

Efficient Inference of $P(W < Z)$ using Ranked Set Sampling: The Inverse Power Half-Logistic Model

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Abstract: Lifetime distributions are an essential statistical tool for modeling different attributes. The statistical literature offers many complex distributions for analyzing such data sets. However, the estimation process is challenging due to the large number of parameters in these distributions. To expand and model possibilities for these datasets, we introduce the inverse power half-logistic distribution (IPHLD) as a novel model. We derive some of its statistical properties and explore its application in stress-strength reliability modeling, a significant topic in the field of statistics. The stress-strength reliability model is defined as $\eta = P[W < Z]$, where W and Z represent the stress and strength random variables, respectively, and η is the reliability parameter. Assuming that W and Z are independent IPHLD with different scale parameters. Using ranked set sampling and simple random sampling, the maximum likelihood and Bayesian estimators of η are considered. The Bayes estimate of η under different loss functions is obtained using gamma priors. It is clear that the Bayesian estimators' explicit form is absent. Therefore, the Markov Chain Monte Carlo method is used to validate the Bayesian estimate. A Monte Carlo simulation study is used to examine the performance of different estimating techniques. In the end, eight real-world data sets from four applications are examined to illustrate the recommended estimation methods.

Keywords: Power half-logistic distribution; inverse transformation; entropy measures; perfect ranking; Bayesian estimation; Metropolis-Hasting method.

1 Introduction

In many real-world applications, lifetime data modeling is done using distributions with support on the positive real line. In the domains of biomedical research, reliability, actuarial sciences, demographics, engineering, public health, and so on, exponential and Weibull distributions, for instance, have been utilized to fit data sets. Furthermore, in practical reliability and lifetime data analysis, the half-logistic distribution (HLD) is another lifetime model that is essential for fitting data with a decreasing failure rate (see [1,2]). The half-logistic random variable X with scale parameter v has the following cumulative distribution function (CDF):

$$F(x) = 1 - \frac{2}{1 + e^{vx}}, \quad x, v > 0. \quad (1)$$

Krishnarani [3] proposed the power HLD (PHLD) with an additional shape parameter in order to improve the characteristics of the HLD. Using the transformation $Y = X^{1/\vartheta}$ where X has the CDF (1) and ϑ is the shape parameter, then the CDF for PHLD is provided by

$$F(y) = 1 - \frac{2}{1 + e^{vy\vartheta}}, \quad y, v, \vartheta > 0. \quad (2)$$

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In an attempt to expand or utilize some of the features of the HLD, a number of generalizations and extensions have been made in recent years (see [4]–[8]).

More broadly, there are a number of ways to change a known distribution into an alternate one. The inverse transformation technique, which uses the standard inverse (or ratio) function, is one of the most important of them. More specifically, the inverse distribution of Z is produced by using the transformation $Z = 1/Y$, where Y is a random variable with an existing distribution. The behavior of the density and hazard rate shapes in these distributions shows a variety of features. They make it possible to apply the phenomenon in a number of fields, such as the biological sciences, engineering sciences, survey sampling, and problems pertaining to life testing. Many adaptable inverted models have been studied by several authors, including inverse Lindley distribution [9], inverse exponentiated Weibull distribution [10], inverse power Lindley distribution [11], new hybrid Weibull-inverse Weibull distribution [12], inverted Nadarajah–Haghighi distribution [13], sine inverse exponentiated Weibull distribution [14], length-biased weighted exponentiated inverted exponential distribution [15], inverse Nakagami- m distribution [16], inverse power Lomax distribution [17], inverse power Ailamujia distribution [18], exponentiated inverted Weibull distribution [19], inverse Pareto distribution [20], inverted Topp–Leone distribution [21], exponentiated inverted Topp–Leone distribution [22], inverse power XLindley distribution [23], unit inverse exponentiated Pareto distribution [24], heavy tailed inverse Lindley Type-I distribution [25] and for more recent models [[26]–[30]].

