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A Hybrid Fractional Model for Whooping Cough-Like Infections

Ali Akgül[1,2,*], *Montasir Qasymeh*[3], *Nauman Ahmed*[4], *Zakaria Che Muda*[5], *Enver Ülgül*[2], *Necibullah Sakar*[2] *and Zafar Iqbal*[5]

[1] Department of Electronics and Communication Engineering, Saveetha School of Engineering, SIMATS, Chennai, Tamilnadu, India
[2] Department of Mathematics, Art and Science Faculty, Siirt University, 56100 Siirt, Turkey
[3] Electrical and Computer Engineering Department, Abu Dhabi University, Abu Dhabi, United Arab Emirates
[4] Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan
[5] Faculty of Engineering and Quantity Surveying, INTI International University, 71800 Nilai, Malaysia

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Abstract: In this study, the dynamics of whooping cough-like infections are examined. For this, an integer-order model of the disease is transformed into a fractional-order nonlinear epidemic model by replacing the classical derivatives with the Caputo derivatives. The whole population is categorized into four sub-classes, namely, susceptible individuals (S), the exposed population (E), the infected class (I), and the secured compartment (R). Two numerical techniques are designed to find the solutions to the underlying model. A comparison between the two numerical methods is furnished to evaluate the better and more realistic method. The boundedness of the nonlinear fractional model is confirmed. In addition, the existence and uniqueness of the solution to the underlying nonlinear model are proven. The basic reproduction number(R) of the model also worked out for predicting the disease dynamics. The nonlinear epidemic model has two equilibrium states, namely, the cough-free equilibrium state and the cough-existing equilibrium state. The stability of the model at both equilibrium states is examined. The role of R_0 in maintaining the stable state is also verified. Two numerical schemes are furnished by using the Grünwald-Letnikov approximation and some other standard results. The boundedness and positivity of the GL-NSFD scheme 2 is ascertained. The simulated graphs are sketched to validate the claims about the numerical scheme. The outcome of the study is summarized in the conclusion section.

Keywords: Epidemiological model, Whooping cough-like infections, Hybrid fractional model, Non-standard finite difference, Grünwald-Letnikov method, United Nation SDG 3 (Good Health and Well-Being).

1 Introduction

Respiratory infections resembling whooping cough, also termed as pertussis-like illnesses or pertussis-like syndromes, exhibit symptoms similar to those of pertussis, a highly contagious bacterial infection caused by Bordetella pertussis, primarily attacking the respiratory system. Despite pertussis being a serious global health concern, especially for infants and young children, other pathogens can induce similar symptoms, complicating accurate diagnosis and management.

The hallmark of whooping cough is intense paroxysmal coughing, often followed by a characteristic ”whoop” sound as the infected individual struggles to breathe after coughing spasms. Although pertussis is the main cause, but whooping cough-like respiratory illnesses can also be brought on by viruses such as respiratory syncytial virus (RSV), adenovirus, influenza virus, and parainfluenza virus, as well as other Bordetella species like Bordetella parapertussis and Bordetella bronchiseptica. Additionally, Mycoplasma pneumoniae and Chlamydophila pneumoniae are recognized for causing pertussis-like symptoms. This research supports the UNSDG 3 (Good Health and Well-Being) by predicting the dynamics of whooping cough infections.

The signs and symptoms of pertussis-like infections varies depending on the causative agent, the individual’s age, and immune status. Typical symptoms include paroxysmal coughing fits, post-tussive vomiting, apnea (particularly in infants), and the characteristic ”whoop” sound during inspiration following coughing episodes. However, the classic whooping sound may be absent in older children and adults, posing diagnostic hurdles.

* Corresponding author e-mail: aliakgul00727@gmail.com

Accurate diagnosis is crucial for proper management and public health measures, including isolation, treatment, contact tracing, and vaccination strategies. Diagnostic approaches involve clinical evaluation, laboratory testing (such as polymerase chain reaction or serological assays), and occasionally imaging studies to assess respiratory involvement.

Management of pertussis-like infections focuses on supportive care to alleviate symptoms and prevent complications. Antibiotics may be prescribed in confirmed bacterial cases to lessen symptom severity and duration and to curb transmission. Vaccination against pertussis remains paramount in prevention, emphasizing the significance of immunization programs in controlling the spread of the disease and its mimicking agents.

In summary, pertussis-like infections encompass various respiratory illnesses resembling whooping cough. While Bordetella pertussis is the primary culprit, other bacterial and viral pathogens can produce similar symptoms, posing challenges in diagnosis and management. Effective diagnostic, therapeutic, and preventive strategies are essential for mitigating the impact of these infections on public health.

The application of mathematical and biological structures, graphs, equations, and diagrams to examine events and issues in the actual world is known as mathematical modeling. It involves creating mathematical connections based on particular circumstances. Mathematical modeling is crucial in the field of epidemiology since it helps to understand the dynamics of disease transmission and develop disease control methods. It helps in the development of efficient control strategies and offers insights into the dynamics of individual disease transmission. The machine learning approaches have been adapted by researchers to predict the transmission and forecasting of the advent of next mutated virus [1] and classification and detection of cancer [2]. In[3], the whooping cough disease's transmission dynamics were examined using both analytical and numerical techniques. Evaluations of positivity, constraints, reproduction numbers, balances, and local and global stabilities were among the analytical evaluations conducted. In [4], investigation focused on the dynamics of the whooping cough and phytoplankton nutrition models. The Atangana-Baleanu-Caputo (ABC) fractional derivative sense was used to define the whooping cough transmission pathways and phytoplankton nutrition models. In [5], An analysis was done on the whooping cough vaccine cost-effectiveness for mothers in Australia by researchers. In [6], A SI model of hyperbolic type comprising of two nonlinear partial differential equations was analyzed to explain the spread of pertussis. In [7], It was suggested to use a whooping cough model to examine how the disease spreads in Nebraska. In[8], incorporated a vaccination rate into the whooping cough model. simulations indicate that as we increase the vaccination rate, the duration of infection decreases. Furthermore, applied optimal control techniques to a whooping cough model.

$$
\begin{aligned}
{}_0^{CPC}D_{\mathfrak{r}}^{\zeta}S(\mathfrak{r}) &= \mu^{\zeta} - \mu^{\zeta}S(\mathfrak{r}) - \beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r}), && \mathfrak{r} \geq 0, && (1)\\
{}_0^{CPC}D_{\mathfrak{r}}^{\zeta}E(\mathfrak{r}) &= \beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r}) - V^{\zeta}E(\mathfrak{r}) - \mu^{\zeta}E(\mathfrak{r}), && \mathfrak{r} \geq 0, && (2)\\
{}_0^{CPC}D_{\mathfrak{r}}^{\zeta}I(\mathfrak{r}) &= V^{\zeta}E(\mathfrak{r}) - \alpha^{\zeta}I(\mathfrak{r}) - \mu^{\zeta}I(\mathfrak{r}), && \mathfrak{r} \geq 0, && (3)\\
{}_0^{CPC}D_{\mathfrak{r}}^{\zeta}R(\mathfrak{r}) &= \alpha^{\zeta}I(\mathfrak{r}) - \mu^{\zeta}R(\mathfrak{r}), && \mathfrak{r} \geq 0, && (4)
\end{aligned}
$$

where $S(0) \geq 0, E(0) \geq 0, I(0) \geq 0, R(0) \geq 0$, and $N = S(\mathfrak{r}) + E(\mathfrak{r}) + I(\mathfrak{r}) + R(\mathfrak{r}) = 1$.

2 Boundedness

The feasible region for the model is given as.

$$\Psi = \left\{(S(\mathfrak{r}), E(\mathfrak{r}), I(\mathfrak{r}), R(\mathfrak{r})) \in \mathbb{R}_+^4 \,|\, N = S(\mathfrak{r}) + E(\mathfrak{r}) + I(\mathfrak{r}) + R(\mathfrak{r}), S(\mathfrak{r}), E(\mathfrak{r}), I(\mathfrak{r}), R(\mathfrak{r}) \in \mathbb{R}^+\right\}$$

Adding eqs. (1)-(4), we obtain

$$
\begin{aligned}
&{}_0^{CPC}D_{\mathfrak{r}}^{\zeta}S(\mathfrak{r}) + {}_0^{CPC}D_{\mathfrak{r}}^{\zeta}E(\mathfrak{r}) + {}_0^{CPC}D_{\mathfrak{r}}^{\zeta}I(\mathfrak{r}) + {}_0^{CPC}D_{\mathfrak{r}}^{\zeta}R(\mathfrak{r}) = \\
&\mu^{\zeta} - \mu^{\zeta}S(\mathfrak{r}) - \beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r}) + \beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r}) - V^{\zeta}E(\mathfrak{r}) - \mu^{\zeta}E(\mathfrak{r}) + V^{\zeta}E(\mathfrak{r}) - \alpha^{\zeta}I(\mathfrak{r}) - \mu^{\zeta}I(\mathfrak{r}) + \alpha^{\zeta}I(\mathfrak{r}) - \mu^{\zeta}R(\mathfrak{r})
\end{aligned} \tag{5}
$$

which gives:

$${}_0^{CPC}D_{\mathfrak{r}}^{\zeta}\left(S(\mathfrak{r}) + E(\mathfrak{r}) + I(\mathfrak{r}) + R(\mathfrak{r})\right) = 0$$

where

$${}_0^{CPC}D_{\mathfrak{r}}^{\zeta}N(\mathfrak{r}) = 0, N(0) = N_0 \geq 0 \tag{6}$$

N is the sum of the populations in the system (1)-(4). The solution of (6) is given in [9] as:

$$N(\mathfrak{r}) = N_0 \exp\left(-\frac{k_1(\zeta)}{k_0(\zeta)}t\right)$$

$\mathfrak{r} \to \infty$, then $N(\mathfrak{r}) \geq 0$.
Hence, the fractional-order system (1)-(4) is bounded.

3 Existence and uniqueness

Here, we will apply the technique described in [19] to show that the solution to the fractional order system exists and is unique.

$$\begin{aligned}
{}_0^{CPC}D_{\mathfrak{r}}^{\zeta}S(\mathfrak{r}) &= \mu^{\zeta} - \mu^{\zeta}S(\mathfrak{r}) - \beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r}),\\
{}_0^{CPC}D_{\mathfrak{r}}^{\zeta}E(\mathfrak{r}) &= \beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r}) - V^{\zeta}E(\mathfrak{r}) - \mu^{\zeta}E(\mathfrak{r}),\\
{}_0^{CPC}D_{\mathfrak{r}}^{\zeta}I(\mathfrak{r}) &= V^{\zeta}E(\mathfrak{r}) - \alpha^{\zeta}I(\mathfrak{r}) - \mu^{\zeta}I(\mathfrak{r}),\\
{}_0^{CPC}D_{\mathfrak{r}}^{\zeta}R(\mathfrak{r}) &= \alpha^{\zeta}I(\mathfrak{r}) - \mu^{\zeta}R(\mathfrak{r}).
\end{aligned}$$

Where $\tau \geq 0$.
From [20] a differential equation

$$\frac{dF}{dt} = G(\mathfrak{r}, F(\mathfrak{r}))$$

If the fractional order system meets both of the following two requirements, it has a unique solution:

1. $|G(\mathfrak{r},F)|^2 < M(1+|F|^2)$.
   - $G(\mathfrak{r},F)$: This represents a function that depends on variables $\mathfrak{r}$ and F. The specifics of G would depend on the context, but it generally maps the pair $(\mathfrak{r},F)$ to a value (often in $\mathbb{R}$ or $\mathbb{R}^n$).
   - M: This is a positive constant. It represents an upper bound for the growth of the function.
2. $\|G(\mathfrak{r},F) - G(\mathfrak{r},F_1)\|_{\infty}^2 \leq K\|F - F_1\|_{\infty}^2$. This condition is known as the *Lipschitz Condition* (or *Lipschitz Continuity*). Here's what it means:
   - $G(\mathfrak{r},F)$ and $G(\mathfrak{r},F_1)$: These are the values of the function G at two different points F and F_1, but with the same parameter $\mathfrak{r}$.
   - $\|\cdot\|_{\infty}$: This denotes the L^{∞} norm, which, for a function, gives the maximum absolute value (or the supremum) of the function over its domain.
   - K: This is a positive constant known as the *Lipschitz constant*.

