Sohag Journal of Mathematics
*An International Journal*

http://dx.doi.org/10.18576/sjm/120301

# Complete Exterior and Interior Solution of Einstein's Field Equation using the Riemannian Golden Metric Tensor for Homogenous Spherical Massive Bodies with a Field that has a Variation in Radial Length and Time

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Received: 27 Apr. 2025, Revised: 24 Jun. 2025, Accepted: 28 Jul. 2025
Published online: 1 Sep. 2025

**Abstract:** In the theory of general relativity, the Einstein's field equation is a tensor field equation which described the geometry of space-time to the distribution of matter within it. These equations were constructed in a form of tensorial field which relate the local space-time curvature with the energy, momentum and stress around that space. In this research work, the Riemannian geometry of space-time was applied to obtain fourteen component of affine connection coefficients, Riemann Christofell tensor, Ricci tensor and exterior Einstein's field equation for spherical field and find solution in form of power series. The result obtained yields a function $f(t,r)$ to the order $c^0$ reduces to Laplacian equation , implying that it agrees with the well-known equivalence principle in Physics and to order of $c^{-2}$ it contains post Newtonian additional correctional terms which are not found in Newton's dynamical theory or Einstein's geometrical theory of gravitation. The consequence of the additional correctional terms is that it can be applied in the detection of the existence of gravitational waves. The interior solution was obtained for a static homogeneous spherical bodies whose tensor field varies with time and radial distance was constructed and solved. It was observed that within the interior field, the solution obtained converges to Newton dynamical scalar potential which is thus an extremely discovery with the reliance on two arbitrary function. The result obtained in the limit of weak field is equal to Laplacian equation which does not differ significantly from Newton dynamical theory of gravitation. The solution further confirms the assumption that Newton dynamical theory of gravitation is a limiting case of Einstein's geometrical gravitational theory of gravitation.

**Keywords:** Riemannian golden metric tensor, Einstein's theory, Newton dynamical scalar potential Einstein's field equation.

## 1 Introduction

Einstein's theory of gravitation formulated in 1915/1916 unifies Special Relativity and Sir Isaac Newton's law of universal gravitation, which describe gravitation as a dynamics of metric field of space-time and unifying structure of geometry of the space-time with gravitation. The curvature is being produced by mass-energy and momentum content of the space-time. General Relativity is the most widely accepted theory of gravitation [1]-[3]. After Einstein's publication of geometrical gravitational field equation in 1915, the search for their exact and analytical solution began for all the gravitational field in nature [2]-[8]. The first approach to construction of exact analytical solution of the Einstein's geometrical gravitational field equations was to find a mapping under which the tensor assumed a simple form, such as the vanishing of the off-diagonal elements. This method led to the first analytical solution, the famous Schwarzschild solution. The second method was to assume that the metric tensor assumed certain symmetries assumed forms of the associated killing vector. The assumption of axially symmetric metric tensor led to the solution found by Weyl and Levi-Cevita. The third method of approach was to require that the metric tensor leads to a particular type of classification of Weyl and Riemann-Christoffell

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tensors. This led to plane fronted wave solution. The fourth method was to seek Taylor series expansion of some initial-value hypersurface, subject to consistent initial value. This method has not proved successful in generating the solution [2].
In this article, we show how exact analytical solution to exterior and interior field equations can be constructed in the limit of $c^{-2}$ and $c^0$ in a gravitational field for spherical massive bodies using the Riemannian golden metric tensor for a field that has a variation in radial length and Time.

## 2 Formulation of Exterior Solution to Einstein's Geometrical Gravitational Field Equation

To formulate the exterior field equation, we consider the astrophysical body in spherical geometry in which the tensor field varies with time and radial distance. The covariant metric tensors for this distribution of mass or pressure is given by [7]

$$g_{00} = -\left[1 + \frac{2}{c^2} f(t,r)\right] \tag{1}$$

$$g_{11} = \left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \tag{2}$$

$$g_{22} = r^2 \left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \tag{3}$$

$$g_{33} = r^2 sin^2\theta \left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \tag{4}$$

$$g_{\mu\nu} = 0, \quad Otherwise, \tag{5}$$

where, $f(t,r)$ is a gravitational scalar potential, determined by the mass or pressure, and possesses the symmetries of the latter. In approximate gravitational field, it is equal to Newton's gravitational scalar potential exterior to the spherical mass distribution.

