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Stochastic Effects on Soliton Dynamics in the KdV Equation with Multiplicative White Noise

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Abstract: This research seeks to investigate the impact of multiplicative white noise on the dynamical properties of soliton solutions (SS) within the outline of the KdV equation. While conventional KdV models describe nonlinear wave propagation in deterministic settings, real-world systems are frequently subjected to stochastic disturbances that can notably alter wave dynamics. By incorporating multiplicative white noise, this study examines the effects of random variations on soliton stability, morphology, and propagation. The Adomian decomposition and the simplest equation techniques are utilized to obtain solitary and SS, which are further validated through numerical simulations. This methodology enables the exploration of stochastic influences on soliton interactions and their long-term evolution. The results suggest that noise induces soliton deformation, amplitude modulation, and potential loss of coherence, underscoring the significant role of stochastic processes in nonlinear wave behavior. The findings highlight the resilience of solitons in the presence of random perturbations, providing valuable insights for applications in fields such as fluid dynamics, optical fiber communications and plasma physics. Additionally, the study discusses the interaction between two solitons using the superposition method.

Keywords: multiplicative white noise; soliton dynamics; nonlinear wave propagation; Stochastic KdV equation; Adomian decomposition technique.

1 Introduction

Nonlinear evolution equations are fundamental tools for modeling wave phenomena across a range of physical contexts. The KdV equation, in particular, has proven pivotal in capturing the dynamics of solitary waves in shallow water, plasmas, and other dispersive environments[1]. Over the years, a variety of analytical techniques such as the inverse scattering transform [1], Bäcklund and Darboux transformations[2,3,4,5,6] , and Hirota's bilinear method[14,15] have facilitated a deeper understanding of the integrable nature and solution structures of such equations. These methods have enabled the construction of diverse nonlinear waveforms, including rogue waves [7,16], breathers[21] , and soliton interactions with periodic or cnoidal backgrounds[18]. Advances in soliton theory have further extended to higher-dimensional, coupled, and discrete systems[3,8, 17,22] , as well as to nonlocal formulations[21]. This growth has been supported by refined solution classification techniques[4,11] and the application of symmetry-based approaches[2,18], together offering a comprehensive framework for analyzing deterministic integrable systems. In realistic scenarios, however, the assumption of purely deterministic dynamics is often untenable. Physical systems are frequently subjected to random perturbations, which can have profound effects on wave evolution. Particularly, the incorporation of multiplicative noise leads to stochastic partial differential equations (SPDEs), where understanding the statistical

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characteristics of solutions becomes essential[23,24]. Such stochastic influences are relevant in a range of fields, including fluid mechanics, optical fiber communication, and climate dynamics. To address the analytical challenges posed by SPDEs, semi-analytical techniques such as the Adomian Decomposition Method (ADM) have been developed and successfully applied to nonlinear stochastic problems[25,26,27]. ADM offers a systematic approach for decomposing nonlinear operators, thereby enabling the investigation of how stochastic perturbations modify key properties of soliton solutions, including their stability and propagation behavior. This work focuses on examining the statistical effects of multiplicative white noise on soliton solutions of the KdV equation. By integrating foundational methods from soliton theory[1,14] with stochastic modeling approaches [23,24], the study aims to uncover how noise impacts soliton dynamics. Through the use of ADM and ensemble-based analysis, this research contributes to a deeper understanding of soliton behavior in stochastic environments and broadens the theoretical framework for noise-perturbed integrable systems. Stochastic modeling provides a means to explore how random disturbances influence wave behavior, especially in situations where deterministic models, such as the noise-free KdV equation, are inadequate. By introducing stochastic terms, it becomes possible to study phenomena such as the formation of random solitons and the effects of noise on soliton stability and interactions. In real-world applications, noise-free, idealized systems are rare, and the inclusion of stochastic elements significantly broadens the relevance of the KdV equation, allowing it to better model environments where randomness and noise are inherent. One key stochastic process widely studied across various scientific fields is the Wiener process, also known as Brownian motion. This process is characterized by continuous, random trajectories that begin at the origin, with increments that are independent and typically normally distributed. A distinguishing feature of the Wiener process is its continuity, which ensures that no sudden jumps or discontinuities occur in the motion. Despite its continuity, the trajectories exhibit irregularity and lack well-defined tangents at any point (see Fig.1). The mathematical properties of the Wiener process, including independent increments, continuous paths, and normally distributed changes, make it a valuable tool for simulating stochastic phenomena. Its applications span across multiple disciplines, including economics, engineering, and physics.

The structure of this paper is organized as follows: Section 2 presents the derivation and theoretical formulation of the stochastic KdV equation. In Section 3, we apply the ADM to obtain SS of the governing equation. Section 4 explores alternative solitary solutions using the simplest equation method. Section 5 investigates the interaction dynamics of two solitary waves through the superposition technique. Finally, Section 6 summarizes the main findings and discusses potential directions for future research.

2 Mathematical investigations

In this section, we present the governing stochastic KdV equation and provide a justification for its formulation as follows:

$$U_t + \alpha U U_x + U_{xxx} = \sigma U W_t \tag{1}$$

where $U(x,t)$ is a real valued function of x and t,σ denote noise intensity,$W(t)$ is the white noise, and $\sigma U W_t$ is a multiplicative noise in the Itô sense. Following subsection, we analyses the governing equation using the wave transformation technique.

2.1 The stochastic KdV equation with wave transformation

The wave equation for stochastic KdV equation (1) is given by considering the following wave transformation:

$$U(x,t) = u(x,t)e^{\sigma W(t) - \sigma^2 t/2} \tag{2}$$

where the function $u(x,t)$ is deterministic. We have

$$\begin{aligned} U_t &= \left\{ u_t + u\left(\sigma W_t + \frac{\sigma^2}{2} - \frac{\sigma^2}{2}\right)\right\} e^{\left[\sigma W(t) - \sigma^2 t/2\right]} \\ &= \{u_t + \sigma u W_t\} e^{\left[\sigma W(t) - \sigma^2 t/2\right]} \\ U_x &= u_x e^{\left[\sigma W(t) - \sigma^2 t/2\right]} \\ U_{xxx} &= u_{xxx} e^{\left[\sigma W(t) - \sigma^2 t/2\right]} \end{aligned} \tag{3}$$

By inserting Equation (2) into Equations (1) using equation (3), we have

$$u_t + \sigma u W_t + \alpha u u_x e^{\left[\sigma W(t) - \sigma^2 t/2\right]} + u_{xxx} = \sigma u W_t \tag{4}$$

this can be simplified as

$$u_t + \alpha u u_x e^{\left[\sigma W(t) - \sigma^2 t/2\right]} + u_{xxx} = 0 \tag{5}$$

Taking the expectation to both sides on equation (5) we have:

$$u_t + \alpha u u_x e^{-\sigma^2 t/2} E(e^{\sigma W(t)}) + u_{xxx} = 0 \tag{6}$$