Thus, we verify the following conditions for each equation in our system.
Let

$$\begin{aligned}
G_1(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})) &= \mu^{\zeta} - \mu^{\zeta}S(\mathfrak{r}) - \beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r}),\\
G_2(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})) &= \beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r}) - V^{\zeta}E(\mathfrak{r}) - \mu^{\zeta}E(\mathfrak{r}),\\
G_3(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})) &= V^{\zeta}E(\mathfrak{r}) - \alpha^{\zeta}I(\mathfrak{r}) - \mu^{\zeta}I(\mathfrak{r}),\\
G_4(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})) &= \alpha^{\zeta}I(\mathfrak{r}) - \mu^{\zeta}R(\mathfrak{r}).
\end{aligned}$$

We start with the linear growth condition:

$$\begin{aligned}
|G_1(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r}))|^2 &= \left|\mu^{\zeta}-\mu^{\zeta}S(\mathfrak{r})-\beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r})\right|^2,\\
&\leqslant 3|\mu|^{2\zeta}+3|\mu|^{2\zeta}|S(\mathfrak{r})|^2+3|\beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r})|^2,\\
&\leqslant 3|\mu|^{2\zeta}+3|\mu|^{2\zeta}|S(\mathfrak{r})|^2+3|\beta|^{2\zeta}\sup_{r\in[0,T]}|I(\mathfrak{r})|^2|S(\mathfrak{r})|^2,\\
&\leqslant 3|\mu|^{2\zeta}+3|\mu|^{2\zeta}|S(\mathfrak{r})|^2+3|\beta|^{2\zeta}\|I(\mathfrak{r})\|_{\infty}^2|S(\mathfrak{r})|^2,\\
&\leqslant 3|\mu|^{2\zeta}+\left(3|\mu|^{2\zeta}+3|\beta|^{2\zeta}\|I(\mathfrak{r})\|_{\infty}^2\right)|S(\mathfrak{r})|^2,\\
&\leqslant \left(3|\mu|^{2\zeta}+3|\beta|^{2\zeta}\|I(\mathfrak{r})\|_{\infty}^2\right)\left(\frac{3|\mu|^{2\zeta}}{3|\mu|^{2\zeta}+3|\beta|^{2\zeta}\|I(\mathfrak{r})\|_{\infty}^2}+|S(\mathfrak{r})|^2\right),\\
|G_1(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r}))|^2 &\leqslant 3|\mu|^{2\zeta}+3|\beta|^{2\zeta}\|I(\mathfrak{r})\|_{\infty}^2\left(1+|S(\mathfrak{r})|^2\right).
\end{aligned}$$

If $\frac{3|\mu|^{2\zeta}}{3|\mu|^{2\zeta}+3|\beta|^{2\zeta}\|I(\mathfrak{r})\|_{\infty}^2}<1$, then

$$|G_1(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r}))|^2 \leqslant M_1\left(1+|S(\mathfrak{r})|^2\right).$$

where $M_1=3\mu^{2\zeta}+3\beta^{2\zeta}\|I(\mathfrak{r})\|_{\infty}^2$.
Similarly

$$\begin{aligned}
|G_2(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r}))|^2 &= \left|\beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r})-V^{\zeta}E(\mathfrak{r})-\mu^{\zeta}E(\mathfrak{r})\right|^2,\\
&\leqslant 3|\beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r})|^2+3|V^{\zeta}E(\mathfrak{r})|^2+3|\mu^{\zeta}E(\mathfrak{r})|^2,\\
&\leqslant 3\beta^{2\zeta}\sup_{\mathfrak{r}\in[0,T]}|I(\mathfrak{r})|^2\sup_{r\in[0,T]}|S(\mathfrak{r})|^2+3V^{2\zeta}|E(\mathfrak{r})|^2+3\mu^{2\zeta}|E(\mathfrak{r})|^2,\\
&\leqslant 3\beta^{2\zeta}\|S(\mathfrak{r})\|_{\infty}^2\|I(\mathfrak{r})\|_{\infty}^2+\left(3V^{2\zeta}+3\mu^{2\zeta}\right)|E(\mathfrak{r})|^2,\\
|G_2(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r}))|^2 &\leqslant 3\beta^{2\zeta}\|S(\mathfrak{r})\|_{\infty}^2\|I(\mathfrak{r})\|_{\infty}^2\left(1+\frac{3V^{2\zeta}+3\mu^{2\zeta}|E(\mathfrak{r})|^2}{3\beta^{2\zeta}\|S(\mathfrak{r})\|_{\infty}^2\|I(\mathfrak{r})\|_{\infty}^2}\right).
\end{aligned}$$

If $\frac{3V^{2\zeta}+3\mu^{2\zeta}|E(\mathfrak{r})|^2}{3\beta^{2\zeta}\|S(\mathfrak{r})\|_{\infty}^2\|I(\mathfrak{r})\|_{\infty}^2}<1$ then, we get the following inequality

$$|G_2(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r}))|^2 \leqslant M_2\left(1+|E(\mathfrak{r})|^2\right).$$

where $M_2=3\beta^{2\zeta}\|S(\mathfrak{r})\|_{\infty}^2\|I(\mathfrak{r})\|_{\infty}^2$.
Now

$$\begin{aligned}
|G_3(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r}))|^2 &= \left|V^{\zeta}E(\mathfrak{r})-\alpha^{\zeta}I(\mathfrak{r})-\mu^{\zeta}I(\mathfrak{r})\right|^2,\\
&\leqslant 3|v^nE(\mathfrak{r})|^2+3|\alpha^{\zeta}I(\mathfrak{r})|^2+3|\mu^{\zeta}I(\mathfrak{r})|^2,\\
&\leqslant 3v^{2n}\sup_{t\in[0,T]}|E(\mathfrak{r})|^2+(3\alpha^{2\zeta}+3\mu^{2n})|I(\mathfrak{r})|^2,\\
|G_3(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r}))|^2 &\leqslant 3v^{2n}\|E(\mathfrak{r})\|_{\infty}^2\left(1+\frac{(3\alpha^{2\zeta}+3\mu^{2n})|I(\mathfrak{r})|^2}{3v^{2n}\|E(\mathfrak{r})\|_{\infty}^2}\right).
\end{aligned}$$

If $\frac{(3\alpha^{2\zeta}+3\mu^{2n})}{3v^{2n}\|E(\mathfrak{r})\|_{\infty}^2}<1$ then, we have

$$|G_3(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r}))|^2 \leqslant M_3\left(1+|E(\mathfrak{r})|^2\right).$$

where $M_3 = 3V^{2\zeta}\|E(\mathfrak{r})\|_\infty^2$.

$$\begin{aligned}|G_4(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r}))|^2 &= \left|\alpha^\zeta I(\mathfrak{r}) - \mu^\zeta R(\mathfrak{r})\right|^2,\\ &\leqslant 2|\alpha^\zeta I(\mathfrak{r})|^2 + 2|\mu^\zeta R(\mathfrak{r})|^2,\\ &\leqslant 2\alpha^{2\zeta}\sup_{t\in[0,T]}|I(\mathfrak{r})|^2 + 2\mu^{2\zeta}|R(\mathfrak{r})|^2,\end{aligned}$$

Hence

$$|G_4(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r}))|^2 \leqslant 2\alpha^{2\zeta}\|I(\mathfrak{r})\|_\infty^2\left(1+\frac{2\mu^{2\zeta}|R(\mathfrak{r})|^2}{2\alpha^{2\zeta}\|I(\mathfrak{r})\|_\infty^2}\right).$$

if $\frac{2\mu^{2\zeta}}{2\alpha^{2\zeta}\|I(\mathfrak{r})\|_\infty^2} < 1$, then

$$|G_4(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r}))|^2 \leqslant M_4\left(1+|R(\mathfrak{r})|^2\right).$$

where $M_4 = 2\alpha^{2\zeta}\|I(\mathfrak{r})\|_\infty^2$.
Therefore, the function satisfies the linear growth condition. Now, we will examine the Lipschitz condition in the following way

$$\begin{aligned}&\left|G_1(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})) - G_1\left(\mathfrak{r},S^1(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right)\right|^2\\ &= \left|\mu^\zeta - \mu^\zeta S(\mathfrak{r}) - \beta^\zeta S(\mathfrak{r})I(\mathfrak{r}) - \left(\mu^\zeta - \mu^\zeta S^1(\mathfrak{r}) - \beta^\zeta S^1(\mathfrak{r})I(\mathfrak{r})\right)\right|^2,\\ &= \left|\mu^\zeta - \mu^\zeta S(\mathfrak{r}) - \beta^\zeta S(\mathfrak{r})I(\mathfrak{r}) - \mu^\zeta + \mu^\zeta S^1(\mathfrak{r}) + \beta^\zeta S^1(\mathfrak{r})I(\mathfrak{r})\right|^2,\\ &= \left|\mu^\zeta\left(S(\mathfrak{r}) - S^1(\mathfrak{r})\right) + \beta^\zeta I(\mathfrak{r})\left(S(\mathfrak{r}) - S^1(\mathfrak{r})\right)\right|^2\\ &\leqslant 2\left|\mu^\zeta\left(S(\mathfrak{r}) - S^1(\mathfrak{r})\right)\right|^2 + 2\left|\beta^\zeta I(\mathfrak{r})\left(S(\mathfrak{r}) - S^1(\mathfrak{r})\right)\right|^2,\\ &\left|G_1(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})) - G_1\left(\mathfrak{r},S^1(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right)\right|^2 \leqslant 2\mu^{2\zeta}\left|S(\mathfrak{r}) - S^1(\mathfrak{r})\right|^2 + 2\beta^{2\zeta}|I(\mathfrak{r})|^2\left|S(\mathfrak{r}) - S^1(\mathfrak{r})\right|^2.\end{aligned}$$

Then

$$\begin{aligned}&\sup_{\mathfrak{r}\in[0,T]}\left|G_1(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})) - G_1\left(\mathfrak{r},S^1(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right)\right|^2\\ &\leqslant 2\mu^{2\zeta}\left|S(\mathfrak{r}) - S^1(\mathfrak{r})\right|^2 + 2\beta^{2\zeta}\sup_{\mathfrak{r}\in[0,T]}|I(\mathfrak{r})|^2\sup_{\mathfrak{r}\in[0,T]}\left|S(\mathfrak{r}) - S^1(\mathfrak{r})\right|^2,\\ &= 2\mu^{2\zeta}\|S(\mathfrak{r}) - S^1(\mathfrak{r})\|_\infty^2 + 2\beta^{2\zeta}\|I(\mathfrak{r})\|_\infty^2\|S(\mathfrak{r}) - S^1(\mathfrak{r})\|_\infty^2,\\ &= \left(2\mu^{2\zeta} + 2\beta^{2\zeta}\|I(\mathfrak{r})\|_\infty^2\|S(\mathfrak{r}) - S^1(\mathfrak{r})\|_\infty^2\right).\end{aligned}$$

Thus

$$\left\|G_1(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})) - G_1\left(\mathfrak{r},S^1(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right)\right\|_\infty^2 < \bar{M}_1\|S(\mathfrak{r}) - S^1(\mathfrak{r})\|_\infty^2$$

where $\bar{M}_1 = 2\mu^{2\zeta} + 2\beta^{2\zeta}\|I(\mathfrak{r})\|_\infty^2$.

$$\begin{aligned}
&\left|G_2\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right) - G_2\left(\mathfrak{r},S(\mathfrak{r}),E^1(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right)\right|^2 \\
&= \left|\beta^\zeta S(\mathfrak{r})I(\mathfrak{r})V^\zeta E(\mathfrak{r}) - \mu^\zeta E(\mathfrak{r}) - \left(\beta^\zeta S(\mathfrak{r})I(\mathfrak{r}) - V^\zeta E^1(\mathfrak{r}) - \mu^\zeta E^1(\mathfrak{r})\right)\right|^2, \\
&= \left|\beta^\zeta S(\mathfrak{r})I(\mathfrak{r})V^\zeta E(\mathfrak{r}) - \mu^\zeta E(\mathfrak{r}) - \beta^\zeta S(\mathfrak{r})I(\mathfrak{r}) + V^\zeta E^1(\mathfrak{r}) + \mu^\zeta E^1(\mathfrak{r})\right|^2, \\
&= \left|V^\zeta\left(E(\mathfrak{r}) - E^1(\mathfrak{r})\right) + \mu^\zeta\left(E(\mathfrak{r}) - E^1(\mathfrak{r})\right)\right|^2, \\
&\leqslant 2\left|V^\zeta\left(E(\mathfrak{r}) - E^1(\mathfrak{r})\right)\right|^2 + 2\left|\mu^\zeta\left(E(\mathfrak{r}) - E^1(\mathfrak{r})\right)\right|^2, \\
&\leqslant 2V^{2\zeta}\left|\left(E(\mathfrak{r}) - E^1(\mathfrak{r})\right)\right|^2 + 2\mu^{2\zeta}\left|\left(E(\mathfrak{r}) - E^1(\mathfrak{r})\right)\right|^2, \\
&\left|G_2\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right) - G_2\left(\mathfrak{r},S(\mathfrak{r}),E^1(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right)\right|^2 \leqslant \left(2V^{2\zeta} + 2\mu^{2\zeta}\right)\left|E(\mathfrak{r}) - E^1(\mathfrak{r})\right|^2.
\end{aligned}$$

