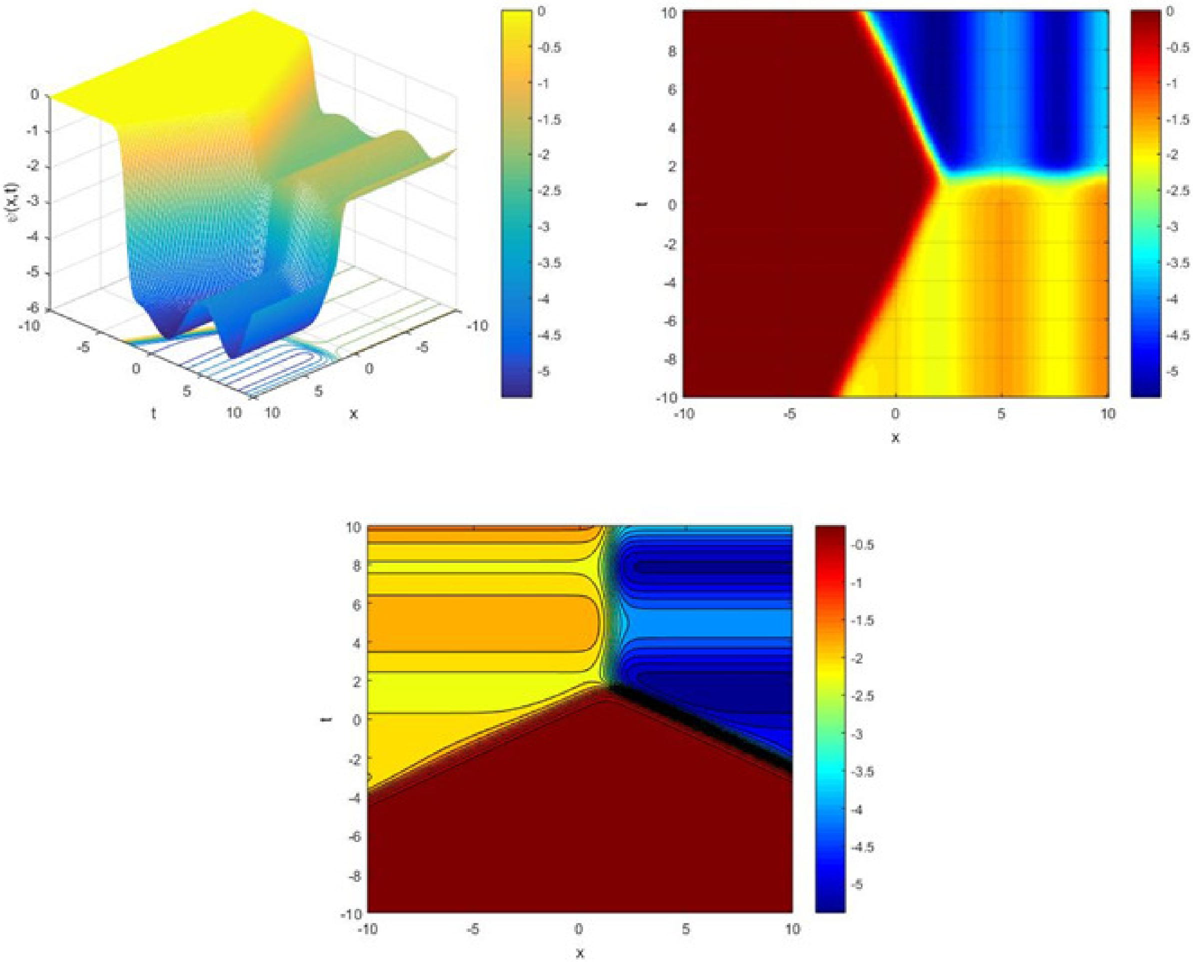

Fig. 12: presents the non-flat anti–two–kink-shaped soliton solution derived from Equation (38). (a) shows the three-dimensional surface plot, (b) depicts the corresponding density plot, (c) displays the heatmap. All visualizations were generated using the parameter values $\sigma = 0.2$, $k_1 = -2$, $k_2 = 2$, $r_1 = -2$ and $r_2 = -5$. These results illustrate the impact of a small, nonzero noise parameter (σ) on the structure and temporal evolution of the anti–two–kink-shaped soliton.

transmission in noisy environments. They also emphasize the importance of parameter optimization (including the tuning of k_1, r_1, and noise-related factors) to improve soliton robustness.