A statistical sampling approach called ranked set sampling (RSS) scheme is used to estimate population parameters more accurately and efficiently than the conventional random sampling (SRS) technique. The RSS technique selects samples according to their relative ranking in a collection of measurements rather than choosing individual samples at random. When population variability is large and conventional random sampling techniques do not adequately represent the underlying variability, this strategy might be very helpful. McIntyre [31] first estimated pasture yield using the RSS approach, which may be regarded as a controlled SRS design. Takahasi and Wakimoto [32] supported it with mathematical theory. Applications of RSS can be found in many fields, such as forestry, agriculture, animal sciences, and medicine. Estimation of different probability distributions has been suggested by several researchers. More recent studies can be found, for instance, Alsadat et al. [33] for the generalized unit half-logistic geometric distribution, Aljohani et al. [34] for the modified kies exponential distribution, Hassan et al. [35] for the inverse power Cauchy distribution, Sabry et al. [36] for the Weibull distribution, Sabry and Almetwally [37] for the exponential Pareto distribution, Hassan and Atia [38] for the inverted Kumaraswamy distribution, Sabry et al. [39] for the Weibull distribution, Metwally et al. [40] for the unit xgamma distribution, Hassan et al. [41] for the Chris-Jerry distribution, Metwally et al. [42] for Ramos–Louzada distribution, among others.

In reliability theory, the estimation issue of the stress-strength reliability (SSR) model based on RSS has caught the attention of many researchers. The aforementioned SSR model, represented by $\eta = P[W < Z]$ describes the life of a system with random strength Z that is exposed to random stress W . When a component receives more stress than its strength, it fails; otherwise, it will continue to work until $W \leq Z$. This model has been used in several technical and life-testing scenarios. The SSR model has been statistically inferred based on RSS and its modification strategies. The determination of SSR for independent exponential distribution, based on RSS, was first discussed by Muttlak [43]. The estimation of SSR for independent Weibull and independent Lindley random samples was discussed [44,45]. The investigation of SSR for an exponentiated Pareto distribution was provided by Al-Omari et al. [46] in the case of median RSS. Akgül et al. [47] covered the estimation of SSR in the case of the Burr Type X distribution. Alsadat et al. [48] provided an efficient estimate for the SSR for the independent unit Gompertz distribution using classical methods. Hassan et al. [49] discussed the SSR for the independent generalized inverse exponential model based on RSS and SRS designs using maximum likelihood (ML) method. In the literature, few studies have considered the Bayesian estimation of the SSR from the probability distribution based on RSS. For example, Basikhasteh et al. [50] examined the SSR of the bathtub-shaped lifetime distribution based on maximum RSS. Using RSS, Esemien et al. [51] dealt with the Bayesian estimation of the SSR drawn from the generalized exponential distribution using Lindley's approximation. Yousef et al. [52] discussed the Bayesian estimation of the SSR from the inverted Topp–Leone distribution using Markov chain Monte Carlo (MCMC) method.

Although several distributions for the analysis of contemporary data are available in the literature, a more flexible distribution that functions effectively in a variety of scenarios is still needed. Furthermore, few studies have addressed the Bayesian estimation problem for SSR modeling. As a result, the first primary goal of this study is to introduce the inverse PHLd (IPHLd) using inverse transformation. Due to its unique characteristics, the proposed two-parameter IPHLd is one of the best methods for handling positively skewed datasets. The second goal is to investigate both the Bayesian and non-Bayesian estimation of the SSR for the IPHLd under RSS and SRS. For the IPHLd, investigations and recommendations are made about simulations and applications. The following succinctly describes the rationale behind the proposed model:

- The IPHLD's wonderful adaptability is seen when matched with the various graphical formats of the probability density function (PDF) and hazard rate function (HRF). The moment's expression, quantile function (QF), incomplete moments (IM), probability weighted moments (PWMs), entropy measures, and SSR parameter are a few of the important statistical features that we generate for the IPHLD.
- The ML estimators (MLEs) of v, ϑ , and η are obtained. The Bayesian estimates (BEs) based on RSS are generated by the use of MCMC methodology. The BEs of v, ϑ, η are determined in informative prior (IF-P) using symmetric (squared error loss function (SQEL)), and asymmetric (linear exponential loss function (LLF)).
- The effectiveness of the several suggested estimators is assessed and compared using Monte Carlo simulations and real-world application scenarios.