Then

$$\begin{aligned}
\sup_{\mathfrak{r}\in[0,T]}\left|G_2\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right) - G_2\left(\mathfrak{r},S(\mathfrak{r}),E^1(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right)\right|^2 &\leqslant \left(2V^{2\zeta} + 2\mu^{2\zeta}\right)\sup_{\mathfrak{r}\in[0,T]}\left|E(\mathfrak{r}) - E^1(\mathfrak{r})\right|^2, \\
&\leqslant \left(2V^{2\zeta} + 2\mu^{2\zeta}\right)\|E(\mathfrak{r}) - E^1(\mathfrak{r})\|_\infty^2,
\end{aligned}$$

Thus

$$\left\|G_2\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right) - G_2\left(\mathfrak{r},S(\mathfrak{r}),E^1(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right)\right\|_\infty^2 < \bar{M}_2\|E(\mathfrak{r}) - E^1(\mathfrak{r})\|_\infty^2.$$

where $\bar{M}_2 = 2V^{2\zeta} + 2\mu^{2\zeta}$.

$$\begin{aligned}
&\left|G_3\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right) - G_3\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I^1(\mathfrak{r}),R(\mathfrak{r})\right)\right|^2 \\
&= \left|V^\zeta E(\mathfrak{r}) - \alpha^\zeta I(\mathfrak{r}) - \mu^\zeta I(\mathfrak{r}) - \left(V^\zeta E(\mathfrak{r}) - \alpha^\zeta I^1(\mathfrak{r}) - \mu^\zeta I^1(\mathfrak{r})\right)\right|^2, \\
&= \left|V^\zeta E(\mathfrak{r}) - \alpha^\zeta I(\mathfrak{r}) - \mu^\zeta I(\mathfrak{r}) - V^\zeta E(\mathfrak{r}) + \alpha^\zeta I^1(\mathfrak{r}) + \mu^\zeta I^1(\mathfrak{r})\right|^2, \\
&= \left|\alpha^\zeta\left(I(\mathfrak{r}) - I^1(\mathfrak{r})\right) + \mu^\zeta\left(I(\mathfrak{r}) - I^1(\mathfrak{r})\right)\right|^2, \\
&\leqslant 2\left|\alpha^\zeta\left(I(\mathfrak{r}) - I^1(\mathfrak{r})\right)\right|^2 + 2\left|\mu^\zeta\left(I(\mathfrak{r}) - I^1(\mathfrak{r})\right)\right|^2, \\
&\left|G_3\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right) - G_3\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I^1(\mathfrak{r}),R(\mathfrak{r})\right)\right|^2 \leqslant \left(2\alpha^{2\zeta} + 2\mu^{2\zeta}\right)\left|I(\mathfrak{r}) - I^1(\mathfrak{r})\right|^2.
\end{aligned}$$

Then

$$\begin{aligned}
\sup_{\mathfrak{r}\in[0,T]}\left|G_3\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right) - G_3\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I^1(\mathfrak{r}),R(\mathfrak{r})\right)\right|^2 &\leqslant \left(2\alpha^{2\zeta} + 2\mu^{2\zeta}\right)\sup_{\mathfrak{r}\in[0,T]}\left|I(\mathfrak{r}) - I^1(\mathfrak{r})\right|^2, \\
&\leqslant \left(2\alpha^{2\zeta} + 2\mu^{2\zeta}\right)\|I(\mathfrak{r}) - I^1(\mathfrak{r})\|_\infty^2.
\end{aligned}$$

Thus

$$\left\|G_3\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})\right) - G_3\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I^1(\mathfrak{r}),R(\mathfrak{r})\right)\right\|_\infty^2 < \bar{M}_3\|I(\mathfrak{r}) - I^1(\mathfrak{r})\|_\infty^2.$$

where $\bar{M}_3 = 2\alpha^{2\zeta} + 2\mu^{2\zeta}$.

$$\begin{aligned}
&\left|G_4(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})) - G_4\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R^1(\mathfrak{r})\right)\right|^2 \\
&= \left|\alpha^{\zeta} I(\mathfrak{r}) - \mu^{\zeta} R(\mathfrak{r}) - \left(\alpha^{\zeta} I(\mathfrak{r}) - \mu^{\zeta} R^1(\mathfrak{r})\right)\right|^2, \\
&= \left|\alpha^{\zeta} I(\mathfrak{r}) - \mu^{\zeta} R(\mathfrak{r}) - \alpha^{\zeta} I(\mathfrak{r}) + \mu^{\zeta} R^1(\mathfrak{r})\right|^2, \\
&= \mu^{2\zeta} \left|\left(\mathfrak{r}(\mathfrak{r}) - R^1(\mathfrak{r})\right)\right|^2, \\
&\leqslant \mu^{2\zeta} \sup_{\mathfrak{r}\in[0,T]} \left|R(\mathfrak{r}) - R^1(\mathfrak{r})\right|^2, \\
&\left|G_4(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})) - G_4\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R^1(\mathfrak{r})\right)\right|^2 = \mu^{2\zeta} \|R(\mathfrak{r}) - R^1(\mathfrak{r})\|_{\infty}^2.
\end{aligned}$$

Thus

$$\sup_{\mathfrak{r}\in[0,T]} \left|G_4(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})) - G_4\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R^1(\mathfrak{r})\right)\right|^2 < \mu^{2\zeta} \sup_{\mathfrak{r}\in[0,T]} |R(\mathfrak{r}) - R^1(\mathfrak{r})|^2,$$

$$\left\|G_4(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R(\mathfrak{r})) - G_4\left(\mathfrak{r},S(\mathfrak{r}),E(\mathfrak{r}),I(\mathfrak{r}),R^1(\mathfrak{r})\right)\right\|_{\infty}^2 < \bar{M}_4 \|R(\mathfrak{r}) - R^1(\mathfrak{r})\|_{\infty}^2.$$

where $\bar{M}_4 = \mu^{2\zeta}$.
Now,under the following condition

$$Max\{M_1, M_2, M_3, M_4\} < 1.$$

The system possesses a unique solution.

4 Basic reproduction number

We consider two matrices, F and V, which represent the matrices of new infection terms and other transfer terms, respectively, and utilize the next generation matrix method to determine the basic reproduction number R_0. We derive

$$F = \begin{pmatrix} \beta^{\zeta} SI \\ 0 \end{pmatrix}, V = \begin{pmatrix} V^{\zeta} E + \mu^{\zeta} E \\ -V^{\zeta} E + \alpha^{\zeta} I + \mu^{\zeta} I \end{pmatrix},$$

$$F = \begin{bmatrix} 0 & \beta^{\zeta} \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} V^{\zeta} + \mu^{\zeta} & 0 \\ -V^{\zeta} & \alpha^{\zeta} + \mu^{\zeta} \end{bmatrix},$$

where

$$E = FV^{-1} = \begin{bmatrix} \frac{\beta^{\zeta} V^{\zeta}}{(V^{\zeta}+\mu^{\zeta})(\alpha^{\zeta}+\mu^{\zeta})} & \frac{\beta^{\zeta}}{\alpha^{\zeta}+\mu^{\zeta}} \\ 0 & 0 \end{bmatrix} \tag{7}$$

Then $|G - \lambda I| = 0$ gives the non trivial eigenvalue as

$$\lambda = \frac{\beta^{\zeta} V^{\zeta}}{(V^{\zeta} + \mu^{\zeta})(\alpha^{\zeta} + \mu^{\zeta})}$$

when there is no epidemic $S = 1$, thus

$$R_0 = \frac{\beta^{\zeta} V^{\zeta}}{(V^{\zeta} + \mu^{\zeta})(\alpha^{\zeta} + \mu^{\zeta})}$$

,

5 Stability of the model

The equilibrium points of the system can be calculated by setting ${}_0^{CPC}D_{\mathfrak{r}}^{\zeta}S(\mathfrak{r})=0$, ${}_0^{CPC}D_{\mathfrak{r}}^{\zeta}E(\mathfrak{r})=0$, ${}_0^{CPC}D_{\mathfrak{r}}^{\zeta}I(\mathfrak{r})=0$ and ${}_0^{CPC}D_{\mathfrak{r}}^{\zeta}R(\mathfrak{r})=0$. This system has two steady-state points, namely disease-free and endemic. The cough-free equilibrium $(CFE-C_1)$ denoted by (S^1,E^1,I^1,R^1) is $(1,0,0,0)$. The cough existing equilibrium $(CEE-C_2)$ denoted by (S^*,E^*,I^*,R^*), we have as follows:

$$
\begin{aligned}
S^* &= \frac{\alpha^{\zeta}V^{\zeta}+\mu^{\zeta}V^{\zeta}-\mu^{\zeta}\alpha^{\zeta}-\mu^{2\zeta}}{\beta^{\zeta}V^{\zeta}},\\
E^* &= \frac{\mu^{\zeta}(\alpha^{\zeta}V^{\zeta}+\mu^{\zeta}V^{\zeta}-\mu^{\zeta}\alpha^{\zeta}-\mu^{2\zeta}-\beta^{\zeta}V^{\zeta})}{\beta^{\zeta}V^{\zeta}(\mu^{\zeta}-V^{\zeta})},\\
I^* &= \frac{\mu^{\zeta}(\alpha^{\zeta}V^{\zeta}+\mu^{\zeta}V^{\zeta}-\mu^{\zeta}\alpha^{\zeta}-\mu^{2\zeta}-\beta^{\zeta}V^{\zeta})}{\beta^{\zeta}(\mu^{\zeta}-V^{\zeta})(\alpha^{\zeta}+\mu^{\zeta})},\\
R^* &= \frac{\alpha^{\zeta}(\alpha^{\zeta}V^{\zeta}+\mu^{\zeta}V^{\zeta}-\mu^{\zeta}\alpha^{\zeta}-\mu^{2\zeta})}{\beta^{\zeta}(\mu^{\zeta}-V^{\zeta})(\alpha^{\zeta}+\mu^{\zeta})}.
\end{aligned}
$$

For simplicity, we let

$$
\begin{aligned}
F(\mathfrak{r},S,E,I,R) &= \mu^{\zeta}-\mu^{\zeta}S(\mathfrak{r})-\beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r}),\\
G(\mathfrak{r},S,E,I,R) &= \beta^{\zeta}S(\mathfrak{r})I(\mathfrak{r})-V^{\zeta}E(\mathfrak{r})-\mu^{\zeta}E(\mathfrak{r}),\\
H(\mathfrak{r},S,E,I,R) &= V^{\zeta}E(\mathfrak{r})-\alpha^{\zeta}I(\mathfrak{r})-\mu^{\zeta}I(\mathfrak{r}),\\
K(\mathfrak{r},S,E,I,R) &= \alpha^{\zeta}I(\mathfrak{r})-\mu^{\zeta}R(\mathfrak{r}).
\end{aligned}
$$

Then

$$
\begin{array}{llll}
\frac{\partial F}{\partial S}=-\mu^{\zeta}-\beta^{\zeta}I(\mathfrak{r}), & \frac{\partial F}{\partial E}=0, & \frac{\partial F}{\partial I}=-\beta^{\zeta}S(\mathfrak{r}), & \frac{\partial F}{\partial R}=0,\\
\frac{\partial G}{\partial S}=\beta^{\zeta}I(\mathfrak{r}), & \frac{\partial G}{\partial E}=-V^{\zeta}-\mu^{\zeta}, & \frac{\partial G}{\partial I}=\beta^{\zeta}S(\mathfrak{r}), & \frac{\partial G}{\partial R}=0,\\
\frac{\partial H}{\partial S}=0, & \frac{\partial H}{\partial E}=V^{\zeta}, & \frac{\partial H}{\partial I}=-\alpha^{\zeta}-\mu^{\zeta}, & \frac{\partial H}{\partial R}=0,\\
\frac{\partial K}{\partial S}=0, & \frac{\partial K}{\partial E}=0, & \frac{\partial K}{\partial I}=\alpha^{\zeta}, & \frac{\partial K}{\partial R}=-\mu^{\zeta}.
\end{array}
$$

The Jacobian matrix for the hybrid fractional model is given as

$$
J=\begin{pmatrix}
-\mu^{\zeta}-\beta^{\zeta}I(\mathfrak{r}) & 0 & -\beta^{\zeta}S(\mathfrak{r}) & 0\\
\beta^{\zeta}I(\mathfrak{r}) & -V^{\zeta}-\mu^{\zeta} & \beta^{\zeta}S(\mathfrak{r}) & 0\\
0 & V^{\zeta} & -\alpha^{\zeta}-\mu^{\zeta} & 0\\
0 & 0 & \alpha^{\zeta} & -\mu^{\zeta}
\end{pmatrix}. \tag{8}
$$

Theorem 1. *The cough-free equilibrium* $(CFE-C_1)=(1,0,0,0)$ *of the hybrid fractional model is locally asymptotically stable if* $R_0<1$ *and unstable if* $R_0>1$.

