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Modeling and Advance Analysis on the Network of Militants with Mittag-Leffler kernel

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Abstract: Many real-world problems are analyzed by using this generalized kind of fractional derivative such as fractal-fractional derivative. In this essay, the detrimental effects on human life and economic considerations are illustrated through an analysis of the militant network's mathematical model. In order to investigate the effects of militants on society, an epidemic model with a time-fractional order was used to characterise a militant network. The suggested model's existence and uniqueness are demonstrated using equilibrium analysis. A qualitative study as well as a sensitivity analysis of the fractional order system are conducted. Another method for assessing the local and global impacts of militancy on society is the Ulam-Hyres stability. Additionally, the Lipschitz condition and linear growth model are employed to satisfy the uniqueness of the exact solution criterion. Two-step Lagrange polynomials with Mittag-Leffler kernels are used to find solutions, which show how the illness affects plants by examining the effects of fractional operators. To comprehend how the militant network model behaves, simulations have been created. In order to reduce the number of militants and the danger of terrorism, decision-makers will be able to create circumstances based on the parameters of the model with the aid of this analysis. This research supports the United Nation SDG 16 (Peace, Justice and Strong Institutions) by developing militant's network model with appropriate level of policing measures.

Keywords: Fractal-fractional derivative; Network of militants; Uniqueness; Sensitivity; Local Stability; Ulam-hyers stability; Mittag-Leffler kernel, UN SDG 16 (Peace, Justice and Strong Institutions).

1 Introduction

Fundamentalism and terrorism are serious issues that harm human beings both physically and psychologically. The attack on September 11th, 2001 [1], demonstrated that terrorism is likely to cause losses worth more than 40bn and the death of roughly 3,000 people, which is more than the deadliest natural catastrophe to hit the United States. To minimize this threat, policymakers constantly attempt to do analysis [2,3,4]. Fundamentalism does not spread by carriers, blood transfusions, or other means like the Hepatitis B virus, Aids, polio, etc. Similar to smoking, fundamentalism spreads via direct interaction with militant groups. A person could occasionally become militant for a variety of reasons. Some turn to terrorism because their family members are terrorists, some people are misled by militants and some are dragged, etc. It is commonly known that mathematical models can address and offer solutions to specific issues related to the condition being researched. For example, epidemiological models are very useful in forecasting the risk of terrorism and the spread of infectious diseases. Statistical regression method [5] and machine learning approach [6] have also been used in predicting the trend and behaviour of the infectious viral diseases. The community is assisted in managing the disorder by recognizing sensitive aspects. To reduce the risk of terrorism in the affected area, mathematical modeling is extremely

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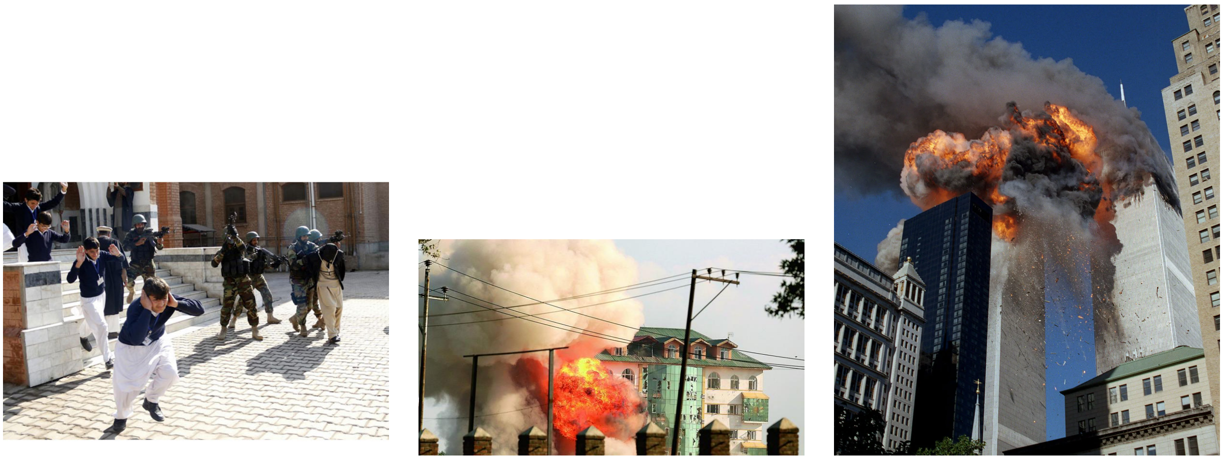
important in analyzing militant's network and in determining the appropriate level of police presence. Both politicians and academics working in related sectors could benefit from the education provided by mathematical models [7]. Some impact of militant network targets can be seen in figure 1(a,b,c), which a re harm and destroy social, economical, and human compatibility.



(a) Effect 1 **(b)** Effect 2 **(c)** Effect 3

Fig. 1: Effect of the Militant Network on Social Life

In this context, Major [1] established a specific function that provides the likelihood of a militant attack strategy and its success. According to Pate et al. [8] a probabilistic model of potential terrorist attacks has been put forth that takes into account the likelihood of various scenarios, the objectives of the United States and terrorists, and their combative dynamics. A probabilistic model of the probability of terrorism and an estimate of the annual damage caused by terrorist strikes were both published by Willis et al. [9]. Additionally, they computed the significant risk reduction. In a case study application inside the United States, a model for estimating regional terrorist risk and its applicability, advantages, and disadvantages is published by Chatterjee et al. [10]. The evolution of British political party membership, which directly influences the political efficacy of such parties, was examined by Jeffs et al [11] using the mathematics of epidemiology. We suggest [12,13] for further information.

I provide a deterministic model with fractal fractional derivatives in this research for the management and propagation of militants.The topic of fractional calculus has garnered a lot of attention recently. At the moment, Atangana [14] has created a novel approach for the fractal fractional derivative. The concept behind this topic is frequently quite beneficial for resolving some challenging problems. This new approach to fractal fraction calculation outperforms the conventional approach [15,16,17,18]. This is so that one may study both fractal dimensions and fraction operators at the same time when dealing with fractal fractional derivatives. You can create a model that explains memory effects in systems more precisely by using this operator, which is a substantial benefit. Furthermore, when comprehending a system's information capability is necessary, there are extra challenges in the real world. Reported some advancements in the use of innovative applications and different kernels in fractal-fractional differential equations [19,20].Fractional calculus is piqueing attention of academics throughout world due to its wide range of advantages and useful applications to problems in physics and engineering. Heritable characteristics, memory, and crossover behavior can only be studied using a model with a fractional-order structure [21,22].

