Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/140508

New Skewed Tail Gompertz Distribution with Different Estimation Methods and Applications to Engineering Data

Saeed E. Hemeda[1], *Gaber Sallam Salem Abdalla* [2], *Khalaf H. Al-Habib*[3], *Hisham A. Gamel*[1], *Ahmed S. Abdelzaher*[4] and *Suleman Nasiru* [5,*]

[1]Department of Basic Sciences, Obour High Institutes for Management, Computer & Informatics, Obour, Egypt
[2]Department of Insurance and Risk Management, College of Business, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia
[3]Ministry of Education, Saladin Education Directorate, Tikrit, Iraq
[4]Department of Financial Sciences, The Applied College, Imam Abdulrahman Bin Faisal University, Dammam 34212, Saudi Arabia
[5]Department of Statistics and Actuarial Science, School of Mathematical Sciences, C. K. Tedam University of Technology and Applied Sciences, Navrongo, Ghana

Received: 2 May 2025, Revised: 2 Jul. 2025, Accepted: 2 Aug. 2025
Published online: 1 Sep. 2025

Abstract: Gompertz is one of the important continuous distributions that is often applied to describe the distribution of adult lifespans by demographers and actuaries. Related fields of science such as biology, and gerontology, also considered the Gompertz distribution for survival analysis. In this study, a new Gompertz distribution with four parameters was proposed according to the truncated Nadarajah-Haghighi-G family to obtain a more flexible distribution, some statistical properties for the latest model where introduced, different shapes for CDF and PDF given with two- and three-dimension plot. To verify the aim of the study, two data sets were applied to illustrate flexibility for the proposed distribution compared with some statistical distributions, published in the literature according to information standards such as AIC, CAIC, BIC, and HQIC. The research produced encouraging results that could help statistical work and data analysis.

Keywords: Entropies, Moments, Truncated Nadarajah-Haghighi distribution, Gompertz distribution, Order Statistics.

1 Introduction

One essential aspect of our lives is the dissemination of statistics. It enables us to comprehend the environment we live in and make wise choices. It also aids in the discovery of possibilities and trends. Over the past few decades, a number of statistical distributions have been extensively utilized in a wide range of disciplines, such as engineering, economics, the medical sciences, demographics, and more. Many researchers, such us, El-Gohary *et al.* [10] proposed the generalized Gompertz distribution; Mazucheli et al. [19] developed Unit-Gompertz distribution; Eghwerido et al.[8] introduced the modified beta Gompertz distribution; Eraikhuemen et al. [11] formulated transmuted power Gompertz distribution; Bantan et al. [7] developed the unit gamma/Gompertz distribution; Ieren et al. [13] developed the power Gompertz distribution; Atanda et al.[6] proposed a new odd Lindley-Gompertz distribution; Meraou et al. [20] formulated the exponential T-X Gompertz model; Mahdy et al. [17] introduced a new bivariate odd generalized exponential Gompertz distribution; Joshi & Kumar [16] proposed Lindley Gompertz distribution; Nzei et al. [21] developed the Topp-Leone Gompertz distribution; Pasupuleti & Pathak [22] studied a particular form of the Gompertz model and its application; and Alizadeh et al. [3] studied some characterizations of the exponentiated Gompertz distribution and employed it to analyze many life phenomena. In this paper, a new Gompertz distribution with four parameters was established. In addition some statistical properties, for instance, the quantile function, moments, incomplete moments, order statistics, and three various types of entropies. The term and concept of entropy is used in a wide range of fields, from classical statistical and

* Corresponding author e-mail: sulemanstat@gmail.com

dynamical physics to quantum physics. In fact, entropy refers to a common concept of uncertainty. It is one of the types with common applications, Rényi entropy, Arimoto entropy and Tsallis entropy. Its importance has been discussed by researchers, such as: Alotaibi *et al.* [4], Habib et al. [12] and Ahsan & Aslam, [1].

Several methods for creating general families of distributions have been created and explored by numerous authors in recent years. By include one or more factors in a baseline model, these families provide greater modeling flexibility for lifetime data. To increase accuracy and adaptability, they are crucial for applied statisticians in fields including economics, environmental sciences, finance, medicine, and reliability studies. The widely recognized generators include the following: Marshall Olkin-G in [23], a new shifted Lomax-X family in [24], sine-exponentiated Weibull-H family in [25], Weibull-G in [26], a new truncated Zubair-G in [27], a new truncated Muth-G in [28], odd-Burr-G in [29], a new logarithmic tangent-U family in [30], odd inverse power generalized Weibull-G in [31], Kumaraswamy-G Poisson family in [32], odd Burr-G Poisson family in [33], a new generalized Burr-G [34], sine-G in [35], unit exponentiated half logistic power series class of distributions in [36], ratio exponentiated-G in [37], compounded Bell- G in [38], alpha power transformed weibull-G in [39], weighted exponentiated -G in [40], sine Burr-G in [41], odd Lomax trigonometric-G in [42], new hyperbolic sine-G in [43], discrete analogue of odd Weibull-G in [44], log-logistic tan generalized family in [45]. For more details see [46]-[59].
The motivation behind choosing and writing the topic is to generate a distribution with four parameters to be more flexible, to describe natural phenomena that classical distributions could not express in a way that models them accurately. In addition, most modern data sets are considered a challenge in data analysis processes. To be a clear contribution to the field of statistics and data analysis by generating a distribution with four parameters capable of dealing with a wide spectrum of modern natural phenomena.

2 The Proposed Model

Al Habib *et al.* [2] proposed and studied a new family for extended distributions which is [0, 1] truncated Nadarajah Haghighi – G family. The new family have the following CDF and PDF respectively:

$$F(x) = \aleph \left(1 - e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x}-1\right)\right)}\right)\right]^{a}} \right). \tag{1}$$

And

$$f(x) = \aleph \left(ab\theta\alpha e^{-\theta\left(e^{\alpha x}-1\right)} e^{-\alpha x} \left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x}-1\right)\right)}\right)\right]^{a-1} e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x}-1\right)\right)}\right)\right]^{a}} \right). \tag{2}$$

Where; $\aleph = \frac{1}{1-\, e^{1-[1+b]^{a}}}$.

Figure 1 shows that the curve of the two-dimensional CDF of the proposed four-parameter Gompertz-type distribution under several parameter settings increases monotonically from 0 to 1, illustrating how parameter changes shift the onset and steepness of accumulation (earlier/later rise) and affect tail behavior. Figure 2 shows that the 3-D surface of the CDF for the proposed four-parameter Gompertz-type model with ($a = 0.80,\ b = 0.50,\ \theta = 1.50,\ \alpha = 0.75$): the surface increases smoothly from 0 to 1 across the support, with a gentle rise near the origin, a steeper accumulation in the mid-range, and clear saturation as $F \to 1$. This parameter setting illustrates a right-skewed behavior with accelerated growth (steeper gradient) at intermediate values and a compressed upper tail. From Fig. 3(A), we notice that the 2-D panel contrasts how the parameters shift the mode and alter peak sharpness and tail weight, while the 3-D surface reveals the density landscape and the ridge around the modal region. Fig. 3(B) shows that the 3-D surface of the PDF for the proposed four-parameter Gompertz-type model ($a = 0.80,\ b = 0.50,\ \theta = 1.50,\ \alpha = 0.75$) is unimodal with a pronounced ridge at mid-range, showing a steep rise toward the mode and a gradual decay thereafter, consistent with moderate right-skewness.

The survival hazard rate functions will be

$$S(x) = \aleph \left(e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x}-1\right)\right)}\right)\right]^{a}} - e^{1-[1+b]^{a}} \right), \tag{3}$$

$$h(x) = \frac{ab\theta\alpha e^{-\alpha x} e^{-\theta\left(e^{\alpha x}-1\right)} \left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x}-1\right)\right)}\right)\right]^{a-1} e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x}-1\right)\right)}\right)\right]^{a}}}{e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x}-1\right)\right)}\right)\right]^{a}} - e^{1-[1+b]^{a}}}. \tag{4}$$