Since $W(t)$is a Gaussian process, we have $E(e^{\sigma W(t)}) = e^{-\sigma^2 t/2}$. Hence, Equation (6) becomes:

$$u_t + \alpha u u_x + u_{xxx} = 0. \tag{7}$$

This represents the KdV equation, one of the most well-known nonlinear partial differential equations

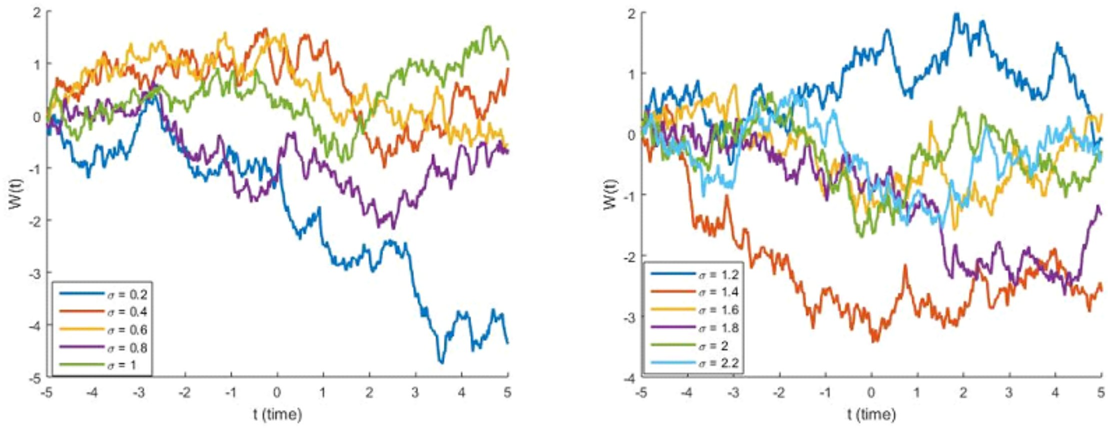

Fig. 1: Illustration of Wiener process trajectories for smooth real random functions on (-5, 5) with different values of from 0.2 to 2.2.

(NLPDEs). Equation (7) describes wave propagation that accounts for both nonlinearity and dispersion. A positive value of α corresponds to forward wave propagation, while a negative value indicates backward wave propagation.

3 Soliton Wave Solutions Using the Adomian Decomposition Technique

George Adomian [25,26,27] developed the Adomian decomposition technique, also referred to as the inverse operator approach, to address both linear and nonlinear problems in mathematical physics. This technique involves decomposing the given equation $Fu = g(t)$ (which may include linear and nonlinear equations, ordinary differential equations, partial differential equations, and integral equations), into a series of components representing solutions of different degrees. Solutions are then determined for each component, and their sum provides an approximation of the true solution with any desired level of accuracy. The process involves several key steps. First, the entire equation is divided into distinct parts, with the linear component further separated into linear and reversible residual components. Second, the Adomian polynomial, which serves as an analogous polynomial, is derived for the nonlinear portion using the splitting principle. Finally, recursive rules are applied to generate the solutions. We assume that the partial differential system is expressed in the operator formula as follows to solve the nonlinear KdV model:

$$\ell_t \mu + \ell_x \mu + M(u) = 0 \tag{8}$$

where the symbolizations represent the nonlinear operator $M(u)$, and the linear differential operators are represented by $\ell_t = \partial/\partial t$ and $\ell_x = \partial/\partial x$. Using Equation (8) and the inverse differential operator $\ell_t^{-1} = \int_0^t (\cdot)\, d\tau$, we have

$$u(t,x) = f(x) - \ell_t^{-1}\left[F(u) + \ell_x u\right] \tag{9}$$

with $u_0(x,0) = f(x)$ is the function of the initial condition that has been given. The decomposition technique assumes that there is an infinite series for an unknown function $u(x,t)$ in the form

$$u(t,x) = \sum_{j=0}^{\infty} u_j(t,x) \tag{10}$$

in addition to nonlinear operators, the expression for is provided by an infinite series of Adomian polynomials. $F(u)$ is

$$F(u) = \sum_{j=0}^{\infty} M_j \tag{11}$$

where the process described in [27] is used to generate the pertinent Adomian polynomials. For the convenience of the reader, we employ the nonlinear term $F(u) = \sum_{j=0}^{\infty} M_j$ of the generic preparation for the Adomian polynomials M_j , which is described by

$$M_j(u_0,\ldots,u_j) = \frac{1}{j!}\frac{d^j}{d\lambda^j}\left[f\left(\sum_{s=0}^{j}\lambda^s u_s\right)\right]_{\lambda=0}, \quad j \geq 1 \tag{12}$$

This approach can be readily employed to guide the computer code in determining the number of polynomials needed for both explicit and numerical solutions. For a comprehensive explanation of the Adomian decomposition method and the general formula for Adomian polynomials, the reader is directed to [27] and the relevant references there in.

Using the decomposition process, the recursive relationship generates the formula for the nonlinear Equation (12)

$$\varphi_0(t,x) = f(x), \quad u_{j+1}(t,x) = -\ell_t^{-1}\left[\ell_x u_j + M_j\right] \tag{13}$$

when the initial condition is specified, the function $f_1(x)$ is derived based on these conditions. Recalling that the zeroth components are more distinct than the other components is crucial, $u_j(t,x), j \geq 1$ could be fully determined, meaning each term could be calculated using the terms that came before it.

Thus, the series solutions are completely determined, and components $u_0, u_1, u_3, \ldots$ are identified. However, obtaining the exact solution in a closed form is often feasible. We created the solution $u(t,x)$ using numerical formulae in the form

$$u(t,x) = \lim_{j\to\infty} \sum_{k=0}^{j} u_k(t,x) \tag{14}$$

and the recurrent relationship is given by Equation (13). Moreover, the solutions of the decomposition series tend to converge rather quickly in truly physical domains. In the next part, we analyze the nonlinear KdV model to show that the ADM is applicable. The nonlinear KdV model that follows is to be considered as

$$u_t + u_{xxx} + \alpha u u_x = 0 \tag{15}$$

under the initial condition

$$f(x) = u_0(x,0) \tag{16}$$

with $f(x)$ is a given function. We redefine Equation (15) in an operator formula to solve it using the Adomian decomposition approach.

$$\ell_t \mu = -\alpha u u_x - u_{xxx} \tag{17}$$

where the differential linear operator is expressed by $\ell_t = \partial/\partial t$ and the inverse fractional operator ℓ_t^{-1} is provided by

$$\ell_t^{-1} = \int_0^t (\cdot)\, d\tau \tag{18}$$

Operating with ℓ_t^{-1} on both sides of Equation (17) give

$$u(t,x) = u(0,x) - \ell_t^{-1}\left(\alpha u u_x + u_{xxx}\right) \tag{19}$$

To characterize the unknown function, the ADM assumes that an infinite series can be employed $u(t,x)$ as

$$u(t,x) = \sum_{j=0}^{\infty} u_j(t,x) \tag{20}$$

Equations (9) can be substituted with Equation (19) to get

$$u_{j+1}(t,x) = u(0,x) - \ell_t^{-1}\left(M + \ell_x u_j\right), \quad M = \alpha u u_x \tag{21}$$