$$g^{00} = -\left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \tag{6}$$

$$g^{11} = \left[1 + \frac{2}{c^2} f(t,r)\right] \tag{7}$$

$$g^{22} = \frac{1}{r^2} \left[1 + \frac{2}{c^2} f(t,r)\right] \tag{8}$$

$$g^{33} = \frac{1}{r^2 sin^2\theta} \left[1 + \frac{2}{c^2} f(t,r)\right] \tag{9}$$

$$g^{\mu\nu} = 0, \quad Otherwise \tag{10}$$

The affine connection coefficient defined by the metric tensors of space and time are determined [1-9] using the tensor equation.

$$\Gamma^{\mu}_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right) \tag{11}$$

And they are found to be given explicitly as follow

$$\Gamma^0_{00} = \frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial t} \tag{12}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial r} \tag{13}$$

$$\Gamma^0_{11} = -\frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-3} \frac{\partial f}{\partial t} \tag{14}$$

$$\Gamma^0_{22} = -\frac{r^2}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-3} \frac{\partial f}{\partial t} \tag{15}$$

$$\Gamma^0_{33} = -\frac{r^2 sin^2\theta}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-3} \frac{\partial f}{\partial t} \tag{16}$$

$$\Gamma^1_{00} = \frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right) \frac{\partial f}{\partial r} \tag{17}$$

$$\Gamma^1_{01} = \Gamma^1_{10} = -\frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial t} \tag{18}$$

$$\Gamma^1_{11} = -\frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial r} \tag{19}$$

$$\Gamma^1_{22} = -r + \frac{r^2}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial r} \tag{20}$$

$$\Gamma^2_{02} = \Gamma^2_{20} = -\frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial t} \tag{21}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} - \frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial r} \tag{22}$$

$$\Gamma^3_{03} = \Gamma^3_{30} = -\frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial t} \tag{23}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} - \frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial r} \tag{24}$$

$$\Gamma^{\mu}_{\alpha\beta} = 0; \quad Otherwise \tag{25}$$

The exterior field equation in this field is given as

$$G_{00} = R_{00} - \tfrac{1}{2} R g_{00} = 0 \tag{26}$$

The choice of this component is because it is observed that all the solutions to the field equation towards the exterior converge in the same way. The expressions for the Ricci tensor $R_{00}$ and the Riemann curvature $R$ in this field are given, respectively, as:

$$R_{00}=\frac{12}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial t}\right)^2-\frac{3}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\frac{\partial^2 f(t,r)}{\partial t^2}$$
$$-\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]\frac{\partial^2 f(t,r)}{\partial r^2}-\frac{2}{c^2 r}\left[1+\frac{2f(t,r)}{c^2}\right]\frac{\partial f(t,r)}{\partial r}+\frac{2}{c^4}\left(\frac{\partial f(t,r)}{\partial r}\right)^2 \tag{27}$$

$$\begin{aligned}R=&\frac{30}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-3}\left(\frac{\partial f(t,r)}{\partial t}\right)^2\\&+\frac{6}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\frac{\partial^2 f(t,r)}{\partial t^2}\\&+\frac{4}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial t}\right)^2\\&-\frac{2}{c^2}\frac{\partial^2 f(t,r)}{\partial r^2}-\frac{4}{c^2 r}\frac{\partial f(t,r)}{\partial r}\\&+\frac{2}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial t}\right)^2\\&+\frac{2}{r^2}\left[1+\frac{2f(t,r)}{c^2}\right]\end{aligned} \tag{28}$$

Substituting equations 1, 27 and 28 into 26 yields the following result;

$$\begin{aligned}G_{00}=&\frac{2}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]\frac{\partial^2 f(t,r)}{\partial r^2}\\&-\frac{4}{c^2 r}\left[1+\frac{2f(t,r)}{c^2}\right]\frac{\partial f(t,r)}{\partial r}+\frac{2}{c^4}\left(\frac{\partial f(t,r)}{\partial r}\right)^2\\&+\frac{1}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial r}\right)^2\\&+\frac{2}{c^4}\left(\frac{\partial f(t,r)}{\partial r}\right)^2-\frac{3}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial r}\right)^2\\&+\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^2=0\end{aligned} \tag{29}$$