The section titled "Introduction" in this endeavour discusses the Introduction. Section 2 of the article covered the fundamentals of the commonly used fractional operator. In section 3, we present a deterministic model for the spread and control of militants with fractal fractional derivative, discussed the positiveness and boundedness of solutions, studied the uniqueness, existence and stability analysis with related theorems. In section 4, On the basis of the fractal fractional derivative, a numerical Atangana-Toufik scheme is presented. In section 5, the model is simulated for various fractional order values to observe how it behaves dynamically. In section 6, We draw a conclusion on the model's findings.

2 Preliminaries

We have some important definition in this section

Definition 21[23] The fractional integral operator of order $\mu > 0$ in Riemann-Liouville sense for $U(t) : (0,\infty) \in R$ is as follow

$$ {}_0^{RL}D_t^{-\mu}U(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}U(\tau)d(\tau), \quad t > 0, \tag{1} $$

or

$$ {}_0^{RL}I_t^{\mu}U(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}U(\tau)d(\tau), \quad t > 0, $$

$$ {}_0^{RL}I_t^{0}U(t) = U(t) \tag{2} $$

where $\mu > 0$and $\Gamma(.)$ is known as gamma function.

Definition 22[23] The fractional derivative of order $\mu > 0$ in Riemann-Liouville sense for $U(t) : (0,\infty) \in R$ is as follow

$$ {}_0^{RL}D_t^{\mu}U(t) = \begin{cases} \frac{1}{\Gamma(l-\mu)}\frac{d^l}{dt^l}\int_0^t \frac{U(\tau)}{(t-\tau)^{\mu-l+1}}d(\tau), & 0 \le l-1 < \mu, l = \lceil \mu \rceil, \\ \frac{d^l}{dt^l}U(t), & \mu = l \in N. \end{cases} \tag{3} $$

Definition 23[23] The fractional derivative of order $\mu > 0$ in Caputo sense for $U(t) : (0,\infty) \in R$ is as follow

$$ {}_0^{RL}D_t^{\mu}U(t) = \begin{cases} \frac{1}{\Gamma(l-\mu)}\int_0^t \frac{\frac{d^l}{d\tau^l}U(\tau)}{(t-\tau)^{\mu-l+1}}d(\tau), & 0 \le l-1 < \mu, l = \lceil \mu \rceil, \\ \frac{d^l}{dt^l}U(t), & \mu = l \in N. \end{cases} \tag{4} $$

Definition 24 For $U \in H^1(x,y)$. The Caputo-Fabrizio fractional derivative [23] is defined as

$$ {}^{CF}D_t^{\mu}\Big(U(t)\Big) = \frac{\aleph(\mu)}{l-\mu}\int_x^t \frac{d^l}{d\tau^l}U(\tau)exp\Big[-\mu\frac{t-\tau}{l-\mu}\Big]d\tau, \; l-1 < \mu < l, \tag{5} $$

where $\aleph(\mu)$ is a normalization function.

Definition 25[23,24] The left and right Antagana-Baleanu derivative in Caputo sense (ABC) is defined as

$$ {}_a^{ABC}D_t^{\mu}(U(t)) = \frac{AB(\mu)}{l-\mu}\int_a^t \frac{d^l}{dt^l}U(\tau)E_{\mu}\Big[-\mu\frac{(t-\tau)^{\mu}}{l-\mu}\Big]d\tau, \; l-1 < \mu < l, \tag{6} $$

and

$$ {}_b^{ABC}D_t^{\mu}(U(t)) = -\frac{AB(\mu)}{l-\mu}\int_b^t \frac{d^l}{dt^l}U(\tau)E_{\mu}\Big[-\mu\frac{(t-\tau)^{\mu}}{l-\mu}\Big]d\tau, \; l-1 < \mu < l, \tag{7} $$

where E_{μ} is the Mittag-Leffler function and $AB(\mu)$ is a normalization function.

Definition 26[23] Atangana-Baleanu fractional integral of order μ of a function $\psi(t)$ can be expressed as

$$ {}_0^{ABC}I_t^{\mu}(U(t)) = \frac{1-\mu}{B-\mu}U(t) + \frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^t U(\tau)(t-\tau)^{\mu-1}d\tau. \tag{8} $$

3 Fractional Order Network of Militants Model

The network of militants may be analyzed mathematically, and the optimal amount of police presence in the affected area can be determined to reduce the threat of terrorism. Politicians as well as academics working in relevant subjects might benefit from mathematical models. The militant network is modeled deterministically in this part. This model consists of four classes. We propose the model [7] in a modified form:

$$ \begin{aligned} {}_0^{ABC}D_t^{\mu}P(t) &= \varepsilon - cP(t)M(t) - \varphi P(t), \\ {}_0^{ABC}D_t^{\mu}O(t) &= cP(t)M(t) - (\chi + d)O(t), \\ {}_0^{ABC}D_t^{\mu}M(t) &= drO(t) - (\psi + e)M(t), \\ {}_0^{ABC}D_t^{\mu}Q(t) &= d(1-r)O(t) + eM(t) - \omega Q(t). \end{aligned} \tag{9} $$

With initial condition

$P(0) \geq 0,\ O(0) \geq 0,\ M(0) \geq 0,\ Q(0) \geq 0$

Here P(t) represents the potential militants class. The function O(t) represents the occasional militant class, and The function M(t) represents the militant class. The function Q(t) represents quit class.

Table 1: The parameters of the recommended model

ε	The constant rate of potential class growth brought on by migration and births
φ	The potential class's inherent mortality rate
χ	the natural mortality rate of the occasionally militant class
ψ	the natural death rate of the militant class
ω	the natural death rate of quit class
c	the transmission coefficient from potential class to the occasional militant class
d	the transition rate from the occasional class to militants and quit class
e	the rate at which militants quit militancy

The positive value of parameter c increases by increasing the activities of militants.

3.1 Boundness and Positiveness of system

We start with the $O(t)$ class. $\forall t \geq 0$, we have

$$O(t) \geq O_0 e^{-(\chi+d)t}, \tag{10}$$

With respect to $M(t)$, $\forall t \geq 0$ we have the following inequality

$$M(t) \geq M_0 e^{-(\psi+e)t}, \tag{11}$$

For $Q(t)$, the inequality is following

$$Q(t) \geq Q_0 e^{-\omega t}, \quad \forall t \geq 0 \tag{12}$$

Norm for model is defined as

$$\| \alpha \|_\infty = \sup_{t\in D_\alpha} |\alpha(t)| \tag{13}$$

where D_α is the domain of α. we have

$$\begin{aligned} \dot{P} &= \varepsilon - cP(t)M(t) - \varphi P(t), \quad \forall t \geq 0 \\ &\geq -(c|M| + \varphi)P, \quad \forall t \geq 0 \\ &\geq -(c \sup_{t\in D_\alpha} |M| + \varphi)P, \quad \forall t \geq 0 \\ &\geq -(c\|M\|_\infty + \varphi)P, \quad \forall t \geq 0 \end{aligned} \tag{14}$$