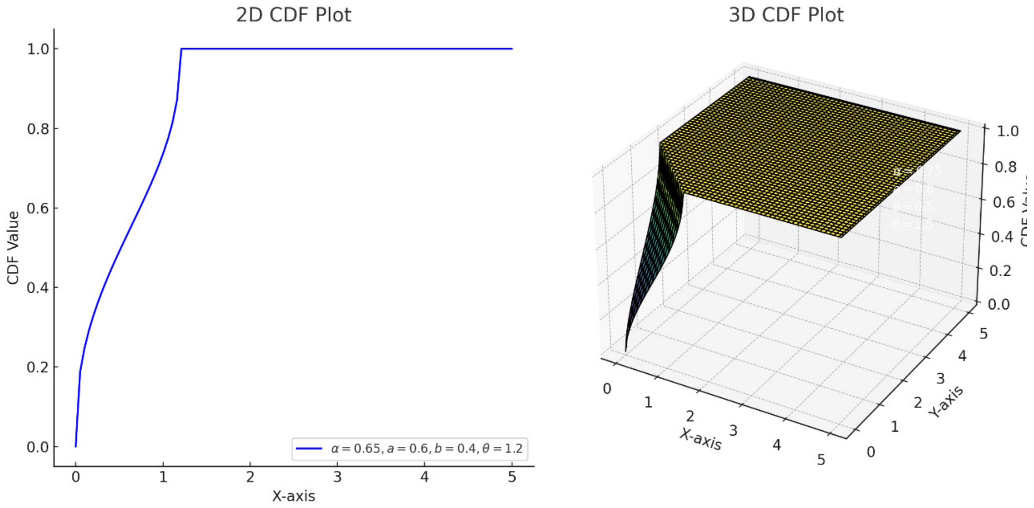


Fig. 1: 2-dim plot for CDF with different parameters

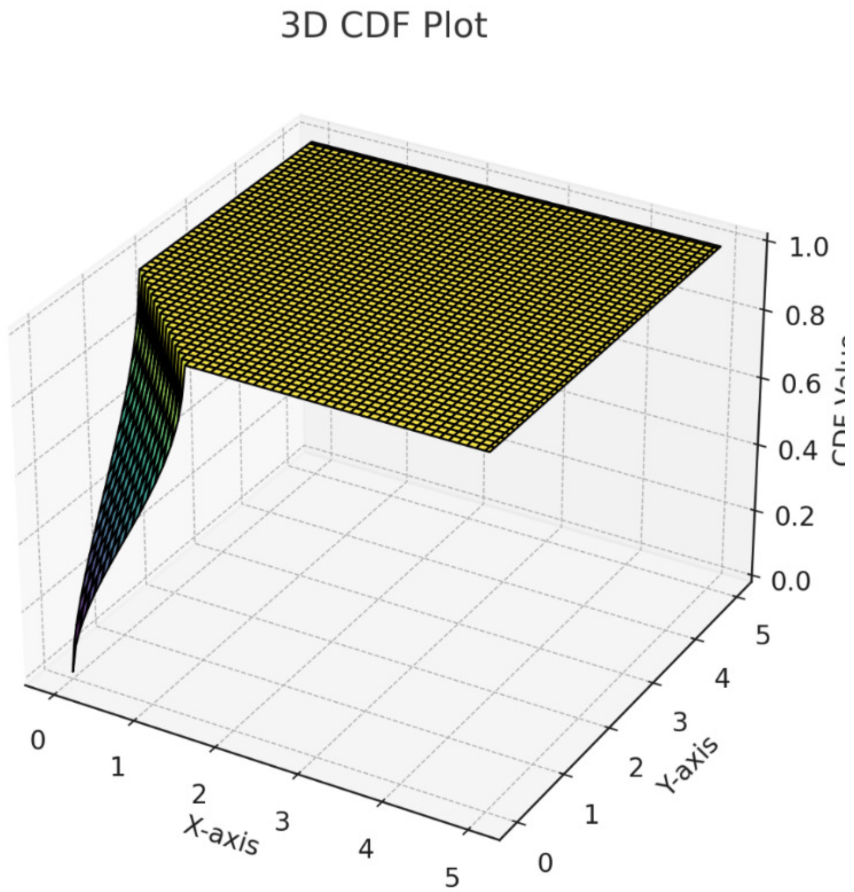


Fig. 2: 3-dim plot for CDF with parameters ($a = 0.80$, $b = 0.50$, $\theta = 1.50$, $\alpha = 0.75$)

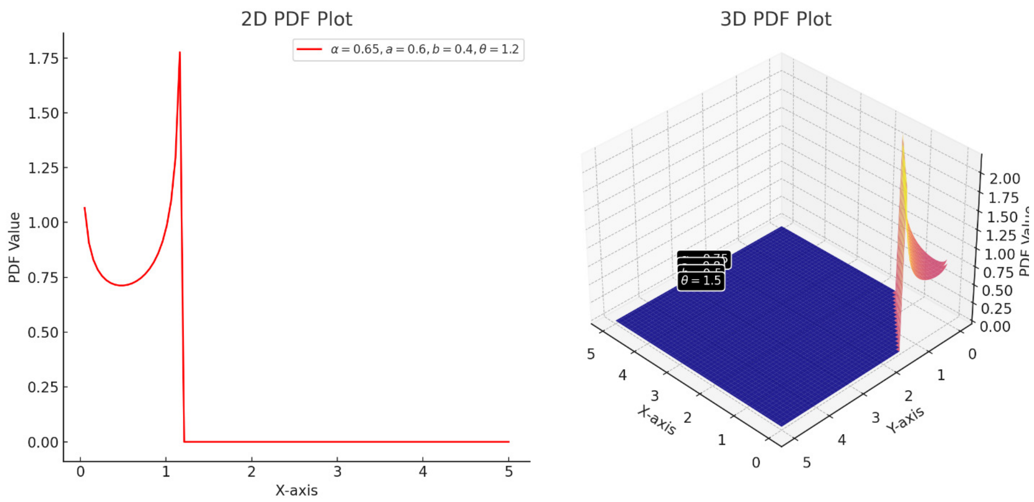


Figure 3 (a) 2-dim and 3-dim plot for PDF with different parameters

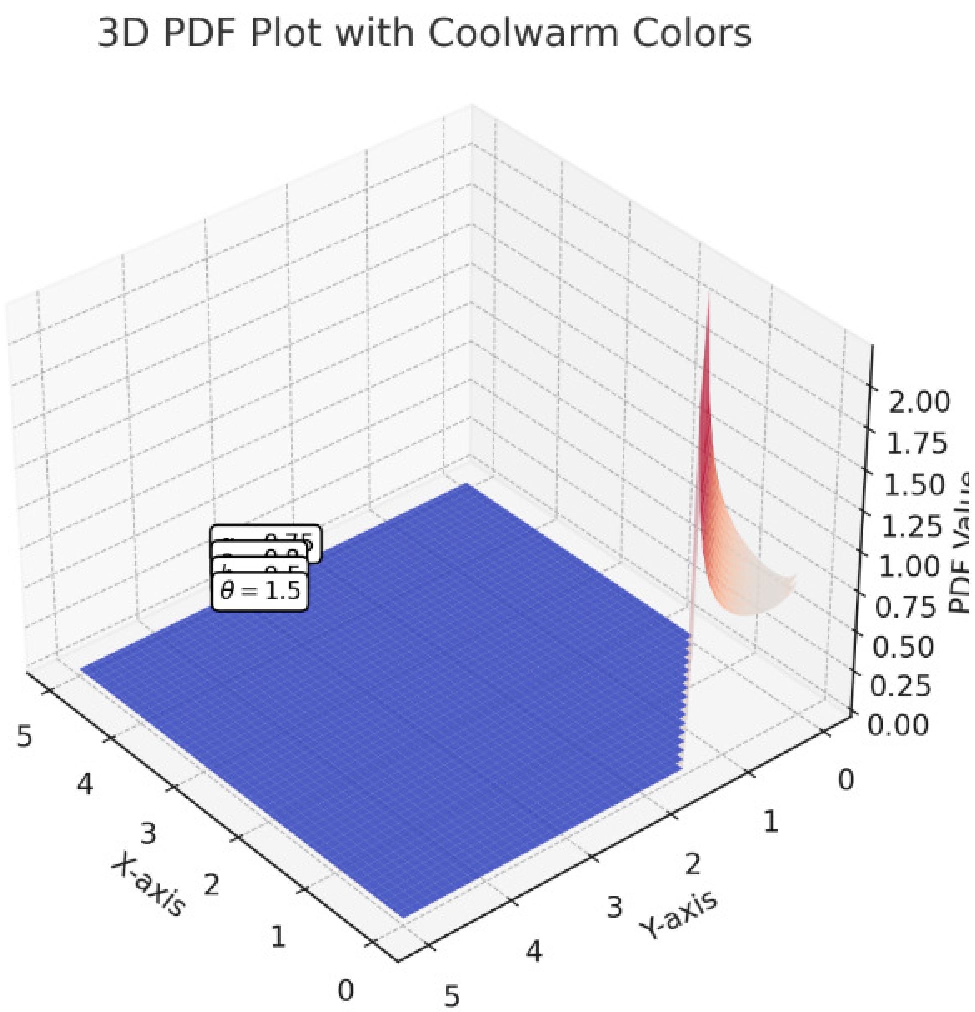


Figure 3 (b) 3-dim plot for PDF with parameters ($a = 0.80$, $b = 0.50$, $\theta = 1.50$, $\alpha = 0.75$)

The mixture representation of the PDF is essential in the derivation of the statistical properties of [0,1] TNHGo distributions. By take eq. (2) and use the expansion exponential formula, moreover, by use the generalized binomial theorem formula; the pdf will be

$$f(x) = \psi\, e^{-(d+1)\alpha x}. \tag{5}$$

Where

$$\psi = \aleph\,(a,b,\theta,\alpha) \sum_{j,k,s,n,m,d=0}^{\infty} \binom{j}{k}\binom{d}{n} \frac{(-1)^{k+s+n+d}\,(-\theta(s+1))^{n}\, b^{m}}{j!\,n!} \binom{ak+a-1}{m} \binom{m(ak+a-1)}{s}.$$

3 Statistical Measures

The document explores several statistical measures related to a newly proposed probability distribution, including the quantile function, moments (mean, variance, skewness, and kurtosis), and the moment-generating function, which are essential for describing the distribution's characteristics. It also examines incomplete moments for applications in risk assessment and order statistics for analyzing sample extremes. Additionally, it derives the Lorenz and Bonferroni curves, commonly used in inequality analysis and reliability studies. Furthermore, the document evaluates entropy measures such as Rényi, Arimoto, and Tsallis entropy, which quantify uncertainty within the distribution. The findings suggest that failure time uncertainty decreases as the entropy index increases while remaining within a non-zero range, providing deeper insights into the behavior of the proposed model.