Multiplying 29 through by $-\frac{2}{c^2}$ and dividing through by $\left[1+\frac{2f(t,r)}{c^2}\right]$ yields

$$\begin{aligned}&\frac{\partial^2 f(t,r)}{\partial r^2}+\frac{\partial f(t,r)}{r\partial r}-\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial r}\right)^2\\&-\frac{1}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial r}\right)^2\\&\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial r}\right)^2\\&+\frac{3}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-3}\left(\frac{\partial f(t,r)}{\partial t}\right)^2+\frac{c^2}{2r^2}\left[1+\frac{2f(t,r)}{c^2}\right]^2=0\end{aligned} \tag{30}$$

Hence, equation 30 could be equivalently written as

$$\begin{aligned}&\nabla^2 f(t,r)-\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial r}\right)^2\\&-\frac{1}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial r}\right)^2\\&-\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial t}\right)^2\\&+\frac{3}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-3}\left(\frac{\partial f(t,r)}{\partial t}\right)^2\\&-\frac{c^2}{2r^2}\left[1+\frac{2f(t,r)}{c^2}\right]=0\end{aligned} \tag{31}$$

In the weak field limit of the order $c^0$, equation 31 reduces to

$$\nabla^2 f(t,r)=0 \tag{32}$$

Now considering limiting equation 31 to the order $c^{-2}$, the field equation becomes

$$\begin{aligned}&\nabla^2 f(t,r)-\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial r}\right)^2\\&-\frac{1}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial r}\right)^2\\&-\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial t}\right)^2\\&+\frac{3}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-3}\left(\frac{\partial f(t,r)}{\partial t}\right)^2\\&-\frac{c^2}{2r^2}\left[1+\frac{2f(t,r)}{c^2}\right]=0\end{aligned} \tag{33}$$

Let us seek a solution of 33 in the form

$$f(t,r)=\sum_{n=0}^{\infty}R_n(r)\exp n\left(f-\frac{r}{c}\right) \tag{34}$$

where $R_n(r)$ is a function of $r$ only

Now taking the partial derivative of equation 35 twice w.r.t (r) yields equation 36

$$\begin{aligned}\frac{\partial^2 f}{\partial r^2}=&R_0^{11}(r)+\left[R_1^{11}(r)-\frac{2}{c}R_1'(r)+\frac{1}{c^2}R_1\right]\exp\left(t-\frac{r}{c}\right)\\&+\left[R_2^{11}(r)-\frac{2r}{c}R_2'(r)+\frac{r^2}{c^2}R_2\right]\exp 2\left(t-\frac{r}{c}\right)\\&+\left[R_3^{11}(r)-\frac{2\cdot 3}{c}R_3'(r)+\frac{3^2}{c^2}R_3\right]\exp 3\left(t-\frac{r}{c}\right)\end{aligned} \tag{35}$$

Taking the partial derivative of equation 35 w.r.t (r) yields equation 37

$$\begin{aligned}\frac{2}{r}\frac{\partial f}{\partial r}=&\frac{2}{r}R_0(r)+\frac{2}{r}R_1(r)\exp\left(t-\frac{r}{c}\right)\\&-\frac{2}{cr}R_1(r)\exp\left(t-\frac{r}{c}\right)\\&+\frac{2}{r}R_2(r)\exp\left(2t-\frac{2r}{c}\right)\\&-\frac{2\cdot 2}{cr}R_2(r)\exp\left(2\left(t-\frac{r}{c}\right)\right)\\&+\frac{2}{r}R_3(r)\exp 3\left(t-\frac{r}{c}\right)-\frac{2\cdot 3}{cr}R_3(r)\exp 3\left(t-\frac{r}{c}\right)\end{aligned} \tag{36}$$

Again partially differentiating equation 35 w.r.t $(r)$ and squaring yields equation 38