This yeilds

$$P(t) \geq P_0 \exp^{-(c\|M\|_\infty+\varphi)t}, \quad \forall t \geq 0 \tag{15}$$

Theorem 31 The recommended mlitants model solution is unique and constrained in R^4_+, along with the initial circumstances.
Proof. By using method given in [18], We have got

$$\begin{aligned} &D_t^{\mu,\sigma}(P(t))_{P=0} = \varepsilon \ \geq 0, \\ &D_t^{\mu,\sigma}(O(t))_{O=0} = cPM \ \geq 0, \\ &D_t^{\mu,\sigma}(M(t))_{M=0} = drO \ \geq 0, \\ &D_t^{\mu,\sigma}(Q(t))_{Q=0} = d(1-r)O + eM \ \geq 0, \end{aligned} \tag{16}$$

If $(P(0), O(0), M(0), Q(0)) \in R^4_+$, then (16) states that the solution is unable to depart from the hyperplane.

3.2 Existence and uniqueness

Here, we will use fixed point theorems to determine whether a solution to our suggested model (9) exists. We can write kernels simply as;

$$\begin{aligned} G_1(t,P(t)) &= \varepsilon - cP(t)M(t) - \varphi P(t), \\ G_2(t,O(t)) &= cP(t)M(t) - (\chi+d)O(t), \\ G_3(t,M(t)) &= drO(t) - (\psi+e)M(t), \\ G_4(t,Q(t)) &= d(1-r)O(t) + eM(t) - \omega Q(t) \end{aligned} \tag{17}$$

using Eq.(8), we have

$$\begin{aligned} P(t)-P(0) &= (\frac{\sigma(1-\mu)t^{\sigma-1}U(t)}{AB(\mu)}) \times [\varepsilon - cP(t)M(t) - \varphi P(t)] \\ &+ (\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1}) \times [\varepsilon - cP(\phi)M(\phi) - \varphi P(\phi)]d\phi \\ O(t)-O(0) &= (\frac{\sigma(1-\mu)t^{\sigma-1}U(t)}{AB(\mu)}) \times [cP(t)M(t) - (\chi+d)O(t)] \\ &+ (\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1}) \times [cP(\phi)M(\phi) - (\chi+d)O(\phi)]d\phi \\ M(t)-M(0) &= (\frac{\sigma(1-\mu)t^{\sigma-1}U(t)}{AB(\mu)}) \times [drO(t) - (\psi+e)M(t)] \\ &+ (\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1}) \times [drO(\phi) - (\psi+e)M(\phi)]d\phi \\ Q(t)-Q(0) &= (\frac{\sigma(1-\mu)t^{\sigma-1}U(t)}{AB(\mu)}) \times [d(1-r)O(t) + eM(t) - \omega Q(t)] \\ &+ (\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1}) \times [d(1-r)O(\phi) + eM(\phi) - \omega Q(\phi)]d\phi \end{aligned} \tag{18}$$

consider that $P(t)$,$O(t)$,$M(t)$,$Q(t)$, and $\overline{P}(t)$,$\overline{O}(t)$,$\overline{M}(t)$,$\overline{Q}(t)$ are functions that are continuous so that

$$\| P(t) \| \le \kappa_1, \| O(t) \| \le \kappa_2, \| M(t) \| \le \kappa_3, \| Q(t) \| \le \kappa_4 \tag{19}$$

Now we will prove that the kernels G_1, G_2, G_3, and G_4 fulfil the Lipschitz condition.

Theorem 32 The Lipschitz condition and contraction are satisfied by the kernel G_1 if the following inequality is true.:

$$0 \le (c\kappa_3 + \varphi) < 1 \tag{20}$$

Proof: Assuming that there are two functions, $P(t)$ and $\overline{P}(t)$, then

$$\begin{aligned} \| G_1(t,P(t)) - G_1(t,\overline{P}(t)) \| &= \| [\varepsilon - cP(t)M(t) - \varphi P(t)] - [\varepsilon - c\overline{P}(t)M(t) - \varphi\overline{P}(t)] \| \\ &\le (c \| M(t) \|) \| P(t) - \overline{P}(t) \| + (\varphi) \| P(t) - \overline{P}(t) \| \\ &\le \Lambda_1 \| P(t) - \overline{P}(t) \| \end{aligned} \tag{21}$$

The Lipschitz conditions for G_2, G_3, G_4 are :

$$\| G_2(t,O(t)) - G_2(t,\overline{O}(t)) \| \le \Lambda_2 \| O(t) - \overline{O}(t) \| \tag{22}$$

$$\| G_3(t,M(t)) - G_3(t,\overline{M}(t)) \| \le \Lambda_2 \| M(t) - \overline{M}(t) \| \tag{23}$$

$$\| G_4(t,Q(t)) - G_4(t,\overline{Q}(t)) \| \le \Lambda_2 \| Q(t) - \overline{Q}(t) \| \tag{24}$$

where

$$\Lambda_2 = (\chi+d), \Lambda_3 = (\psi+e), \Lambda_4 = \omega \tag{25}$$

If $0 < (\chi+d) < 1$, then G_2 is a contraction.
If $0 < (\psi+e) < 1$, then contraction G_3 exist.
If $0 < (\omega) < 1$, then contraction G_4 exist.

Theorem 33 Suppose

$$\aleph = max\{\Lambda_i\} < 1\ ,\ i = 1,2,3,4 \tag{26}$$

Hence proved.
Proof: Suppose that

$$\begin{aligned} \delta_{1n}(t) &= P_{n+1}(t) - P(t) \\ \delta_{4n}(t) &= Q_{n+1}(t) - QU(t) \end{aligned} \tag{27}$$

we obtain, similarly for others

$$\begin{aligned} \| \delta_{1n}(t) \| &\leq [\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)}] \times \| G_1(t,P_n(t)) - G_1(t,P(t)) \| \\ &+ [\frac{\mu\sigma}{AB(\mu)} \int_0^t (t-\phi)\phi^{\mu-1}] \times \| G_1(t,P_n(t)) - G_1(t,P(t)) \| d\phi \\ &\leq (\frac{\sigma(1-\mu)t^{\sigma-1} + \mu\sigma}{AB(\mu)})\Lambda_1 \| P_n - P \| \\ &\leq (\frac{\sigma(1-\mu)t^{\sigma-1} + \mu\sigma}{(AB(\mu))})^n\ \aleph^n \| P - P_1 \| \end{aligned} \tag{28}$$