The remaining text has been split into the following sections: The formation of the IPHLD and a few of its characteristics are shown in Sections 2 and 3. Section 4 outlines the MLEs of $\eta = P[W < Z]$ when stress and strength random variables following IPHLD using RSS and SRS. Section 5 provides the Bayesian estimators of SSR under SQEL and LLF. In Section 6, a Monte Carlo simulation is used to examine the efficacy of the proposed estimation methods. Four real data applications with eight real datasets were examined to illustrate the importance of the IPHLD in Section 7. We provided some findings in Section 8.

2 Inverted Power Half-Logistic Distribution

Applications in econometrics, biological and engineering sciences, medical research, and life testing are among the many fields where inverted distributions are important. Presenting the IPHLD is the primary goal here.

2.1 Description of the Model

Using the inverse transformation $Z = 1/Y$, where Y has the CDF (2), hence the CDF of IPHLD is given by

$$F(z) = \frac{2}{1 + e^{vz^{-\vartheta}}}, \quad z, v, \vartheta > 0. \quad (3)$$

The PDF of IPHLD associated with CDF (3) is given by:

$$f(z) = \frac{2v\vartheta z^{-\vartheta-1} e^{vz^{-\vartheta}}}{(1 + e^{vz^{-\vartheta}})^2}, \quad z, v, \vartheta > 0. \quad (4)$$

The HRF of the IPHLD is as follows:

$$h(z) = \frac{2v\vartheta z^{-\vartheta-1} e^{vz^{-\vartheta}}}{e^{2vz^{-\vartheta}} - 1}.$$

Note that for $\vartheta = 1$, the inverse HLD is produced as a new model. In Figure 1, the PDF (left panel) and HRF (right panel) are displayed graphically for different values of v and ϑ . As seen from Figure 1, the PDF can take different asymmetric shapes as well as the HRF might be either decreasing, or upside-down, which is required for reliability assessment.

2.2 Extensions of the IPHLD

The IPHLD allows for the creation of several kinds of extensions. A few of them bear a strong connection to distributions that are currently in the literature. Extending the ones in Krishnarani [3], a few more log transformations of the IPHLD are now examined.

Let Z be a random variable following the IPHLD.

1. Suppose that $T = e^Z$, where Z has the IPHLD, then the PDF of T is

$$f(t) = \frac{2v\vartheta (\log t)^{-\vartheta-1} e^{v(\log t)^{-\vartheta}}}{t(1 + e^{v(\log t)^{-\vartheta}})^2}; \quad t > 1.$$

Hence, the random variable T has the log positive inverse power half-logistic distribution. This model can analyze financial ratios, which are frequently right-skewed and usually greater than 1. Additionally, it is suitable for measuring the concentrations of pollutants detected beyond a specific safety or detection threshold, such as in parts per million readings where a value of 1 represents a significant baseline.