Where $M(u,u_x) = \sum_{j=0}^{\infty} M_j$ the components $M_j$can be computed by using the formula

$$\begin{aligned} M &= \sum_{j=0}^{\infty} M_j = u u_x \\ &= \left(u_0 + u_1\lambda + u_2\lambda^2 + \ldots\right)\frac{\partial}{\partial x}\left(u_0 + u_1\lambda + u_2\lambda^2 + \ldots\right) \end{aligned} \tag{22}$$

Since

$$M_j = \frac{1}{j!}\frac{d^j}{d\lambda^j}\left[\left(u_0 + u_1\lambda + u_2\lambda^2 + \ldots\right)\frac{\partial}{\partial x}\left(u_0 + u_1\lambda + u_2\lambda^2 + \ldots\right)\right]_{\lambda=0} \tag{23}$$

The first four terms can be written as

$$\begin{aligned} M_0 &= 2u_0u_{0x}, \\ M_1 &= 2u_0u_{1x} + 2u_1u_{0x}, \\ M_2 &= 2u_0u_{2x} + 2u_1u_{1x} + 2u_2u_{0x}, \\ M_3 &= 2u_0u_{3x} + 2u_1u_{2x} + 2u_2u_{1x} + 2u_3u_{0x}. \end{aligned} \tag{24}$$

For a given

$$u_0(0,x) = \frac{3\omega}{\alpha}\operatorname{sech}^2\left(\sqrt{\frac{\omega}{4}}x\right) \tag{25}$$

with ω is the speed velocity of the wave. The remainder components $u_j(t,x), j > 0$ can be found using the recursive relations in the following manner

$$u_0(t,x) = \frac{3\omega}{\alpha}\operatorname{sech}^2\left(\sqrt{\frac{\omega}{4}}x\right) \tag{26}$$

$$\begin{aligned} u_1(t,x) = {} & \frac{3\alpha\omega^2}{\alpha^2}\sqrt{\omega}\tanh\left(\sqrt{\frac{\omega}{4}}x\right) \\ & \times\left[\alpha\operatorname{sech}^2\left(\sqrt{\frac{\omega}{4}}x\right) + 3(2-\alpha)\operatorname{sech}^4\left(\sqrt{\frac{\omega}{4}}x\right)\right]t \end{aligned} \tag{27}$$

and so forth. The decomposition of a domain solutions of $u(t,x)$ so

$$\begin{aligned} u(t,x) = {} & \frac{3\omega}{\alpha}\operatorname{sech}^2\left(\sqrt{\frac{\omega}{4}}x\right) \\ & + \frac{3\alpha\omega^2}{\alpha^2}\sqrt{\omega}\tanh\left(\sqrt{\frac{\omega}{4}}x\right)\left[\alpha\operatorname{sech}^2\left(\sqrt{\frac{\omega}{4}}x\right)\right. \\ & \left. + 3(2-\alpha)\operatorname{sech}^4\left(\sqrt{\frac{\omega}{4}}x\right)\right]t + \ldots \end{aligned} \tag{28}$$

This can be succinctly expressed as

$$u(t,x) = \frac{3\omega}{\alpha}\operatorname{sech}^2\left(\sqrt{\frac{\omega}{4}}(x-\omega t)\right) \tag{29}$$

Also, for a given equation

$$u_0(0,x) = f(x) = \frac{3\omega}{\alpha} - \frac{3\omega}{\alpha}\tanh^2\left(\sqrt{\frac{\omega}{4}}x\right) \tag{30}$$

the remainder components$u_j(t,x), j >$ may be discovered in the same way using the recursive relations, and it can be compactly stated as

$$u(t,x) = \frac{3\omega}{\alpha} - \frac{3\omega}{\alpha}\tanh^2\left(\sqrt{\frac{\omega}{4}}(x-\omega t)\right) \tag{31}$$

The impacts of white noise on the SS of the stochastic KdV equation (1) are discussed here. Focuses on how the noise influence on the dynamical behaviors of the model, particularly in the reference of deterministic model which impacts crucial for analyzing how model evolve over time or in long-term behaviors represented by equation (29). We will introduce various graphical representations to depict the solutions of the stochastic KdV equation (1) under varying conditions, particularly emphasizing the impact of noise strength (σ). These illustrations help us for good understanding the fine details of how different noise levels affect the dynamics of the model. Figure 2 represented the deterministic bell soliton solution when $(\sigma = 0)$, gives the flat bell soliton solution. Figure 3 represented the non-deterministic solution when $(\sigma = 0.4)$, gives the non-flat bell soliton solution. Also Figure 4 represented the non-deterministic solution when $(\sigma = 1.3)$, gives the non-flat bell soliton solution. Figure $5-a$ represented the cross section 2-dimnsional at constant $x = 3$ with different choose of $(\sigma = 0.1, 0.3, 0.5, 0.7, 0.9)$. In addition, Figure $5-b$ represented the 2-dimnsional at constant $x = 3$ with different choose of $(\sigma = 1.2, 1.4, 1.6, 1.8, 2)$. In Figure 6 which illustrates the backward wave propagation for $\alpha = -6$. Each subplot is labeled sequentially: $(6-a)$ represents the deterministic case (a special case of the SDE) for a one-soliton solution when $(\sigma = 0)$; $(6-b)$ shows the SDE for a one-soliton solution with $(\sigma = 0.3)$; $(6-c)$ presents the SDE for a one-soliton solution with $(\sigma = 1.1)$; and $(6-d)$ displays the 2D cross-section at $x = 1$ for diverse values of $(\sigma = 0.1, 0.3, 0.5, 0.7, 0.9)$.

4 Solitary solution based on simplest technique

The proposed approach is grounded in the principle of the simplest equation method (SEM). In this context, we provide a brief overview of the SEM algorithm, Initially, assuming a traveling wave solution $\xi = kx - \omega t$, where k represented wave number and ω denoted wave speed, the equation is subsequently reduced to a nonlinear ordinary differential equation. Next, it is assumed that the exact solution can be expressed as a finite series

$$u(\xi) = \sum_{s=0}^{N} a_s [\psi(\xi)]^s \tag{32}$$

with a_s are determined later, $\psi(\xi)$ is explicit solution of Burgers model referred to as simplest equation, N is a constant to be determined, which represents the power of the specified solution function finite series, which can be computed by balancing between the highest order derivative term and the highest order nonlinear term appearing in the nonlinear PDE. The simplest equation is of lower order than the nonlinear PDE and its general solution is will know.