Proof. The Jacobian matrix evaluated at $(CFE-C_1)=(1,0,0,0)$ gives

$$
J=\begin{pmatrix}
-\mu^{\zeta} & 0 & -\beta^{\zeta} & 0\\
0 & -V^{\zeta}-\mu^{\zeta} & \beta^{\zeta} & 0\\
0 & V^{\zeta} & -\alpha^{\zeta}-\mu^{\zeta} & 0\\
0 & 0 & \alpha^{\zeta} & -\mu^{\zeta}
\end{pmatrix}, \tag{9}
$$

$$
|J(CFE)-\lambda I|=\begin{vmatrix}
-\mu^{\zeta}-\lambda & 0 & -\beta^{\zeta} & 0\\
0 & -V^{\zeta}-\mu^{\zeta}-\lambda & \beta^{\zeta} & 0\\
0 & V^{\zeta} & -\alpha^{\zeta}-\mu^{\zeta}-\lambda & 0\\
0 & 0 & \alpha^{\zeta} & -\mu^{\zeta}-\lambda
\end{vmatrix}=0.
$$

it implies that $|J(CFE)-\lambda I|=0$ and simple calculations give the eigenvalues of λ as

$$\lambda_1=-\mu^{\zeta}<0,\lambda_2=-\mu^{\zeta}<0,$$

$$|J(CFE)-\lambda I|=\begin{vmatrix}-V^{\zeta}-\mu^{\zeta}-\lambda & \beta^{\zeta}\\ V^{\zeta} & -\alpha^{\zeta}-\mu^{\zeta}-\lambda\end{vmatrix}=0,$$

$$\lambda^2+\lambda(V^{\zeta}+2\mu^{\zeta}+\alpha^{\zeta})+\alpha^{\zeta}V+\mu^{\zeta}V+\alpha^{\zeta}\mu^{\zeta}+\mu^{2\zeta}-\beta^{\zeta}V=0,$$
$$\lambda^2+a_1\lambda+a_0=0.$$

By applying the Routh-Hurwitz criterion for second-order equations, the coefficients of second-order polynomials must be positive. Where $a_1=V^{\zeta}+2\mu^{\zeta}+\alpha^{\zeta}>0$ and $a_0=\alpha^{\zeta}V^{\zeta}+\mu^{\zeta}V^{\zeta}+\alpha^{\zeta}\mu^{\zeta}+\mu^{2\zeta}-\beta^{\zeta}V^{\zeta}>0$, if $R_0<1$

$$\alpha^{\zeta}V^{\zeta}+\mu^{\zeta}V^{\zeta}+\alpha^{\zeta}\mu^{\zeta}+\mu^{2\zeta}-\beta^{\zeta}V^{\zeta}>0, \tag{10}$$
$$\alpha^{\zeta}V^{\zeta}+\mu^{\zeta}V^{\zeta}+\alpha^{\zeta}\mu^{\zeta}+\mu^{2\zeta}>\beta^{\zeta}V^{\zeta}, \tag{11}$$
$$1>\frac{\beta^{\zeta}V^{\zeta}}{\alpha^{\zeta}V^{\zeta}+\mu^{\zeta}V^{\zeta}+\alpha^{\zeta}\mu^{\zeta}+\mu^{2\zeta}}, \tag{12}$$
$$1>\frac{\beta^{\zeta}V^{\zeta}}{(V^{\zeta}+\mu^{\zeta})(\alpha^{\zeta}+\mu^{\zeta})}, \tag{13}$$
$$R_0=\frac{\beta^{\zeta}V^{\zeta}}{(V^{\zeta}+\mu^{\zeta})(\alpha^{\zeta}+\mu^{\zeta})}<1. \tag{14}$$

Therefore, employing the Routh-Hurwitz criterion for second-order equations, C_1 is locally asymptotically stable.

Theorem 2.*The cough existing equilibrium* $(CEE-C_2)=(S^*,E^*,I^*,R^*)$ *is locally asymptotically stable if* $R_0>1$.

*Proof.*The Jacobian matrix assessed at C_2 is as follows.

$$J=\begin{pmatrix}-\mu^{\zeta}-\beta^{\zeta}I^*(\mathfrak{r}) & 0 & -\beta^{\zeta}S^*(\mathfrak{r}) & 0\\ \beta^{\zeta}I^*(\mathfrak{r}) & -V^{\zeta}-\mu^{\zeta} & \beta^{\zeta}S^*(\mathfrak{r}) & 0\\ 0 & V^{\zeta} & -\alpha^{\zeta}-\mu^{\zeta} & 0\\ 0 & 0 & \alpha^{\zeta} & -\mu^{\zeta}\end{pmatrix}, \tag{15}$$

$$|J(CEE)-\lambda I|=\begin{vmatrix}-\mu^{\zeta}-\beta^{\zeta}I^*(\mathfrak{r})-\lambda & 0 & -\beta^{\zeta}S^*(\mathfrak{r}) & 0\\ \beta^{\zeta}I^*(\mathfrak{r}) & -V^{\zeta}-\mu^{\zeta}-\lambda & \beta^{\zeta}S^*(\mathfrak{r}) & 0\\ 0 & V^{\zeta} & -\alpha^{\zeta}-\mu^{\zeta}-\lambda & 0\\ 0 & 0 & \alpha^{\zeta} & -\mu^{\zeta}-\lambda\end{vmatrix}=0,$$

$$\lambda_1=-\mu^{\zeta}<0,$$

$$\lambda^3+\left(3\mu^{\zeta}+\beta^{\zeta}I^*(\mathfrak{r})+V^{\zeta}+\alpha^{\zeta}\right)\lambda^2+\left(2V^{\zeta}\mu^{\zeta}+3\mu^{2\zeta}+2\alpha^{\zeta}\mu+V^{\zeta}\beta^{\zeta}I^*(\mathfrak{r})+2\mu^{\zeta}\beta I^*(\mathfrak{r})+V^{\zeta}\alpha^{\zeta}-V^{\zeta}\beta^{\zeta}S^*(\mathfrak{r})\right)\lambda$$
$$+\left(V^{\zeta}\alpha^{\zeta}\mu^{\zeta}+V^{\zeta}\mu^{2\zeta}+\alpha^{\zeta}\mu^{2\zeta}+\mu^{3\zeta}-V^{\zeta}\mu^{\zeta}\beta^{\zeta}S^*(\mathfrak{r})+V^{\zeta}\alpha^{\zeta}\beta^{\zeta}I^*(\mathfrak{r})+V^{\zeta}\mu^{\zeta}\beta^{\zeta}I^*(\mathfrak{r})+\alpha^{\zeta}\beta^{\zeta}I^*(\mathfrak{r})\mu^{\zeta}+\mu^{2\zeta}\beta^{\zeta}I^*(\mathfrak{r})\right)=0$$

$$\lambda^3+a_2\lambda^2+a_1\lambda+a_0=0,$$

where $a_2=3\mu^{\zeta}+\beta^{\zeta}I^*(\mathfrak{r})+V^{\zeta}+\alpha^{\zeta}$, $a_1=2V^{\zeta}\mu^{\zeta}+3\mu^{2\zeta}+2\alpha^{\zeta}\mu+V^{\zeta}\beta^{\zeta}I^*(\mathfrak{r})+2\mu^{\zeta}\beta I^*(\mathfrak{r})+V^{\zeta}\alpha^{\zeta}-V^{\zeta}\beta^{\zeta}S^*(\mathfrak{r})$,
$a_0=V^{\zeta}\alpha^{\zeta}\mu^{\zeta}+V^{\zeta}\mu^{2\zeta}+\alpha^{\zeta}\mu^{2\zeta}+\mu^{3\zeta}-V^{\zeta}\mu^{\zeta}\beta^{\zeta}S^*(\mathfrak{r})+V^{\zeta}\alpha^{\zeta}\beta^{\zeta}I^*(\mathfrak{r})+V^{\zeta}\mu^{\zeta}\beta^{\zeta}I^*(\mathfrak{r})+\alpha^{\zeta}\beta^{\zeta}I^*(\mathfrak{r})\mu^{\zeta}+\mu^{2\zeta}\beta^{\zeta}I^*(\mathfrak{r})$
Therefore, by applying the Routh-Hurwitz criterion for third-order equations,
$a_2>0,a_0>0,a_2a_1>a_0$, if $R_0>1$.
Hence, C_2 is locally asymptotically stable.

6 Discretization using Grünwald-Letnikov approximation

We consider the non-integer differential equation with constant proportional Caputo fractional operator as.

$$ {}_0^{CPC}D_{\mathfrak{r}}^{\zeta} y(\mathfrak{r}) = f(\mathfrak{r}, y(\mathfrak{r})), \quad 0 < \zeta < 1, \quad y(\mathfrak{r}_0) = y_0 \tag{16} $$

and we assume there exists a unique solution $y = y(\mathfrak{r})$ in the interval $[0, T]$
Now, from the relation, (16) can be written as.

$$ {}_0^{CPC}D_{\mathfrak{r}}^{\zeta} y(\mathfrak{r}) = \frac{1}{\Gamma(1-\zeta)} \left(\int_0^{\mathfrak{r}} (y(\tau)K_1(\zeta) + y_1(\tau)K_0(\zeta))(\mathfrak{r}-\tau)^{-\zeta} d\tau \right) = K_1(\zeta) {}_0^{R}D_{\mathfrak{r}}^{1-\zeta} y(\mathfrak{r}) + K_0(\zeta) {}_0^{C}D_{\mathfrak{r}}^{\zeta} y(\mathfrak{r}). $$

Therefore, equation (16) becomes

$$ K_1(\zeta) {}_0^{RL}D_{\mathfrak{r}}^{1-\zeta} y(\mathfrak{r}) + K_0(\zeta) {}_0^{C}D_{\mathfrak{r}}^{\zeta} y(\mathfrak{r}) = f(\mathfrak{r}, y(\mathfrak{r})). \tag{17} $$

Let $\mathfrak{r}_n = n\tau$, $\tau = \frac{T_f}{N_n}$, $N_n \in \mathbb{N}$, $\mathfrak{r}_{n+1} - \mathfrak{r}_n = \tau$, $0 = \mathfrak{r}_0 < \mathfrak{r}_1 < \mathfrak{r}_2 < \cdots < \mathfrak{r}_{n+1} = T_f$, and let y_k denote the approximate value to the true solution $y(\mathfrak{r}_k)$. Now, (17) can be discretized at $r = \mathfrak{r}_{n+1}$ using the Grunwald-Letnikov approximation to both the Riemann-Liouville and the Caputo fractional derivatives as given in [21] to obtain the explicit and implicit Grunwald-Letnikov approximation to the constant proportional Caputo fractional operator (17) respectively as:

$$ \frac{K_1(\zeta)}{(\tau)^{\zeta-1}} \left(y_{n+1} + \sum_{i=1}^{n+1} w_i y_{n+1-i} \right) + \frac{K_0(\zeta)}{(\tau)^{\zeta}} \left(y_{n+1} - \sum_{i=1}^{n+1} \mu_i y_{n+1-i} - q_{n+1} y_0 \right) = f(\mathfrak{r}_n, y(\mathfrak{r}_n)). \tag{18} $$

$$ \frac{K_1(\zeta)}{(\tau)^{\zeta-1}} \left(y_{n+1} + \sum_{i=1}^{n+1} w_i y_{n+1-i} \right) + \frac{K_0(\zeta)}{(\tau)^{\zeta}} \left(y_{n+1} - \sum_{i=1}^{n+1} \mu_i y_{n+1-i} - q_{n+1} y_0 \right) = f(\mathfrak{r}_{n+1}, y(\mathfrak{r}_{n+1})). \tag{19} $$

where, $w_0 = 1$, $w_i = (1-\zeta_i) w_{i-1}$, $\mu_i = (-1)^{i-1}\zeta_i$, $\mu_1 = \zeta$, $q_i = i\zeta \frac{\Gamma(1-\zeta)}{\Gamma(2-\zeta)}$, and $i = 1, 2, 3, \ldots, n+1$. Also, the following conditions are satisfied.

$$ 0 < \mu_{i+1} < \mu_i < \cdots < \mu_1 = \zeta < 1, $$

$$ 0 < q_{i+1} < q_i < \cdots < q_1 = \frac{1}{1-\zeta}, $$

$$ \sum_{i=1}^{\infty} \mu_i = 1 \quad \text{and} \quad \sum_{i=1}^{\infty} w_i = 1. $$

7 Numerical methods

Numerical modeling serves as a valuable tool for understanding the dynamics of infectious disease transmission and quantifying its impact. It relies on our understanding of how infections propagate within a population. In this study, we focus on investigating a mathematical model for whooping cough-like and developing an unconditionally stable nonstandard finite difference (NSFD) scheme tailored to this model. The NSFD numerical scheme demonstrates dynamic reliability and preserves the positivity of the solutions [10]-[18]. Notably, for each time step size, the NSFD method converges to the equilibrium points of the model. The results presented in this paper highlight that NSFD schemes offer a detailed depiction and a clear representation of the continuous model. In the subsequent subsection, we proceed to construct the Grunwald-Letnikov approximations and the NSFD Grunwald-Letnikov scheme for the hybrid fractional system (1)-(4) at $t = t_{n+1}$ as follows.