3.1 Quantile function

The quantile of a random variable X is defined by solving the following equation:

$u = \dfrac{1-e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x}-1\right)\right)}\right)\right]^{a}}}{1-e^{1-[1+b]^{a}}}$, and therefore:

$$x = \frac{-1}{\alpha}\ln\left\{\frac{-1}{\theta} ln\left\{1-\frac{1}{b}\left\{1-\left(1-ln\left\{1-u\left[1-e^{1-[1+b]^{a}}\right]\right\}\right)^{\frac{1}{a}}\right\}\right\}+1\right\}. \tag{6}$$

3.2 The r^{th} moments

The r^{th} moment of a random variable X can be expressed as $\mu_r = \int_{-\infty}^{\infty} x^r f(x)\,dx$, where $f(x)$ given in (5) so that:

$$\mu_r = \int_{-\infty}^{\infty} x^r \psi\, e^{-(d+1)\alpha x} dx = \psi\left(\frac{1}{(d+1)\,\alpha}\right)^{r+1} \Gamma(r+1). \tag{7}$$

Table 1 shows some numerical values of moments for different values of parameters.

Table 1: The first fourth moments for the new model with variance, Skewness and kurtosis

a	b	θ	α	m_1	m_2	m_3	m_4	Var(X)	Skew	kurtosis
0.2	**0.5**	**0.3**	**2**	0.400	0.303	0.324	0.432	0.143	1.942	4.705
0.2	**0.5**	**0.3**	**3**	0.276	0.149	0.115	0.113	0.072	1.999	5.089
0.2	**1.5**	**0.4**	**2**	0.341	0.232	0.224	0.271	0.115	2.004	5.034
0.2	**1.5**	**0.5**	**2**	0.331	0.214	0.194	0.217	0.104	1.959	4.738
1.2	**0.5**	**0.3**	**2**	0.385	0.286	0.301	0.398	0.137	1.967	4.865
2.2	**0.5**	**0.3**	**2**	0.346	0.238	0.239	0.308	0.118	2.058	5.437
0.6	**0.7**	**0.3**	**2**	0.382	0.284	0.299	0.397	0.138	1.975	4.922
0.6	**1.2**	**0.3**	**3**	0.352	0.251	0.259	0.34	0.127	2.059	5.396

3.3 Moment generating function

The moment generating function for a random variable X can be expressed as follows:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_{-\infty}^{\infty} x^r \; f(x) dx$$

Where $f(x)$ was defined in (5) we get

$$M_X(t) = \psi \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r \psi \, e^{-(d+1)\alpha x} dx,$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \psi \left(\frac{1}{(d+1)\,\alpha} \right)^{r+1} \blacksquare(r+1). \tag{8}$$

3.4 Incomplete moments

The incomplete moments of a random variable X can be expressed as $M_r(y) = \int_{-\infty}^{y} x^r f(x)\, dx$, where $f(x)$ given in (5), we have

$$M_r(y) = \psi \int_0^y x^r \; e^{-(d+1)\alpha x} dx = \psi \left(\frac{1}{(d+1)\,\alpha} \right)^{r+1} \gamma(r+1, (d+1)\,\alpha y). \tag{9}$$

3.5 Order statistics

Let $X_1, X_2, X_3, \ldots, X_n$ have [0,1] TNHGo distribution with CDF, PDF defined in (3),(4) respectively and let $X_{1:n}, X_{2:n}, X_{3:n}, \ldots, X_{n:n}$ be the order statistic calculated using this sample. To extract the probability density function of the p^{th} order statistic from the [0,1] TNHGo distribution, use (3) and (4) in the equation below.

$$f_{X_{p:n}}(x) = \frac{n!}{(\mathrm{p}-1)!(n-p)!} \, [F(x)]^{p-1} [1-F(x)]^{n-p} \; f(x)$$

$$f_{X_{p:n}}(x) = \sum_{s=0}^{n-p} d(-1)^s \binom{n-p}{s} [F(x)]^{p+s-1} \; f(x)$$

Where $d = \frac{n!}{(\mathrm{p}-1)!(n-p)!}$

That is

$$f_{X_{p:n}}(x) = \sum_{s=0}^{n-p} d(-1)^s \binom{n-p}{s} \left(\frac{1 - e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x}-1\right)\right)}\right)\right]^a}}{1 - e^{1-[1+b]^a}} \right)^{p+s-1}$$

$$\times \frac{ab\theta\alpha\, e^{-\theta\left(e^{\alpha x}-1\right)} e^{-\alpha x} \left[1 + b\left(1 - e^{\left(-\theta\left(e^{-\alpha x}-1\right)\right)}\right)\right]^{a-1} e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x}-1\right)\right)}\right)\right]^a}}{1 - e^{1-[1+b]^a}} \tag{10}$$

3.6 Lorenz curve

The Lorenz curve for a random variable X can be described as

$$L_F(y) = \frac{1}{\mu} \int_{-\infty}^{y} x f(x) dx.$$

Using the PDF in (5) we obtain the Lorenz curve of the new model as follows.

$$L_F(y) = \frac{\psi}{\mu} \int_0^y x \; e^{-(d+1)\alpha x} dx,$$

$$L_F(y) = \psi \left(\frac{1}{(d+1)\,\alpha} \right)^2 \gamma(r+1, (d+1)\,\alpha y). \tag{11}$$

3.7 Bonferroni curve

The Bonferroni Curve for a random variable X has been described as $B_F(y)=\frac{L_F(y)}{F(y)}$, where $L_F(y)$ as in (11) and $F(y)$ was defined in (3) with respect to y, we have

$$B_F(y)=\frac{1-e^{1-[1+b]^a}}{1-e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha y}-1\right)\right)}\right)\right]^a}}\psi\left(\frac{1}{(d+1)\alpha}\right)^2\gamma(r+1,(d+1)\alpha y). \tag{12}$$

3.8 Rényi entropy

Rényi entropy $I_R(c)$ for the random variable X is calculated by

$$I_R(c)=\frac{1}{1-c}\log\left[\int_{-\infty}^{\infty}f^c(x)\,dx\right],\ c\neq 1,\ c>0,$$

So

$$I_R(c)=\frac{1}{1-c}\log\left[\frac{\psi}{-(d+1)\alpha}e^{-(d+1)\alpha x}\right],\ c\neq 1,\ c>0. \tag{13}$$

3.9 Arimoto entropy

Arimoto entropy (ARE) of the new distribution is explained as follows:

$$A_c=\frac{c}{1-c}\left(\left(\int_0^{\infty}f^c(x)\,dx\right)^{\frac{1}{c}}-1\right)\quad,\ c\neq 1,\ c>0,$$

so it will be

$$A_c=\frac{c}{1-c}\left(\left(\frac{\Omega}{-(d+1)\alpha}e^{-(d+1)\alpha x}\right)^{\frac{1}{c}}-1\right)\quad,\ c\neq 1,\ c>0. \tag{14}$$

3.10 Tsallis entropy

The Tsallis entropy (TSE) of the new distribution can be described as follows $T_c=\frac{1}{1-c}\left(1-\int_0^{\infty}f^c(x)\,dx\right)\quad,\ c\neq 1, c>0$, and after some calculations, it becomes

$$T_c=\frac{1}{1-c}\left(1+\frac{\Omega}{(d+1)\alpha}e^{-(d+1)\alpha x}\right). \tag{15}$$

Table 2: Analysis of Rényi Entropy, Arimoto Entropy and Havrda and Charvát Entropy for $a=b=\theta=\alpha=0.3$

c-value	RE	ARE	HCE
1.5	1.491	1.175	-0.162
2.0	1.439	1.026	-0.513
2.5	1.402	0.948	-1.040

Through the three entropy tables 2- 4 above, it is clear that the measure of uncertainty for the phenomenon of failure times decreases when the parameter values remain constant and the value of the entropy index increases. It also maintains the same hypothetical context for the measure when the entropy index remains constant and the parameter values increase successively, and it does not reach the state of constant (0).