similarly,

$$\begin{aligned} \| \delta_{2n}(t) \| &\leq [\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)}] \times \| G_2(t,O_n(t)) - G_2(t,O(t)) \| \\ &+ [\frac{\mu\sigma}{AB(\mu)} \int_0^t (t-\phi)\phi^{\mu-1}] \times \| G_2(t,O_n(t)) - G_2(t,O(t)) \| d\phi \\ &\leq (\frac{\sigma(1-\mu)t^{\sigma-1} + \mu\sigma}{AB(\mu)})\Lambda_2 \| O_n - O \| \\ &\leq (\frac{\sigma(1-\mu)t^{\sigma-1} + \mu\sigma}{(AB(\mu))})^n\ \aleph^n \| O - O_1 \| \end{aligned} \tag{29}$$

$$\begin{aligned} \| \delta_{3n}(t) \| &\leq [\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)}] \times \| G_3(t,M_n(t)) - G_3(t,M(t)) \| \\ &+ [\frac{\mu\sigma}{AB(\mu)} \int_0^t (t-\phi)\phi^{\mu-1}] \times \| G_3(t,M_n(t)) - G_3(t,M(t)) \| d\phi \\ &\leq (\frac{\sigma(1-\mu)t^{\sigma-1} + \mu\sigma}{AB(\mu)})\Lambda_3 \| M_n - M \| \\ &\leq (\frac{\sigma(1-\mu)t^{\sigma-1} + \mu\sigma}{(AB(\mu))})^n\ \aleph^n \| M - M_1 \| \end{aligned} \tag{30}$$

$$\begin{aligned} \| \delta_{4n}(t) \| &\leq [\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)}] \times \| G_4(t,Q_n(t)) - G_4(t,Q(t)) \| \\ &+ [\frac{\mu\sigma}{AB(\mu)} \int_0^t (t-\phi)\phi^{\mu-1}] \times \| G_4(t,Q_n(t)) - G_4(t,Q(t)) \| d\phi \\ &\leq (\frac{\sigma(1-\mu)t^{\sigma-1} + \mu\sigma}{AB(\mu)})\Lambda_4 \| Q_n - Q \| \\ &\leq (\frac{\sigma(1-\mu)t^{\sigma-1} + \mu\sigma}{(AB(\mu))})^n\ \aleph^n \| Q - Q_1 \| \end{aligned} \tag{31}$$

For $n = 1,2,3,4$, we observe that $\delta_{in}(t) \to 0$ as $n \to \infty$.

3.3 Analysis of equilibrium points

I present an analysis of equilibrium locations in this subsection. Militant free equilibrium point is

$$E_0(P,O,M,Q) = (\frac{\varepsilon}{\varphi},0,0,0) \tag{32}$$

as well as the endemic equilibrium point being $E_1(P^*,O^*,M^*,Q^*)$, where

$$\begin{aligned} P^* &= \frac{(d+\chi)(e+\psi)}{cdr} \\ O^* &= \frac{\varepsilon}{d+\chi} - \frac{\varphi(e+\psi)}{cdr} \\ M^* &= -\frac{\varphi}{c} + \frac{d\varepsilon r}{(d+\chi)(e+\psi)} \\ Q^* &= \frac{(\varphi(d+\chi)(e+\psi) - cd\varepsilon r)(e+\psi-\psi r)}{c(d+\chi)(e+\psi)\omega r} \end{aligned}$$

3.4 Reproductive number

To continue with our prior analysis, here we shall derive the reproductive number. Reproduction number play an important role in the field of epidemiological modeling, as it helps to understand the stability conditions.The vectors F and V stand in for the genesis of new illnesses and the spread of already-existing infections, respectively. Reproduction number is calculated as

$$R_0 = \rho(\mathbb{F}\mathbb{V}^{-1}) \tag{33}$$

we have

$$F = \begin{pmatrix} cPM \\ 0 \end{pmatrix}, V = \begin{pmatrix} (\chi+d)O \\ -drO + (\psi+e)M \end{pmatrix}$$

The Jacobians of F and V are calculated as

$$\mathbb{F} = \begin{pmatrix} 0 & cP \\ 0 & 0 \end{pmatrix} \textit{and} \quad \mathbb{V} \begin{pmatrix} \chi+d & 0 \\ -dr & \psi+e \end{pmatrix}$$

Now

$$\mathbb{F}\mathbb{V}^{-1} = \begin{pmatrix} \frac{cdrP}{(\chi+d)(\psi+e)} & \frac{cP}{\psi+e} \\ 0 & 0 \end{pmatrix}$$

Thus $\det(\mathbb{F}\mathbb{V}^{-1} - \lambda I) = -\lambda \left[\frac{cdrP}{(\chi+d)(\psi+e)} - \lambda \right]$
so $\lambda_1 = 0$ and $\lambda_2 = \frac{cdrP}{(\chi+d)(\psi+e)}$
Due to the fact that R_0 is a dominating eigenvalue of $\mathbb{F}\mathbb{V}^{-1}$

$$R_0 = R(E_0) = \frac{cdr\varepsilon}{\varphi(\chi+d)(\psi+e)} \tag{34}$$

3.5 Sensitivity analysis

Sensitivity analysis is used to evaluate how different parameters will affect a model's stability given uncertain data. This study may also identify the key factors.

If we take into account the partial derivative of the threshold for the relevant parameters, we can investigate the sensitivity of R_0 as follows:

$$\begin{aligned}
\frac{\partial R_0}{\partial c} &= \frac{dr\varepsilon}{\varphi(\chi+d)(\psi+e)} > 0\\
\frac{\partial R_0}{\partial r} &= \frac{cd\varepsilon}{\varphi(\chi+d)(\psi+e)} > 0\\
\frac{\partial R_0}{\partial \varepsilon} &= \frac{cdr}{\varphi(\chi+d)(\psi+e)} > 0\\
\frac{\partial R_0}{\partial d} &= \frac{crd\chi}{\varphi(\chi+d)^2(\psi+e)} > 0\\
\frac{\partial R_0}{\partial \varphi} &= -\frac{cdr\varepsilon}{\varphi^2(\chi+d)(\psi+e)} < 0\\
\frac{\partial R_0}{\partial \chi} &= -\frac{crd\varepsilon}{\varphi(\chi+d)^2(\psi+e)} < 0\\
\frac{\partial R_0}{\partial \psi} &= -\frac{crd\varepsilon}{\varphi(\chi+d)(\psi+e)^2} < 0\\
\frac{\partial R_0}{\partial e} &= -\frac{crd\varepsilon}{\varphi(\chi+d)(\psi+e)^2} < 0
\end{aligned}$$

We can see that R_0 is very sensitive when we alter the parameters. In this work, c,r,d,ε are expanding while φ,χ,ψ,e are decreasing. Therefore, prevention is better to control the militants.