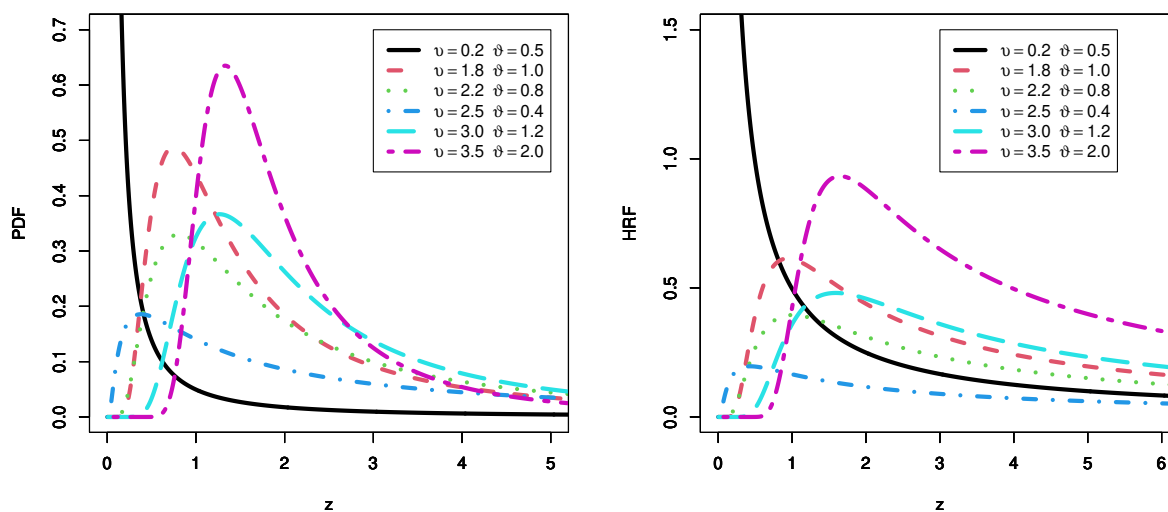


Fig. 1: PDF (left panel) and HRF (right panel) for IPHLD with different shapes.

2. Suppose that $K = e^{-Z}$, then the PDF of K is

$$f(k) = \frac{2v\vartheta(-\log k)^{-\vartheta-1}e^{v(-\log k)^{-\vartheta}}}{k(1+e^{v(-\log k)^{-\vartheta}})^2}; \quad 0 < k < 1.$$

In this case T has the log inverse power half-logistic distribution with domain $(0,1)$. It is a brand-new unit distribution that can be employed in several statistical uses. This model can be used to model market share data, rates, or fractions. For instance, the percentage of a budget allocated to research and development, the percentage of a day that a machine is in operation.

3. Suppose that $T = e^{vZ^{-\vartheta}}$, then the PDF of T is

$$f(t) = \frac{2}{(1+t)^2}; \quad t > 1.$$

4. Suppose that $K = e^{-vZ^{-\vartheta}}$, then the PDF of T is

$$f(k) = \frac{2}{(1+k)^2}; \quad 0 < k < 1.$$

Thus, we have two distributions that share the same structure but are specified at two disjoint intervals: $[0, 1]$ and $[1, \infty)$. This property of a nonnegative random variable with regard to log transformations is what we receive. These take the same shape as the inverse half-logistic distribution that we change using the transformations $T = e^Z$, and $K = e^{-Z}$.

2.3 Quantile Function

The following formula yields the QF of the IPHLD, represented by $Q(u)$: For every $u \in (0, 1)$, $F(Q(u)) = Q(F(u)) = u$. Following a little algebra, an analytical equation is produced, and it is

$$Q(u) = \left[\frac{1}{v} \log \left(\frac{2-u}{u} \right) \right]^{-1/\vartheta}, \quad u \in (0, 1). \quad (5)$$

The median is given by $\text{Med} = Q(0.5)$, first quantile $Q_1 = Q(0.25)$, third quantile $Q_3 = Q(0.75)$, and interquartile range, $\text{IQR} = Q(0.75) - Q(0.25)$.

With U having a uniform distribution $(0, 1)$, the following formula uses the QF (5) to produce random data sets:

$$z_i = \left[\frac{1}{v} \log \left(\frac{2-u_i}{u_i} \right) \right]^{-1/\vartheta}, \quad i = 0, 1, 2, \dots, n. \quad (6)$$

In addition, the result below may be used to generate random variables from IPHLD.

Result: If $X \sim U(0, 1)$, then the random variable $T = \left[\frac{1}{v} \log \frac{a-x}{x} \right]^{-1/\vartheta}$ has truncated IPHLD (TIPHLD).