Table 3: Analysis of Rényi Entropy, Arimoto Entropy and Havrda and Charvát Entropy for $a = b = c = \alpha = 1.5$

θ-value	**RE**	**ARE**	**HCE**
0.3	-0.420	-0.450	0.062
0.5	-0.454	-0.491	0.067
0.7	-0.487	-0.529	0.073

Table 4: Analysis of Rényi Entropy, Arimoto Entropy and Havrda and Charvát Entropy for $a = b = c = \theta = 0.3$

α-value	**RE**	**ARE**	**HCE**
1.3	0.979	3.784	-3.394
2.3	0.558	1.148	-1.030
3.3	0.269	0.375	-0.336

4 Different Estimation Technique

Parameter estimation techniques are essential to the analysis of data and the production of precise and trustworthy findings in mathematical statistics. The goal of parameter estimation is to use the available data to estimate the unknown values of parameters in a family of statistical distributions. The parameter of the [0,1] TNHGo distribution will be estimated in this section using five different techniques: the maximum likelihood technique, the Cramér–von Mises technique, the weighted least squares technique, the right-tail Anderson-Darling technique, and the ordinary least squares technique.

4.1 Ordinary Least Squares

The ordinary least square estimation [60,61] parameters of the [0,1] TNHGo parameters are obtained by minimizing the equation:

$$\sum_{s=1}^{k}\left(\aleph\left(1-e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x_s}-1\right)\right)}\right)\right]^{a}}\right)-\frac{s}{k+1}\right)^{2}.$$

4.2 Weighted Least-Squares

We can obtain weighted least square estimation [62,63] for [0.1] TNHGo parameters can obtained by
Minimizing the following function with respect to the parameters:

$$\sum_{s=1}^{k}\frac{(n+1)^2(n+2)}{s(n-s+1)}\left(\aleph\left(1-e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x_s}-1\right)\right)}\right)\right]^{a}}\right)-\frac{s}{n+1}\right)^{2}.$$

4.3 Cramér–Von Mises

Cramér-von Mises estimation [64,65]for [0,1] TNHGo parameters obtained by minimizing.

$$\frac{1}{12k}+\sum_{s=1}^{k}\left(\aleph\left(1-e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x_s}-1\right)\right)}\right)\right]^{a}}\right)-\frac{2s-1}{2k}\right)^{2}.$$

4.4 Right-Tail Anderson-Darling

The Right Anderson Darling estimation [66,67] of [0,1] TNHGo parameters acquired by minimizing:

$$\frac{k}{2}-2\sum_{s=1}^{k}\aleph\left(1-e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x_s}-1\right)\right)}\right)\right]^{a}}\right)$$
$$-\frac{1}{k}\sum_{s=1}^{k}(2s-1)\ln\left[1-\aleph\left(1-e^{1-\left[1+b\left(1-e^{\left(-\theta\left(e^{-\alpha x_s}-1\right)\right)}\right)\right]^{a}}\right)\right]..$$

4.5 Maximum likelihood

The log likelihood function form of a sample n observations, $(X_1, X_2, ..., X_n)$ for the TNHGo distribution is given as

$$L(a,b,\theta,\alpha)=\prod_{i=1}^{n}\eta\, a\, b\, \theta\, \alpha\, e^{-[\theta(e^{\alpha x_i}-1)+\alpha x_i]}\left(1+b(1-e^{-\theta(e^{\alpha x_i}-1)})\right)^{a-1}e^{1-[1+b(1-e^{-\theta(e^{\alpha x_i}-1)})]}.$$

$$\ln L(a,b,\theta,\alpha)=\sum_{i=1}^{n}\ln\left[\eta\, a\, b\, \theta\, \alpha\, e^{-[\theta(e^{\alpha x_i}-1)+\alpha x_i]}\left(1+b(1-e^{-\theta(e^{\alpha x_i}-1)})\right)^{a-1}e^{1-[1+b(1-e^{-\theta(e^{\alpha x_i}-1)})]}\right].$$

The normal equations are obtained through calculating the partial derivatives with regard to a, b, α, θ and equating them to zero.

$$\frac{\partial \ln L}{\partial a}=n\cdot\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)^{(1-a)}$$
$$\cdot\left\{a\alpha b\theta\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)^{(a-1)}e^{-b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)}e^{\alpha x_i-\theta(e^{\alpha x_i}-1)}\right.$$
$$\cdot\ln\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)+\alpha b\theta\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)^{(a-1)}$$
$$\left.\cdot e^{-b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)}e^{\alpha x_i-\theta(e^{\alpha x_i}-1)}\right\}=0,$$

$$\frac{\partial \ln L}{\partial b}=n\cdot\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)^{(1-a)}$$
$$\cdot\left\{a\alpha\theta\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)^{(a-1)}e^{-b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)}e^{\alpha x_i-\theta(e^{\alpha x_i}-1)}\right.$$
$$\cdot\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)\ln\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)+\alpha\theta\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)^{(a-1)}$$
$$\left.\cdot e^{-b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)}e^{\alpha x_i-\theta(e^{\alpha x_i}-1)}\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)\right\}=0,$$

$$\frac{\partial \ln L}{\partial \alpha}=n\cdot\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)^{(1-a)}$$
$$\cdot\left\{ab\theta x_i e^{\alpha x_i}\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)^{(a-1)}e^{-b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)}e^{\alpha x_i-\theta(e^{\alpha x_i}-1)}\right.$$
$$\cdot\left[\theta e^{\alpha x_i}\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right]\ln\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)$$
$$+b\theta x_i e^{\alpha x_i}\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)^{(a-1)}e^{-b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)}e^{\alpha x_i-\theta(e^{\alpha x_i}-1)}$$
$$\left.\cdot\left[\theta e^{\alpha x_i}\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right]\right\}=0,$$

$$\frac{\partial \ln L}{\partial \theta} = n \cdot \left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)^{(1-a)} \cdot \left\{ a\alpha b\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)^{(a-1)} e^{-b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)} e^{\alpha x_i-\theta(e^{\alpha x_i}-1)} \cdot (e^{\alpha x_i}-1)\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)\ln\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right) + \alpha b\left(b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)+1\right)^{(a-1)} e^{-b\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right)} e^{\alpha x_i-\theta(e^{\alpha x_i}-1)} \cdot (e^{\alpha x_i}-1)\left(1-e^{-\theta(e^{\alpha x_i}-1)}\right) \right\} = 0.$$

Since there is no closed-form solution to this system of equations, we will solve for $\hat{a}, \hat{b}, \hat{\alpha}$ and $\hat{\theta}$ iteratively, using R package.

5 Simulation Study

Estimation methods in section are used to compare the simulation study in terms of the averages of the three quantities: absolute Bais $|Bais\tau| = \frac{1}{N}\sum_{i=1}^{N}|\hat{\tau}-\tau|$, Root mean square error (RMSE), $RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\tau}-\tau)^2}$ and mean relative error (MRE), $MRE = \frac{1}{N}\sum_{i=1}^{N}|\hat{\tau}-\tau|/\tau$.
We generate N = 1,000 random samples, and sample size *n = 50, 100,* and *200* from the [0,1] TNHGo distribution with six different sets of initial parameters. All numerical results are provided in Tables 5-10.

Table 5: Simulation results for sample size $n = 50$ with 1000 iterations and initial values $a = 0.1$, $b = 0.8$, $\alpha = 0.7$, $\theta = 3$.

Name	Initial	Mean	RMSE	Bias
a_mle	0.1	0.3206198	0.3553930	0.2206198
b_mle	0.8	0.5575577	0.3508849	0.2424423
α_mle	0.7	0.5371202	0.4830373	0.1628798
θ_mle	3	10.318117	4.882625	7.318117
a_olse	0.1	0.01312539	14.882625	0.11312539
b_lose	0.8	0.74495164	0.43077586	0.05504836
α_olse	0.7	0.5937770	0.29393114	0.1062230
θ_olse	3	4.248083	0.7551407	1.248083
a_wlse	0.1	0.0292198	3.216459	0.0707802
b_wlse	0.8	0.76831561	0.3427389	0.03168439
α_wlse	0.7	0.60306699	0.30870450	0.09693301
θ_wlse	3	4.288619	0.74474119	1.288619
a_cvme	0.1	0.090130395	0.410042275	0.009869605
b_cvme	0.8	0.73426475	0.30330520	0.06573525
α_cvme	0.7	0.68220148	0.70723977	0.01779852
θ_cvme	3	3.707371	2.301032	0.707371
a_rade	0.1	0.08021415	0.30327098	0.01978585
b_rade	0.8	0.71474981	0.32249087	0.08525019
α_rade	0.7	0.5464788	0.7506912	0.1535212
θ_rade	3	4.692142	4.092967	1.692142

6 Applications

In this section, R software employed to apply a real data sets of the model [0,1] Truncated Nadarajah Haghighi Gompertz distribution. In order to obtain the best result using the following statistical criteria (-LL, AIC, AIC, BIC, HQIC) compared to other models, such as Beta Gompertz (BeGo) Jafari et al. [15]. Kumaraswamy Gompertz (KuGo) Rocha et al. [68].