7.1 Grunwald Letnikov scheme

The explicit GL approximation for the fractional system is followed by (18) and the model equations. Consider the model equations (1)-(4), which yields the following system of equations.

$$\frac{K_1(\zeta)}{(\tau)^{\zeta-1}}\left(S^{n+1}+\sum_{i=1}^{n+1}w_iS^{n+1-i}\right)+\frac{K_0(\zeta)}{(\tau)^{\zeta}}\left(S^{n+1}-\sum_{i=1}^{n+1}\mu_iS^{n+1-i}-q_{n+1}S^0\right)=\mu^{\zeta}-\mu^{\zeta}S^n(\mathfrak{r})-\beta^{\zeta}S^n(\mathfrak{r})I^n(\mathfrak{r}),$$
$$\frac{K_1(\zeta)}{(\tau)^{\zeta-1}}\left(E^{n+1}+\sum_{i=1}^{n+1}w_iE^{n+1-i}\right)+\frac{K_0(\zeta)}{(\tau)^{\zeta}}\left(E^{n+1}-\sum_{i=1}^{n+1}\mu_iE^{n+1-i}-q_{n+1}E^0\right)=\beta^{\zeta}S^n(\mathfrak{r})I^n(\mathfrak{r})-V^{\zeta}E^n(\mathfrak{r})-\mu^{\zeta}E^n(\mathfrak{r}),$$
$$\frac{K_1(\zeta)}{(\tau)^{\zeta-1}}\left(I^{n+1}+\sum_{i=1}^{n+1}w_iI^{n+1-i}\right)+\frac{K_0(\zeta)}{(\tau)^{\zeta}}\left(I^{n+1}-\sum_{i=1}^{n+1}\mu_iI^{n+1-i}-q_{n+1}I^0\right)=V^{\zeta}E^n(\mathfrak{r})-\alpha^{\zeta}I^n(\mathfrak{r})-\mu^{\zeta}I^n(\mathfrak{r}),$$
$$\frac{K_1(\zeta)}{(\tau)^{\zeta-1}}\left(R^{n+1}+\sum_{i=1}^{n+1}w_iR^{n+1-i}\right)+\frac{K_0(\zeta)}{(\tau)^{\zeta}}\left(R^{n+1}-\sum_{i=1}^{n+1}\mu_iR^{n+1-i}-q_{n+1}R^0\right)=\alpha^{\zeta}I^n(\mathfrak{r})-\mu^{\zeta}R^n(\mathfrak{r}).$$

The implicit GL approximation for the fractional system is obtained by equation (19) model equations (1), (2), (3) and (4), which is expressed in the form of system of equations as stated below.

$$\frac{K_1(\zeta)}{(\tau)^{\zeta-1}}\left(S^{n+1}+\sum_{i=1}^{n+1}w_iS^{n+1-i}\right)+\frac{K_0(\zeta)}{(\tau)^{\zeta}}\left(S^{n+1}-\sum_{i=1}^{n+1}\mu_iS^{n+1-i}-q_{n+1}S^0\right)=\mu^{\zeta}-\mu^{\zeta}S^{n+1}(\mathfrak{r})-\beta^{\zeta}S^{n+1}(\mathfrak{r})I^{n+1}(\mathfrak{r}),$$
$$\frac{K_1(\zeta)}{(\tau)^{\zeta-1}}\left(E^{n+1}+\sum_{i=1}^{n+1}w_iE^{n+1-i}\right)+\frac{K_0(\zeta)}{(\tau)^{\zeta}}\left(E^{n+1}-\sum_{i=1}^{n+1}\mu_iE^{n+1-i}-q_{n+1}E^0\right)=\beta^{\zeta}S^{n+1}(\mathfrak{r})I^{n+1}(\mathfrak{r})-V^{\zeta}E^{n+1}(\mathfrak{r})-\mu^{\zeta}E^{n+1}(\mathfrak{r}),$$
$$\frac{K_1(\zeta)}{(\tau)^{\zeta-1}}\left(I^{n+1}+\sum_{i=1}^{n+1}w_iI^{n+1-i}\right)+\frac{K_0(\zeta)}{(\tau)^{\zeta}}\left(I^{n+1}-\sum_{i=1}^{n+1}\mu_iI^{n+1-i}-q_{n+1}I^0\right)=V^{\zeta}E^{n+1}(\mathfrak{r})-\alpha^{\zeta}I^{n+1}(\mathfrak{r})-\mu^{\zeta}I^{n+1}(\mathfrak{r}),$$
$$\frac{K_1(\zeta)}{(\tau)^{\zeta-1}}\left(R^{n+1}+\sum_{i=1}^{n+1}w_iR^{n+1-i}\right)+\frac{K_0(\zeta)}{(\tau)^{\zeta}}\left(R^{n+1}-\sum_{i=1}^{n+1}\mu_iR^{n+1-i}-q_{n+1}R^0\right)=\alpha^{\zeta}I^{n+1}(\mathfrak{r})-\mu^{\zeta}R^{n+1}(\mathfrak{r}).$$

7.2 Non standart finite difference scheme

In the section, we formulate a non-standart finite difference scheme as stated in [21], also known is weakly non-standart finite difference scheme, in which traditional denominator τ is replace by $\phi(\tau)=\tau+O(\tau^2)$. In the first order discrete derivative. where $\phi(\tau)$ is a non-negative function. The hybrid fractional model (1)-(4) with parameter τ is converted into a discretized system by adobting the rules of R.E. Mickens [22]. Therefore, the explicit and implicit NSFD numerical method for eq.(18) and eq.(19) is respectively given by.

$$\frac{K_1(\zeta)}{\phi(\tau)^{\zeta-1}}\left(y_{n+1}+\sum_{i=1}^{n+1}w_iy_{n+1-i}\right)+\frac{K_0(\zeta)}{\phi(\tau)^{\zeta}}\left(y_{n+1}-\sum_{i=1}^{n+1}\mu_iy_{n+1-i}-q_{n+1}y_0\right)=f(\mathfrak{r}_n,y(\mathfrak{r}_n)),$$
$$\frac{K_1(\zeta)}{\phi(\tau)^{\zeta-1}}\left(y_{n+1}+\sum_{i=1}^{n+1}w_iy_{n+1-i}\right)+\frac{K_0(\zeta)}{\phi(\tau)^{\zeta}}\left(y_{n+1}-\sum_{i=1}^{n+1}\mu_iy_{n+1-i}-q_{n+1}y_0\right)=f(\mathfrak{r}_{n+1},y(\mathfrak{r}_{n+1})).$$

7.3 GL-NSFD scheme 1

Here, we present the expilicit GL-NSFD scheme for computing the numarical solutions to the hybrid model, expresed by eqs. (1)-(4). For this, we consider the susceptible class of the model.

$$\frac{K_1(\zeta)}{\phi(\tau)^{\zeta-1}}\left(S^{n+1}+\sum_{i=1}^{n+1}w_iS^{n+1-i}\right)+\frac{K_0(\zeta)}{\phi(\tau)^{\zeta}}\left(S^{n+1}-\sum_{i=1}^{n+1}\mu_iS^{n+1-i}-q_{n+1}S^0\right)=\mu^{\zeta}-\mu^{\zeta}S^n-\beta^{\zeta}S^nI^n$$

$$\frac{K_1(\zeta)}{\phi(\tau)^{\zeta-1}}S^{n+1}+\frac{K_0(\zeta)}{\phi(\tau)^{\zeta}}S^{n+1}=\frac{K_0(\zeta)}{\phi(\tau)^{\zeta}}\left(\sum_{i=1}^{n+1}\mu_iS^{n+1-i}+q_{n+1}S^0\right)-\frac{K_1(\zeta)}{\phi(\tau)^{\zeta-1}}\left(\sum_{i=1}^{n+1}w_iS^{n+1-i}\right)$$
$$+\mu^{\zeta}-\mu^{\zeta}S^n-\beta^{\zeta}S^nI^n$$

Let $A=\frac{K_1(\zeta)}{\phi(\tau)^{\zeta-1}}$ and $B=\frac{K_0(\zeta)}{\phi(\tau)^{\zeta}}$ then

$$S^{n+1}=\frac{1}{A+B}\left(B\sum_{i=1}^{n+1}\mu_iS^{n+1-i}+Bq_{n+1}S^0-A\sum_{i=1}^{n+1}w_iS^{n+1-i}+\mu^{\zeta}-\mu^{\zeta}S^n-\beta^{\zeta}S^nI^n\right) \tag{20}$$

similarly, for the exposed class of the model, we obtain the expression represented by eqn.(21)

$$E^{n+1}=\frac{1}{A+B}\left(B\sum_{i=1}^{n+1}\mu_iE^{n+1-i}+Bq_{n+1}E^0-A\sum_{i=1}^{n+1}w_iE^{n+1-i}+\beta^{\zeta}S^nI^n-V^{\zeta}E^n-\mu^{\zeta}E^n\right) \tag{21}$$

On the same lines, we reach at the expresions (22) and (23)

$$I^{n+1}=\frac{1}{A+B}\left(B\sum_{i=1}^{n+1}\mu_iI^{n+1-i}+Bq_{n+1}I^0-A\sum_{i=1}^{n+1}w_iI^{n+1-i}+V^{\zeta}E^n-\alpha^{\zeta}I^n-\mu^{\zeta}I^n\right) \tag{22}$$

$$R^{n+1}=\frac{1}{A+B}\left(B\sum_{i=1}^{n+1}\mu_iR^{n+1-i}+Bq_{n+1}R^0-A\sum_{i=1}^{n+1}w_iR^{n+1-i}+\alpha^{\zeta}I^n-\mu^{\zeta}R^n\right) \tag{23}$$

7.4 GL-NSFD scheme 2

In this section, we will construct the numerical scheme, named an GL-NSFD scheme 2. While formulating this scheme non-local approximation of the non-linear term and other ruler obtained by R.E. Mickens [22], are applied.
The impilicit GL-NSFD scheme for hybrid model. Takes the form as expressed by eqn.(24)

$$\frac{K_1(\zeta)}{\phi(\tau)^{\zeta-1}}\left(S^{n+1}+\sum_{i=1}^{n+1}w_iS^{n+1-i}\right)+\frac{K_0(\zeta)}{\phi(\tau)^{\zeta}}\left(S^{n+1}-\sum_{i=1}^{n+1}\mu_iS^{n+1-i}-q_{n+1}S^0\right)=\mu^{\zeta}-\mu^{\zeta}S^{n+1}-\beta^{\zeta}S^{n+1}I^n$$

$$AS^{n+1}+BS^{n+1}+\mu^{\zeta}S^{n+1}+\beta^{\zeta}S^{n+1}I^n=\mu^{\zeta}-A\sum_{i=1}^{n+1}w_iS^{n+1-i}+B\sum_{i=1}^{n+1}\mu_iS^{n+1-i}+Bq_{n+1}S^0$$

$$S^{n+1}=\frac{\mu^{\zeta}-A\sum_{i=1}^{n+1}w_iS^{n+1-i}+B\sum_{i=1}^{n+1}\mu_iS^{n+1-i}+Bq_{n+1}S^0}{A+B+\mu^{\zeta}+\beta^{\zeta}I^n} \tag{24}$$

similarly

$$E^{n+1}=\frac{\beta^{\zeta}S^{n+1}I^n-A\sum_{i=1}^{n+1}w_iE^{n+1-i}+B\sum_{i=1}^{n+1}\mu_iE^{n+1-i}+Bq_{n+1}E^0}{A+B+V^{\zeta}+\mu^{\zeta}} \tag{25}$$

and

$$I^{n+1}=\frac{V^{\zeta}E^{n+1}-A\sum_{i=1}^{n+1}w_iI^{n+1-i}+B\sum_{i=1}^{n+1}\mu_iI^{n+1-i}+Bq_{n+1}I^0}{A+B+\alpha^{\zeta}+\mu^{\zeta}} \tag{26}$$

finaly, we reach at

$$R^{n+1}=\frac{\alpha^{\zeta}I^{n+1}-A\sum_{i=1}^{n+1}w_iR^{n+1-i}+B\sum_{i=1}^{n+1}\mu_iR^{n+1-i}+Bq_{n+1}R^0}{A+B+\mu^{\zeta}} \tag{27}$$

7.5 Positivity of the GL-NSFD scheme 2

In epidemic models, positivity in the significant characteristic that should be maintained for every moment of time. Equivalently one can state that, populace in each compartment must be non- negative. As, negative value of any state varible in physical meaningless.
In this section, we ensure the positivity property of the numerical scheme given by equations (24)-(27).