4.1 The simplest equation technique based on Burgers' equation

The Burgers' model is

$$\psi_t(t,x) - 2\psi(t,x)\psi_x(t,x) - \psi_{xxx}(t,x) = 0 \tag{33}$$

The solitary solution of Equation (33) is:

$$\psi(t,x) = \frac{ke^{kx-\alpha t}}{1+e^{kx-\alpha t}} \tag{34}$$

with k_i and ω_i are random constants and N positive number. The traveling wave ansatz is $\xi = kx - \omega t$, so that $\psi(t,x) = \psi(\xi)$. The Burgers' model (33) is thus changed into

$$-\omega\psi' - 2k\psi\psi' - k^2\psi'' = 0 \tag{35}$$

Upon performing the first integration of Equation (25) with respect to ξ , and supposing that the constant of integration is zero, the resulting expression simplifies to

$$\psi_\xi = -\frac{1}{k}\psi^2 - \frac{\omega}{k^2}\psi \tag{36}$$

It is widely recognized that the dispersion relation of Equation (33) is

$$\omega = -k^2 \tag{37}$$

Consequently, Equation (36) can be expressed as.

$$\psi_\zeta = \psi - \frac{1}{k}\psi^2 \tag{38}$$

and its solitary solution is constructed as:

$$\psi(t,x) = \frac{ke^{kx-\alpha t}}{1+e^{kx-\alpha t}} \tag{39}$$

Building upon the aforementioned procedures, the use of the SEM, with the assistance of Equations (39) and (38), proves effective in solving the $(1+1)$-dimensional nonlinear PDEs.

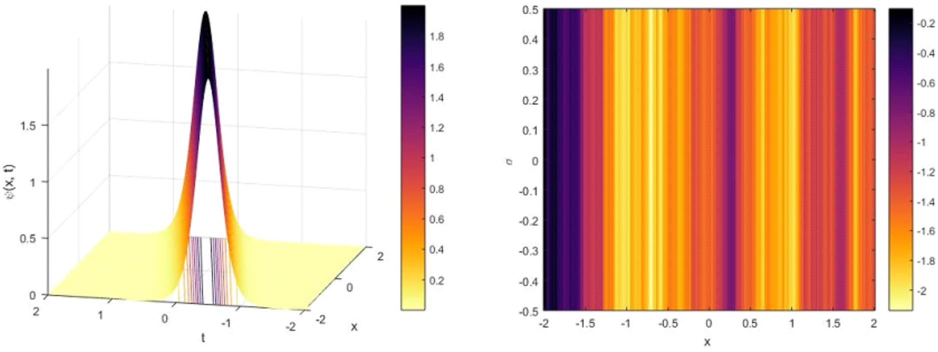

Fig. 2: Evolution of flat bell SS when $\sigma = 0$, $\omega = 4$, and $\alpha = 6$.

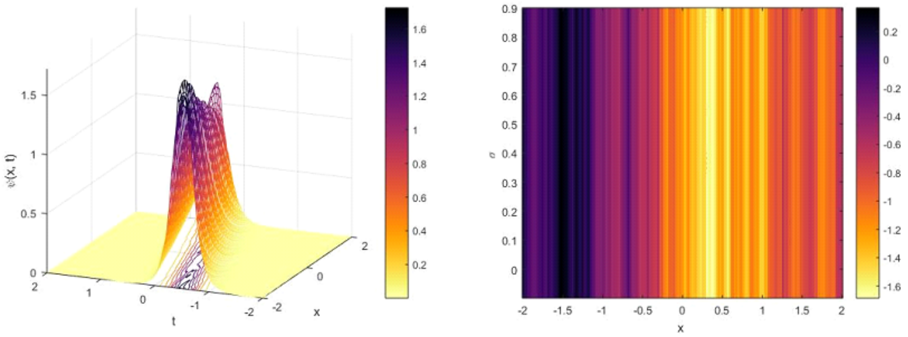

Fig. 3: Evolution of non-flat one SS when $\sigma = 0.4$, $\omega = 4$, and $\alpha = 6$.

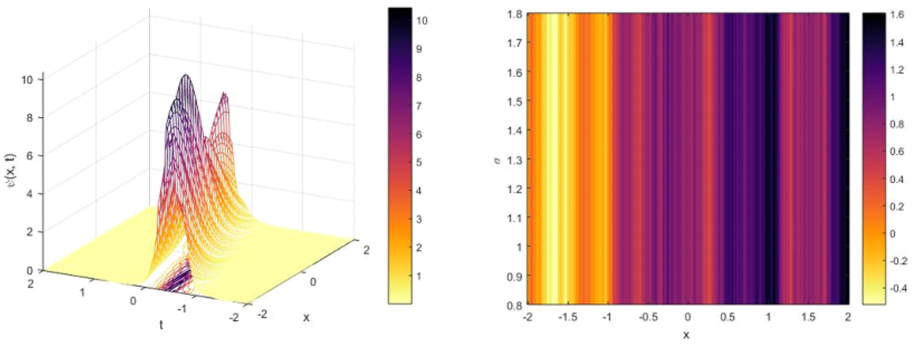

Fig. 4: Evolution of non-flat one SS when $\sigma = 1.3$, $\omega = 4$, and $\alpha = 6$.

4.2 New SS of the KdV model (7)

To demonstrate the accuracy of the aforementioned method, the KdV equation is examined. Let us consider the KdV equation as follows:

$$u_t + \alpha u u_x + u_{xxx} = 0 \tag{40}$$

By introducing the traveling wave variable $\xi = kx - \omega t$, then Equation (40) is transformed into

$$-\omega u_\xi + \alpha k u u_\xi + k^3 u_{\xi\xi\xi} = 0 \tag{41}$$

By integrating with respect to ξ and supposing the constant of integration is zero, the result is

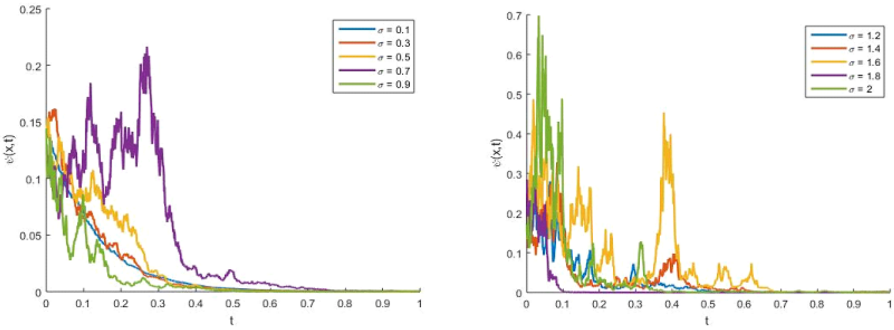

Fig. 5: Cross section 2-dimnsional at constant $x = 1$ with different choose of σ.(a) when $\sigma = 0.1, 0.3, 0.5, 0.7, 0.9$. (b) when $\sigma = 1.2, 1.4, 1.6, 1.8, 2$.