3.6 Local stability of the model

Theorem 34 The suggested fractional-order model of militants has a disease-free equilibrium (DF) that is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.
proof. At E_0 , we have

$$J = \begin{pmatrix} -cM-\varphi & 0 & -cP & 0\\ cM & -(\chi+d) & cP & 0\\ 0 & dr & -(\psi+e) & 0\\ 0 & d(1-r) & e & -\omega \end{pmatrix}$$

At disease free equilibrium point, we have

$$J_0 = \begin{pmatrix} -\varphi & 0 & \frac{-c\varepsilon}{\varphi} & 0\\ 0 & -(\chi+d) & \frac{c\varepsilon}{\varphi} & 0\\ 0 & dr & -(\psi+e) & 0\\ 0 & d(1-r) & e & -\omega \end{pmatrix}$$

The DFE is locally asymptotically stable if all eigenvalues $\lambda_i, i = 1,2,3,4$ of the matrix J_0 fulfil the requirement

$$\left|arg\Big(eig(J_0)\Big)\right| = \left|arg(\lambda_i)\right| > \upsilon\frac{\Pi}{2}, \quad i = 1,2,3,4 \tag{35}$$

eigenvalues can be find out by solving the characteristic equation

$$|J_0 - \lambda I| = 0 \tag{36}$$

eigen values are

$$\lambda_1 = -\varphi$$
$$\lambda_2 = \frac{-(\chi+d+\psi+e)+\sqrt{(\chi+d+\psi+e)^2-4\left((\chi+d)(\psi+e)-\frac{cd\varepsilon r}{\varphi}\right)}}{2}$$
$$\lambda_3 = \frac{-(\chi+d+\psi+e)-\sqrt{(\chi+d+\psi+e)^2-4\left((\chi+d)(\psi+e)-\frac{cd\varepsilon r}{\varphi}\right)}}{2}$$
$$\lambda_4 = -\omega \quad (37)$$

We observe that the real part of each eigenvalue is exclusively negative for time t if and only if $R_0 < 1$. Hence, the model (9) is locally asymptotically stable at point E_0 if $R_0 < 1$ and is unstable when $R_0 > 1$.

Theorem 35 The proposed fractional order model of militants' EE point exhibits locally asymptotically stable behaviour if $R_0 > 1$ and unstable otherwise.
proof. The jacobian matrix at EE point is given as

$$J_1 = \begin{pmatrix} -\varphi R_0 & 0 & -\varphi(R_0-1) & 0 \\ \varphi(R_0-1) & -(\chi+d) & \varphi(R_0-1) & 0 \\ 0 & dr & -(\psi+e) & 0 \\ 0 & d(1-r) & e & -\omega \end{pmatrix}$$

By considering the characteristic equation $|J_1 - \lambda I| = 0$, we have the following form

$$(-\omega-\lambda)\Big[(-\varphi R_0-)\Big((-b_1-\lambda)(-c_1-\lambda)-dr\varphi(R_0-1)\Big)-(\varphi(R_0-1))^2 dr\Big] = 0 \quad (38)$$

where $b_1 = \chi + d$ and $c_1 = \psi + e$. It is discovered through the use of the software Matlab that, if $R_0 > 1$, the real part of all the eigenvalues are negative. Hence this complete the proof.

3.7 *Ulam-Hyers Stability*

The system's is Ulam-Hyers stable if there are constants $\Omega_i > 0$ $(i = 1,2,3,4)$ fulfilling for every $\rho_i > 0$ $(i = 1,2,3,4)$.

$$| P(t) - (\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)})G_1(t,P(t)) - (\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1})G_1(t,P(t))d\phi | \leq \rho_1$$
$$| O(t) - (\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)})G_2(t,O(t)) - (\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1})G_2(t,O(t))d\phi | \leq \rho_2$$
$$| M(t) - (\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)})G_3(t,M(t)) - (\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1})G_3(t,M(t))d\phi | \leq \rho_3$$
$$| Q(t) - (\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)})G_4(t,Q(t)) - (\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1})G_4(t,Q(t))d\phi | \leq \rho_4 \quad (39)$$

there exist $\overline{P}(t)$, $\overline{O}(t)$, $\overline{M}(t)$, $\overline{Q}(t)$ satisfying

$$\overline{P}(t) = [\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)}]G_1(t,\overline{P}(t)) + [\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1}]G_1(t,\overline{P}(t))d\phi$$
$$\overline{O}(t) = [\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)}]G_2(t,\overline{O}(t)) + [\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1}]G_2(t,\overline{O}(t))d\phi$$
$$\overline{M}(t) = [\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)}]G_3(t,\overline{M}(t)) + [\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1}]G_3(t,\overline{M}(t))d\phi$$
$$\overline{Q}(t) = [\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)}]G_4(t,\overline{Q}(t)) + [\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1}]G_4(t,\overline{Q}(t))d\phi \quad (40)$$

such that

$$\begin{aligned} P(t)-\overline{P}(t) &\leq \rho_1\Omega_1 \\ O(t)-\overline{O}(t) &\leq \rho_2\Omega_2 \\ M(t)-\overline{M}(t) &\leq \rho_3\Omega_3 \\ Q(t)-\overline{Q}(t) &\leq \rho_4\Omega_4 \end{aligned} \tag{41}$$

Theorem 36 Consider that $P(t)$,$O(t)$,$M(t)$,$Q(t)$, and $\overline{P}(t)$,$\overline{O}(t)$,$\overline{M}(t)$,$\overline{Q}(t)$ are functions that are continuous so that

$$\| P(t) \| \leq \Xi_1, \| O(t) \| \leq \Xi_2, \| M(t) \| \leq \Xi_3, \| Q(t) \| \leq \Xi_4 \tag{42}$$

If so, then Ulam-Hyers stability for the fractional model (9) exists.
Proof : We have found a solution for $P(t)$, $O(t)$, $M(t)$, and $Q(t)$. Let's say the approximate solutions to our system are $\overline{P}(t)$, $\overline{O}(t)$, $\overline{M}(t)$, $\overline{Q}(t)$, and they meet the following criteria.