Based on the results obtained from Equation (31) and utilizing the relationship defined in Equation (3), the corresponding expressions for M can be written as

$$m(x,y,t) = \frac{r_1\, e^{k_1 x + r_1 y + k_1^2 t} + r_2\, e^{k_2 x + r_2 y + k_2^2 t}}{1 + e^{k_1 x + r_1 y + k_1^2 t} + e^{k_2 x + r_2 y + k_2^2 t}} e^{\sigma W(t) - \sigma^2 t/2}. \tag{38}$$

The effect of multiplicative noise on the two-kink soliton solution of the stochastic (2+1)-dimensional Burgers equation was investigated through comprehensive numerical simulations. Solution profiles were evaluated under varying noise intensities to assess soliton robustness. Figure 8 depicts the deterministic two-kink soliton profiles derived from Equation (38) under noise-free conditions ($\sigma = 0$), which serve as the baseline for stability analysis. Figures 8, 9 and 10 present the system's response to low ($\sigma = 0.2$) and intermediate ($\sigma = 0.5$) noise levels, respectively. Under weak stochastic perturbations, the two-kink soliton largely retains its structural integrity, exhibiting only slight amplitude and phase modulations. As the noise intensity increases, however, more significant distortions and nonlinear interactions appear, reflecting the solution increased sensitivity to external fluctuations

Figures 11, 12, and 13 display the mirrored soliton structures obtained by applying the negative signs of r_1