Theorem 3(Positivity). *Suppose that in* (24)-(27) $S^0>0, E^0>0, I^0>0$ *and* $R^0>0$*, more over, all the parameters involved in the scheme are positive i.e.* $\mu^\zeta>0$*,* $\beta^\zeta>0$*,* $\alpha^\zeta>0$*,* $V^\zeta>0$ *then* $S^{n+1}>0, E^{n+1}>0, I^{n+1}>0$ *and* $R^{n+1}>0, \forall n, \quad n=1,2,3$ *if* $Aw_i<B\mu_i$ *for* $i=1,2,...,n+1$.

*Proof.*The proof is done by induction. From (24)-(27) we have for $n=0$

$$\begin{aligned}
S^1 &= \frac{(B\mu_1-Aw_1)S^0+Bq_1S^0+\mu^\zeta}{A+B+\mu^\zeta+\beta^\zeta I^0}>0,\\
E^1 &= \frac{(B\mu_1-Aw_1)E^0+Bq_1E^0+\beta^\zeta S^1E^0}{A+B+V^\zeta+\mu^\zeta}>0,\\
I^1 &= \frac{(B\mu_1-Aw_1)I^0+Bq_1I^0+V^\zeta E^1}{A+B+\alpha^\zeta+\mu^\zeta I^0}>0,\\
R^1 &= \frac{(B\mu_1-Aw_1)R^0+Bq_1R^0+\alpha^\zeta I^1}{A+B+\mu^\zeta+\beta^\zeta I^0}>0.
\end{aligned}$$

$S^1>0, E^1>0, I^1>0$ and $R^1>0$, If $Aw_1<B\mu_1$, suppose that $S^n>0, E^n>0, I^n>0$ and $R^n>0$, then

$$\begin{aligned}
S^{n+1} &= \frac{\mu^\zeta-A\sum_{i=1}^{n+1}w_iS^{n+1-i}+B\sum_{i=1}^{n+1}\mu_iS^{n+1-i}+Bq_{n+1}S^0}{A+B+\mu^\zeta+\beta^\zeta I^n}>0\\
E^{n+1} &= \frac{\beta^\zeta S^{n+1}I^n-A\sum_{i=1}^{n+1}w_iE^{n+1-i}+B\sum_{i=1}^{n+1}\mu_iE^{n+1-i}+Bq_{n+1}E^0}{A+B+V^\zeta+\mu^\zeta}>0,\\
I^{n+1} &= \frac{V^\zeta E^{n+1}-A\sum_{i=1}^{n+1}w_iI^{n+1-i}+B\sum_{i=1}^{n+1}\mu_iI^{n+1-i}+Bq_{n+1}I^0}{A+B+\alpha^\zeta+\mu^\zeta}>0,\\
R^{n+1} &= \frac{\alpha^\zeta I^{n+1}-A\sum_{i=1}^{n+1}w_iR^{n+1-i}+B\sum_{i=1}^{n+1}\mu_iR^{n+1-i}+Bq_{n+1}R^0}{A+B+\mu^\zeta}>0.
\end{aligned}$$

7.6 Boundedness of the GL-NSFD scheme 2

Like positivity, boundedness is also an important trait of the state variables involved in the model, i.e. λ sum of all the sub-populations must not exceed the total population. Because, if the numarical method fails to preserve the boundedness, then, it is not reliable. Here we will investigate the boundedness of the numerical scheme.

Theorem 4.*Let* $S^0+E^0+I^0+R^0=1, \mu^\zeta>0, V^\zeta>0, \alpha^\zeta>0, \beta^\zeta>0, \phi(\tau)^\zeta>0, \phi(\tau)^{\zeta-1}>0, \zeta\in(0,1)$ *Then* $S^{n+1}+E^{n+1}+I^{n+1}+R^{n+1}$ *are bounded for* $n=0,1,2,...,N$

Proof.

$$\begin{aligned}
&AS^{n+1}+BS^{n+1}+\mu^\zeta S^{n+1}+\beta^\zeta I^nS^{n+1}+AE^{n+1}+BE^{n+1}+V^\zeta E^{n+1}+\mu^\zeta E^{n+1}+AI^{n+1}+BI^{n+1}\\
&+\alpha^\zeta I^{n+1}+\mu^\zeta I^{n+1}+AR^{n+1}+BR^{n+1}+\mu^\zeta R^{n+1}=B\sum_{i=1}^{n+1}\mu_i\left(S^{n+1-i}+E^{n+1-i}+I^{n+1-i}+R^{n+1-i}\right)\\
&+Bq_{n+1}\left(S^0+E^0+I^0+R^0\right)-A\sum_{i=1}^{n+1}w_i\left(S^{n+1-i}+E^{n+1-i}+I^{n+1-i}+R^{n+1-i}\right)\\
&+\mu^\zeta+\beta^\zeta S^{n+1}I^n+V\zeta E^{n+1}+\alpha^\zeta I^{n+1}
\end{aligned}$$

for $n=0$

$$\begin{aligned}
&\left(A+B+\mu^{\zeta}+\beta^{\zeta}I^{0}\right)S^{1}+\left(A+B+V^{\zeta}+\mu^{\zeta}\right)E^{1}+\left(A+B+\alpha^{\zeta}+\mu^{\zeta}\right)I^{1}+\left(A+B+\mu^{\zeta}\right)R^{1}\\
&=B\mu_1\left(S^{0}+E^{0}+I^{0}+R^{0}\right)+Bq_1-Aw_1\left(S^{0}+E^{0}+I^{0}+R^{0}\right)+\mu^{\zeta}+\beta^{\zeta}S^{1}I^{0}+V^{\zeta}E^{1}+\alpha^{\zeta}I^{1}\\
&=B\mu_1+Bq_1-Aw_1+\mu\zeta+\beta^{\zeta}I^{0}S^{1}+V^{\zeta}E^{1}+\alpha^{\zeta}I^{1}\\
&=B\zeta+\frac{B}{\Gamma(1-\zeta)}-A(1-\zeta)+\mu^{\alpha}+\beta^{\zeta}I^{0}S^{1}+V^{\zeta}E^{1}+\alpha^{\zeta}I^{1}\\
&\left(A+B+\mu^{\zeta}+\beta^{\zeta}I^{0}\right)S^{1}+\left(A+B+V^{\zeta}+\mu^{\zeta}\right)E^{1}+\left(A+B+\alpha^{\zeta}+\mu^{\zeta}\right)I^{1}+\left(A+B+\mu^{\zeta}\right)R^{1}\\
&<(A+B)\zeta+\frac{B}{\Gamma(1-\zeta)}+\mu^{\zeta}+\beta^{\zeta}I^{0}S^{1}+V^{\zeta}E^{1}+\alpha^{\zeta}I^{1}
\end{aligned}$$

it implies that

$$\left(A+B+\mu^{\zeta}\right)S^{1}+\left(A+B+\mu^{\zeta}\right)E^{1}+\left(A+B+\mu^{\zeta}\right)I^{1}+\left(A+B+\mu^{\zeta}\right)R^{1}<(A+B)\zeta+\frac{B}{\Gamma(1-\zeta)}+\mu^{\zeta}=M_1 \quad (28)$$

thus

$$S^{1},E^{1},I^{1},R^{1}<\frac{M_1}{A+B+\mu^{\zeta}}.$$

Similarly, for $n=1$

$$\begin{aligned}
&\left(A+B+\mu^{\zeta}+\beta^{\zeta}I^{1}\right)S^{2}+\left(A+B+\mu^{\zeta}+V^{\zeta}\right)E^{2}+\left(A+B+\mu^{\zeta}+\alpha^{\zeta}\right)I^{2}+\left(A+B+\mu^{\zeta}\right)R^{2}\\
&=B\mu_1\left(S^{1}+E^{1}+I^{1}+R^{1}\right)+B\mu_2\left(S^{0}+E^{0}+I^{0}+R^{0}\right)+Bq_2-Aw_1\left(S^{1}+E^{1}+I^{1}+R^{1}\right)\\
&-Aw_2\left(S^{0}+E^{0}+I^{0}+R^{0}\right)+\mu^{\zeta}+\beta^{\zeta}I^{1}S^{2}+V^{\zeta}E^{2}+\alpha^{\zeta}I^{2},
\end{aligned}$$

then

$$\begin{aligned}
&\left(A+B+\mu^{\zeta}\right)S^{2}+\left(A+B+\mu^{\zeta}\right)E^{2}+\left(A+B+\mu^{\zeta}\right)I^{2}+\left(A+B+\mu^{\zeta}\right)R^{2}\\
&=B\mu_1\left(S^{1}+E^{1}+I^{1}+R^{1}\right)+B\mu_2+Bq_2-Aw_1\left(S^{1}+E^{1}+I^{1}+R^{1}\right)-Aw_2+\mu^{\zeta},\\
&<B\zeta\left(S^{1}+E^{1}+I^{1}+R^{1}\right)+B\zeta+\frac{B}{\Gamma(1-\zeta)}-A(1-\zeta)\left(S^{1}+E^{1}+I^{1}+R^{1}\right)-A(1-\zeta)+\mu^{\zeta},\\
&<B\zeta\left(S^{1}+E^{1}+I^{1}+R^{1}\right)+A\zeta\left(S^{1}+E^{1}+I^{1}+R^{1}\right)+B\zeta+A\zeta+\frac{B}{\Gamma(1-\zeta)}+\mu^{\zeta},\\
&<B\zeta 4M_1+A\zeta 4M_1+\zeta(A+B)+\frac{B}{\Gamma(1-\zeta)}+\mu^{\zeta},\\
&<(A+B)\zeta 4M_1+\zeta(A+B)+\frac{B}{\Gamma(1-\zeta)}+\mu^{\zeta}=M_2.
\end{aligned}$$

which leads to

$$S^{2},E^{2},I^{2},R^{2}<\frac{M_2}{A+B+\mu^{\zeta}}$$

since

$$S^{1},E^{1},I^{1},R^{1}<\frac{M_1}{A+B+\mu^{\zeta}} \quad \text{and} \quad S^{2},E^{2},I^{2},R^{2}<\frac{M_2}{A+B+\mu^{\zeta}}$$

Suppose it is true for $n=N-1$ i.e.

