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A Laplace Transform Approach to the Biparametric V-Derivative with Applications in Generalized Calculus

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Abstract: This article introduces the definition of the Laplace transform with two parameters, developed within the framework of a new generalized local derivative of two parameters. Its fundamental properties are analyzed, and its application in solving differential equations with two parameters is explored.

Keywords: biparametric derivative, biparametric integral, biparametric Laplace transform.

1 Preliminaries

In mathematics, the Laplace transform is a widely used technique that converts functions defined in the time domain, τ, into functions in the complex frequency domain, s, through the integral:

$$\mathcal{L}\{g(\tau)\} = \mathcal{G}(s) = \int_0^\infty g(\tau)e^{-s\tau}\, d\tau.$$

From an applied perspective, the Laplace transform facilitates the analysis and design of systems by providing an alternative representation that simplifies computations. For instance, by transforming time-domain functions into the frequency domain, it converts differential equations into algebraic ones, and convolution operations into multiplications, which proves particularly useful.

One of the main advantages of the Laplace transform lies in its ability to incorporate initial conditions directly into the algebraic solution, thus avoiding additional calculations. Moreover, it is especially valuable for analyzing linear systems such as electrical circuits, mechanical systems, or control models.

Fractional calculus, which extends the concepts of derivatives and integrals to non-integer orders, originates from a letter written by Leibniz to L'Hôpital in 1695. In that correspondence, Leibniz introduced the idea of derivatives of fractional order, thereby opening a new perspective in mathematical analysis. During the 18th and 19th centuries, mathematicians such as Euler, Fourier, and Liouville developed formalisms and applications that consolidated this area. Initially of theoretical interest, fractional calculus today finds practical applications in disciplines such as physics, biology, and signal processing.

Fractional derivatives of non-integer order can be understood from two main approaches: global (or classical) and local. Global derivatives, such as those of Riemann–Liouville and Caputo, are defined via integral transforms like Fourier or Mellin, which endow them with a non-local nature characterized by "memory". These are linked to the origins of fractional calculus, developed by mathematicians such as Euler, Laplace, Fourier, Abel, and Liouville. Recently, these notions have been extended and applied in numerous areas (see [3], [2], [7], [13], [19], [23], [29]). There are also efforts to extend the classical notion of the Laplace transform to the fractional setting (see [25]).

In contrast, local derivatives—such as the conformable fractional derivative introduced by Khalil [15]—are based on an incremental approach and are defined via the following limit:

$$D_\zeta(\hbar)(\tau) = \lim_{\varepsilon\to 0} \frac{\hbar(\tau+\varepsilon\tau^{1-\zeta}) - \hbar(\tau)}{\varepsilon}, \quad \zeta \in (0,1),\ \tau > 0.$$

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Abdeljawad extended this theory (see [1]) by developing concepts such as left and right derivatives, higher-order integrals (for $\zeta > 1$), Taylor series, chain rule, and integration by parts formulas. Additionally, there have been numerous advances in fractional calculus (see [4], [5],, [9], [10], [11], [13], [18], [19], [21], [20], [22], [24], [27], [28], [29], [31], [30], [32], [34], [36]).

Moreover, considerable effort has been made to extend the Laplace transform to the fractional context, where such tools have become fundamental in modern applications (see [8], [12], [14], [16], [17], [23], [26], [33]).

The following introduces the definition of the biparametric derivative and its fundamental properties (see [35]), which are essential to establish the main results of this paper.

Definition 1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function, with $\zeta \geq 0$ and $\chi > 0$. The biparametric derivative of g is defined as*

$$V^{\zeta,\chi}(g(\tau)) := \lim_{h\to 0} \frac{(\chi + h(\chi - \zeta))\, g\left(\tau + h\frac{\zeta}{\chi}\right) - \chi g(\tau)}{\chi \cdot h},$$

provided that the limit exists.

Remark. If g is differentiable, then

$$V^{\zeta,\chi}(g(\tau)) = \frac{\zeta}{\chi} g'(\tau) + \frac{\chi - \zeta}{\chi} g(\tau),$$

where $g'(\tau) = \lim_{h\to 0} \dfrac{g(\tau + h) - g(\tau)}{h}$ denotes the classical derivative of g.

If the biparametric derivative $V^{\zeta,\chi}$ exists, the function g is said to be (ζ, χ)-differentiable.

The chain rule for biparametric derivatives, along with several additional results, can be found in [35] and are stated below.

Theorem 1. *If g is a (ζ, χ)-differentiable function and f is differentiable at $g(\tau)$, then $f \circ g$ is (ζ, χ)-differentiable, and*

$$V^{\zeta,\chi}(f \circ g(\tau)) = \frac{\zeta}{\chi} f'(g(\tau)) g'(\tau) + \frac{(\chi - \zeta)}{\chi} f(g(\tau)).$$

Theorem 2. *The operator $V^{\zeta,\chi}$ satisfies the following properties:*

a) $V^{\zeta,\chi}(v f(\tau) + w g(\tau)) = v V^{\zeta,\chi}\{(\tau) + w V^{\zeta,\chi} g(\tau)$

b) $V^{\zeta,\chi}(f \cdot g)(\tau) = \dfrac{\zeta}{\chi}[f'(\tau)g(\tau) - f(\tau)g'(\tau)] + \dfrac{\chi - \zeta}{\chi} f(\tau)g(\tau)$

c) $V^{\zeta,\chi}\left(\dfrac{f}{g}\right)(\tau) = \dfrac{\zeta}{\chi}\left[\dfrac{f'(\tau)g(\tau) - f(\tau)g'(\tau)}{[g(\tau)]^2}\right] + \dfrac{\chi - \zeta}{\chi}\left[\dfrac{f(\tau)}{g(\tau)}\right], \quad g(\tau) \neq 0$

d) $V^{\zeta,\chi}(k) = \dfrac{\chi - \zeta}{\chi} k, \quad k$ *constant*

e) $V^{\zeta,\chi}(\tau^n) = n\dfrac{\zeta}{\chi}\tau^{n-1} + \dfrac{\chi - \zeta}{\chi}\tau^n, \quad n \in \mathbb{R}$

f) $V^{\zeta,\chi}(e^\tau) = e^\tau$

g) $V^{\zeta,\chi}(\sin(\tau)) = \dfrac{\zeta}{\chi}\cos(\tau) + \dfrac{\chi - \zeta}{\chi}\sin(\tau)$

h) $V^{\zeta,\chi}(\cos(\tau)) = -\dfrac{\zeta}{\chi}\sin(\tau) + \dfrac{\chi - \zeta}{\chi}\cos(\tau)$

We now introduce the definition of the biparametric integral, as defined in [35].

Definition 1.2 Let g be a continuous function defined on $[u, v]$. The (ζ, χ)-integral, or biparametric integral, denoted by $I_u^{\zeta,\chi}(g)$, is defined by the integral

$$I_u^{\zeta,\chi}(g(\tau)) := \frac{\chi}{\zeta} e^{-\frac{(\chi-\zeta)}{\zeta}\tau} \int_u^\tau e^{\frac{(\beta-\alpha)}{\alpha}y} g(y)dy$$

The following result is analogous to that of classical calculus.

Theorem 3. *Let g be a continuous function defined on $[u, v]$. Then, $I_u^{\zeta,\chi}(g)$ is (ζ, χ)-differentiable on (u, v), and the following holds:*

$$V_u^{\zeta,\chi}\left(I_u^{\zeta,\chi} g(y)\right) = g(y).$$

Moreover, if h is a continuous function and h is the (ζ, χ)-derivative of g on (u, v), that is, $h = V_u^{\zeta,\chi}(g)$, then we have

$$I_u^{\zeta,\chi}\left(V_u^{\zeta,\chi} g(\tau)\right) = g(\tau) - g(u)e^{\frac{(\chi-\zeta)}{\zeta}(u-\tau)}$$

We now present the biparametric integrals of some functions (see [35]).