$$\begin{aligned}\frac{2}{r}\left(\frac{\partial f}{\partial r}\right)^2=&\frac{1}{c^2}\left(R_0^1\right)^2(r)+\frac{1}{c^2}\left(R_1^1(r)\right)^2\exp\left(t-\frac{r}{c}\right)\\&-\frac{2}{c^3}R_1^1(r)R_1(r)\exp 2\left(t-\frac{r}{c}\right)+\\&\frac{1}{c^4}R_1^2(r)R_1(r)\exp 2\left(t-\frac{r}{c}\right)\\&+\frac{1}{c^2}\left(R_2^1\right)^2(r)\exp 4\left(t-\frac{r}{c}\right)\\&-\frac{4}{c^3}R_2^1(r)R_2(r)\exp 4\left(t-\frac{r}{c}\right)+\\&\frac{1}{c^2}\left(R_3^1\right)(r)\exp 6\left(t-\frac{r}{c}\right)\\&-\frac{6}{c^3}R_3^1(r)R_3(r)\exp 6\left(t-\frac{r}{c}\right)\\&+\frac{3}{c^4}R_3^1(r)\exp 6\left(t-\frac{r}{c}\right)\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{1}{2c^2}\frac{\partial f}{\partial t}=&\frac{1}{2c^2}R(r)\exp\left(t-\frac{r}{c}\right)\\&+\frac{2}{c^2}R_2\exp 2\left(t-\frac{r}{c}\right)+\frac{3}{2c^2}R_3(r)\exp 3\left(r-\frac{r}{c}\right)\end{aligned} \tag{38}$$

Partially differentiating equation 33 w.r.t $(t)$ and squaring yields

$$\begin{aligned}\frac{1}{c^2}\left[\frac{\partial f}{\partial t}\right]^2=&\frac{1}{c^2}R_1^2(r)\exp\left(2\left(t-\frac{r}{c}\right)\right)\\&+\frac{4}{c^2}R_2^2(r)\exp\left(4\left(t-\frac{r}{c}\right)\right)+\frac{9}{c^2}R_3^2(r)\exp 6\left(t-\frac{r}{c}\right)\end{aligned} \tag{39}$$

Partially differentiating equation 35 w.r.t (t) and squaring yields

$$\frac{3}{2c^2}\left[\frac{\partial f}{\partial t}\right]^2 = \frac{3}{2c^2}R_1^2(r)\exp 2\left(t-\frac{r}{c}\right) + \frac{6}{c^2}R_2^2(r)\exp 4\left(t-\frac{r}{c}\right) + \frac{27}{2c^2}R_3^2(r)\exp 6\left(t-\frac{r}{c}\right) \quad (40)$$

Equating the coefficient of exp(0) we get

$$R_0^{11}(r)\frac{2}{+r}R_0^1\frac{1}{+c^2}\left(r_0^1\right)^2(r) = 0 \quad (41)$$

To the limit of $c^0$ equation 42 becomes

$$R_0^{11}(r)\frac{2}{+r}R_0^1 = 0 \quad (42)$$

Solving the second order partial differential equation 42 we obtain the auxiliary solution as the equation below

$$R_0(r) = -\frac{2}{r} \quad (43)$$

But according to Newton's dynamical theory, Newton's gravitational scalar potential exterior to a distribution of mass or pressure is given by

$$f(r) = -\frac{Gm_0}{r} \quad (44)$$

where,
G = Universal Gravitational constant
$m_0$ = Total mass of the spherical body
$r$ = Distance of the spherical body $k = v$
Now comparing equation 43 with the Newton's gravitational scalar potential 44 we then choose the most convenient astrophysical solution of equation 41 as

$$R_0 \approx -\frac{k}{r} \quad (45)$$

Then our gravitational scalar potential obtained is given as

$$f(t.r) \approx -\frac{k}{r} \quad (46)$$

Comparing the coefficient of $\exp\left(t-\frac{r}{c}\right)$ yields

$$R^{11}(r) + 2\left[\frac{1}{r}-\frac{1}{c}\right]R_1^1(r) + \frac{1}{c}\left[1-\frac{2}{r}+\frac{1}{2c}\right]R_1(r) = 0 \quad (47)$$

This is our exact differential equation for $R_1$ which is our solution for $R_0$ Thus, the solution in the order of $c0$, reduces to

$$f(t,r) \approx -\frac{k}{r}\exp\left(t-\frac{r}{c}\right) \quad (48)$$

Interestingly, we discover that the solution obtained, that is equation 48 has a particular link to the pure Newtonian gravitational scalar potential for the gravitational field and hence put Einstein's geometrical gravitational field on the same foot with the Newtonian dynamical theory of gravitation as obtained by [3, 5, 6, 14] Equation 48 contains an arbitrary function which is a function of time and radial distance equal to Newton's dynamical scalar potential, our single dependent function $f(t,r)$ which is our physically and mathematically most satisfactory solution contains unknown post Newtonian terms or pure Einsteinian gravitational terms in order of, and hence, this research work has shown that the Einstein's geometrical field equation can be obtained as a generalization or completion of Newton's dynamical gravitational field equations