$$\begin{aligned} \| P(t)-\overline{P}(t) \| &\leq [\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)}]\ \Xi_1(t,\overline{P}(t)) + [\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1}]\ \Xi_1(t,\overline{P}(t))d\phi \\ &\leq (\frac{\sigma(1-\mu)t^{\sigma-1}+\mu\sigma}{AB(\mu)})\ \Xi_1 \| P-\overline{P} \| \end{aligned} \tag{43}$$

consider $\Xi_i = \rho_i$ and $\Omega_i = \frac{\sigma(1-\mu)t^{\sigma-1}+\mu\sigma}{AB(\mu)}$ for $i = 1,2,3,4$, we have

$$\| P(t)-\overline{P}(t) \| \leq \rho_1\Omega_1 \tag{44}$$

similarly,

$$\begin{aligned} \| O(t)-\overline{O}(t) \| &\leq [\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)}]\ \Xi_2(t,\overline{O}(t)) + [\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1}]\ \Xi_2(t,\overline{O}(t))d\phi \\ &\leq (\frac{\sigma(1-\mu)t^{\sigma-1}+\mu\sigma}{AB(\mu)})\ \Xi_2 \| O-\overline{O} \| \leq \rho_2\Omega_2 \end{aligned} \tag{45}$$

$$\begin{aligned} \| M(t)-\overline{M}(t) \| &\leq [\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)}]\ \Xi_3(t,\overline{M}(t)) + [\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1}]\ \Xi_3(t,\overline{M}(t))d\phi \\ &\leq (\frac{\sigma(1-\mu)t^{\sigma-1}+\mu\sigma}{AB(\mu)})\ \Xi_3 \| M-\overline{M} \| \leq \rho_3\Omega_3 \end{aligned} \tag{46}$$

$$\begin{aligned} \| Q(t)-\overline{Q}(t) \| &\leq [\frac{\sigma(1-\mu)t^{\sigma-1}}{AB(\mu)}]\ \Xi_4(t,\overline{Q}(t)) + [\frac{\mu\sigma}{AB(\mu)}\int_0^t (t-\phi)\phi^{\mu-1}]\ \Xi_4(t,\overline{Q}(t))d\phi \\ &\leq (\frac{\sigma(1-\mu)t^{\sigma-1}+\mu\sigma}{AB(\mu)})\ \Xi_4 \| Q-\overline{Q} \| \leq \rho_4\Omega_4 \end{aligned} \tag{47}$$

These discrepancies demonstrate the Ulam-Hyers stability of the system.

4 Numerical scheme

In this section, The nonlinear fractional differential equation with respect to the ABC fractional derivative of order *mu* is presented below, and this approach can be utilized to derive an approximation of the solution:

$${}_0^{ABC}D_t^{\mu}U(t) = g(t,U(t)) \tag{48}$$

with the following initial conditions

$$U^{\upsilon}(0) = U_0^{\upsilon}, \quad \upsilon = 0,1,2,3,...,\lceil\mu\rceil - 1. \tag{49}$$

By utilizing the fundamental theorem of FC, Eq,(48) may be changed into a fractional integral equation as

$$U(t)=U(0)+\frac{1-\mu}{ABC(\mu)}g(t,U(t))+\frac{\mu}{\Gamma(\mu)ABC(\mu)}\times\int_0^t g(\tau,U(\tau))(t-\tau)^{\mu-1}d\tau \tag{50}$$

At a given point $t_{n+1}, n=1,2,3,...$ the above equation can be written as

$$U(t_{n+1})=U(0)+\frac{1-\mu}{ABC(\mu)}g(t_n,U(t_n))+\frac{\mu}{\Gamma(\mu)ABC(\mu)}\times\int_0^{t_{n+1}} g(\tau,U(\tau))(t_{n+1}-\tau)^{\mu-1}d\tau \tag{51}$$

$$U(t_{n+1})=U(0)+\frac{1-\mu}{ABC(\mu)}g(t_n,U(t_n))+\frac{\mu}{\Gamma(\mu)ABC(\mu)}\times\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} g(\tau,U(\tau))(t_{n+1}-\tau)^{\mu-1}d\tau \tag{52}$$

Using the two-step Lagrange polynomial interpolation, the function $g(t,U(t))$ with in the interval $[t_j,t_{j+1}]$can be approximated as follows:

$$\begin{aligned}
G_k(\tau) &= \frac{\tau-t_{j-1}}{t_j-t_{j-1}}g(t_j,U(t_j))-\frac{\tau-t_j}{t_j-t_{j-1}}g(t_{j-1},U(t_{j-1}))\\
G_k(\tau) &= \frac{g(t_j,U(t_j))}{h}(\tau-t_{j-1})-\frac{g(t_{j-1},U(t_{j-1}))}{h}(\tau-t_j)\\
&\cong \frac{g(t_j,U_j)}{h}(\tau-t_{j-1})-\frac{g(t_{j-1},U_{j-1})}{h}(\tau-t_j)
\end{aligned} \tag{53}$$

substitute the above equation into equation [52], we get

$$\begin{aligned}
U(t_{n+1}) &= U(0)+\frac{1-\mu}{ABC(\mu)}g(t_n,U(t_n))+\frac{\mu}{\Gamma(\mu)ABC(\mu)}\times\sum_{j=0}^{n}\Big(\frac{g(t_j,U_j)}{h}\int_{t_j}^{t_{j+1}}(\tau-t_{j-1})(t_{n+1}-\tau)^{\mu-1}d\tau\\
&\quad -\frac{g(t_{j-1},U_{j-1})}{h}\int_{t_j}^{t_{j+1}}(\tau-t_j)(t_{n+1}-\tau)^{\mu-1}d\tau\Big)
\end{aligned} \tag{54}$$

we get

$$\begin{aligned}
U(t_{n+1}) &= U(0)+\frac{1-\mu}{ABC(\mu)}g(t_n,U(t_n))+\frac{\mu}{ABC(\mu)}\times\\
&\sum_{j=0}^{n}\Big(h^{\mu}\frac{g(t_j,U_j)}{\Gamma(\mu+2)}(b_1b_2-b_3b_4)-h^{\mu}\frac{g(t_{j-1},U_{j-1})}{\Gamma(\mu+2)}(b_5-b_3b_6)\Big)
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
b_1 &= (n-j+1)^{\mu},\\
b_2 &= (n-j+2+\mu),\\
b_3 &= (n-j)^{\mu},\\
b_4 &= (n-j+2+2\mu),\\
b_5 &= (n-j+1)^{\mu+1},\\
b_6 &= (n-j+1+\mu),
\end{aligned}$$