Table 6: Simulation results for sample size $n = 100$ with 1000 iterations and initial values $a = 0.1$, $b = 0.8$, $\alpha = 0.7$, $\theta = 3$.

Name	**Initial**	**Mean**	**RMSE**	**Bias**
a_mle	0.1	0.2568874	0.25931998	0.15688740
b_mle	0.8	0.5957603	0.3251745	0.2042397
α_mle	0.7	0.5365248	0.5257647	0.1634752
θ_mle	3	7.454992	8.054750	4.454992
a_olse	0.1	0.0271732	0.32799876	0.07282679
b_lose	0.8	0.7424295	0.29484341	0.05757041
α_olse	0.7	0.6176859	0.66080161	0.08231403
θ_olse	3	3.8185499	2.3512357	0.8185499
a_wlse	0.1	0.04864219	0.22825957	0.05135781
b_wlse	0.8	0.76550603	0.29373853	0.03449397
α_wlse	0.7	0.5922188	0.6077939	0.1077812
θ_wlse	3	3.9258939	2.4237520	0.9258939
a_cvme	0.1	0.0987317	0.308651866	0.0012682
b_cvme	0.8	0.73464221	0.27646493	0.065357
α_cvme	0.7	0.5860080	0.6091458	0.1139920
θ_cvme	3	3.9162082	2.5251657	0.9162082
a_rade	0.1	0.099154965	0.209329816	0.000845035
b_rade	0.8	0.75914271	0.28460847	0.04085729
α_rade	0.7	0.5210859	0.6360715	0.1789141
θ_rade	3	4.009638	2.589439	1.009638

Table 7: Simulation results for sample size $n = 200$ with 1000 iterations and initial values $a = 0.1$, $b = 0.8$, $\alpha = 0.7$, $\theta = 3$.

Name	**initial**	**Mean**	**RMSE**	**Bias**
a_mle	0.1	0.22865196	0.21110604	0.12865196
b_mle	0.8	0.6282245	0.3186117	0.1717755
α_mle	0.7	0.475875	0.4975663	0.2241249
θ_mle	3	6.592611	6.029430	3.592611
a_olse	0.1	0.03790415	0.24199623	0.06209585
b_lose	0.8	0.76110513	0.24721069	0.03889487
α_olse	0.7	0.65330624	0.50153302	0.04669376
θ_olse	3	3.4798055	0.50153302	0.4798055
a_wlse	0.1	0.106679188	0.148471138	0.006679188
b_wlse	0.8	0.78290065	0.23117688	0.01709935
α_wlse	0.7	0.4634423	0.5029453	0.2365577
θ_wlse	3	4.079788	2.203073	1.079788
a_cvme	0.1	0.09351699	0.20976394	0.00648301
b_cvme	0.8	0.76257052	0.22889760	0.03742948
α_cvme	0.7	0.60829096	0.50462026	0.09170904
θ_cvme	3	3.6848975	2.2807533	0.6848975
a_rade	0.1	0.108180904	0.154915364	0.008180904
b_rade	0.8	0.78856193	0.22733890	0.01143807
α_rade	0.7	0.4402423	0.6094888	0.2597577
θ_rade	3	3.9297066	2.2564728	0.9297066

Exponential Generalized Gompertz (EGGo), weibull Gompertz (WeGo) El-Bassiouny et al. [9]. Gompertez Gompertz (GoGo). The first data set (failure times of 20 components) were studied by Ismael, & AL-Bairmani. [14], New extension for Chen distribution and Arshad et al. [5]. A comprehensive review of data sets. Also the second data set (failure times of 84 Aircraft Windshield) proposed by many research like, Al-Sadat N. [69]. A new modified model with application to engineering data sets, Maqbool et al. [18]. Modified-Weibull distribution with Applications. Some descriptive analysis for the real data sets are provided in Table 11.

Table 8: Simulation results for sample size $n = 50$ with 1000 iterations and initial values $a = 0.9$, $b = 0.6$, $\alpha = 1.2$, $\theta = 3.5$.

Name	**Initial**	**Mean**	**RMSE**	**Bias**
a_mle	0.9	1.2156419	0.6684944	0.3156419
b_mle	0.6	0.8417091	0.4468618	0.2417091
α_mle	1.2	0.8020091	0.7892512	0.3979909
θ_mle	3.5	7.072269	12.259250	3.572269
a_olse	0.9	1.2955448	0.9605318	0.3955448
b_lose	0.6	0.8232932	0.4550286	0.2232932
α_olse	1.2	0.7715597	0.8300767	0.4284403
θ_olse	3.5	4.631767	2.954303	1.131767
a_wlse	0.9	1.0481989	0.8673047	0.1481989
b_wlse	0.6	0.8851117	0.5529052	0.2851117
α_wlse	1.2	0.8654644	0.7330694	0.3345356
θ_wlse	3.5	4.0078161	1.9013252	0.5078161
a_cvme	0.9	1.4664666	1.1275919	0.5664666
b_cvme	0.6	0.7730599	0.3942963	0.1730599
α_cvme	1.2	0.9783706	0.7706341	0.2216294
θ_cvme	3.5	4.0943671	2.2406407	0.5943671
a_rade	0.9	0.97521001	0.60783491	0.07521001
b_rade	0.6	0.8517561	0.4764981	0.2517561
α_rade	1.2	0.7938293	0.8100914	0.4061707
θ_rade	3.5	4.2617916	1.9187803	0.7617916

Table 9: Simulation result for sample size 100, iterations 1000 and initial value 0.9 0.6 1.2 3.5

Name	Initial	Mean	RMSE	Bias
a_mle	0.9	1.2203702	0.5802119	0.3203702
b_mle	0.6	0.8771945	0.4353599	0.2771945
α_mle	1.2	0.7371512	0.6996620	0.4628488
θ_mle	3.5	5.466114	8.839013	1.966114
a_olse	0.9	1.1141102	0.6388656	0.2141102
b_lose	0.6	0.8368312	0.4065096	0.2368312
α_olse	1.2	0.7745948	0.7166976	0.4254052
θ_olse	3.5	4.2174506	1.7004163	0.7174506
a_wlse	0.9	1.0394679	0.6703240	0.1394679
b_wlse	0.6	0.8686625	0.4946095	0.2686625
α_wlse	1.2	0.8474542	0.6967895	0.3525458
θ_wlse	3.5	4.1729091	1.8707508	0.6729091
a_cvme	0.9	1.3432334	0.8126958	0.4432334
b_cvme	0.6	0.7943335	0.3457040	0.1943335
α_cvme	1.2	0.8248041	0.6917055	0.3751959
θ_cvme	3.5	4.3377077	2.2745812	0.8377077
a_rade	0.9	0.9562398	0.4542371	0.0562398
b_rade	0.6	0.8044597	0.4033533	0.2044597
α_rade	1.2	0.8593951	0.6972537	0.3406049
θ_rade	3.5	4.0198615	1.4850917	0.5198615

It is evident from the values displayed in Tables (12,13,14,15) that the distribution performs better than the comparison distributions. Because the proposed extended distribution has the biggest p-value and the lowest values based on informational and statistical criteria, it offers an appropriate depiction.

Table 10: Simulation result for sample size 200, iterations 1000 and initial value 0.9 0.6 1.2 3.5

Name	Initial	Mean	RMSE	Bias
a_mle	0.9	1.0920209	0.4117221	0.1920209
b_mle	0.6	0.8713124	0.4454021	0.2713124
α_mle	1.2	0.8627722	0.5639909	0.3372278
θ_mle	3.5	3.9840451	2.2961452	0.4840451
a_olse	0.9	1.2335683	0.6083143	0.3335683
b_lose	0.6	0.8096269	0.3364980	0.2096269
α_olse	1.2	0.7532842	0.6799320	0.4467158
θ_olse	3.5	4.2815115	1.4433577	0.7815115
a_wlse	0.9	1.0630597	0.5592175	0.1630597
b_wlse	0.6	0.8257620	0.4330405	0.2257620
α_wlse	1.2	0.8468562	0.6416692	0.3531438
θ_wlse	3.5	4.2029553	1.9925126	0.7029553
a_cvme	0.9	1.2650991	0.6299762	0.3650991
b_cvme	0.6	0.8313567	0.3965089	0.2313567
α_cvme	1.2	0.7559804	0.6929347	0.4440196
θ_cvme	3.5	4.1014559	1.3755361	0.6014559
a_rade	0.9	1.0351232	0.4803006	0.1351232
b_rade	0.6	0.8449523	0.4652328	0.2449523
α_rade	1.2	0.8417786	0.6199099	0.3582214
θ_rade	3.5	3.8543586	1.4061220	0.3543586

Table 11: Descriptive statistic for the two datasets.

Var	n	mean	median	min	max	skew	kurtosis
Data1	20	2.1	2.21	0.48	3.22	- 0.83	0.93
Data2	85	2.56	2.38	0.04	4.66	0.09	- 0.69

Table 12: The p-value and K-S, W value of the failure times of 20 components.