## 3 Formulation of Interior Solution to Einstein's Geometrical Gravitational Field Equation

The Einstein's field equation (EFE) interior to a homogeneous spherical distribution of mass is given by [5-13]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{4\pi GT_{\mu\nu}}{c^4} \quad (49)$$

Where $T_{\mu\nu}$ is the energy-momentum tensor due to any distribution of mass or pressure $G$ is the universal gravitational constant
Consider a homogeneous mass distribution in a weak field limit. We can neglect the contribution from the source, the energy-momentum tensor given by

$$T_{\mu\nu} = \frac{1}{2}\rho_0 c^2 \quad (50)$$

where,
$\rho_0$ is the density
$c$ is the speed of light in vacuum
Now substituting equation 50 into 49 yields

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{4\pi G\rho_0}{c^2} \quad (51)$$

It is observed in 12 that the exterior field equations along the $G_{11}$, $G_{22}$ and $G_{33}$ converge within the exterior field, similarly along the interior field. For mathematical convenience, we choose $G_{00}$
Hence the field equation is given by

$$G_{00} = R_{00} - \frac{1}{2}Rg_{00} = \frac{4\pi G\rho_0}{c^2} \quad (52)$$

Substituting equation 27, 28 and 1 into 52 equation gives

$$\begin{aligned}\nabla^2 f(t,r) &- \frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial r}\right)^2 \\ &-\frac{1}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial r}\right)^2 \\ &-\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial t}\right)^2 \\ &+\frac{3}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-3}\left(\frac{\partial f(t,r)}{\partial t}\right)^2 \\ &-\frac{c^2}{2r^2}\left[1+\frac{2f(t,r)}{c^2}\right] = \frac{4\pi G\rho_0}{c^2}\end{aligned} \tag{53}$$

In the weak field limit of the order $c^0$, equation 53 becomes

$$\nabla^2 f(t,r) = \frac{4\pi G\rho_0}{c^2} \tag{54}$$

## 4 Conclusion

From the result obtained in equation 54, we have established the fact that for a weak gravitational field, the result of Einstein's geometrical gravitational field equations does not differ significantly from Newton dynamical theory of gravitation. But for intense gravitational field, the result diverges from that of Newton's gravitational theory because of additional correctional terms which are not found in the existing once, thus equation, is the Newton dynamical scalar field equation. It is indeed a profound discovery, it confirms the assumption made by 14 that Newton dynamical theory of gravitation (NDTG) is a limiting case of Einstein's geometrical gravitational field equations (EGGFE), and this gives more light on the report of 15. It Experimentally shows equivalence principle of physics with the dependency of the gravitational scalar function on time and radial distance only.

. Furthermore, the obtained result in equation 48 differ from [3, 5, 6, 14] in the sense that 3 is for a hypothetical system which varies with azimuthal angle only, whereas 5 is for static homogenous oblate spheroidal systems, and 14 is for a static astrophysical system which varies with radial distance and azimuthal angle only and 6 is exterior to a spherical mass with varying potential whose tensor varies with time, radial distance and polar angle. Our resulting space-time is spherically symmetric and varies with time when solved with a tensor field that depends on both time and radial distance, far more complicated solution is obtained in 6 by including azimuthal angle which breaks spherical symmetry and could be used to explain rotating or asymmetric gravitational fields, The major difference is the extra challenge of simulating non-spherical space-time and its dynamics. Spherical symmetry, which states that the gravitational field is the same in all directions at a given radius and time, is assumed by a time- and radial-dependent solution in equation 48. This symmetry is eliminated in 6 by including the azimuthal angle $(\theta)$, which introduces variance around a rotating axis or another specified direction. Time $(t)$ and radial distance $(r)$ may be the only starting dependencies of the metric tensor, which describes the geometry of space-time. Cross-terms that connect these coordinates may appear, though, if it also depends on t, r, and the polar angle $\theta$. Because there are more variables and less symmetry, this increased complexity makes the equations harder to solve. For the description of systems with non-spherical mass distributions or revolving black holes, such a formulation is required. Although it adds another degree of freedom and allows for the portrayal of more complex and diverse gravitational effects, the azimuthal angle also increases the difficulty of the mathematics.

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