Theorem 4. *Let $\zeta \geq 0$, $\chi > 0$, then we have:*

1) $I_u^{\zeta,\chi}(\sin(\tau)) = \dfrac{\zeta\chi}{(\zeta^2 + \chi^2) + \zeta^2}\left[\left(\dfrac{\chi - \zeta}{\zeta}\sin(\tau) - \cos(\tau)\right) + \left(\cos(u) + \dfrac{\chi - \zeta}{\zeta}\sin(u)\right)e^{\frac{\chi-\zeta}{\zeta}(u-\tau)}\right]$

2) $I_u^{\zeta,\chi}(e^\tau) = e^\tau - e^u e^{\frac{\chi-\zeta}{\zeta}(u-\tau)}$

3) $I_u^{\zeta,\chi}(\lambda) = \dfrac{\chi\lambda}{\chi - \zeta}\left[1 - e^{\frac{\chi-\zeta}{\zeta}(u-\tau)}\right]$

provided that $\zeta \neq \chi$

2 New Results

Definition 2. *Let $\zeta \in (0,1)$, $\chi \in (0,1)$, and c be a real number. We define the (ζ,χ)-exponential function as follows:*

$$E_{\zeta,\chi}(c,\tau) = e^{c\tau} e^{\frac{-(\chi-\zeta)}{\zeta}\tau}$$

Definition 3. *A function is said to be of generalized (ζ,χ)-exponential order if there exist constants M and a such that*

$$|g(\tau)| \leq M E_{\zeta,\chi}(v,\tau)$$

for sufficiently large τ.

We now define the biparametric Laplace Transform, also referred to as the (ζ,χ)-Laplace Transform.

Definition 4. *Let $\zeta \in (0,1)$, $\chi \in (0,1)$, $s \in \mathbb{C}$. Let g be a function defined for $\tau \geq 0$.*
If the integral

$$I_0^{\zeta,\chi}(E_{\zeta,\chi}(-s,\tau)g(\tau))(+\infty) = \int_0^{+\infty} E_{\zeta,\chi}(-s,\tau)g(\tau)\,d\tau$$
$$= \int_0^{+\infty} e^{-s\tau} e^{-\frac{(\chi-\zeta)}{\zeta}\tau} g(\tau)\,d\tau$$

converges for a given value of s, then we define the function $\mathcal{G}$ by the expression

$$\mathcal{G}(s) = I_0^{\zeta,\chi}(E_{\zeta,\chi}(-s\tau)g(\tau))(+\infty),$$

and we write $\mathcal{G} = \mathcal{L}_{\zeta,\chi}(g)$.

The operator $\mathcal{L}_{\zeta,\chi}$ is called the biparametric Laplace Transform, and we say that $\mathcal{G}$ is the (ζ,χ)-Laplace Transform of g. Likewise, g is the (ζ,χ)-inverse Laplace Transform of $\mathcal{G}$, denoted by $g = \mathcal{L}_{\zeta,\chi}^{-1}(\mathcal{G})$, where $\mathcal{L}_{\zeta,\chi}^{-1}$ is the (ζ,χ)-inverse Laplace Transform operator and is defined as:

$$\mathcal{L}_{\zeta,\chi}^{-1}\left[\frac{1}{s+v}\right](\tau) = e^{-v\tau} e^{\frac{(\chi-\zeta)}{\zeta}\tau}$$

In order for Definition 4 to be meaningful, the following conditions must be satisfied:

– g must be piecewise continuous on the interval $(0,T]$ for any $T \in (0,+\infty)$.
– g must be of generalized (ζ,χ)-exponential order; that is, there exist positive constants M and v such that Definition 3 is satisfied with $\mathrm{Re}(v-c) < \dfrac{\chi-\zeta}{\zeta}$ and

$$|g(\tau)| \leq M E_{\zeta,\chi}(v\tau)$$

for all τ and $\zeta,\chi \in (0,1)$.

Therefore, the (ζ,χ)-Laplace Transform $\mathcal{G}(s)$ of g exists for $s > v - \dfrac{(\chi-\zeta)}{\zeta}$.

Indeed, since g is of generalized (ζ,χ)-exponential order, there exist constants $T > 0$, $K > 0$, and $v \in \mathbb{R}$ such that

$$|g(\tau)| \leq K E_{\zeta,\chi}(v,\tau)$$

for all $t \geq T$ and $\zeta,\chi \in (0,1)$. We write

$$\begin{aligned} I &= I_0^{\zeta,\chi}(E_{\zeta,\chi}(-s,\tau)g(\tau))(+\infty) \\ &= I_0^{\zeta,\chi}(E_{\zeta,\chi}(-s,\tau)g(\tau))(T) + \\ &\qquad I_T^{\zeta,\chi}(E_{\zeta,\chi}(-s,\tau)g(\tau))(+\infty) \\ &= I_1 + I_2 \end{aligned}$$

Since g is piecewise continuous, I_1 exists. For the integral I_2, note that for all $t \geq T$:

$$|E_{\zeta,\chi}(-s,\tau)g(\tau)| \leq K E_{\zeta,\chi}(-(s-v),\tau)$$

Therefore,

$$I_T^{\zeta,\chi}(E_{\zeta,\chi}(-s,\tau)g(\tau))(+\infty) \leq$$
$$K I_T^{\zeta,\chi}(E_{\zeta,\chi}(-(s-v),\tau))(+\infty) = \frac{K}{\left(s - v + \dfrac{(\chi-\zeta)}{\zeta}\right)},$$

whenever $s - v + \frac{\chi-\zeta}{\zeta} > 0$.

Thus, the integral I_2 converges absolutely for $s > v - \frac{(\chi-\zeta)}{\zeta}$, and since both I_1 and I_2 exist, it follows that I exists for $s > v - \frac{(\chi-\zeta)}{\zeta}$.

Therefore, g is said to be a (ζ,χ)-transformable function.

Theorem 5. *Let $\zeta,\chi \in (0,1)$. Then we have:*

a) $\mathcal{L}_{\zeta,\chi}[1](s) = \dfrac{1}{s + \dfrac{\chi-\zeta}{\zeta}}$, *from which it follows that* $\mathcal{L}_{\zeta,\chi}(c) = c\mathcal{L}_{\zeta,\chi}(1)$, *for all* $c \in \mathbb{R}$

b) $\mathcal{L}_{\zeta,\chi}[\tau^w](s) = \dfrac{\Gamma(w+1)}{\left(s + \dfrac{\chi-\zeta}{\zeta}\right)^{w+1}}$,
where the Gamma function is defined by:
$\Gamma(v,x) = \displaystyle\int_x^{\infty} e^{-\tau}\tau^{v-1}d\tau$,
$\Gamma(v,0) := \Gamma(v)$ *and* $w > -1$

c) $\mathcal{L}_{\zeta,\chi}[E_{\zeta,\chi}(c,\tau)](s) = \dfrac{1}{s - c + \dfrac{2(\chi-\zeta)}{\zeta}}$,
provided that $s - c + \dfrac{2(\chi-\zeta)}{\zeta} > 0$

d) $\mathcal{L}_{\zeta,\chi}[\sin(w\tau)](s) = \dfrac{w}{\left(s+\dfrac{\chi-\zeta}{\zeta}\right)^2 + w^2}$,

provided that $s + \dfrac{\chi-\zeta}{\zeta} > 0$

e) $\mathcal{L}_{\zeta,\chi}[\cos(w\tau)](s) = \dfrac{s+\dfrac{\chi-\zeta}{\zeta}}{\left(s+\dfrac{\chi-\zeta}{\zeta}\right)^2 + w^2}$,

provided that $s + \dfrac{\chi-\zeta}{\zeta} > 0$

f) $\mathcal{L}_{\zeta,\chi}[\sinh(w\tau)](s) = \dfrac{w}{\left(s+\dfrac{\chi-\zeta}{\zeta}\right)^2 + w^2}$,

provided that $s > \max\left(w - \dfrac{(\chi-\zeta)}{\zeta}, -w - \dfrac{\chi-\zeta}{\zeta}\right)$

g) $\mathcal{L}_{\zeta,\chi}[\cosh(w\tau)](s) = \dfrac{s+\dfrac{\chi-\zeta}{\zeta}}{\left(s+\dfrac{\chi}{\zeta}\right)^2 + w^2}$,

provided that $s > \max\left(w - \dfrac{(\chi-\zeta)}{\zeta}, -w - \dfrac{(\chi-\zeta)}{\zeta}\right)$

h) $\mathcal{L}_{\zeta,\chi}[e^{v\tau}](s) = \dfrac{1}{s - v + \dfrac{\chi-\zeta}{\zeta}}$

provided that $s - v + \dfrac{\chi-\zeta}{\zeta} > 0$

i) $\mathcal{L}_{\zeta,\chi}[e^{v\tau}\tau](s) = \dfrac{1}{\left(s - v + \dfrac{\chi-\zeta}{\zeta}\right)^2}$

provided that $s - v + \dfrac{\chi-\zeta}{\zeta} > 0$

j) $\mathcal{L}_{\zeta,\chi}[e^{v\tau}\tau^w](s) = \dfrac{\Gamma(w+1)}{\left(s - v + \dfrac{\chi-\zeta}{\zeta}\right)^{w+1}} = \dfrac{w!}{\left(s - v + \dfrac{\chi-\zeta}{\zeta}\right)^{w+1}}$