Model	W	A	K-S	p-value
[0,1]TNHGo	0.0443	0.2996	0.1100	0.9471
BeGo	0.0606	0.4013	0.1205	0.9001
KuGo	0.1039	0.6417	0.2062	0.3174
EGGo	0.0607	0.4018	0.1209	0.8984
WeGo	0.0626	0.4231	0.1281	0.8570
GoGo	0.0860	0.5973	0.1838	0.4551

Table 13: Values of statistical criteria for the failure times of 20 components.

Model	-LL	AIC	CAIC	BIC	HQIC
[0,1]TNHGo	15.82	39.65	42.31	43.63	40.42
BeGo	16.31	40.63	43.30	44.620	41.41
KuGo	17.53	43.07	45.73	47.05	43.84
EGGo	16.32	40.64	43.31	44.627	41.42
WeGo	16.43	40.87	43.54	44.85	41.65
GoGo	17.82	43.64	46.31	74.63	44.42

7 Conclusion

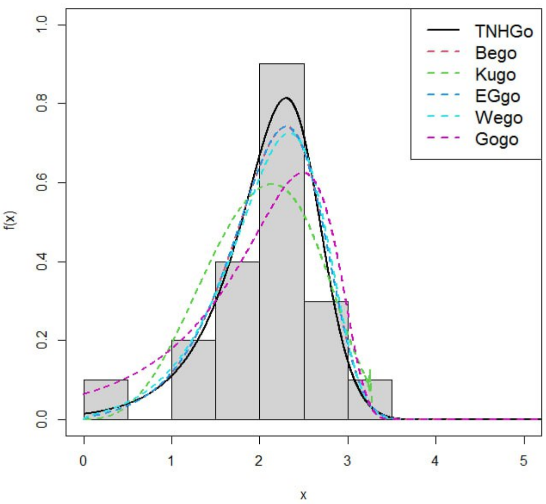
This paper introduced a new four-parameter Gompertz distribution (THNGo) generated within the truncated Nadarajah Haghighi–G (TNHG) family. The model accommodates a wide spectrum of shapes for the PDF and CDF including pronounced right-skewness and varying tail behavior while retaining analytical tractability. In particular, we derived a closed-form quantile function, enabling straightforward random-variate generation and facilitating simulation-based inference (e.g., Monte Carlo experiments and bootstrap accuracy assessments). We established core distributional properties (moments and incomplete moments, order statistics) and examined entropy-based measures (Rényi, Arimoto,

Table 14: The p-value and K-S, W value of the failure times of 84 Aircraft Windshield.

Model	W	A	K-S	p-value
[0,1]TNHGo	0.0908	0.5996	0.0845	0.5782
BeGo	0.0971	0.6607	0.0855	0.5623
KuGo	0.1021	0.6868	0.0859	0.5570
EGGo	0.1023	0.6876	0.0854	0.5636
WeGo	0.1048	0.7015	0.0850	0.5707
GoGo	0.1916	1.1464	0.0896	0.5009

Table 15: Values of statistical criteria for the failure times of 84 Aircraft Windshield.

Model	-LL	AIC	CAIC	BIC
[0,1]TNHGo	126.96	261.93	261.43	271.70
BeGo	128.21	264.42	264.92	247.19
KuGo	128.31	264.62	265.12	274.39
EGGo	128.36	264.72	265.22	274.49
WeGo	128.38	264.77	265.27	274.54
GoGo	129.35	266.71	267.21	276.48



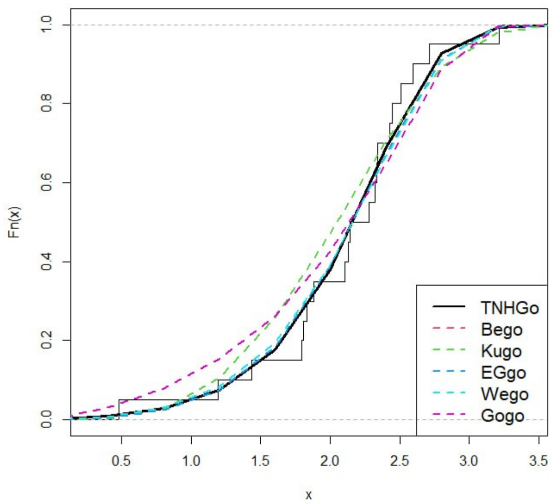


Fig. 4: Fitted THNGo pdf and cdf, respectively for the failure times of 20 components data.

and Tsallis). Numerical evaluations of these entropies across parameter ranges exhibit coherent trends that, within the examined settings, indicate higher certainty/stability for larger parameter values and higher entropy orders. On the empirical side, case studies on the failure times of 20 components and 84 aircraft windshields show that the proposed model delivers flexible and competitive fits relative to five benchmark alternatives Beta-Gompertz, Kumaraswamy-Gompertz, Exponentiated Generalized Gompertz, Weibull-Gompertz, and Gompertz-Gompertz as reflected by standard diagnostics (log-likelihood, AIC/BIC, and KS-based measures). Overall, the THNGo distribution provides a versatile alternative for reliability and survival analyses where skewness and tail behavior matter. Promising directions include (i) regression and accelerated-life formulations, (ii) Bayesian estimation and uncertainty quantification, (iii) multivariate coupled constructions via copulas, (iv) stress strength reliability, and (v) enhanced goodness-of-fit diagnostics under model misspecification.

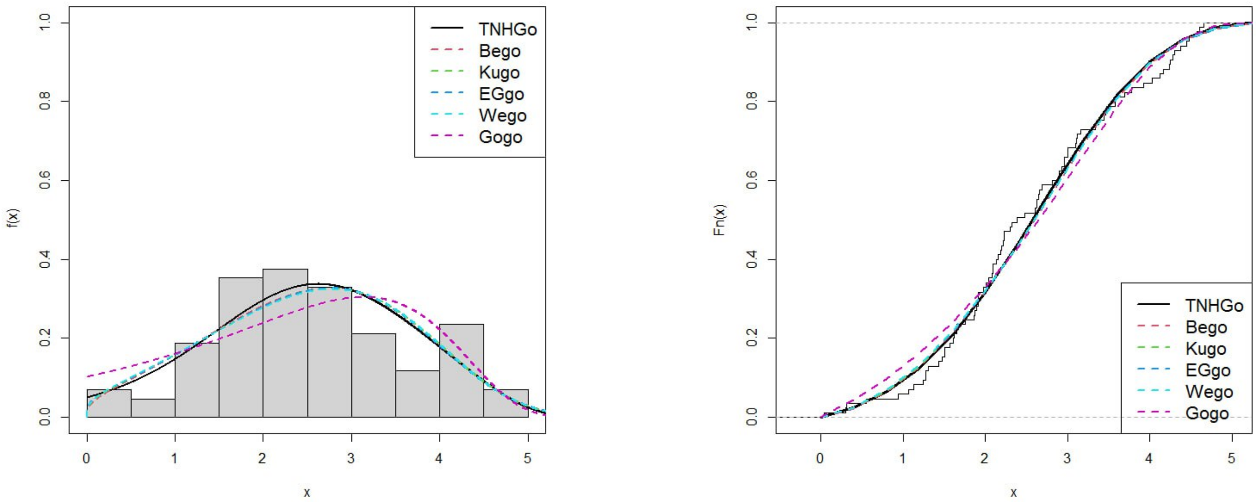


Fig. 5: Fitted THNGo pdf and cdf for the failure times of 84 Aircraft Windshield data.

References

[1] Ahsan ul Haq, M., & Aslam, M. Comparison of Entropy Measures for Gompertz and Truncated Gompertz Distribution. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, 93(1), 113–120, 2023.

[2] Al-Habib, K. H., Khaleel, M. A., & Al-Mofleh, H. A new family of truncated nadarajah-haghighi-g properties with real data applications. *Tikrit Journal of Administrative and Economic Sciences*, 19(61), 2, 2023.

[3] Alizadeh, M., Tahmasebi, S., Kazemi, M. R., Nejad, H. S. A., & Hamedani, G. H. G. The odd log-logistic gompertz lifetime distribution: properties and applications. *Studia Scientiarum Mathematicarum Hungarica*, 56(1), 55–80, 2019.

[4] Alotaibi, N., Al-Moisheer, A. S., Elbatal, I., Shrahili, M., Elgarhy, M., & Almetwally, E. M. Half logistic inverted Nadarajah–Haghighi distribution under ranked set sampling with applications. *Mathematics*, 11(7), 1693, 2023.

[5] Arshad, M. Z., Iqbal, M. Z., & Al Mutairi, A. A comprehensive review of datasets for statistical research in probability and quality control. *Journal of Mathematical and Computer Science*, 11(3), 3663–3728, 2021.

[6] Atanda, O. D., Mabur, T. M., & Onwuka, G. I. A New Odd Lindley-Gompertz Distribution: Its Properties and Applications. *Asian Journal of Probability and Statistics*, 29–47, 2020.