Proof: If $X \sim U(0, 1)$, then

$$P(T \leq t) = P \left\{ \left[\frac{1}{v} \log \frac{a-x}{x} \right]^{-1/\vartheta} \leq t \right\} = \frac{a}{1 + e^{vt^{-\vartheta}}},$$

which is a new distribution with a density function

$$f(t) = \frac{va\vartheta t^{-\vartheta-1} e^{vt^{-\vartheta}}}{(1 + e^{vt^{-\vartheta}})^2}, \quad t > \left[\frac{1}{v} \log(a-1) \right]^{-1/\vartheta}, \quad v, \vartheta > 0, a \geq 2. \quad (7)$$

The PDF (7) is referred to as the truncated inverse half-logistic distribution. Note that for $\vartheta = 1$ in (7) the truncated inverse half-logistic distribution is presented. The IHLD is produced when $a = 2$, $\vartheta = 1$ and IPHLD is produced when $a = 2$, $\vartheta > 0$.

3 Some Statistical Properties

This section gives some of the statistical properties, including moments, IM, PWMs, entropy measures, and the SSR parameter.

3.1 Moments and Associated Measures

Representations of the moments of the IPHLD are given. The m -th moment of Z is given by

$$\mu'_m = \int_0^\infty z^m \frac{2v\vartheta z^{-\vartheta-1} e^{vz^{-\vartheta}}}{(1 + e^{vz^{-\vartheta}})^2} dz = \int_0^\infty z^m \frac{2v\vartheta z^{-\vartheta-1} e^{-vz^{-\vartheta}}}{(1 + e^{-vz^{-\vartheta}})^2} dz. \quad (8)$$

The generalized binomial expansion is defined by:

$$(1+x)^{-\alpha} = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha+k-1}{k} x^k. \quad (9)$$

Then by employing (9) in (8), gives

$$\mu'_m = \sum_{k=0}^{\infty} (-1)^k 2(k+1) \int_0^\infty z^{m-\vartheta-1} v\vartheta z^{-\vartheta-1} e^{-v(k+1)z^{-\vartheta}} dz$$

Then after some simplification, we get

$$\mu'_m = \sum_{k=0}^{\infty} (-1)^k 2[v(k+1)]^{m/\vartheta} \Gamma\left(1 - \frac{m}{\vartheta}\right), \quad \vartheta > m. \quad (10)$$

From μ'_m , various quantities can be obtained. Specifically, the mean of Z is determined for $m=1$ in (10). Additionally, m -th central moment is provided by

$$\mu_m = E(Z - \mu'_1)^m = \sum_{j=0}^m \binom{m}{j} (-1)^j (\mu'_1)^j \mu'_{m-j}.$$

Based on the previous equation, several measures can be obtained, such as variance and measures of skewness and kurtosis.

The m -th incomplete moment is obtained by using PDF (4) and binomial expansion as below:

$$\varphi_m(t) = \sum_{k=0}^{\infty} (-1)^k 2(k+1) \int_0^t z^{m-\vartheta-1} v \vartheta z^{-\vartheta-1} e^{-v(k+1)z^{-\vartheta}} dz.$$

Hence, the m -th incomplete moment is given by

$$\varphi_m(t) = \sum_{k=0}^{\infty} (-1)^k 2[v(k+1)]^{m/\vartheta} \Gamma\left(1 - \frac{m}{\vartheta}, [v(k+1)t]^{-\vartheta}\right), \quad \vartheta > m, \quad (11)$$

where, $\Gamma(.,x)$ is the upper incomplete gamma function. For $m=1$ in (11), the first incomplete moment is determined. The important application of the $\varphi_1(t)$ is related to the Bonferroni and Lorenz curves, defined by $L(p) = \varphi_1(t)/\mu'_1$, and $B(p) = \varphi_1(t)/p\mu'_1$, where t_p can be evaluated numerically using Equation (5) for a given probability p . A further application of the $\varphi_1(t)$ is to the mean waiting time, represented by $m_1(t) = \frac{1-\varphi_1(t)}{1-F(t)} - t$, and the mean residual life which is provided by $m_1^*(t) = 1 - \frac{\varphi_1(t)}{F(t)}$.