provided that $s - v + \dfrac{\chi-\zeta}{\zeta} > 0$

Proof. a) By definition, we have

$$\mathcal{L}_{\zeta,\chi}[1](s) = \int_0^{+\infty} e^{-\left(s+\frac{\chi-\zeta}{\zeta}\right)\tau}\, d\tau = \frac{1}{s+\frac{\chi-\zeta}{\zeta}},$$

provided that $s + \frac{\chi-\zeta}{\zeta} > 0$
b) The integral is related to the Gamma function:

$$\mathcal{L}_{\zeta,\chi}(s) = \int_0^{+\infty} e^{-\left(s+\frac{\chi-\zeta}{\zeta}\right)\tau}\tau^w\, d\tau = \frac{\Gamma(w+1)}{\left(s+\frac{\chi-\zeta}{\zeta}\right)^{w+1}},$$

provided that $s + \frac{\chi-\zeta}{\zeta} > 0$
c) Using the definition, we have

$$\begin{aligned}\mathcal{L}_{\zeta,\chi}[E_{\zeta,\chi}(c,\tau)](s) &= \\ &\int_0^{+\infty} e^{-s\tau}e^{-\frac{(\chi-\zeta)}{\zeta}\tau}e^{-c\tau}e^{-\frac{(\chi-\zeta)}{\zeta}\tau}\, d\tau \\ &= \int_0^{+\infty} e^{-\left(s-c+2\frac{(\chi-\zeta)}{\zeta}\right)\tau}\, d\tau \\ &= \frac{1}{s-c+2\frac{(\chi-\zeta)}{\zeta}},\end{aligned}$$

provided that $s - c + 2\frac{(\chi-\zeta)}{\zeta} > 0$
d) Taking into account that the classical Laplace transform of $sin(w\tau)$ is given by

$$\int_0^{+\infty} e^{-v\tau}\sin(w\tau)\, d\tau = \frac{w}{v^2+w^2},$$

then, applying this result in the context of the (ζ,χ)-Laplace transform, we obtain

$$\begin{aligned}\mathcal{L}_{\zeta,\chi}[\sin(w\tau)](s) &= \int_0^{+\infty} e^{-\left(s+\frac{\chi-\zeta}{\zeta}\right)\tau}\sin(w\tau)\, d\tau \\ &= \frac{w}{\left(s+\dfrac{\chi-\zeta}{\zeta}\right)^2 + w^2},\end{aligned}$$

provided that $s + \dfrac{\chi-\zeta}{\zeta} > 0$
e) It is obtained similarly to d)
f) It is obtained in a similar manner to what will be shown in g)
g) To derive the result, we note that the classical Laplace transform of $cosh(w\tau)$ is given by

$$\int_0^{+\infty} e^{-v\tau}\cosh(w\tau)\, d\tau = \frac{v}{v^2-w^2},$$

for our (ζ,χ)-Laplace transform of $\cosh(w\tau)$, we have

$$\mathcal{L}_{\zeta,\chi}[\cosh(w\tau)](s) = \int_0^{+\infty} e^{-\left(s+\frac{\chi-\zeta}{\zeta}\right)\tau}\cosh(w\tau)\, d\tau$$
$$= \frac{s+\frac{\chi-\zeta}{\zeta}}{\left(s+\frac{\chi-\zeta}{\zeta}\right)^2 - w^2},$$

provided that $s > \max\left(w - \frac{\chi-\zeta}{\zeta}, -w - \frac{\chi-\zeta}{\zeta}\right)$.
h) This result is obtained directly from the definition.
i) This result corresponds to a special case of the formula to be established in item j).
j) The result follows by using the relation of the integral

with the Gamma function

$$\mathcal{L}_{\zeta,\chi}\left[e^{v\tau}\tau^{w}\right](s) = \int_0^\infty e^{-\left(s+\frac{\chi-\zeta}{\zeta}\right)\tau} e^{-v\tau}\tau^{w}\,d\tau$$
$$= \int_0^\infty e^{-(s+v-\frac{\chi-\zeta}{\zeta})\tau}\tau^{w}d\tau = \frac{\Gamma(w+1)}{\left(s-v+\frac{\chi-\zeta}{\zeta}\right)^{w+1}}$$
$$= \frac{w!}{\left(s-v+\frac{\chi-\zeta}{\zeta}\right)^{w+1}},$$

provided that $s - v + \frac{\chi-\zeta}{\zeta} > 0$

*Remark.*If $\zeta = \chi$, the (ζ,χ)-Laplace transform coincides with the classical Laplace transform, that is,

$$\mathcal{L}_{\zeta,\chi}[g(\tau)](s) = \mathcal{L}[g(\tau)](s)$$

Proposition 1. *If the functions f and g are (ζ,χ)-transformable, then the (ζ,χ)-transform of their sum exists and equals the sum of their respective (ζ,χ)-transforms, that is:*

$$\mathcal{L}_{\zeta,\chi}[f+g](s) = \mathcal{L}_{\zeta,\chi}[f](s) + \mathcal{L}_{\zeta,\chi}[g](s)$$

Proposition 2. *If the function g is (ζ,χ)-transformable and λ is a real number, then the (ζ,χ)-transform of the product λg exists and equals the product of λ and the (ζ,χ)-transform of g, that is,*

$$\mathcal{L}_{\zeta,\chi}[\lambda g](s) = \lambda\mathcal{L}_{\zeta,\chi}[g](s)$$

*Remark.*In view of the previous propositions, the operator $\mathcal{L}_{\zeta,\chi}$ is said to be linear.

Proposition 3.*First translation or shifting property*
If the function g is (ζ,χ)-transformable and

$$\mathcal{L}_{\zeta,\chi}[g(\tau)](s) = \mathcal{G}_{\zeta,\chi}\left(s+\frac{\chi-\zeta}{\zeta}\right),$$

then

$$\mathcal{L}_{\zeta,\chi}[e^{c\tau}g(\tau)](s) = \mathcal{G}_{\zeta,\chi}\left(s+\frac{\chi-\zeta}{\zeta}-c\right).$$

*Proof.*We compute:

$$\mathcal{L}_{\zeta,\chi}[e^{c\tau}g(\tau)](s) = \int_0^{+\infty} e^{-s\tau}e^{-\frac{\chi-\zeta}{\zeta}\tau}e^{c\tau}g(\tau)\,d\tau$$
$$= \int_0^{+\infty} e^{-(s-c+\frac{\chi-\zeta}{\zeta})\tau}g(\tau)\,d\tau$$
$$= \mathcal{G}_{\zeta,\chi}\left(s+\frac{\chi-\zeta}{\zeta}-c\right).$$

Proposition 4.*Second translation or shifting property*
If the function g is (ζ,χ)-transformable and

$$\mathcal{L}_{\zeta,\chi}[g(\tau)](s) = \mathcal{G}_{\zeta,\chi}\left(s+\frac{\chi-\zeta}{\zeta}\right),$$

and

$$E_{\zeta,\chi}(a,\tau) = \begin{cases} g(\tau-a), & \tau > a, \\ 0, & \tau < a, \end{cases}$$

then

$$\mathcal{L}_{\zeta,\chi}[E_{\zeta,\chi}(a,\tau)](s) = e^{-(s+\frac{\chi-\zeta}{\zeta})a}\mathcal{G}_{\zeta,\chi}\left(s+\frac{\chi-\zeta}{\zeta}\right).$$

*Proof.*From the definition, we get

$$\mathcal{L}_{\zeta,\chi}[E_{\zeta,\chi}(a,\tau)](s) = \int_0^{a} e^{-s\tau}e^{-\frac{\chi-\zeta}{\zeta}\tau}\cdot 0\,d\tau$$
$$+ \int_a^{+\infty} e^{-s\tau}e^{-\frac{\chi-\zeta}{\zeta}\tau}g(\tau-a)\,d\tau$$
$$= \int_a^{+\infty} e^{-s\tau}e^{-\frac{\chi-\zeta}{\zeta}\tau}g(\tau-a)\,d\tau.$$

Substitute $u = \tau - a$, $d\tau = du$:

$$= \int_0^{+\infty} e^{-(s+\frac{\chi-\zeta}{\zeta})(u+a)}g(u)\,du$$
$$= e^{-(s+\frac{\chi-\zeta}{\zeta})a}\int_0^{+\infty} e^{-(s+\frac{\chi-\zeta}{\zeta})u}g(u)\,du$$
$$= e^{-(s+\frac{\chi-\zeta}{\zeta})a}\mathcal{G}_{\zeta,\chi}\left(s+\frac{\chi-\zeta}{\zeta}\right).$$