$$S^{N-1},E^{N-1},I^{N-1},R^{N-1}<\frac{M_{N-1}}{\left(A+B+\mu^{\zeta}\right)}$$

which follows that

$$S^{N}<M_N,E^{N}<M_N,I^{N}<M_N,R^{N}<M_N$$

where

$$M_N = \zeta(A+B) + \frac{B}{\Gamma(1-\zeta)} + 4\zeta(A+B)\left(M_1 + M_2 + \cdots + M_{N-1}\right) \tag{29}$$

then, for $n = N$ we have

$$\begin{aligned}
&\left(A+B+\mu^{\zeta}+\beta^{\zeta}I^{N}\right)S^{N+1} + \left(A+B+\mu^{\zeta}+V^{\zeta}\right)E^{N} + \left(A+B+\mu^{\zeta}+\alpha^{\zeta}\right)I^{N+1} + \left(A+B+\mu^{\zeta}\right)R^{N+1} \\
&= B\mu_1\left(S^{N}+E^{N}+I^{N}+R^{N}\right) + B\mu_2\left(S^{N-1}+E^{N-1}+I^{N-1}+R^{N-1}\right) + \cdots + B\mu_N\left(S^{0}+E^{0}+I^{0}+R^{0}\right) + Bq_{N+1} \\
&- Aw_1\left(S^{N}+E^{N}+I^{N}+R^{N}\right) - Aw_2\left(S^{N-1}+E^{N-1}+I^{N-1}+R^{N-1}\right) - \cdots - Aw_{N+1}\left(S^{0}+E^{0}+I^{0}+R^{0}\right) \\
&+ \mu^{\zeta} + \beta^{\zeta}I^{N}S^{N+1} + V^{\zeta}E^{N+1} + \alpha^{\zeta}I^{N+1} \\
&< B\zeta 4M_N + B\zeta 4M_{N-1} + \cdots + B\zeta + \frac{B}{\Gamma(1-\zeta)} - A\zeta 4M_N - A\zeta 4M_{N-1} - \cdots - A\zeta + \mu^{\zeta} + \beta^{\zeta}I^{0}S^{1} + V^{\zeta}E^{1} + \alpha^{\zeta}I^{1} \\
&< B\zeta 4M_N + B\zeta M_{N-1} + \cdots + B\zeta 4M_1 + B\zeta + \frac{B}{\Gamma(1-\zeta)} + A\zeta 4M_N + A\zeta 4M_{N-1} + \cdots + A\zeta 4M_1 + A\zeta + \mu^{\zeta} \\
&= 4\zeta(A+B)M_N + 4\zeta(A+B)M_{N-1} + \cdots + 4\zeta(A+B)M_1 + 4\zeta(A+B) + \frac{B}{\Gamma(1-\zeta)} + \mu^{\zeta} \\
&= (A+B)\zeta + \frac{B}{\Gamma(1-\zeta)} + 4\zeta(A+B)\left(M_N + M_{N-1} + \cdots + M_1\right) + \mu^{\zeta} = M_{N+1}
\end{aligned}$$

thus

$$\left(A+B+\mu^{\zeta}\right)S^{N+1}, \left(A+B+\mu^{\zeta}\right)E^{N+1}, \left(A+B+\mu^{\zeta}\right)I^{N+1}, \left(A+B+\mu^{\zeta}\right)R^{N+1} < M_{N+1}$$

and

$$S^{N+1}, E^{N+1}, I^{N+1}, R^{N+1} < \frac{M_{N+1}}{\left(A+B+\mu^{\zeta}\right)}$$

this gives

$$\begin{aligned}
&S^{N+1}, E^{N+1}, I^{N+1}, R^{N+1} < M_{N+1} \\
&S^{n+1}, E^{n+1}, I^{n+1}, R^{n+1} < M_{n+1} \quad \text{for} \quad n = 0, 1, 2, \ldots, N
\end{aligned}$$

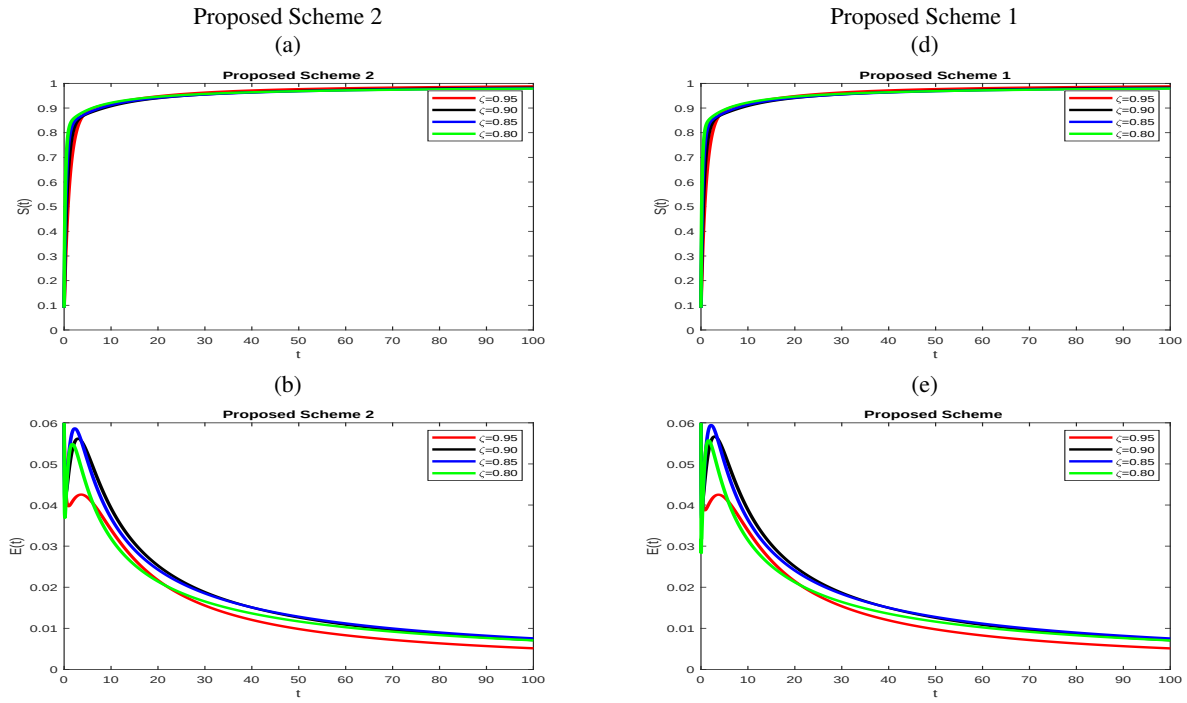

Fig. 1: (a) & (b) Graphs of susceptible and exposed in the model (1)-(4) by using proposed scheme 2 with step size 0.05 for disease free case. (d) & (e) Graphs of susceptible and exposed in the model (1)-(4) by using proposed scheme 1 with step size 0.05 for disease free case.

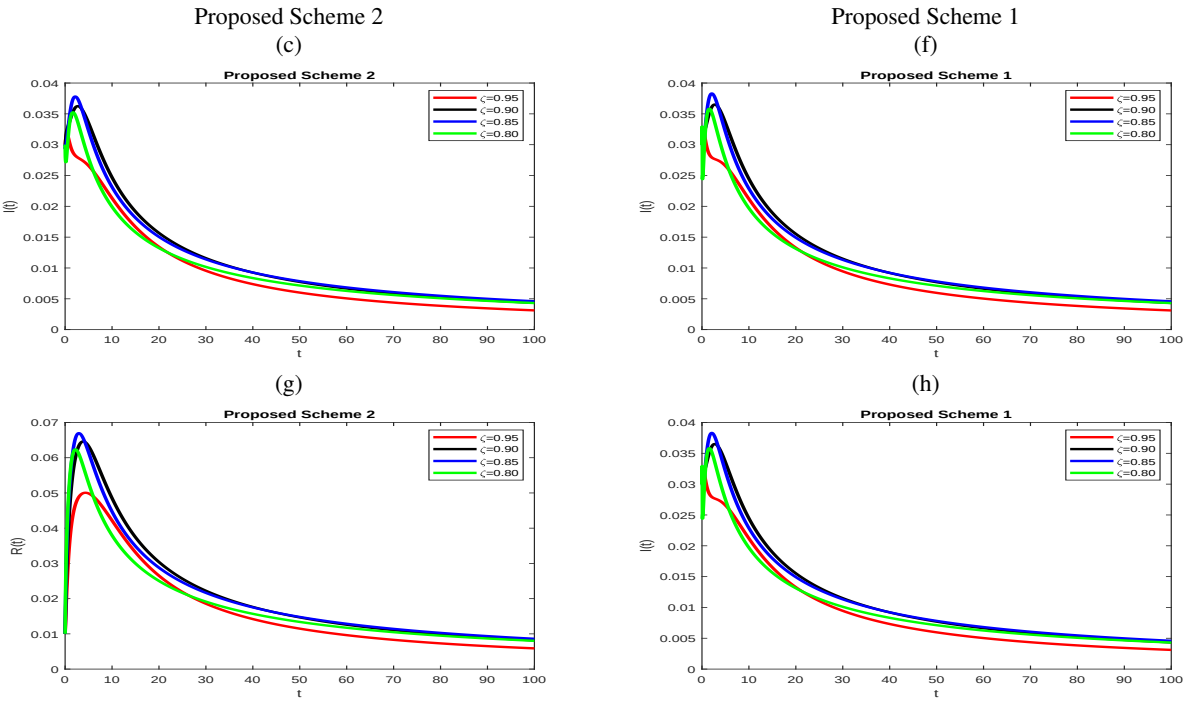

Fig. 2: (c) & (g) Graphs of infected and recovered in the model (1)-(4) by using proposed scheme 2 with step size 0.05 for disease free case. (f) & (h) Graphs of infected and recovered in the model (1)-(4) by using proposed scheme 1 with step size 0.05 for disease free case.

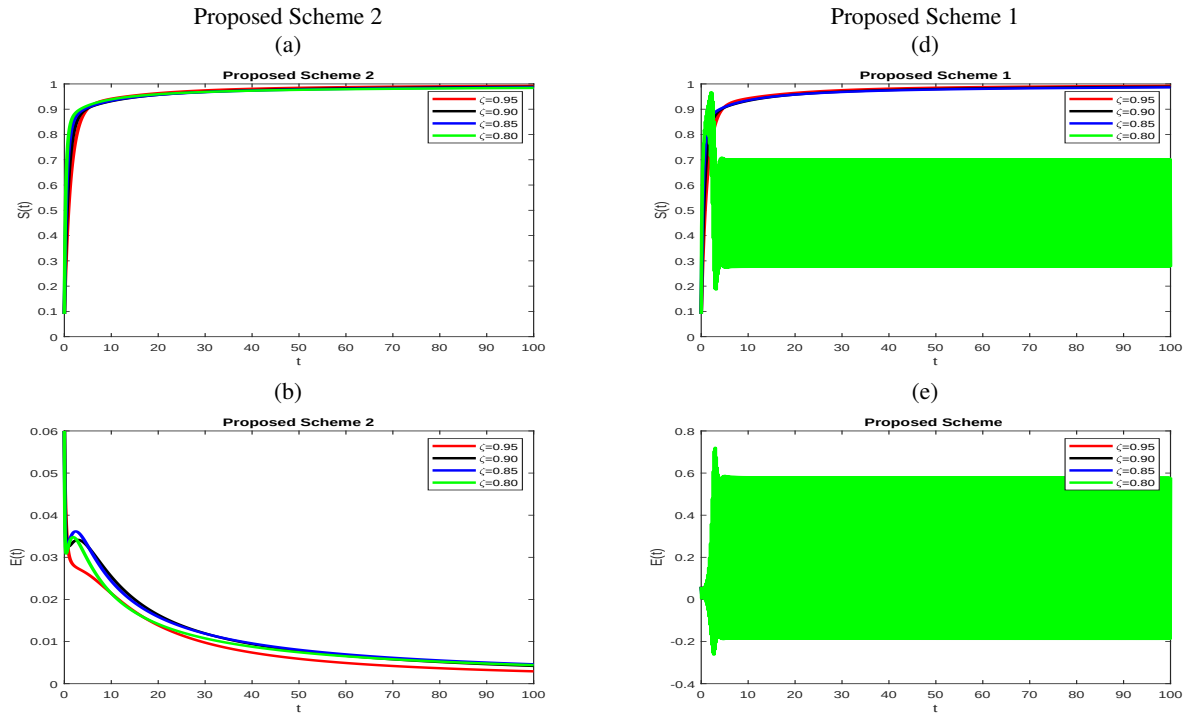

Fig. 3: (a) & (b) Graphs of susceptible and exposed in the model (1)-(4) by using proposed scheme 2 with step size 0.1 for disease free case. (d) & (e) Graphs of susceptible and exposed in the model (1)-(4) by using proposed scheme 1 with step size 0.1 for disease free case.