For model [],9 we have

$$\begin{aligned}
P_{n+1} &= P(0)+\frac{1-\mu}{ABC(\mu)}g(t_n,P(t_n))+\frac{\mu}{ABC(\mu)}\times\\
&\sum_{j=0}^{n}\Big(h^{\mu}\frac{g(t_j,P_j)}{\Gamma(\mu+2)}(b_1b_2-b_3b_4)-h^{\mu}\frac{g(t_{j-1},P_{j-1})}{\Gamma(\mu+2)}(b_5-b_3b_6)\Big)
\end{aligned} \tag{56}$$

$$O_{n+1} = O(0) + \frac{1-\mu}{ABC(\mu)} g(t_n, O(t_n)) + \frac{\mu}{ABC(\mu)} \times \sum_{j=0}^{n} \Big(h^{\mu} \frac{g(t_j, O_j)}{\Gamma(\mu+2)} (b_1 b_2 - b_3 b_4) - h^{\mu} \frac{g(t_{j-1}, O_{j-1})}{\Gamma(\mu+2)} (b_5 - b_3 b_6) \Big) \tag{57}$$

$$M_{n+1} = M(0) + \frac{1-\mu}{ABC(\mu)} g(t_n, M(t_n)) + \frac{\mu}{ABC(\mu)} \times \sum_{j=0}^{n} \Big(h^{\mu} \frac{g(t_j, M_j)}{\Gamma(\mu+2)} (b_1 b_2 - b_3 b_4) - h^{\mu} \frac{g(t_{j-1}, M_{j-1})}{\Gamma(\mu+2)} (b_5 - b_3 b_6) \Big) \tag{58}$$

$$Q_{n+1} = Q(0) + \frac{1-\mu}{ABC(\mu)} g(t_n, Q(t_n)) + \frac{\mu}{ABC(\mu)} \times \sum_{j=0}^{n} \Big(h^{\mu} \frac{g(t_j, Q_j)}{\Gamma(\mu+2)} (b_1 b_2 - b_3 b_4) - h^{\mu} \frac{g(t_{j-1}, Q_{j-1})}{\Gamma(\mu+2)} (b_5 - b_3 b_6) \Big) \tag{59}$$

5 Numerical results sand Discussion

This section discusses how the suggested militant model was numerically simulated utilizing a fractal-fractional approach. For model with given beginning circumstances, we thus employed the fractal-fractional derivative. A nonlinear system's results are derived using fractional values at given initial condition and parameters values in [5] which are . The following are the parameters and beginning conditions:

$c = 0.00001576507, d = 0.20635, e = 0.0022, \varphi = 0.008, \chi = 0.008, \psi = 0.008, \omega = 0.008, \varepsilon = 0.015, r = 0.7$ $P(0) = 439140, \quad O(0) = 9000, \quad M(0) = 1300 \quad and \quad Q(0) = 560.$

In Figures 1 to 4, we displayed simulation results with a range of fractional order (μ) values. As shown in Figures 1-4, we graphically illustrate the network of militants model using the suggested numerical approach at various fractional order $\mu = 0.85, 0.9, 0.95$ values and contrast results for integer order and fractal-fractional order. The trajectories all follow the same path and come together at actual EE point. The dynamics of the potential population of militants are shown in figure 1 for each trajectory against a particular value of μ. Each graph in Figure 1 has a distinct rate of convergence (per μ), which is obvious, but each curve ends up in a steady state. Figure 1 shows that as the order μ is raised, the potential population of militants becomes less valuable. Fig.2, shows that when we increase the value of fractional order (μ) , there is also an increase in occasional militants class to a certain point after that start decreasing. Hence the population of the militants will grow as shown in fig.3, and after reaching the equilibrium point start decreasing. Since the population of occasional and militants is decreasing after reaching an equilibrium point. As a result, the population of quit class is increasing as represented in fig.4. The graphical results demonstrate how well it achieved the intended results. Exams are conducted for particular parameters at different fractional levels in order to ascertain the impact of the fractional order model. All of the compartments' solutions reduce fractional values to provide the required precision and dependability. The memory effect of the fractional derivative is illustrated graphically in order to observe the dynamic behaviour of the model's compartments.