Proposition 5.*Change of scale property*
If the function g is (ζ,χ)-transformable and

$$\mathcal{L}_{\zeta,\chi}[g(\tau)](s) = \mathcal{G}_{\zeta,\chi}\left(s+\frac{\chi-\zeta}{\zeta}\right),$$

then

$$\mathcal{L}_{\zeta,\chi}[g(a\tau)](s) = \frac{1}{a}\mathcal{G}_{\zeta,\chi}\left(\frac{1}{a}\left(s+\frac{\chi-\zeta}{\zeta}\right)\right).$$

*Proof.*By direct computation, we obtain:

$$\mathcal{L}_{\zeta,\chi}[g(a\tau)](s) = \int_0^{+\infty} e^{-s\tau}e^{-\frac{\chi-\zeta}{\zeta}\tau}g(a\tau)\,d\tau$$
$$= \int_0^{+\infty} e^{-(s+\frac{\chi-\zeta}{\zeta})\tau}g(a\tau)\,d\tau.$$

Substitute $u = a\tau$, $d\tau = \frac{1}{a}du$:

$$= \int_0^{+\infty} e^{-\frac{1}{a}(s+\frac{\chi-\zeta}{\zeta})u}g(u)\cdot\frac{1}{a}\,du$$
$$= \frac{1}{a}\mathcal{G}_{\zeta,\chi}\left(\frac{1}{a}\left(s+\frac{\chi-\zeta}{\zeta}\right)\right).$$

Proposition 6. *If g is a (ζ,χ)-transformable function then the $V^{\zeta,\chi}(g)$-derivative is (ζ,χ)-transformable and we have*

$$\mathcal{L}_{\zeta,\chi}[V^{\zeta,\chi}(g(\tau)](s) = \left(\frac{\zeta}{\chi}s + 2\frac{\chi-\zeta}{\chi}\right)\mathcal{L}_{\zeta,\chi}[g(\tau)](s) - \frac{\zeta}{\chi}g(0).$$

Proof. Since $V^{\zeta,\chi}(g(\tau)) = \frac{\zeta}{\chi}g'(\tau) + \frac{\chi-\zeta}{\chi}g(\tau)$, we first compute the (ζ,χ)-transform of $g'(\tau)$ and obtain

$$\mathcal{L}_{\zeta,\chi}[g'(\tau)](s) = \int_0^{+\infty} e^{-s\tau}e^{-\frac{\chi-\zeta}{\zeta}\tau}g'(\tau)\,d\tau$$

if $u = e^{-\left(s+\frac{\chi-\zeta}{\zeta}\right)\tau}$ and $dv = g'(\tau)\,d\tau$, then

$$\mathcal{L}_{\zeta,\chi}[g'(\tau)](s) = e^{-\left(s+\frac{\chi-\zeta}{\zeta}\right)\tau}g(\tau)\Big|_0^{+\infty} - \int_0^{+\infty} g(\tau)e^{-\left(s+\frac{\chi-\zeta}{\zeta}\right)\tau}\left(-\left(s+\frac{\chi-\zeta}{\zeta}\right)\right)d\tau$$

$$= -g(0) + \left(s+\frac{\chi-\zeta}{\zeta}\right)\int_0^{+\infty} e^{-\left(s+\frac{\chi-\zeta}{\zeta}\right)\tau}g(\tau)\,d\tau$$
$$= -g(0) + \left(s+\frac{\chi-\zeta}{\zeta}\right)\mathcal{L}_{\zeta,\chi}[g(\tau)](s).$$

Then

$$\mathcal{L}_{\zeta,\chi}[V^{\zeta,\chi}(g(\tau))](s) = \frac{\zeta}{\chi}\mathcal{L}_{\zeta,\chi}[g'(\tau)](s) + \frac{\chi-\zeta}{\chi}\mathcal{L}_{\zeta,\chi}[g(\tau)](s)$$
$$= \left[\frac{\zeta}{\chi}s + 2\frac{\chi-\zeta}{\chi}\right]\mathcal{L}_{\zeta,\chi}[g(\tau)](s) - \frac{\zeta}{\chi}g(0).$$

We now present the (ζ,χ)-transform of the second (ζ,χ)-derivative.

Proposition 7. *The (ζ,χ)-transform of the second (ζ,χ)-derivative is given by*

$$\mathcal{L}_{\zeta,\chi}\left[V^{\zeta,\chi}\left(V^{\zeta,\chi}(g(\tau))\right)\right](s) = -\left(\frac{\zeta^2}{\chi^2}s + \frac{3(\chi-\zeta)\zeta}{\chi^2}\right)g(0) - \frac{\zeta^2}{\chi^2}g'(0) + \left(\frac{\zeta}{\chi}s + \frac{(\chi-\zeta)^2}{\chi}\right)\mathcal{L}_{\zeta,\chi}[g(\tau)]$$

Proof. Applying the $V^{\zeta,\chi}$-derivative to the $V^{\zeta,\chi}$-derivative, we obtain

$$V^{\zeta,\chi}(V^{\zeta,\chi}(g(\tau))) = \frac{\zeta}{\chi}\left(\frac{\zeta}{\chi}g''(\tau) + \frac{(\chi-\zeta)}{\chi}g'(\tau)\right) + \frac{(\chi-\zeta)}{\chi}\left(\frac{\zeta}{\chi}g'(\tau) + \frac{(\chi-\zeta)}{\chi}g(\tau)\right)$$
$$= \frac{\zeta^2}{\chi^2}g''(\tau) + 2\frac{\zeta(\chi-\zeta)}{\chi^2}g'(\tau) + \frac{(\chi-\chi)^2}{\chi^2}g(\tau)$$

Now,

$$\mathcal{L}_{\zeta,\chi}[g''(\tau)](s) = \int_0^{\infty} e^{-\left(s+\frac{\chi-\zeta}{\zeta}\right)\tau}g''(\tau)d\tau$$
$$= -g'(0) - g(0)\left(s+\frac{\chi-\zeta}{\zeta}\right) + \left(s+\frac{\chi-\zeta}{\zeta}\right)^2\mathcal{L}_{\zeta,\chi}[g(\tau)](s).$$

Then,

$$\mathcal{L}_{\zeta,\chi}\left[V^{\zeta,\chi}\left(V^{\zeta,\chi}(g(\tau))\right)\right](s) = \frac{\zeta^2}{\chi^2}\left[-g'(0) - g(0)\left(s+\frac{\chi-\zeta}{\zeta}\right) + \left(s+\frac{\chi-\zeta}{\zeta}\right)^2\mathcal{L}_{\zeta,\chi}[g(\tau)](s).\right]$$
$$+ 2\frac{\zeta(\chi-\zeta)}{\chi^2}\left[-g(0) + \left(s+\frac{\chi-\zeta}{\zeta}\right)\mathcal{L}_{\zeta,\chi}[g(\tau)](s)\right] + \frac{(\chi-\zeta)^2}{\chi^2}\mathcal{L}_{\zeta,\chi}[g(\tau)](s)$$
$$= -\frac{\zeta^2}{\chi^2}g'(0) - \left(\frac{\zeta^2}{\chi^2}s + 3\frac{\zeta(\chi-\zeta)}{\chi^2}\right)g(0) + \left\{\frac{\zeta^2}{\chi^2}\left(s+\frac{\chi-\zeta}{\zeta}\right) + 2\frac{\zeta(\chi-\zeta)}{\chi^2}\left(s+\frac{\chi-\zeta}{\zeta}\right) + \frac{(\chi-\zeta)^2}{\chi^2}\right\}\mathcal{L}_{\zeta,\chi}[g(\tau)](s)$$
$$= \frac{\zeta^2}{\chi^2}g'(0) - \left(\frac{\zeta^2}{\chi^2}s + 3\frac{\zeta(\chi-\zeta)}{\chi^2}\right)g(0) + \left(\frac{\zeta}{\chi}s + \frac{\chi-\zeta}{\chi}\right)^2\mathcal{L}_{\zeta,\chi}[g(\tau)](s)$$

Proposition 8. *Let g be a (ζ,χ)-exponential order function, continuous for $\tau \geq 0$.*

Then,

$$\begin{aligned}
&\mathcal{L}_{\zeta,\chi}\left[I_u^{\zeta,\chi}(g(\tau))\right](s)\\
&\qquad= \mathcal{L}_{\zeta,\chi}\left[\frac{\zeta}{\chi}e^{-\frac{(\chi-\zeta)}{\zeta}\tau}\int_u^{+\infty} e^{\frac{(\chi-\zeta)}{\zeta}y} g(y)\,dy\right](s)\\
&\qquad= \frac{\chi}{\zeta\left(s+2\frac{(\chi-\zeta)}{\zeta}\right)}\mathcal{L}_{\zeta,\chi}[g(\tau)](s)
\end{aligned}$$