3.2 Entropy Measures

This subsection is devoted to deriving the Renyi and Tsallis entropies of the IPHLD. Entropy has been applied to fields such as engineering, medicine, and physics. For a random variable Z , entropy is a measure of the variation in uncertainty. Using the following formula, the Renyi of order s , where $s > 0$, and $s \neq 1$, for the IPHLD may be found

$$R_Z(s) = \frac{1}{1-s} \left[\log \int_0^{\infty} (f(z))^s dz \right] = \frac{1}{1-s} \left[\log \int_0^{\infty} \frac{(2v\vartheta)^s z^{-s(\vartheta+1)} e^{svz^{-\vartheta}}}{(1+e^{vz^{-\vartheta}})^{2s}} dz \right]. \quad (12)$$

Using the binomial expansion (9) in (12), we have

$$R_Z(s) = \frac{1}{1-s} \log \left[(2v\vartheta)^s \sum_{k=0}^{\infty} (-1)^k \binom{2s+k-1}{k} \int_0^{\infty} z^{-s(\vartheta+1)} e^{-(s+k)vz^{-\vartheta}} dz \right].$$

Then after simplified form, we get

$$R_Z(s) = \frac{1}{1-s} \log \left[(2v)^s \vartheta^{s-1} \sum_{k=0}^{\infty} (-1)^k \binom{2s+k-1}{k} ((s+k)v)^{\frac{1-s(\vartheta+1)}{\vartheta}} \Gamma\left(\frac{s(\vartheta+1)-1}{\vartheta}\right) \right].$$

Additionally, the Tsallis entropy of order s , where $s > 0$, and $s \neq 1$, is defined by:

$$T_Z(s) = \frac{1}{s-1} \left[1 - \int_0^{\infty} (f(z))^s dz \right]. \quad (13)$$

Then by using PDF (4) and binomial expansion (9) in (13), then

$$T_Z(s) = \frac{1}{s-1} \left[1 - (2v)^s \vartheta^{s-1} \sum_{k=0}^{\infty} (-1)^k \binom{2s+k-1}{k} ((s+k)v)^{\frac{1-s(\vartheta+1)}{\vartheta}} \Gamma\left(\frac{s(\vartheta+1)-1}{\vartheta}\right) \right].$$

3.3 Probability Weighted Moments

Generally speaking, the PWMs are regarded as superior to the regular moments. At extreme levels, the PWMs are less sensitive. They are used in rare cases when MLEs are hard to come by. For a given random variable Z , the class of PWM is defined as follows.

$$\omega_{r,h} = \mathbb{E}[Z^r F(Z)^h] = \int_{-\infty}^{\infty} z^r [F(z)]^h f(z) dz, \quad (14)$$

where h , and r are integers. Setting (3) and (4) in (14), then the PWM of the IPHLD is

$$\omega_{r,h} = \int_0^\infty z^r \frac{2^{h+1} v \vartheta z^{-\vartheta-1} e^{vz^{-\vartheta}}}{(1 + e^{vz^{-\vartheta}})^{h+2}} dz. \quad (15)$$

Using the expansion (9) in (15) leads to

$$\begin{aligned} \omega_{r,h} &= \sum_{k=0}^{\infty} (-1)^k \binom{h+1+k}{k} v \vartheta 2^{h+1} \int_0^\infty z^{r-\vartheta-1} e^{-(k+1)vz^{-\vartheta}} dz \\ &= \sum_{k=0}^{\infty} \binom{h+1+k}{k} \frac{(-1)^k 2^{h+1} v^{r/\vartheta}}{(1+k)^{1-r/\vartheta}} \Gamma\left(1 - \frac{r}{\vartheta}\right), \quad \vartheta > r. \end{aligned}$$