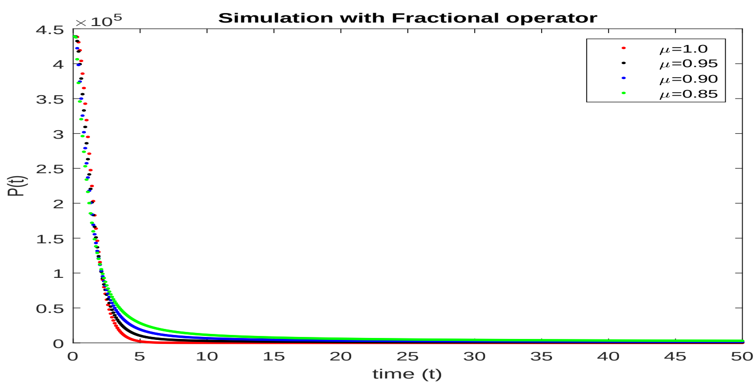


Fig. 2: Modeling of P(t) with fractional operator for proposed model

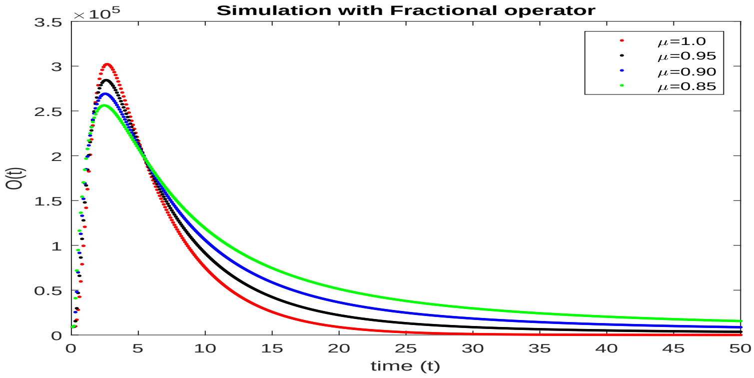


Fig. 3: Modeling of O(t) with fractional operator for proposed model

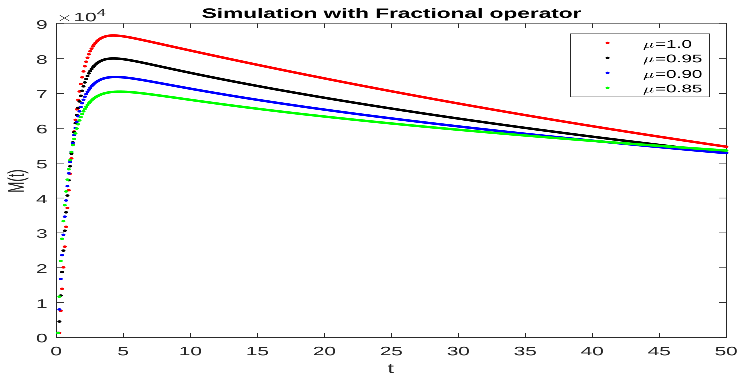


Fig. 4: Modeling of M(t) with fractional operator for proposed model

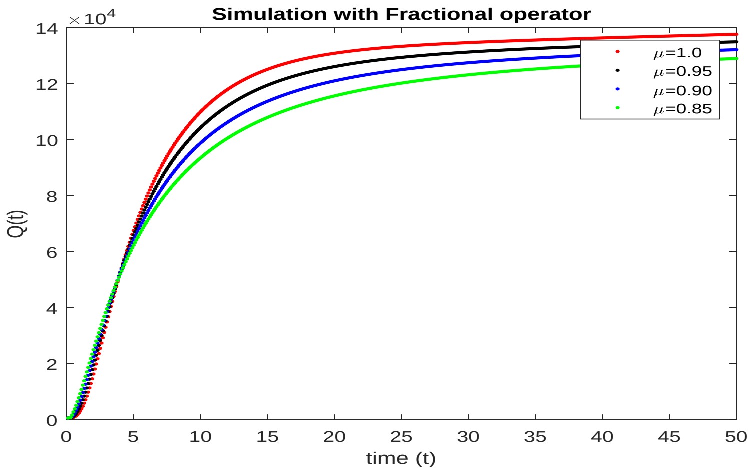


Fig. 5: Modeling of Q(t) with fractional operator for proposed model

6 Conclusion

In this paper, we provided a fractal-fractional derivative model of the militant network. In contrast to the classical model, it should be noted that the fractional order model's result have an effect on memory in the epidemic model. The solutions' positivity and boundedness is carried out. We decided to conduct a thorough investigation on the existence and originality of answer. The stability analysis has been discussed using the Ulam-Hyers technique after getting reproduction number and equilibrium points. It has been discussed how stability analysis can help with practical problems involving the fractional operator's memory effect. There have been built numerical simulation for various fractional order values (μ).It has been compared between fractional order and integer order to understand how the suggested model behaves dynamically. Policymakers will find the findings from our study useful in their upcoming work. The memory aspects of fractal fractional derivative, which are not possible with integer-order derivative, should be highlighted since they investigate the hidden dynamics of mathematical models.

References

[1] Major, J. A. (2002). Advanced techniques for modeling terrorism risk. The Journal of Risk Finance.
[2] Ezell, B. C., Bennett, S. P., Von Winterfeldt, D., Sokolowski, J., and Collins, A. J. (2010). Probabilistic risk analysis and terrorism risk. Risk Analysis: An International Journal, 30(4), 575-589.
[3] Willis, H. H., Morral, A. R., Kelly, T. K., and Medby, J. J. (2006). Estimating terrorism risk. Rand Corporation.
[4] Koller, G. (2000). Risk modeling for determining value and decision making. Chapman and Hall/CRC.
[5] Hussain, S. (2019). Dynamical Behavior of Mathematical Model on the Network of Militants. *Punjab University Journal of Mathematics*, 51(1), 51–60.
[6] Mishra, S., Kaur, M., Bushi, G., Shabil, M., Syed, R., Puri, S., Kumar, S., Ansar, S., Sah, S., Jena, D., Zahiruddin, Q.S., Goh, K.W. (2025). Burden of acute hepatitis E virus in South Asia: Insights from Global Burden of Disease study 2021. *Diagnostic Microbiology and Infectious Disease*, 112(3).
[7] Hussain, S. (2019). Dynamical Behavior of Mathematical Model on the Network of Militants. Punjab Uiversity Journal of mathematics, 51(1), 51-60.
[8] Paté-Cornell, E., and Guikema, S. (2002). Probabilistic modeling of terrorist threats: A systems analysis approach to setting priorities among countermeasures. Military Operations Research, 5-23.
[9] Willis, H. H., and LaTourrette, T. (2008). Using probabilistic terrorism risk modeling for regulatory benefit-cost analysis: Application to the western hemisphere travel initiative in the land environment. Risk Analysis: An International Journal, 28(2), 325-339.
[10] Chatterjee, S., and Abkowitz, M. D. (2011). A methodology for modeling regional terrorism risk. Risk Analysis: An International Journal, 31(7), 1133-1140.
[11] Jeffs, R. A., Hayward, J., Roach, P. A., and Wyburn, J. (2016). Activist model of political party growth. Physica A: Statistical Mechanics and its Applications, 442, 359-372.
[12] Hausken, K., Bier, V. M., and Zhuang, J. (2009). Defending against terrorism, natural disaster, and all hazards. In Game theoretic risk analysis of security threats (pp. 65-97). Springer, Boston, MA.
[13] Quade, E. S. (1964). Analysis for military decisions. RAND CORP SANTA MONICA CA.
[14] Atangana, A. (2017). Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system. Chaos, solitons and fractals, 102, 396-406.
[15] Farman, M., Saleem, M. U., Ahmad, A., Imtiaz, S., Tabassum, M. F., Akram, S., and Ahmad, M. O. (2020). A control of glucose level in insulin therapies for the development of artificial pancreas by Atangana Baleanu derivative. Alexandria Engineering Journal, 59(4), 2639-2648.
[16] Xu, C., Farman, M., Hasan, A., Akgül, A., Zakarya, M., Albalawi, W., and Park, C. (2022). Lyapunov Stability and Wave Analysis of Covid-19 Omicron Variant of Real Data with Fractional Operator. Alexandria Engineering Journal.
[17] Aslam, M., Farman, M., Akgül, A., Ahmad, A., and Sun, M. (2021). Generalized form of fractional order COVID-19 model with Mittag-Leffler kernel. Mathematical Methods in the Applied Sciences, 44(11), 8598-8614.
[18] Farman, M., Amin, M., Akgül, A., Ahmad, A., Riaz, M. B., and Ahmad, S. (2022). Fractal fractional operator for COVID-19 (Omicron) variant outbreak with analysis and modeling. Results in Physics, 105630.
[19] Farman, M., Akgül, A., Nisar, K. S., Ahmad, D., Ahmad, A., Kamangar, S., and Saleel, C. A. (2022). Epidemiological analysis of fractional order COVID-19 model with Mittag-Leffler kernel. AIMS Mathematics, 7(1), 756-783.
[20] Yao, S. W., Ahmad, A., Inc, M., Farman, M., Ghaffar, A., and Akgul, A. L. I. (2022). Analysis of fractional order diarrhea model using fractal fractional operator.
[21] Rashid, S., Sultana, S., Karaca, Y., Khalid, A., and Chu, Y. M. (2022). Some further extensions considering discrete proportional fractional operators. Fractals, 30(01), 2240026.
[22] He, Z. Y., Abbes, A., Jahanshahi, H., Alotaibi, N. D., and Wang, Y. (2022). Fractional-order discrete-time SIR epidemic model with vaccination: Chaos and complexity. Mathematics, 10(2), 165.

[23] Naik, P. A., Yavuz, M., Qureshi, S., Zu, J., and Townley, S. (2020). Modeling and analysis of COVID-19 epidemics with treatment in fractional derivatives using real data from Pakistan. The European Physical Journal Plus, 135(10), 1-42.
[24] Abdeljawad, T., and Baleanu, D. (2018). On fractional derivatives with generalized Mittag-Leffler kernels. Advances in Difference Equations, 2018(1), 1-15.