Proof.

$$\begin{aligned}
&\mathcal{L}_{\zeta,\chi}\left[\frac{\zeta}{\chi}e^{-\frac{(\chi-\zeta)}{\zeta}\tau}\int_u^{+\infty} e^{\frac{(\chi-\zeta)}{\zeta}y} g(y)\,dy\right](s) =\\
&\int_u^{+\infty} e^{-s\tau}e^{-\frac{\chi-\zeta}{\zeta}\tau}\left[\frac{\zeta}{\chi}e^{-\frac{(\chi-\zeta)}{\zeta}\tau}\int_u^{+\infty} e^{\frac{(\chi-\zeta)}{\zeta}y} g(y)\,dy\right]d\tau\\
&= \frac{\zeta}{\chi}\int_u^{+\infty} e^{\frac{\chi-\zeta}{\zeta}y} g(y)\left[\int_y^{+\infty} e^{-s\tau}e^{-2\frac{(\chi-\zeta)}{\zeta}\tau}\,d\tau\right]dy\\
&= \frac{\zeta}{\chi}\int_u^{+\infty} e^{\frac{(\chi-\zeta)}{\zeta}y} g(y)\frac{e^{-\left(s+2\frac{(\chi-\zeta)}{\zeta}\right)y}}{s+2\frac{(\chi-\zeta)}{\zeta}}\,dy\\
&= \frac{\chi}{\zeta\left(s+2\frac{(\chi-\zeta)}{\zeta}\right)}\int_u^{+\infty} e^{-s\tau}e^{-\frac{(\chi-\zeta)}{\zeta}\tau} g(y)\,dy\\
&= \frac{\chi}{\zeta\left(s+2\frac{(\chi-\zeta)}{\zeta}\right)}\mathcal{L}_{\zeta,\chi}[g(\tau)](s)
\end{aligned}$$

The following result establishes the relationship between the (ζ,χ)-Laplace transform and the classical Laplace transform.

Theorem 6. *Let ζ and $\chi \in (0,1)$, and let g be a (ζ,χ)-transformable function. Then, the following identity holds:*

$$\mathcal{L}_{\zeta,\chi}[g](s) = \mathcal{L}\left[e^{-\frac{\chi-\zeta}{\zeta}\tau}g(\tau)\right](s).$$

Proof. The result follows directly from the definition. Indeed, we have:

$$\begin{aligned}
\mathcal{L}_{\zeta,\chi}[g(\tau)](s) &= \int_0^{\infty} e^{-s\tau}e^{-\left(\frac{\chi-\zeta}{\zeta}\right)\tau} g(\tau)\,d\tau\\
&= \int_0^{\infty} e^{-s\tau}\left[e^{-\left(\frac{\chi-\zeta}{\zeta}\right)\tau} g(\tau)\right]d\tau\\
&= \mathcal{L}\left[e^{-\left(\frac{\chi-\zeta}{\zeta}\right)\tau} g(\tau)\right](s)
\end{aligned}$$

The following result presents an analogue of the convolution theorem for the classical Laplace transform.

Theorem 7. *Let $\zeta,\chi \in (0,1)$, and let f and g be real functions defined on $[0,+\infty[$, such that f and g are (ζ,χ)-transformable.*
If we denote their (ζ,χ)-Laplace transforms by

$$F(s) = \mathcal{L}_{\zeta,\chi}\left[f\left(e^{-\frac{\chi-\zeta}{\zeta}\tau}\right)\right](s)\ \text{and}\ G(s) = \mathcal{L}_{\zeta,\chi}[g(\tau)](s)$$

then the following identity holds:

$$\mathcal{L}_{\zeta,\chi}[f * g](s) = F(s)\cdot G(s)$$

where

$$[f * g]_{\zeta,\chi}(\tau) = \int_0^{\tau} f\left(e^{-\frac{\chi-\zeta}{\zeta}(\tau-r)}\right) g(r)dr$$

Proof. From the definition, we have

$$\begin{aligned}
&\mathcal{L}_{\zeta,\chi}\left[(f * g)_{\zeta,\chi}\right](s)\\
&= \int_0^{+\infty} e^{-s\tau}e^{-\frac{(\chi-\zeta)}{\zeta}\tau}\left[\int_0^{\tau} f\left(e^{-\frac{\chi-\zeta}{\zeta}(\tau-r)}\right) g(r)dr\right]d\tau\\
&= \int_0^{+\infty} g(r)\left[\int_r^{+\infty} e^{-(s+\frac{(\chi-\zeta)}{\zeta})\tau} f\left(e^{-\frac{\chi-\zeta}{\zeta}(\tau-r)}\right)d\tau\right]dr
\end{aligned}$$

if $u = \tau - r$, then $du = d\tau$, and we obtain

$$\begin{aligned}
&\mathcal{L}_{\zeta,\chi}\left[(f * g)_{\zeta,\chi}\right](s) =\\
&\int_0^{+\infty} g(r)e^{-\left(s+\frac{(\chi-\zeta)}{\zeta}\right)r}\left[\int_0^{+\infty} e^{-(s+\frac{(\chi-\zeta)}{\zeta})u} f\left(e^{-\frac{\chi-\zeta}{\zeta}u}\right)d\mu\right]dr\\
&= \int_0^{+\infty} g(r)e^{-\left(s+\frac{(\chi-\zeta)}{\zeta}\right)r}\mathcal{L}_{\zeta,\chi}\left[f\left(e^{-\frac{(\chi-\zeta)}{\zeta}u}\right)\right](s)\,dr\\
&= F(s)\cdot\int_0^{+\infty} g(r)e^{-\left(s+\frac{(\chi-\zeta)}{\zeta}\right)r}\,dr = F(s)\cdot G(s)
\end{aligned}$$

The following result guarantees the existence and boundedness of the (ζ,χ)-Laplace transform under certain conditions on the function.

Theorem 8. *If g is piecewise continuous on $[0,\infty)$ and is (ζ,χ)-exponentially bounded, then*

$$\lim_{s\to\infty} G_{\zeta,\chi}(s) = 0,$$

where $G_{\zeta,\chi}(s) = \mathcal{L}_{\zeta,\chi}[g(\tau)](s)$.

Proof. Since g is of (ζ,χ)-exponential order, there exist t_0, M_1, and c such that

$$|g(\tau)| \le M_1 E_{\zeta,\chi}(c,\tau)$$

for all $\tau \ge t_0$. As g is piecewise continuous on $[0,t_0]$, then g is bounded, hence there exists M_2 such that $|g(\tau)| \le M_2$ for all $\tau \in [0,t_0]$.

If we choose $M = \max\{M_1, M_2\}$, we obtain:

$$|g(\tau)| \le M E_{\zeta,\chi}(c,\tau) \quad \text{for } \tau \ge 0.$$

Now we have,

$$\begin{aligned}
\left|\int_0^r E_{\zeta,\chi}(-s,\tau)\,g(\tau)\,d\tau\right| &\le \int_0^r |E_{\zeta,\chi}(-s,\tau)\,g(\tau)|\,d\tau\\
&\le M\int_0^r E_{\zeta,\chi}(-s+c,\tau)\,d\tau\\
&= M\int_0^r e^{(-s+c)\tau}e^{-\frac{(\chi-\zeta)}{\zeta}\tau}\,d\tau\\
&\le M\cdot\frac{1}{s-c+\frac{\chi-\zeta}{\zeta}}
\end{aligned}$$

Therefore,

$$\lim_{r\to\infty}\left|\int_0^r E_{\zeta,\chi}(-s,\tau)\,g(\tau)d\tau\right| \leq \frac{M}{s-c+\frac{\chi-\zeta}{\zeta}} = 0,$$

This concludes the proof

3 Examples and Applications

Example 1. Consider the V-ordinary differential equation of order (ζ,χ)

$$V^{\zeta,\chi}x(\tau) = \lambda x(\tau), \quad x(0) = x_0 \qquad (3.1)$$

If $\zeta = \chi$, we obtain an ordinary differential equation under the assumption that the growth rate of the function $x(\tau)$ is proportional to its current value. This leads to the classical population growth model of Kermack and McKendrick, which is written as

$$x'(\tau) = \lambda x(\tau), \quad x(0) = x_0, \quad \text{whose solution is}$$
$$x(\tau) = x_0 e^{\lambda\tau}$$