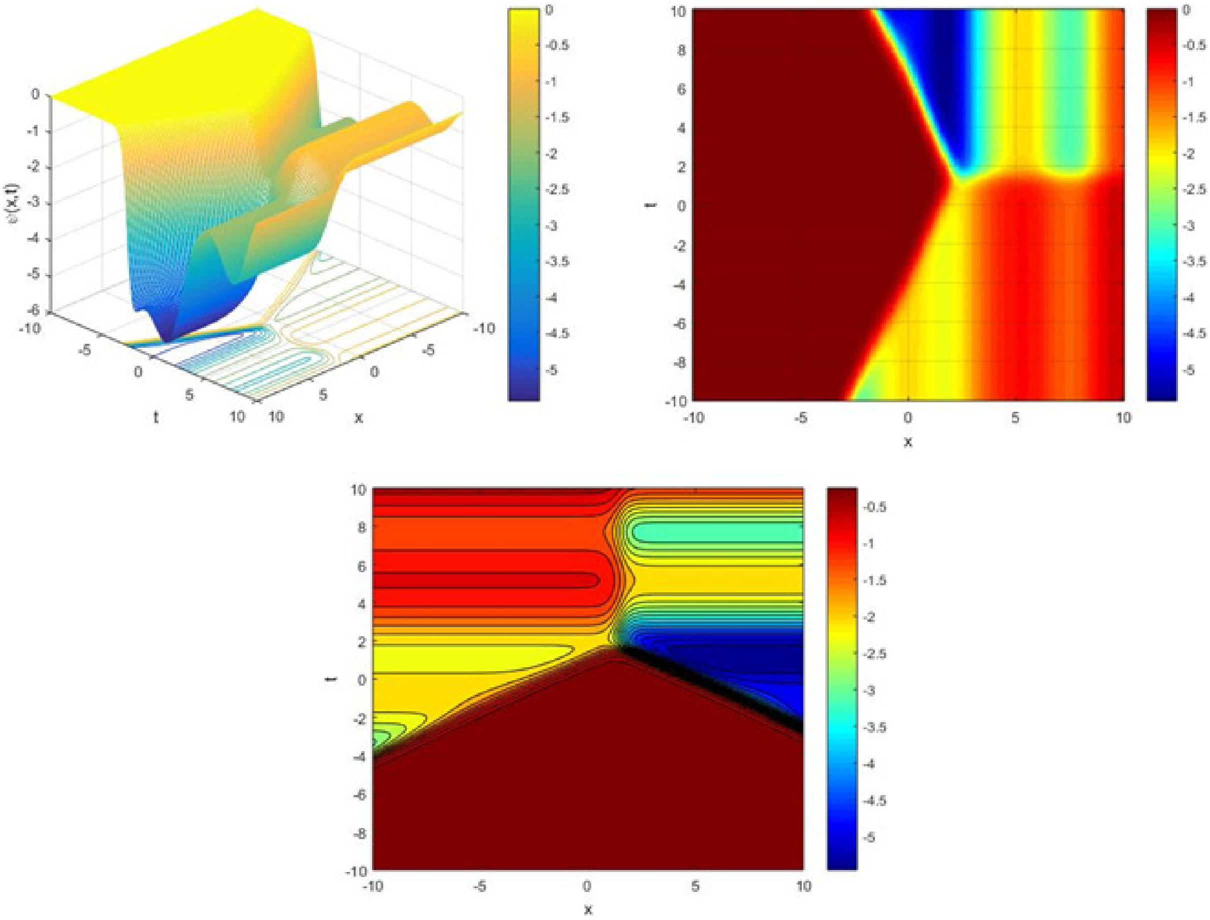

Fig. 13: presents the non-flat anti–two–kink-shaped soliton solution derived from Equation (38). (a) shows the three-dimensional surface representation, (b) illustrates the corresponding density plot, (c) depicts the heatmap. All visualizations were generated using the parameter values $\sigma = 0.5$, $k_1 = -2$, $k_2 = 2$, $r_1 = -2$ and $r_2 = -5$. The results highlight the increased structural deformation of the anti–two–kink soliton as the noise intensity (σ) increases.

and r_2, showing the same noise-dependent behavior. All simulatio

ns were conducted using the fixed parameter values $k_1 = -2, k_2 = 2, r_1 = 2$ and $r_2 = 5$. The results demonstrate that weak noise introduces minor perturbations without compromising coherence, whereas stronger noise significantly alters the soliton morphology and may, in some cases, lead to partial loss of coherence. This highlights the dual role of multiplicative noise in nonlinear systems: low-level fluctuations are tolerable, but higher noise intensities disrupt the delicate balance between nonlinearity and dispersion necessary for soliton stability. These findings have important implications for systems reliant on two-kink structures, such as domain walls in condensed matter physics, spin-transfer torque devices, and signal transmission in noisy environments. Furthermore, the results underscore the importance of parameter optimization (including the tuning of k_1, k_2, r_1, r_2 , and noise-related factors) to enhance soliton robustness.

6 Conclusion and Future Work

In this study, the stochastic (2+1)-dimensional Burgers equation was thoroughly analyzed to investigate the interplay between nonlinearity, dispersion, and stochastic perturbations. Explicit solutions for the deterministic model were derived using the Singular Manifold Method and Multiple-Kink Wave Solution techniques, providing a versatile framework for describing complex nonlinear behaviors such as kink and multi-kink solitons. Numerical simulations confirmed the accuracy of the analytical results and revealed how variations in viscosity, noise intensity, and initial wave configurations influence the stability and evolution of localized structures.

The simulation results demonstrated that kink solitons retain structural integrity under weak stochastic perturbations but exhibit significant amplitude distortions and phase fluctuations as the noise intensity increases. This sensitivity underscores the dual role of multiplicative noise, which can act as a minor perturbation at low levels but induces severe disruptions at higher intensities. These findings have direct implications for physical systems that rely on the persistence of kink-like structures, including domain wall dynamics in condensed matter physics, spin-transfer torque devices, and energy transport in noisy environments. The results also emphasize the importance of parameter tuning for enhancing soliton robustness, particularly with respect to system parameters k_i, r_i, and noise characteristics.

Future research could extend the current work in several directions. First, incorporating alternative stochastic processes such as colored noise or LÃ©vy noise would provide deeper insights into how different noise structures influence soliton persistence and energy redistribution. Second, exploring higher-order extensions and coupled systems could further enrich the understanding of multi-scale interactions in stochastic nonlinear wave dynamics. Third, conducting a broader parameter-space analysis would help identify critical thresholds where solitons undergo qualitative changes or destabilization. Finally, the integration of advanced control strategies could be investigated to mitigate noise-induced instabilities, thereby enhancing the applicability of the stochastic Burgers equation in modeling and engineering real-world systems.

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