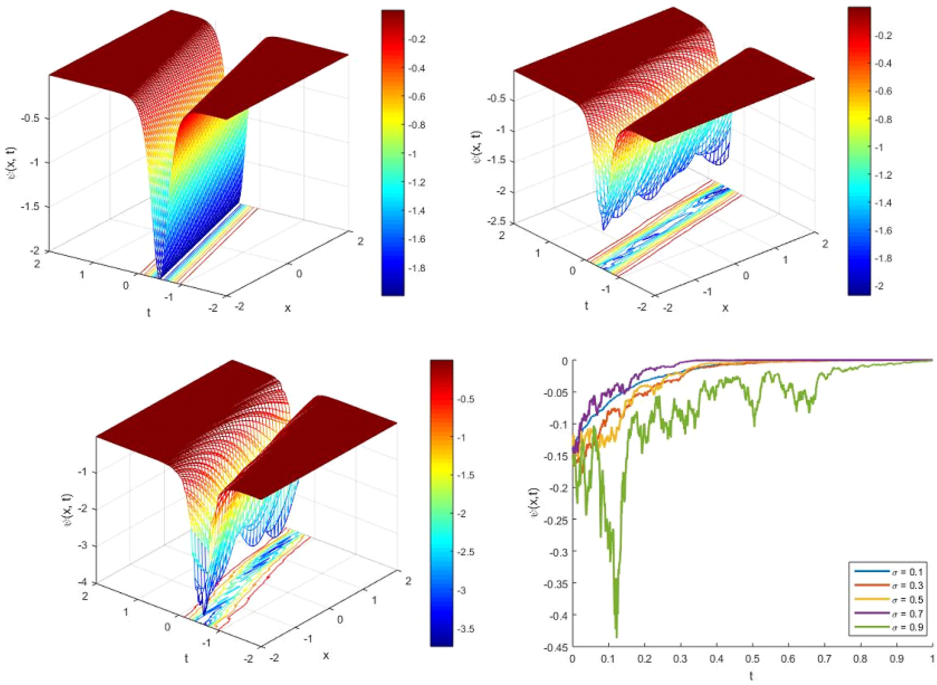

Fig. 6: Represented backward wave propagation with $\alpha = -6, \omega = 4.6 - a)$ evolution of flat one soliton solution when $\sigma = 0, 6 - b)$ non-flat one soliton solution when $\sigma = 0.4.(6 - c)$ non-flat one soliton solution when $\sigma = 1.3.6 - d)$ cross section 2-dimnsional at constant $x = 1$ with different choose of σ.

$$-\omega u + \frac{\alpha k}{2} u^2 + k^3 u_{\xi\xi} = 0 \tag{42}$$

Substituting Equations (38) and (39) into (42) and utilizing the balance equation, balancing $u_{\xi\xi}$ with u^2 yields $N = 2$. Thus, the exact solution to Equation (42) is assumed to be

$$u = a_0 + a_1\psi + a_2\psi^2 \tag{43}$$

Since $\psi_\xi = \psi - \frac{1}{k}\psi^2$ so we have

$$\begin{aligned} u_\xi &= (a_1 + 2a_2\psi)\psi_\xi \\ &= (a_1 + 2a_2\psi)\left(\psi - \frac{1}{k}\psi^2\right) \\ &= a_1\psi - \left(\frac{a_1}{k} - 2a_2\right)\psi^2 - \frac{2a_2}{k}\psi^3 \end{aligned} \tag{44}$$

$$u_{\xi\xi} = a_1\psi + \left(-\frac{3a_1}{k} + 4a_2\right)\psi^2 + \left(\frac{2a_1}{k^2} - \frac{10a_2}{k}\right)\psi^3 + \frac{6a_2}{k^2}\psi^4 \tag{45}$$

Substituting Equations (45) and (43) into (42), we have

$$-\omega(a_0 + a_1\psi + a_2\psi^2) + \frac{\alpha k}{2}\left(a_0^2 + 2a_0a_1\psi + (2a_0a_2 + a_1^2)\psi^2\right) + \frac{\alpha k}{2}\left(2a_1a_2\psi^3 + a_2^2\psi^4\right) + k^3\left(a_1\psi + \left(-\frac{3a_1}{k} + 4a_2\right)\psi^2\right) + k^3\left(\left(\frac{2a_1}{k^2} - \frac{10a_2}{k}\right)\psi^3 + \frac{6a_2}{k^2}\psi^4\right) = 0$$

Setting like power coefficient of $\psi(\zeta)$ to zero, gives

$$\begin{aligned}
\psi^0 &: \quad -\omega a_0 k^2 + \frac{\alpha k}{2}a_0^2 = 0,\\
\psi^1 &: \quad -\omega a_1 + \frac{\alpha k}{2}\cdot 2a_0a_1 + a_1k^3 = 0,\\
\psi^2 &: \quad -\omega a_2 + \frac{\alpha k}{2}(2a_0a_2 + a_1^2) + k^3\left(-\frac{3a_1}{k} + 4a_2\right) = 0,\\
\psi^3 &: \quad \frac{\alpha k}{2}\cdot 2a_1a_2 + k^3\left(\frac{2a_1}{k^2} - \frac{10a_2}{k}\right) = 0,\\
\psi^4 &: \quad \frac{\alpha k}{2}a_2^2 + k^3\cdot\frac{6a_2}{k^2} = 0.
\end{aligned}$$

Solving this system we have

$$a_0 = 0, \quad a_1 = \frac{12k}{\alpha}, \quad a_2 = -\frac{12}{\alpha}, \quad \omega = k^3 \tag{46}$$

So, the soliton solution constructed as:

$$u = \frac{12}{\alpha}\left(\frac{k^2e^{kx-k^3t}}{1+e^{kx-k^3t}}\right) - \frac{12}{\alpha}\left(\frac{ke^{kx-k^3t}}{1+e^{kx-k^3t}}\right)^2 \tag{47}$$

The influences of white noise on the SS of the stochastic KdV equation (1) are debated here. Emphases on how the noise effect on the dynamical performances of the model, particularly in the reference of deterministic model which impacts crucial for analyzing how model evolve over time or in long-term behaviors represented by equation (47). We will introduce various graphical representations to depict the soliton solutions of the stochastic KdV equation (1) under varying conditions, particularly highlighting the influence of noise strength (σ). These illustrations help us for good understanding the fine details of how different noise levels affect the dynamics of the model. Figure 7 represented the deterministic soliton solution when ($\sigma = 0$), gives the flat periodic soliton solution. Figure8 represented the non-deterministic solution when ($\sigma = 0.4$), gives the non-flat soliton solution. Also Figure 9 represented the non-deterministic solution when ($\sigma = 1.3$), gives the non-flat soliton solution. Figure10-a represented

the cross section 2-dimnsional at constant x=1 with different choose of ($\sigma = 0.1, 0.3, 0.5, 0.7, 0.9$). In addition, Figure10-b represented the 2-dimnsional at constant $x = 1$ with different choose of ($\sigma = 1.2, 1.4, 1.6, 1.8, 2$). In Figure11 which illustrates the backward wave propagation for $\alpha = -6$. Each subplot is labeled sequentially: $(11-a)$ represents the deterministic case (a special case of the SDE) for a one-SS when ($\sigma = 0$); $(11-b)$ shows the SDE for a SS with ($\sigma = 0.3$); $(11-c)$ presents the SDE for a SS with ($\sigma = 1.3$); and $(11-d)$ displays the 2D cross-section at $x = 1$ for diverse values of ($\sigma = 0.1, 0.3, 0.5, 0.7, 0.9$).