Applying the (ζ,χ)-Laplace transform to both sides of the equation $V^{\zeta,\chi}x(\tau) = \lambda x(\tau)$, we obtain

$$\left(\frac{\zeta}{\chi}s + 2\frac{(\chi-\zeta)}{\chi}\right)\mathcal{L}_{\zeta,\chi}[x(\tau)](s) - \frac{\zeta}{\chi}x(0) = \lambda\mathcal{L}_{\zeta,\chi}[x(\tau)](s),$$

which implies

$$\left(\frac{\zeta}{\chi}s + 2\frac{(\chi-\zeta)}{\chi} - \lambda\right)\mathcal{L}_{\zeta,\chi}[x(\tau)](s) = \frac{\zeta}{\chi}x_0,$$

from which it follows that

$$\mathcal{L}_{\zeta,\chi}[x(\tau)](s) = \frac{\frac{\zeta}{\chi}x_0}{\frac{\zeta}{\chi}s + 2\frac{(\chi-\zeta)}{\chi} - \lambda} = \frac{x_0}{s + 2\frac{(\chi-\zeta)}{\zeta} - \frac{\chi}{\zeta}\lambda}$$

Applying the biparametric inverse Laplace transform, we obtain:

$$x(\tau) = \mathcal{L}^{-1}_{\zeta,\chi}\left[\frac{x_0}{s + 2\frac{(\chi-\zeta)}{\zeta} - \frac{\chi}{\zeta}\lambda}\right]$$
$$= x_0\, e^{-\left(2\frac{(\chi-\zeta)}{\zeta} - \frac{\chi}{\zeta}\lambda\right)\tau} e^{\frac{\chi-\zeta}{\zeta}\tau} = x_0 e^{\left(\frac{\chi}{\zeta}\lambda - \frac{(\chi-\zeta)}{\zeta}\right)\tau}$$

The solution of equation (3.1), obtained using the (ζ,χ)-Laplace transform method, is illustrated in Figure 1 for the parameter values $\zeta = \frac{3}{10}$ and $\chi = \frac{1}{5}$; and $\zeta = \frac{2}{5}$ and $\chi = \frac{3}{5}$.

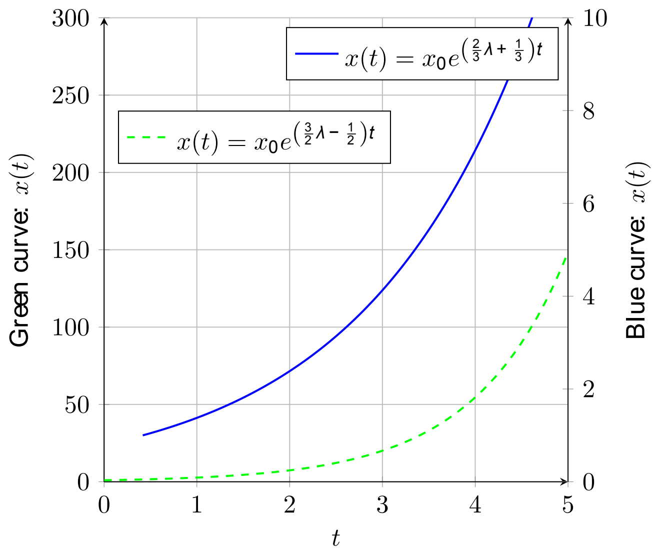

Fig. 1: Graphs of $x(\tau) = x_0 e^{\left(\frac{2}{3}\lambda+\frac{1}{3}\right)\tau}$ and $x(\tau) = x_0 e^{\left(\frac{3}{2}\lambda-\frac{1}{2}\right)\tau}$.

Example 2. Solve the V-ordinary differential equation of order (ζ,χ), which was studied in [35] in the absence of an initial condition.

$$V^{\frac{1}{4},\frac{1}{2}}x(\tau) + x(\tau) = \tau e^{-\tau}, \quad x(0) = x_0 \qquad (3.2)$$

We apply the (ζ,χ)-Laplace transform to both sides of the equation,

$$\mathcal{L}_{\frac{1}{4},\frac{1}{2}}\left[V^{\frac{1}{4},\frac{1}{2}}x(\tau)\right](s) + \mathcal{L}_{\frac{1}{4},\frac{1}{2}}[x(\tau)](s) = \mathcal{L}_{\frac{1}{4},\frac{1}{2}}[\tau e^{-\tau}](s)$$

which implies

$$\left(\frac{\frac{1}{4}}{\frac{1}{2}}s + 2\frac{\left(\frac{1}{2}-\frac{1}{4}\right)}{\frac{1}{2}}\right)\mathcal{L}_{\frac{1}{4},\frac{1}{2}}[x(\tau)](s) - \frac{\frac{1}{4}}{\frac{1}{2}}x(0) + \mathcal{L}_{\frac{1}{4},\frac{1}{2}}[x(\tau)](s) = \mathcal{L}_{\frac{1}{4},\frac{1}{2}}[\tau e^{-\tau}](s),$$

from which it follows that

$$\left(\frac{1}{2}s + 2\right)\mathcal{L}_{\frac{1}{4},\frac{1}{2}}[x(\tau)](s) = \frac{1}{2}x(0) + \int_0^\infty \tau e^{-s\tau}e^{-2\tau}\,d\tau,$$

and hence,

$$\mathcal{L}_{\frac{1}{4},\frac{1}{2}}[x(\tau)](s) = \frac{x(0)}{(s+4)} + \frac{2}{(s+4)(s+2)^2}$$
$$= \frac{x(0)}{(s+4)} + \frac{\frac{1}{2}}{(s+4)} + \frac{-\frac{1}{2}}{(s+2)} + \frac{1}{(s+2)^2}$$

Applying the inverse (ζ,χ)-Laplace transform, we obtain

$$\begin{aligned}x(\tau) &= \mathcal{L}^{-1}_{\frac{1}{4},\frac{1}{2}}\left[\frac{x(0)}{(s+4)}+\frac{\frac{1}{2}}{(s+4)}+\frac{-\frac{1}{2}}{(s+2)}+\frac{1}{(s+2)^2}\right]\\ &= x_0e^{-3\tau}+\frac{1}{2}e^{-3\tau}-\frac{1}{2}e^{-\tau}+\tau e^{-\tau}\\ &= \left(x_0+\frac{1}{2}\right)e^{-3\tau}+\left(\tau-\frac{1}{2}\right)e^{-\tau}\end{aligned}$$

This result coincides with that obtained in Example 5 of [35], with the difference that the constants are now specified by the initial condition.

*Example 3.*To solve the V-ordinary differential equations of order (ζ,χ) studied in Example 6 of [35]:

$$V^{\frac{1}{9},\frac{1}{3}}x(\tau)+3x(\tau)=4\tau e^{-5\tau},\quad x(0)=x_0 \tag{3.3}$$

the (ζ,χ)-Laplace transform is applied to both sides of the equation, yielding

$$\mathcal{L}_{\frac{1}{9},\frac{1}{3}}\left[V^{\frac{1}{9},\frac{1}{3}}x(\tau)\right](s)+3\mathcal{L}_{\frac{1}{9},\frac{1}{3}}[x(\tau)](s)=\mathcal{L}_{\frac{1}{9},\frac{1}{3}}[4\tau e^{-5\tau}](s)$$

which implies

$$\left(\frac{1}{3}s+\frac{13}{3}\right)\mathcal{L}_{\frac{1}{9},\frac{1}{3}}[x(\tau)](s)=\frac{1}{3}x(0)+4\frac{1}{(s+7)^2}$$

then

$$\mathcal{L}_{\frac{1}{9},\frac{1}{3}}[x(\tau)](s)=\frac{x(0)}{(s+13)}+12\frac{1}{(s+13)(s+7)^2}$$

Applying the (ζ,χ)-inverse Laplace transform, we get:

$$\begin{aligned}x(\tau) &= x_0e^{-13\tau}e^{2\tau}+\frac{1}{3}e^{-13\tau}e^{2\tau}-\frac{1}{3}e^{-7\tau}e^{2\tau}+2\tau e^{-7\tau}e^{2\tau}\\ &= \left(x_0+\frac{1}{3}\right)e^{-11\tau}+2\left(\tau-\frac{1}{6}\right)e^{-5\tau}\end{aligned}$$

This result coincides with that obtained in [35], with the difference that the constants are now specified by the initial condition.

The solution of equation (3.3), obtained using the (ζ,χ)-Laplace transform method, is illustrated in Figure 3 for the initial conditions $x_0=1$, $x_0=2$, and $x_0=4$.