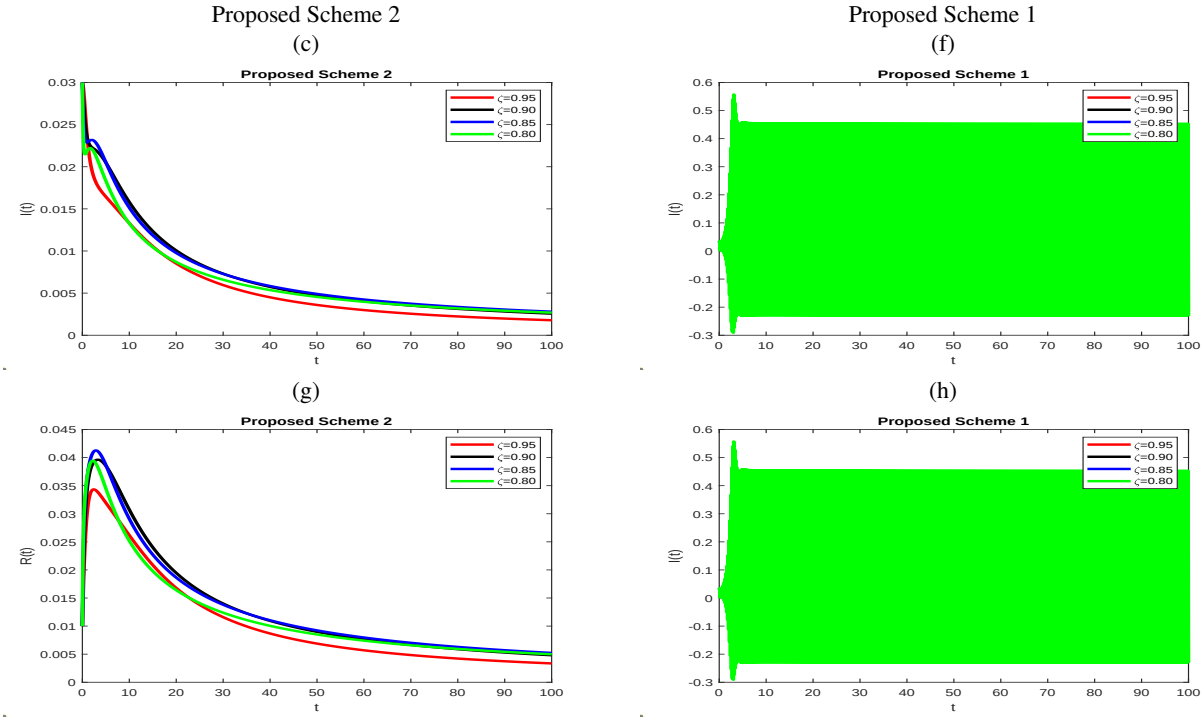

Fig. 4: (c) & (g) Graphs of infected and recovered in the model (1)-(4) by using proposed scheme 2 with step size 0.1 for disease free case. (f) & (h) Graphs of infected and recovered in the model (1)-(4) by using proposed scheme 1 with step size 0.1 for disease free case.

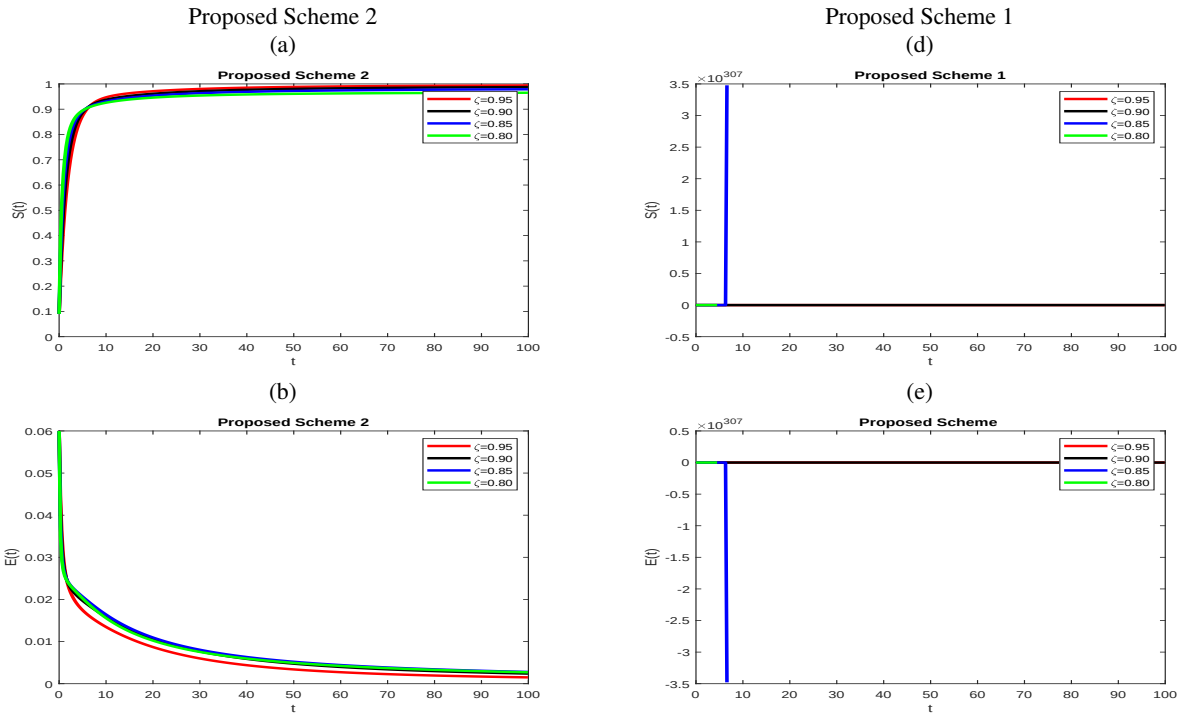

Fig. 5: (a) & (b) Graphs of susceptible and exposed in the model (1)-(4) by using proposed scheme 2 with step size 0.3 for disease free case. (d) & (e) Graphs of susceptible and exposed in the model (1)-(4) by using proposed scheme 1 with step size 0.3 for disease free case.

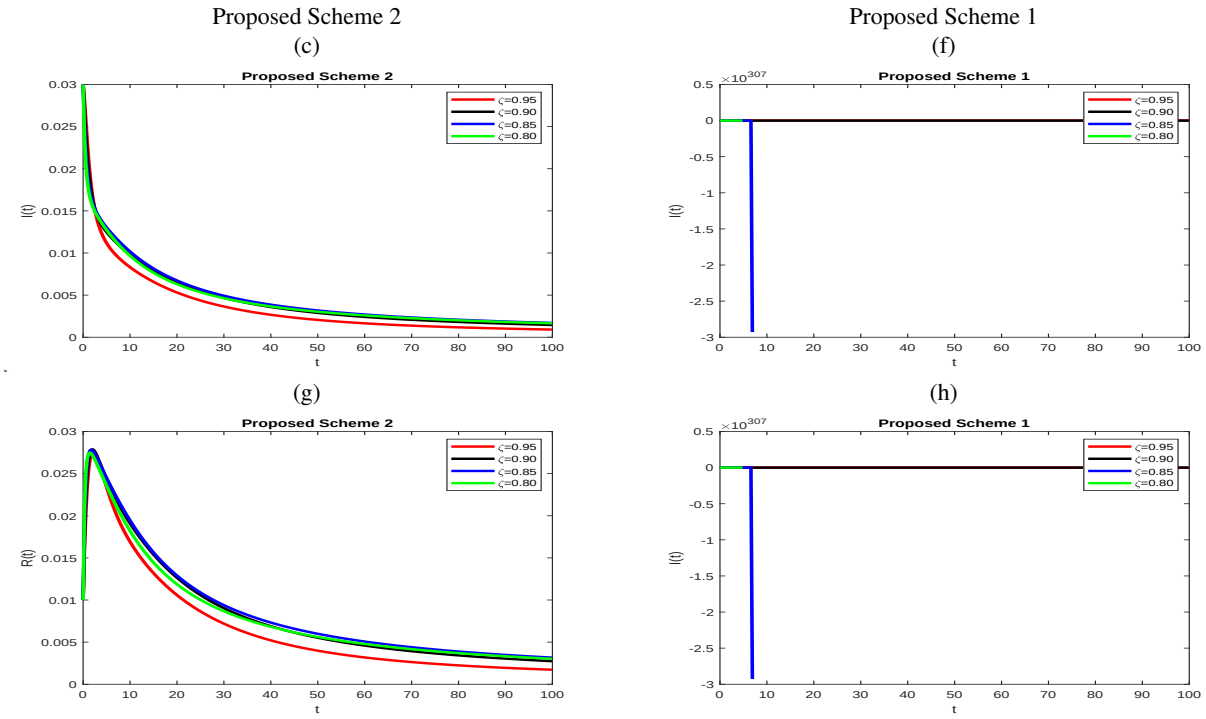

Fig. 6: (c) & (g) Graphs of infected and recovered in the model (1)-(4) by using proposed scheme 2 with step size 0.3 for disease free case. (f) & (h) Graphs of infected and recovered in the model (1)-(4) by using proposed scheme 1 with step size 0.3 for disease free case.

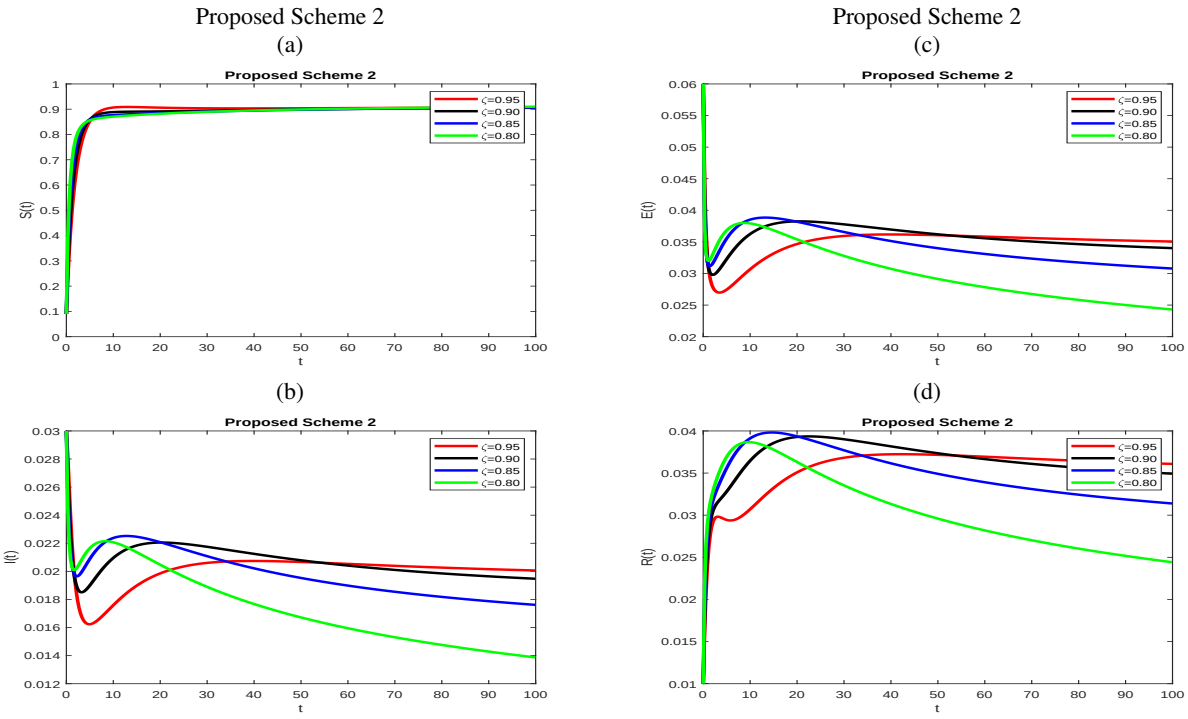

Fig. 7: (a), (b), (c) & (d) Graphs of susceptible, exposed, infected and recovered in the model (1)-(4) by using proposed scheme 2 with step size 0.3 for endemic case.

The graph depicts the evolution of the population infected with whooping cough-like infections over time for various fractional orders. This visual representation provides insights into how changes in the fractional order affect the efficacy and rate of strategies aimed at addressing whooping cough-like infections. By comparing different fractional orders, this visualization enables an assessment of the model's sensitivity to variations in the fractional order, thereby offering valuable insights into the dynamics of the system under different conditions.

8 Conclusion

In this research, whooping cough, also referred to as pertussis, is explored. Pertussis stands as a highly transmissible airborne ailment caused by bacteria. An constant proportional Caputo fractional model for pertussis cough contagion instigated by pertussis bacteria has been proposed to grasp the transmission and dissemination of the ailment within a community. Two numerical approaches were devised to scrutinize the model numerically. Equilibrium junctures were attained analytically and corroborated via numerical simulations. The presence and singularity of the solution were examined, local steadiness assessment for the model was acquired at both ailment-free and endemic balance junctures, and steadiness assessment for numerical approaches was acquired employing the constraint theorem. R_0 is utilized to probe the existence or nonexistence of ailment in a populace. The visual outcomes of the numerical approaches align with the analytical outcome. Assorted ζ values was employed to probe each subset of the populace.

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