5 Interaction of two SS based on superposition technique

The superposition method is a mathematical technique employed to analyze the interactions between two or three solitons, which are solitary waveforms that maintain their shape and velocity after interacting with one another. This approach is particularly valuable, as solitons exhibit the unique property of retaining their integrity post-interaction, unlike regular waves that tend to scatter or dissipate. The core concept behind the superposition method is that, in specific circumstances, the interaction of multiple solitons can be represented as the sum of individual soliton solutions. Solitons are a distinct category of waves that retain their form while propagating at constant speeds, and their interactions exhibit characteristics that differ significantly from linear waves. Whereas linear waves simply superpose upon interaction, solitons interact in a nonlinear manner. The superposition principle facilitates the modeling of these interactions, accounting for the persistence of shape and velocity, even after the solitons collide.

The mathematical treatment of solitons often involves solving nonlinear PDEs, such as the KdV or the sine-Gordon equations. In these cases, solutions representing multiple solitons can often be expressed as a sum of individual SS, particularly when the interaction is analyzed in the asymptotic regime, before or after the actual interaction occurs. A notable feature of solitons is their ability to collide and pass through one another without altering their identity. The superposition technique models this behavior effectively, allowing for the representation of solitons both before and after their interactions. While the positions of the solitons may shift during the interaction, they retain their original shape and speed, akin to an elastic collision. The superposition method is advantageous due to the mathematical structure of the equations governing solitons, which often permit

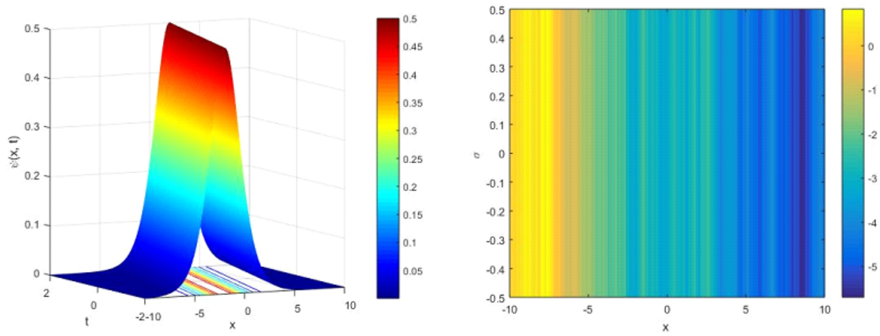

Fig. 7: Evolution of flat SS when $\sigma = 0$, $k = 1$, and $\alpha = 6$

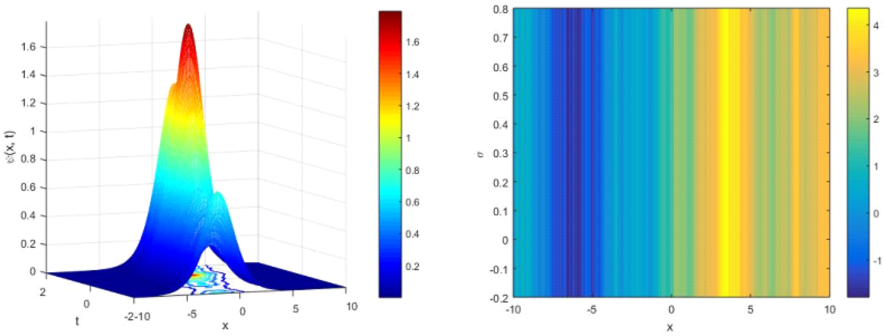

Fig. 8: Evolution of non-flat SS when $\sigma = 0.4$, $k = 1$, and $\alpha = 6$

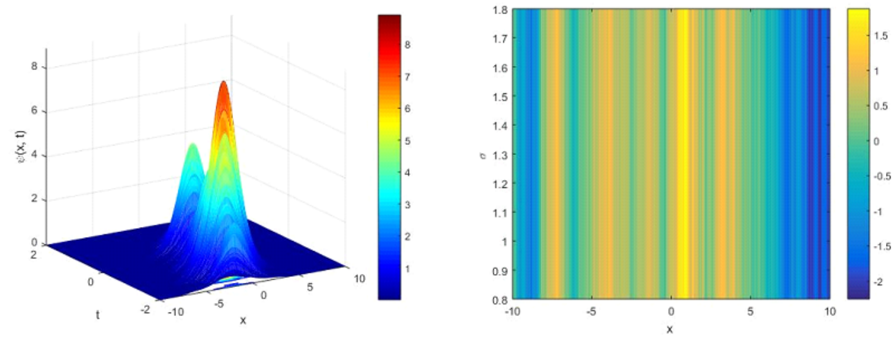

Fig. 9: Evolution of non-flat SS when $\sigma = 1.3$, $k = 1$, and $\alpha = 6$

solutions that can be algebraically combined. Following an interaction, the solitons can be represented as shifted versions of their original forms. In scenarios involving more than two solitons, such as three or more, the superposition method remains applicable because solitons preserve their integrity through collisions. As a result, each soliton continues to move independently after interacting with others, thereby maintaining its individual characteristics. Overall, the superposition method simplifies the analysis of complex nonlinear interactions between solitons, providing a powerful tool for modeling and understanding the behavior of solitons in various contexts. This method aligns with the inherent nature of soliton solutions, which retain their form and energy during interactions.

To apply the superposition technique for two solitons, the solution is obtained by adding the individual soliton

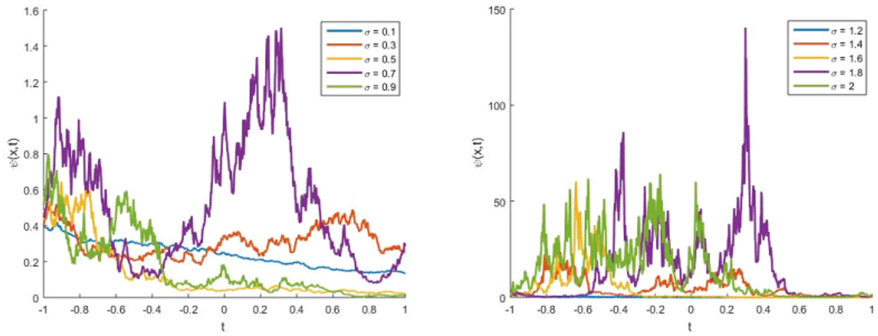

Fig. 10: Cross section 2-dimnsional at constant $x = 1$ with diverse choose of σ. a) when $\sigma = 0.1, 0.3, 0.5, 0.7, 0.9$. b) when $\sigma = 1.2, 1.4, 1.6, 1.8, 2$.