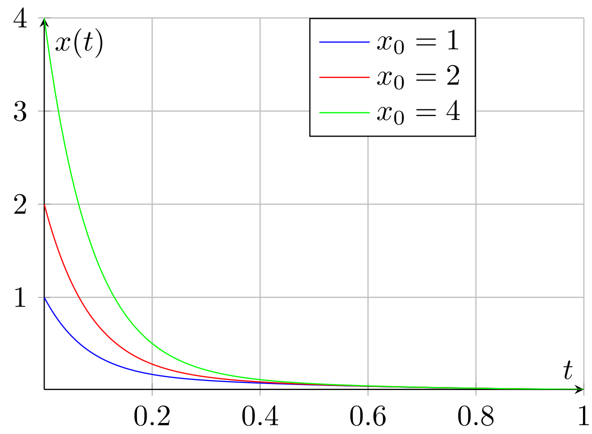

Fig. 2: Graphs of $x(\tau)=(x_0+\frac{1}{3})e^{-11\tau}+2(\tau-\frac{1}{6})e^{-5\tau}$ for $x_0=1$, $x_0=2$, and $x_0=4$.

Example 4. Solve the V-ordinary differential equation of order (ζ,χ).

$$V^{\zeta,\chi}x(t)=x^{2/3}(\tau)-x(\tau),\quad x(0)=x_0 \tag{3.4}$$

We solve it using the change of variable $z=3x^{1/3}$, and we have

$$\begin{aligned}&\frac{\zeta}{\chi}x'(\tau)+\frac{(\chi-\zeta)}{\chi}x(\tau)=x^{2/3}(\tau)-x(\tau)\\ &\frac{\zeta}{\chi}\frac{z^2(\tau)}{9}+\frac{(\chi-\zeta)}{\chi}\frac{z^3(\tau)}{27}=\left(\frac{z(\tau)}{3}\right)^2-\left(\frac{z(\tau)}{3}\right)^3,\end{aligned}$$

which implies

$$\frac{\zeta}{\chi}+\frac{(\chi-\zeta)}{\chi}\frac{z(\tau)}{3}=1-\frac{z(\tau)}{3},$$

then

$$V^{\zeta,\chi}\left(\frac{z(\tau)}{3}\right)=1-\frac{z(\tau)}{3}-\frac{2}{3}\frac{\zeta}{\chi},$$

where we apply the (ζ,χ)-Laplace transform, and we obtain

$$\left(\frac{\zeta}{3\chi}s+\frac{2(\chi-\zeta)}{3\chi}+\frac{1}{3}\right)\mathcal{L}_{\zeta,\chi}[z(\tau)]=\frac{\zeta}{3\chi}z_0+\frac{3\chi-2\zeta}{3\chi\left(s+\frac{\chi-\zeta}{\chi}\right)},$$

from which,

$$\mathcal{L}_{\zeta,\chi}[z(\tau)]=\frac{z_0}{\left(s+\frac{3\chi-2\zeta}{\zeta}\right)}+\frac{\frac{3\chi-2\zeta}{2\chi-\zeta}}{\left(s+\frac{\chi-\zeta}{\zeta}\right)}+\frac{\frac{3\chi-2\zeta}{\zeta-2\chi}}{\left(s+\frac{3\chi-2\zeta}{\zeta}\right)},$$

therefore,

$$z(\tau) = z_0 e^{-\frac{(3\chi-2\zeta)}{\zeta}\tau} e^{\frac{\chi-\zeta}{\zeta}\tau} + \frac{3\chi-2\zeta}{2\chi-\zeta} e^{\frac{-(\chi-\zeta)}{\zeta}\tau} e^{\frac{\chi-\zeta}{\zeta}\tau} + \frac{3\chi-2\zeta}{\zeta-2\chi} e^{\frac{\zeta-2\chi}{\zeta}\tau}$$

then,

$$x^{\frac{1}{3}}(\tau) = x_0^{\frac{2}{3}} e^{\frac{\zeta-2\chi}{3\zeta}\tau} + \frac{3\chi-2\zeta}{2\chi-\zeta} + \frac{3\chi-2\zeta}{\zeta-2\chi} e^{\frac{\zeta-2\chi}{3\zeta}\tau}$$

and thus,

$$x(\tau) = \left[\frac{3\chi-2\zeta}{2\chi-\zeta} + \left(x_0^{\frac{2}{3}} + \frac{3\chi-2\zeta}{\zeta-2\chi}\right) e^{\frac{\zeta-2\chi}{3\zeta}\tau}\right]^3$$

The solution of equation (3.4), obtained using the (ζ,χ)-Laplace transform method for the parameter values $\zeta = \frac{1}{4}, \chi = \frac{1}{5}$; $\zeta = \frac{1}{5}, \chi = \frac{1}{6}$; and $\zeta = \frac{1}{6}, \chi = \frac{1}{7}$, and for the initial conditions $x_0 = 1$ and $x_0 = 2$, is illustrated in Figures 3 and 4.

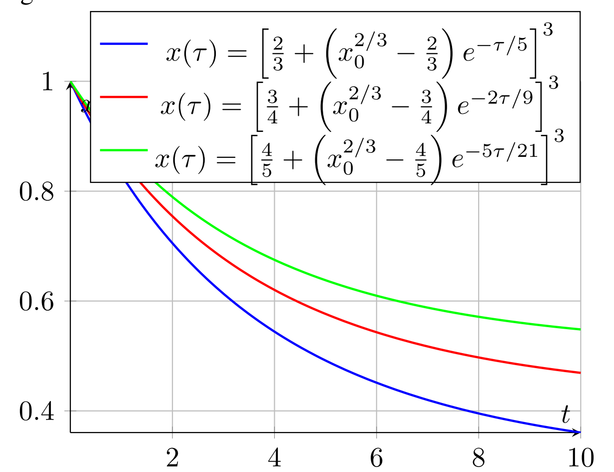

Fig. 3: Solutions $x(\tau)$ for different pairs (ζ,χ) with $x_0 = 1$.

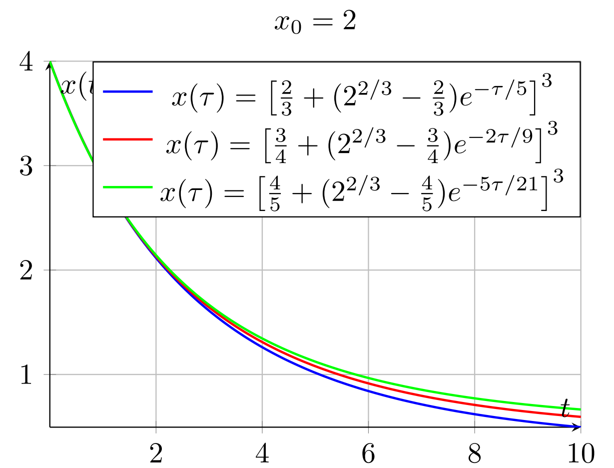

Fig. 4: Solutions $x(t)$ for different (ζ,χ) values with $x_0 = 2$.

Example 5. Let us consider the V-ordinary differential equation of order (ζ,χ).

$$V^{\zeta,\chi}(V^{\zeta,\chi}(x(\tau))) + cx(\tau) = 0, \qquad (3.5)$$

with initial conditions $x(0) = x_0$, $V^{\zeta,\chi}x(0) = 0$, c=$\sqrt{\frac{g}{L}}$