[7] Bantan, R. A., Jamal, F., Chesneau, C., & Elgarhy, M. Theory and applications of the unit gamma/Gompertz distribution. *Mathematics*, 9(16), 1850, 2021.

[8] Eghwerido, J. T., Nzei, L. C., & Agu, F. I. The alpha power Gompertz distribution: characterization, properties, and applications. *Sankhya A*, 83(1), 449–475, 2021.

[9] El-Bassiouny, A. H., Medhat, E. L., Mustafa, A., & Eliwa, M. S. Characterization of the generalized Weibull-Gompertz distribution based on the upper record values. 55, 7, 2015.

[10] El-Gohary, A., Alshamrani, A., & Al-Otaibi, A. N. The generalized Gompertz distribution. *Applied Mathematical Modelling*, 37(1-2), 13–24, 2013.

[11] Eraikhuemen, I. B., Chama, A. A. F., Asongo, A. I., Yakura, B. S., & Bala, A. H. Properties and applications of a transmuted power Gompertz distribution. *Asian Journal of Probability and Statistics*, 7(1), 41–58, 2020.

[12] Habib, K. H., Salih, A. M., Khaleel, M. A., & Abdal-hammed, M. K. OJCA Rayleigh distribution, Statistical Properties with Application. *Tikrit Journal of Administration and Economics Sciences*, 19(Special Issue part 5), 2023.

[13] Ieren, T. G., Kromtit, F. M., Agbor, B. U., Eraikhuemen, I. B., & Koleoso, P. O. A power Gompertz distribution: Model, properties and application to bladder cancer data. *Asian Research Journal of Mathematics*, 15(2), 1–14, 2019.

[14] Ismael, A. A., & AL-Bairmani, Z. A. A. New extension for Chen distribution based on [0,1] Truncated Nadarajah-Haghighi-G family with two real data applications, 2023.

[15] Jafari, A. A., Tahmasebi, S., & Alizadeh, M. The beta-Gompertz distribution. *arXiv preprint arXiv:1407.0743*, 2014.

[16] Joshi, R. K., & Kumar, V. Lindley Gompertz distribution with properties and applications. *International Journal of Statistics and Applied Mathematics*, 5(6), 28–37, 2020.

[17] Mahdy, M., Fathy, E., & Eltelbany, D. S. A New Bivariate Odd Generalized Exponential Gompertz Distribution, 2023.

[18] Maqbool, A., Ali, Z., & Rani, A. Modified-Weibull Distribution with Applications on Real Life Data Sets. *Journal of Statistics*, 27, 103, 2023.

[19] Mazucheli, J., Menezes, A. F., & Dey, S. Unit-Gompertz distribution with applications. *Statistica*, 79(1), 25–43, 2019.
[20] Meraou, M. A., Alshahrani, F., Almanjahie, I. M., & Attouch, M. K. The Exponential TX Gompertz Model for Modeling Real Lifetime Data: Properties and Estimation. *Chiang Mai Journal of Science*, 50(5), 2023.
[21] Nzei, L. C., Eghwerido, J. T., & Ekhosuehi, N. Topp-Leone Gompertz Distribution: Properties and Applications. *Journal of Data Science*, 18(4), 782–794, 2020.
[22] Pasupuleti, S. S., & Pathak, P. Special form of Gompertz model and its application. *Genus*, 66(2), 2011.
[23] Marshall, A. W., & Olkin, I. A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. *Biometrika*, 84(3), 641–652, 1997.
[24] Atchadé, M. N., Agbahide, A. A., Otodji, T., Bogninou, M. J., & Moussa Djibril, A. A new shifted Lomax-X family of distributions: Properties and applications to actuarial and financial data. *Computational Journal of Mathematical and Statistical Sciences*, 4(1), 41–71, 2025.
[25] Alyami, S. A., Elbatal, I., Alotaibi, N., Almetwally, E. M., & Elgarhy, M. Modeling total factor productivity of the United Kingdom food chain: Using a new lifetime-generated family of distributions. *Sustainability*, 14(14), 8942, 2022.
[26] Bourguignon, M., Silva, R. B., & Cordeiro, G. M. The Weibull-G family of probability distributions. *Journal of Data Science*, 12(1), 53–68, 2014.
[27] Soliman, D., Hegazy, M., AL-Dayian, G., & EL-Helbawy, A. Statistical properties and applications of a new truncated Zubair-generalized family of distributions. *Computational Journal of Mathematical and Statistical Sciences*, 4(1), 222–257, 2025.
[28] Almarashi, A. M., Jamal, F., Chesneau, C., & Elgarhy, M. A new truncated Muth generated family of distributions with applications. *Complexity*, 2021(1), 1211526, 2021.
[29] Alizadeh, M., Cordeiro, G. M., Nascimento, A. D., Lima, M. D. C. S., & Ortega, E. M. Odd-Burr generalized family of distributions with some applications. *Journal of Statistical Computation and Simulation*, 87(2), 367–389, 2017.
[30] Alsolmi, M. A new logarithmic tangent-U family of distributions with reliability analysis in engineering data. *Computational Journal of Mathematical and Statistical Sciences*, 4(1), 258–282, 2025.
[31] Al-Moisheer, A. S., Elbatal, I., Almutiry, W., & Elgarhy, M. Odd inverse power generalized Weibull generated family of distributions: Properties and applications. *Mathematical Problems in Engineering*, 2021(1), 5082192, 2021.
[32] Ramos, M. W. A., Marinho, P. R. D., Cordeiro, G. M., da Silva, R. V., & Hamedani, G. G. The Kumaraswamy-G Poisson family of distributions. *Journal of Statistical Theory and Applications*, 14(3), 222–239, 2015.
[33] Nasir, M. A., Jamal, F., Silva, G. O., & Tahir, M. H. Odd Burr-G Poisson family of distributions. *Journal of Statistics Applications and Probability*, 7(1), 9–28, 2018.
[34] Nasir, M. A., Tahir, M. H., Jamal, F., & Ozel, G. A new generalized Burr family of distributions for the lifetime data. *Journal of Statistics Applications and Probability*, 6(2), 401–417, 2017.
[35] Benchiha, S., Sapkota, L. P., Al Mutairi, A., Kumar, V., Khashab, R. H., Gemeay, A. M., ... & Nassr, S. G. A new sine family of generalized distributions: Statistical inference with applications. *Mathematical and Computational Applications*, 28(4), 83, 2023.
[36] Alghamdi, S. M., Shrahili, M., Hassan, A. S., Mohamed, R. E., Elbatal, I., & Elgarhy, M. Analysis of milk production and failure data: Using unit exponentiated half logistic power series class of distributions. *Symmetry*, 15(3), 714, 2023.
[37] Bantan, R. A., Jamal, F., Chesneau, C., & Elgarhy, M. On a new result on the ratio exponentiated general family of distributions with applications. *Mathematics*, 8(4), 598, 2020.
[38] Alsadat, N., Imran, M., Tahir, M. H., Jamal, F., Ahmad, H., & Elgarhy, M. Compounded Bell-G class of statistical models with applications to COVID-19 and actuarial data. *Open Physics*, 21(1), 20220242, 2023.
[39] Elbatal, I., Elgarhy, M., & Kibria, B. G. Alpha power transformed Weibull-G family of distributions: Theory and applications. *Journal of Statistical Theory and Applications*, 20(2), 340–354, 2021.
[40] Ahmad, Z., Hamedani, G., & Elgarhy, M. The weighted exponentiated family of distributions: Properties, applications and characterizations. *Journal of the Iranian Statistical Society*, 19(1), 209–228, 2022.
[41] Elbatal, I., Khan, S., Hussain, T., Elgarhy, M., Alotaibi, N., Semary, H. E., & Abdelwahab, M. M. A new family of lifetime models: Theoretical developments with applications in biomedical and environmental data. *Axioms*, 11(8), 361, 2022.
[42] Bakr, M. E., Al-Babtain, A. A., Mahmood, Z., Aldallal, R. A., Khosa, S. K., Abd El-Raouf, M. M., ... & Gemeay, A. M. Statistical modelling for a new family of generalized distributions with real data applications. *Mathematical Biosciences and Engineering*, 19, 8705–8740, 2022.
[43] Ahmad, A., Alsadat, N., Atchade, M. N., ul Ain, S. Q., Gemeay, A. M., Meraou, M. A., ... & Hussam, E. New hyperbolic sine-generator with an example of Rayleigh distribution: Simulation and data analysis in industry. *Alexandria Engineering Journal*, 73, 415–426, 2023.
[44] El-Morshedy, M., Eliwa, M. S., & Tyagi, A. A discrete analogue of odd Weibull-G family of distributions: Properties, classical and Bayesian estimation with applications to count data. *Journal of Applied Statistics*, 49(11), 2928–2952, 2022.
[45] Zaidi, S. M., Sobhi, M. M. A., El-Morshedy, M., & Afify, A. Z. A new generalized family of distributions: Properties and applications. *Aims Mathematics*, 6(1), 456–476, 2021.
[46] Elgarhy, M., Hassan, A. S., Alsadat, N., Balogun, O. S., Shawki, A. W., & Ragab, I. E. A heavy tailed model based on power XLindley distribution with actuarial data applications. *Computer Modeling in Engineering & Sciences*, 142(3), 2547–2583, 2025.
[47] Elgarhy, M., Hassan, A. S., & Nagy, H. Parameter estimation methods and applications of the power Topp–Leone distribution. *Gazi University Journal of Science*, 35(2), 731–746, 2022.
[48] Elgarhy, M., Elbatal, I., Hamedani, G. G., & Hassan, A. S. On the exponentiated Weibull Rayleigh distribution. *Gazi University Journal of Science*, 32(3), 1060–1081, 2019.