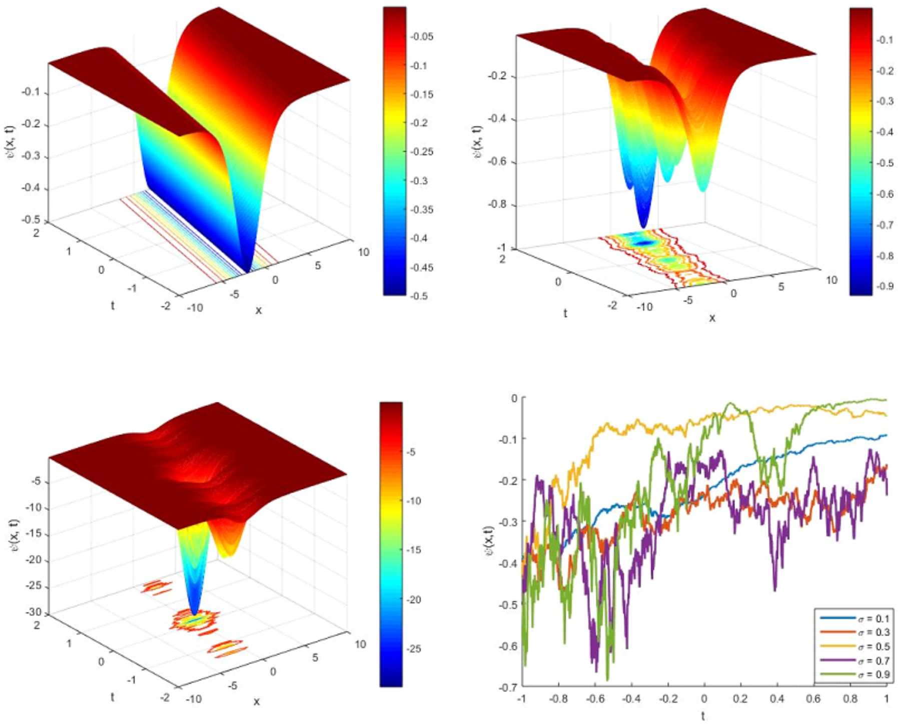

Fig. 11: Represented backward wave propagation with $\alpha = -6$, . a) evolution of flat one SS when $\sigma = 0$. b) non-flat one SS when $\sigma = 0.4$. c) non-flat SS when $\sigma = 1.3$. d) cross section 2-dimnsional at constant $x = 1$ with diverse choose of σ.

solutions. The general form for the two-SS to the deterministic KdV model is

$$u(t,x) = \frac{3\omega_1}{\alpha}\,\mathrm{sech}^2\left(\sqrt{\frac{\omega_1}{4}}(x - \omega_1 t)\right) + \frac{3\omega_2}{\alpha}\,\mathrm{sech}^2\left(\sqrt{\frac{\omega_2}{4}}(x - \omega_2 t)\right) \quad (48)$$

However, this form is only an approximation of the true solution. In the case of soliton interactions, the exact solution involves more complex interactions, and the solitons retain their shape after the interaction. Figure12 represented the interaction of the deterministic two SS when $(\sigma = 0)$ with $\alpha = 6, \omega_1 = 1, \omega_1 = 5$. Figure13 shows the interaction of the non-deterministic two SS when $(\sigma = 0.3)$. Figure14 shows the interaction of the non-deterministic two SS when $(\sigma = 1.1)$ and Figure15

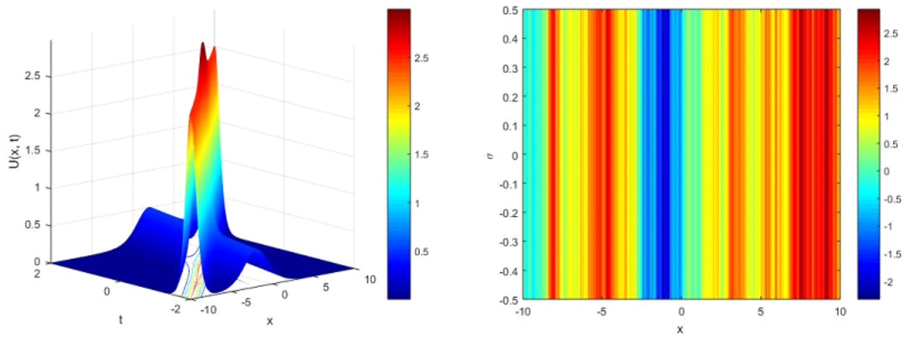

Fig. 12: the interaction of deterministic two SS with $\sigma = 0$, $\alpha = 6, \omega_1 = 1, \omega_2 = 5$.

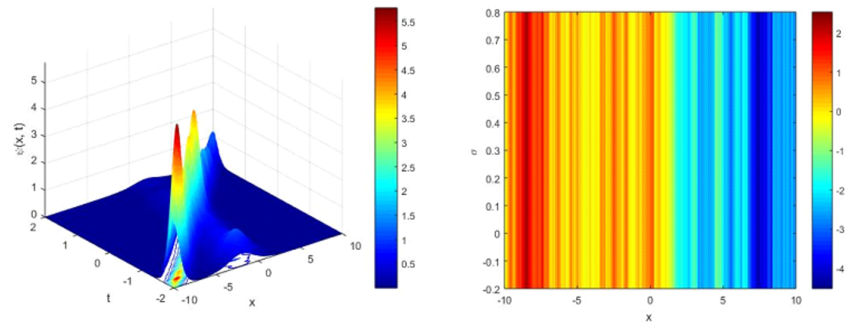

Fig. 13: the interaction of non-deterministic two SS with $\sigma = 0.3$, $\alpha = 6, \omega_1 = 1, \omega_2 = 5$.

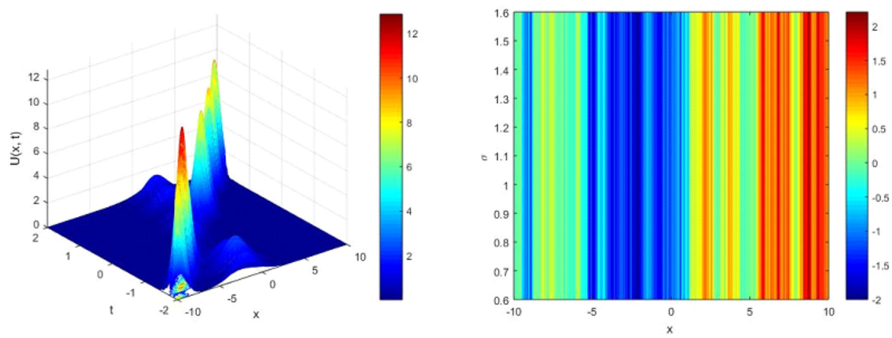

Fig. 14: the interaction of non-deterministic two SS with $\sigma = 1.1$, $\alpha = 6, \omega_1 = 1, \omega_2 = 5$.

shows the cross section 2-dimnsional at constant $x = 1$ with diverse choose of $(\sigma = 0.1, 0.3, 0.5, 0.7, 0.9)$ and $(\sigma = 1.2, 1.4, 1.6, 1.8, 2)$.