When $\zeta = \chi$, the above equation becomes $x''(\tau) + cx(\tau) = 0$, which describes small oscillations of a pendulum. Its exact solution is:

$$x(\tau) = x_0 \cos\left(\sqrt{\frac{g}{L}}\,\tau\right),$$

if we apply the (ζ,χ)-Laplace transform to both sides of the (ζ,χ)-differential equation, we obtain:

$$\left(\frac{\zeta}{\chi}s + \frac{(\chi-\zeta)^2}{\chi}\right)\mathcal{L}_{\zeta,\chi}[x(\tau)] - \left(\frac{\zeta^2}{\chi^2}s + \frac{3(\chi-\zeta)\zeta}{\chi^2}\right)x_0 + c\mathcal{L}_{\zeta,\chi}[x(\tau)] = 0,$$

which implies:

$$\mathcal{L}_{\zeta,\chi}[x(\tau)] = \frac{x_0\left(s + \frac{3(\chi-\zeta)}{\zeta}\right)}{\left(s + \frac{(\chi-\zeta)}{\zeta}\right)^2 + \frac{\chi^2}{\zeta^2}c}$$
$$= \frac{x_0\left(s + \frac{\chi-\zeta}{\zeta}\right)}{\left(s + \frac{(\chi-\zeta)}{\zeta}\right)^2 + \frac{\chi^2}{\zeta^2}c} + \frac{\frac{2}{\sqrt{c}}x_0\frac{\chi}{\zeta}\sqrt{c}}{\left(s + \frac{(\chi-\zeta)}{\zeta}\right)^2 + \frac{\chi^2}{\zeta^2}c} - \frac{2\frac{\zeta}{\chi\sqrt{c}}x_0\frac{\chi}{\zeta}\sqrt{c}}{\left(s + \frac{(\chi-\zeta)}{\zeta}\right)^2 + \frac{\chi^2}{\zeta^2}c}$$

then, applying the inverse (ζ,χ)-Laplace transform, we have:

$$x(\tau) = x_0\cos\left(\frac{\chi}{\zeta}\sqrt{c}\,\tau\right)e^{\frac{\chi-\zeta}{\zeta}\tau} + \frac{2}{\sqrt{c}}x_0\sin\left(\frac{\chi}{\zeta}\sqrt{c}\,\tau\right)e^{\frac{\chi-\zeta}{\zeta}\tau} - 2\frac{\zeta}{\chi\sqrt{c}}x_0\sin\left(\frac{\chi}{\zeta}\sqrt{c}\,\tau\right)e^{\frac{\chi-\zeta}{\zeta}\tau}$$
$$= x_0 e^{\frac{\chi-\zeta}{\zeta}\tau}\left[\cos\left(\frac{\chi}{\zeta}\sqrt{c}\,\tau\right) + \left(\frac{2}{\sqrt{c}} - 2\frac{\zeta}{\chi\sqrt{c}}\right)\sin\left(\frac{\chi}{\zeta}\sqrt{c}\,\tau\right)\right]$$

The solution of equation (3.5), obtained using the (ζ,χ)-Laplace transform method, is illustrated in Figure 5 for the parameter values $\zeta = \frac{1}{7}$ and $\chi = \frac{1}{8}$; $\zeta = \frac{1}{9}$ and $\chi = \frac{1}{10}$; and $\zeta = \frac{1}{5}$ and $\chi = \frac{1}{4}$. These results correspond to the case where the gravitational acceleration is set to $g = 9.8$ and the pendulum length is $L = 1$.

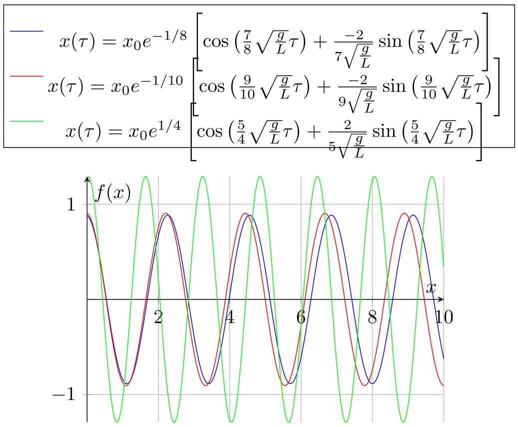

Fig. 5: Graphs of $x(\tau)$ for $(\zeta,\chi) = (\frac{1}{7},\frac{1}{8})$, $(\frac{1}{9},\frac{1}{10})$, and $(\frac{1}{5},\frac{1}{4})$.

*Example 6.*Consider the RL circuit with step input, governed by the V-ordinary differential equation:

$$LV^{\zeta,\chi} i(\tau) + R\, i(\tau) = V_0\, u(\tau), \quad i(0) = 0$$

Applying the (ζ,χ)-Laplace transform, we obtain:

$$\mathcal{L}\left[\left(\frac{\zeta}{\chi}s + 2\frac{(\chi-\zeta)}{\chi}\right)\mathcal{L}_{\zeta,\chi}[i(\tau)](s) - \frac{\zeta}{\chi}i(0)\right] + R\mathcal{L}_{\zeta,\chi}[i(\tau)](s) = V_0\,\frac{1}{s+\frac{\chi-\zeta}{\zeta}}$$

from which it follows that

$$\mathcal{L}_{\zeta,\chi}[i(\tau)](s) = V_0 \cdot \frac{1}{\left(s+\frac{\chi-\zeta}{\zeta}\right)\left[L\left(\frac{\zeta}{\chi}s + 2\frac{(\chi-\zeta)}{\chi}\right) + R\right]}$$

hence

$$\begin{aligned}\mathcal{L}_{\zeta,\chi}[i(\tau)](s) &= V_0\left[\frac{1}{\left(L\frac{\zeta}{\chi}s + 2L\frac{(\chi-\zeta)}{\chi} + R\right)\left(s+\frac{\chi-\zeta}{\zeta}\right)}\right] \\ &= V_0\left[\frac{1}{\left(L\frac{(\chi-\zeta)}{\chi} + R\right)}\cdot\frac{1}{\left(s+\frac{\chi-\zeta}{\zeta}\right)} - \frac{\chi}{\chi R + L(\chi-\zeta)}\cdot\frac{1}{\left(s+2\frac{\chi-\zeta}{\zeta}+\frac{\chi R}{\zeta L}\right)}\right]\end{aligned}$$

Applying the inverse (ζ,χ)-Laplace transform, we obtain:

$$i(\tau) = V_0\left[\frac{1}{\left(L\frac{(\chi-\zeta)}{\chi}+R\right)} - \frac{\chi}{\chi R + L(\chi-\zeta)}\exp^{\left(\frac{(\chi-\zeta)}{\zeta}+\frac{\chi R}{\zeta L}\right)\tau}\right]$$

*Example 7.*To solve the V-ordinary differential equation of order (ζ,χ) modeling the cooling of a body:

$$V^{\zeta,\chi}T(\tau) + kT(\tau) = kT_{\mathrm{amb}}, \quad T(0) = T_0$$

the (ζ,χ)-Laplace transform is applied to both sides, yielding:

$$\left[\left(\frac{\zeta}{\chi}s + 2\frac{\chi-\zeta}{\chi}\right)\mathcal{L}_{\zeta,\chi}[T(\tau)](s) - \frac{\zeta}{\chi}T_0\right] + k\mathcal{L}_{\zeta,\chi}[T(\tau)](s) = \frac{kT_{\mathrm{amb}}}{s+\frac{\chi-\zeta}{\zeta}}$$

from which we get:

$$\mathcal{L}_{\zeta,\chi}[T(\tau)](s) = \frac{\frac{\zeta}{\chi}T_0}{\left(\frac{\zeta}{\chi}s + 2\frac{\chi-\zeta}{\chi} + k\right)} + \frac{kT_{\mathrm{amb}}}{\left(s+\frac{\chi-\zeta}{\zeta}\right)\left(\frac{\zeta}{\chi}s + 2\frac{\chi-\zeta}{\chi} + k\right)}$$

Applying partial fraction decomposition:

$$\begin{aligned}\mathcal{L}_{\zeta,\chi}[T(\tau)](s) = &\frac{T_0}{s+2\frac{(\chi-\zeta)}{\zeta}+k\frac{\chi}{\zeta}} + \frac{kT_{\mathrm{amb}}}{\left(k+\frac{\chi-\zeta}{\chi}\right)}\cdot\frac{1}{s+\frac{\chi-\zeta}{\zeta}} \\ &- \frac{kT_{\mathrm{amb}}}{\left(k+\frac{\chi-\zeta}{\chi}\right)}\cdot\frac{1}{s+2\frac{(\chi-\zeta)}{\zeta}+k\frac{\chi}{\zeta}}\end{aligned}$$

Finally, applying the inverse (ζ,χ)-Laplace transform, we obtain:

$$T(\tau) = \frac{kT_{\mathrm{amb}}}{k+\frac{\chi-\zeta}{\chi}} + \left(T_0 - \frac{kT_{\mathrm{amb}}}{k+\frac{\chi-\zeta}{\chi}}\right)e^{-\left(\frac{\chi-\zeta}{\zeta}+k\frac{\chi}{\zeta}\right)\tau}$$

This provides the solution for the body's temperature over time, expressed through the (ζ,χ)-fractional model.

4 Conclusion

The incorporation of the parameters ζ and χ not only broadens the scope of application of the Laplace transform, but also establishes a natural connection with the foundations of fractional calculus. This generalization turns the transform into a more flexible and powerful tool for analyzing complex dynamical systems, particularly those that cannot be adequately described by classical methods.

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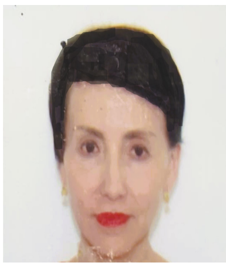

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