[49] Elgarhy, M., Haq, M., Elbatal, I., & Hassan, A. S. Transmuted Kumaraswamy quasi Lindley distribution with applications. *Annals of Data Science*, doi:10.1007/s40745-018-0153-4, 2018.
[50] Elgarhy, M., Haq, M., & Perveen, I. Type II half-logistic exponential distribution with applications. *Annals of Data Science*, doi:10.1007/s40745-018-0154-4, 2018.
[51] Elgarhy, M., Nasir, M. A., Jamal, F., & Ozel, G. Type II Topp–Leone generated family of distributions with applications. *Journal of Statistics and Management Systems*, 21(8), 1529–1551, 2018.
[52] Elgarhy, M., Sharma, V. K., & Elbatal, I. Transmuted Kumaraswamy Lindley distribution with application. *Journal of Statistics & Management Systems*, 21(6), 1083–1104, 2018.
[53] Elgarhy, M., Alsadat, N., Hassan, A. S., Chesneau, C., & Abdel-Hamid, A. H. A new asymmetric modified Topp–Leone distribution: Classical and Bayesian estimations under progressive type-II censored data with applications. *Symmetry*, 15, 1396, 2023.
[54] Elgarhy, M., Elbatal, I., Kayid, M., & Muhammad, M. A new extended Fréchet model with different estimation methods and applications. *Heliyon*, 10(16), e36348, 2024. doi:10.1016/j.heliyon.2024.e36348.
[55] Elgarhy, M., Metwally, D. S., Ragab, I. E., Bashiru, S. O., Semary, H. E., & Gemeay, A. M. The DUS-transformed generalized linear failure rate distribution: Properties, estimation, and applications. *Scientific African*, e02891, 2025.
[56] Elgarhy, M., Abdalla, G. S. S., Obulezi, O. J., & Almetwally, E. M. A tangent DUS family of distributions with applications to the Weibull model. *Scientific African*, e02843, 2025.
[57] Elgarhy, M., Abdalla, G. S. S., Almetwally, E. M., Jobarteh, M., & Mubarak, A. E. Reliability analysis and optimality for a new extended Topp–Leone distribution based on progressive censoring with binomial removal. *Engineering Reports*, 7(6), e70239, 2025.
[58] Elgarhy, M., Metwally, D. S., Hassan, A. S., & Shawki, A. W. A new generalization of power Chris–Jerry distribution with different estimation methods, simulation and applications. *Scientific African*, e02769, 2025.
[59] Elgarhy, M., Abdalla, G., Hassan, A., & Almetwally, E. Reliability inference of the inverted power Burr X distribution under the hybrid censoring approach with applications. *Journal of Applied Mathematics and Computational Mechanics*, 24(3), 46–75, 2025. doi:10.17512/jamcm.2025.3.04.
[60] Onyekwere, C. K., Aguwa, O. C., & Obulezi, O. J. An Updated Lindley Distribution: Properties, Estimation, Acceptance Sampling, Actuarial Risk Assessment and Applications. *Innovation in Statistics and Probability*, 1(1), 1–27, 2025.
[61] Saboor, A., Jamal, F., Shafq, S., & Mumtaz, R. On the Versatility of the Unit Logistic Exponential Distribution: Capturing Bathtub, Upside-Down Bathtub, and Monotonic Hazard Rates. *Innovation in Statistics and Probability*, 1(1), 28–46, 2025.
[62] Noori, N. A., Khaleel, M. A., Khalaf, S. A., & Dutta, S. Analytical Modeling of Expansion for Odd Lomax Generalized Exponential Distribution in Framework of Neutrosophic Logic: A Theoretical and Applied on Neutrosophic Data. *Innovation in Statistics and Probability*, 1(1), 47–59, 2025.
[63] Husain, Q. N., Qaddoori, A. S., Noori, N. A., Abdullah, K. N., Suleiman, A. A., & Balogun, O. S. New Expansion of Chen Distribution According to the Neutrosophic Logic Using the Gompertz Family. *Innovation in Statistics and Probability*, 1(1), 60–75, 2025.
[64] Orji, G. O., Etaga, H. O., Almetwally, E. M., Igbokwe, C. P., Aguwa, O. C., & Obulezi, O. J. A New Odd Reparameterized Exponential Transformed-X Family of Distributions with Applications to Public Health Data. *Innovation in Statistics and Probability*, 1(1), 88–118, 2025.
[65] Gemeay, A. M., Moakofi, T., Balogun, O. S., Ozkan, E., & Hossain, M. M. Analyzing Real Data by a New Heavy-Tailed Statistical Model. *Modern Journal of Statistics*, 1(1), 1–24, 2025.
[66] Sapkota, L. P., Kumar, V., Tekle, G., Alrweili, H., Mustafa, M. S., & Yusuf, M. Fitting Real Data Sets by a New Version of Gompertz Distribution. *Modern Journal of Statistics*, 1(1), 25–48, 2025.
[67] M. A., M., Al-Kandari, N. M., & Mohammad, R. Z. Fundamental Properties of the Characteristic Function using the Compound Poisson Distribution as the Sum of the Gamma Model. *Modern Journal of Statistics*, 1(1), 49–57, 2025.
[68] Rocha, R., Nadarajah, S., Tomazella, V., Louzada, F., & Eudes, A. New defective models based on the Kumaraswamy family of distributions with application to cancer data sets. *Statistical Methods in Medical Research*, 26(4), 1737–1755, 2017.
[69] Al-Sadat, N. A new modified model with application to engineering data sets, 2023.

Saeed E. Hemeda is Associate Professor of Statistics at Al-Obour University and Al-Obour Higher Institute for Management, Computers, and Information Systems - Faculty of Computers and Artificial Intelligence. He holds a Ph.D. in Mathematical Statistics from Cairo University. His research interests include mathematical statistics, parameter estimation, sampling, and compound distributions and their applications in various fields. He has numerous published papers in scientific journals.

Gaber Sallam Salem Abdalla is an Assistant Professor in the Department of Insurance and Risk Management, Faculty of Business, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia. His professional work focuses on insurance, risk management, and actuarial science, with active engagement in teaching, curriculum development, and applied research that connects academic theory to industry practice. His research interests include risk modelling, insurance economics, loss distribution analysis, and enterprise risk management. He has contributed to peer-reviewed publications and participates in academic service and collaborative projects aimed at improving insurance education and practice.

Khalaf H. Al-Habib is Lecturer of Mathematical Statistics, Tikrit University, College of Computer Science and Mathematics. Research interests: Mathematical Statistics, Statistical probability distributions, Different methods for estimating parameters.I had published many research papers in Statistics.

Hisham A. Gamelan IT Consultant at Obour Higher Institutes with 15+ years' experience in teaching, training, e-learning systems, and data analytics. specializes in data analysis and computer science, has authored 15+ published papers.

Ahmed S. Abdelzaheris an Assistant Professor in the Financial Sciences Department at The Applied College, Imam Abdulrahman Bin Faisal University. He is committed to delivering high-quality education in finance and financial management, focusing on practical student training through projects and applied internships. He is involved in academic supervision and curriculum development aligned with labor-market needs, and contributes to applied research that supports the financial sector and wider community.

Suleman Nasiru is a distinguished Ghanaian academic and Professor of Statistics at the C. K. Tedam University of Technology and Applied Sciences (CKT-UTAS), where he has served in various capacities since 2020. He holds a PhD in Mathematics (Statistics) from the Pan African University, Institute for Basic Sciences, Technology and Innovation (PAUISTI), Kenya, an MSc in Applied Statistics, and a First Class Honours BSc in Mathematical Science (Statistics) from the University for Development Studies, Ghana. He is a recipient of the Best PhD Thesis Award in Mathematics from PAUISTI. His research spans statistical distributions, regression modeling, extreme value analysis, and data science. He is an active member of professional bodies including the Ghana Statistical Association, Statistics Without Borders and Ghana Young Academy, and serves as a reviewer for numerous international journals. Professor Nasiru has participated in and delivered keynote addresses at several international conferences and has contributed significantly to academic leadership, curriculum development, and tertiary education accreditation in Ghana. He continues to mentor students and advance research in statistics and data science.