In Figure 16 which illustrates the backward wave propagation of the interaction of two SS for $\alpha = -6$, $\omega_1 = 1, \omega_2 = 5$. Each subplot is labeled sequentially: (16-a) represents the deterministic case (a special case of the SDE) for interaction of two SS when $(\sigma = 0)$; (16-b) shows the SDE for interaction of two SS with $(\sigma = 0.3)$; (16-c) presents the SDE for interaction of two SS with $(\sigma = 1.1)$; and (16-d) displays the 2D cross-section at $x = 1$ for different values of $(\sigma = 0.1, 0.3, 0.5, 0.7, 0.9)$.

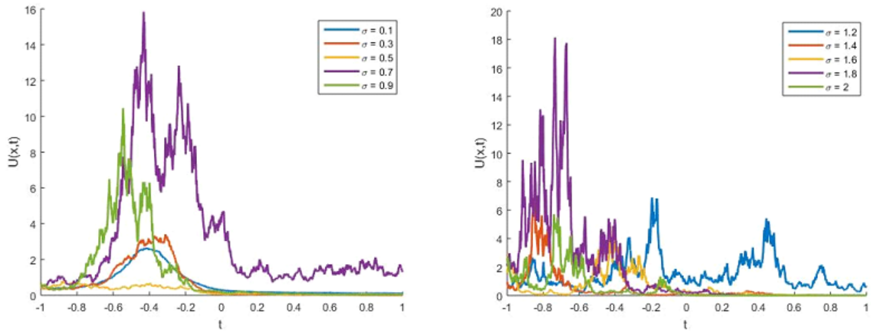

Fig. 15: Cross section 2-dimnsional at constant $x = 1$ with diverse choose of σ. a) when $\sigma = 0.1, 0.3, 0.5, 0.7, 0.9$. b) when $\sigma = 1.2, 1.4, 1.6, 1.8, 2$.

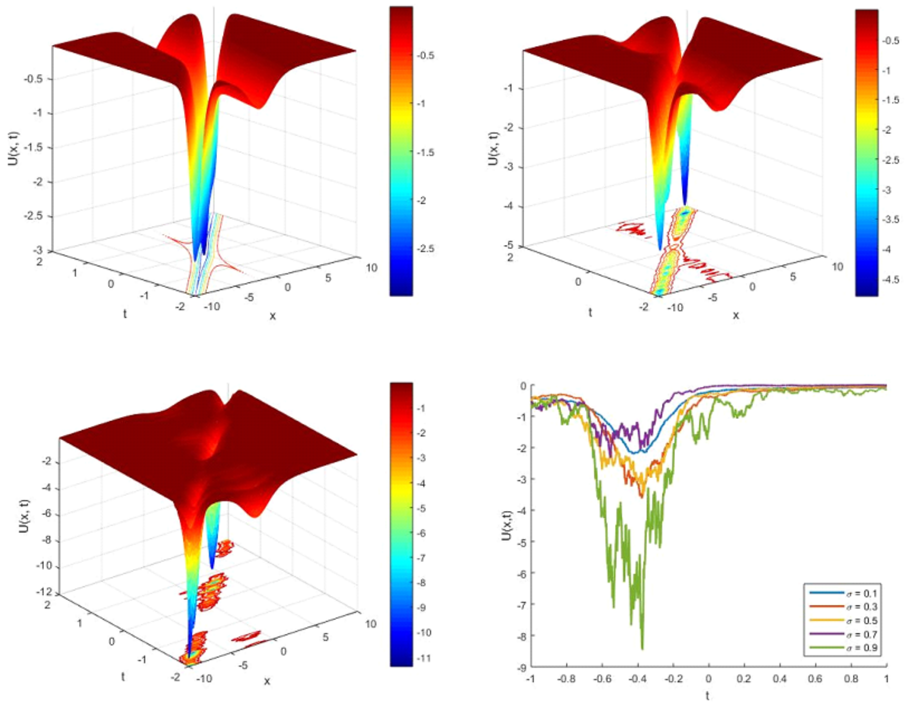

Fig. 16: Represented backward wave propagation of the interaction of two SS with $\alpha = -6$, $\omega_1 = 1, \omega_2 = 5$. a) when $\sigma = 0$. b) when $\sigma = 0.3$. c) when $\sigma = 1.1$. d) cross section 2-dimnsional at constant $x = 1$ with different choose of σ.

6 Concluding remarks and scope for further research

This study highlights the significant impact of multiplicative white noise on the properties of soliton solutions, revealing that random fluctuations can lead to soliton deformation, changes in amplitude, phase shifts, and alterations in wave velocity. These findings challenge the conventional view of solitons as purely stable structures in deterministic systems, suggesting that they are more sensitive to stochastic disturbances than previously anticipated. The analytical and numerical methods employed have provided valuable insights into how noise influences both individual solitons and their interactions, emphasizing the need to account for stochastic effects in models of real-world systems that rely on soliton solutions.

By demonstrating that noise affects soliton dynamics and interactions, this study opens the door to a deeper understanding of how complex, random processes shape the behavior of nonlinear wave systems. The results underscore the importance of incorporating stochastic models to more accurately reflect the dynamics observed in natural and engineered systems.

The results from this study suggest several directions for future investigation. One promising avenue is the exploration of other forms of stochastic processes, such as non-Gaussian or colored noise, and their impact on soliton behavior. These noise models could provide further insights into the influence of different noise structures on soliton interactions and stability. Additionally, extending the study to examine higher-dimensional soliton systems or more complex nonlinear evolution equations could deepen our understanding of how stochasticity affects soliton dynamics in various contexts.

Further research could also focus on the long-term evolution of solitons under the influence of multiplicative noise, particularly in practical applications such as fluid dynamics, plasma physics, and optical fibers, where solitons play a critical role. Investigating the role of noise in the formation and interaction of soliton trains or the stability of multi-soliton solutions could provide important contributions to the modeling of real-world systems. Finally, examining the interplay between noise and external forcing factors could lead to new insights into the robustness and adaptability of solitons in dynamic environments.

Considering the widespread application of solitons in areas like optical fiber communications and plasma physics, it is crucial to explore practical strategies for reducing the effects of noise on soliton propagation. Future research could concentrate on developing optimized communication techniques aimed at minimizing errors caused by noise, thereby enhancing the reliability and efficiency of systems that rely on soliton-based solutions.

Data Availability Statement:

All data generated or analyzed during this study are included within the manuscript. Additional data, if necessary, can be provided upon reasonable request.

Author Contributions:

All authors contributed equally to the conceptualization, methodology, formal analysis, writing, and revision of this manuscript. Each author has read and approved the final version of the paper.

Conflicts of Interest:

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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