3.4 Stress-Strength Reliability

Suppose that W and Z are two independent random variables that have the IPHLD with a common shape parameter and different scale parameters, that is, $Z \sim \text{IPHLD}(v_1, \vartheta)$ and $W \sim \text{IPHLD}(v_2, \vartheta)$, respectively. The associated SSR parameter is then determined by using the formula $\eta = P[W < Z]$ as follows:

$$\eta = \int_0^\infty f(z; v_1, \vartheta) F_W(z, v_2, \vartheta) dz = \int_0^\infty \frac{2v_1 \vartheta z^{-\vartheta-1} e^{v_1 z^{-\vartheta}}}{(1 + e^{v_1 z^{-\vartheta}})^2} \times \frac{2}{(1 + e^{v_2 z^{-\vartheta}})} dz. \quad (16)$$

Using expansion (9) in (16), twice time, we get

$$\begin{aligned} \eta &= \sum_{j_1, j_2=0}^{\infty} (-1)^{j_1+j_2} (j_1+1) 4v_1 \vartheta \int_0^\infty z^{-\vartheta-1} e^{-[v_1(j_1+1)+v_2 j_2]z^{-\vartheta}} dz \\ &= \sum_{j_1, j_2=0}^{\infty} \frac{(-1)^{j_1+j_2} (j_1+1) 4v_1}{v_1(j_1+1) + v_2 j_2}. \end{aligned} \quad (17)$$

Note that the SSR parameter depends on the parameters v_1 , and v_2 . The 3D plots of the SSR parameter is represented in Figure 2 for selected values of parameters.

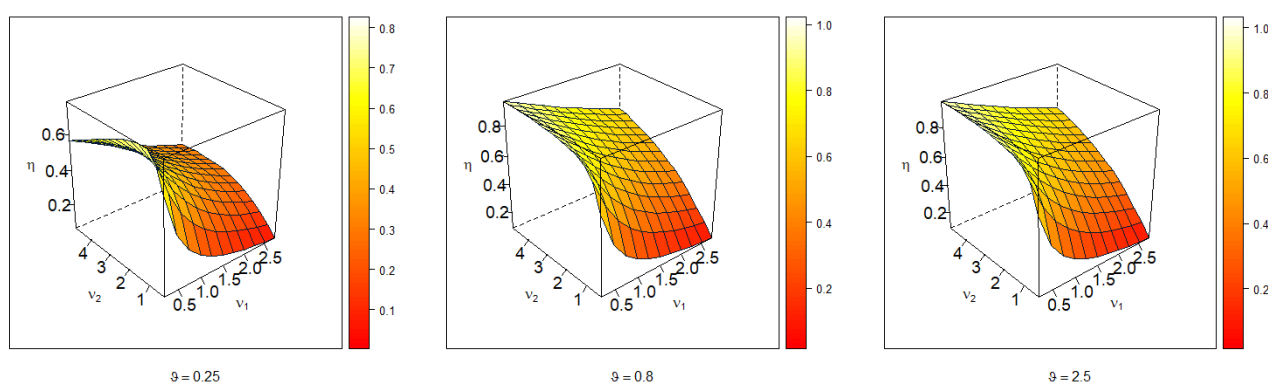


Fig. 2: Stress-Strength reliability with different values of parameters

4 Classical Estimators under Two Sampling Techniques

This section gives a description of the RSS method. Also, the MLEs of η are discussed under RSS and SRS.

4.1 Description of RSS Method

In situations where it is impractical or costly to exactly quantify an observation, the RSS is a powerful method of gathering data. The RSS is more beneficial than the SRS under perfect ranking (see [53]). The protocol for selecting samples based on the RSS is as follows:

- Select h^2 units at random from the specified population of interest.
- Randomly partition the h^2 units into h sets, with h units in each set.
- Regarding the variable of interest, the m units of each set are rated visually or using any low-cost technique.
- To pick a sample of measurements, take the lowest-ranking unit from the first set, the second lowest-ranking unit from the second set, and so on, until the highest-ranking unit is chosen from the final set.
- The whole process can be repeated s times, or as many times as necessary, to generate an RSS sample of size $M^\bullet = h \times s$, where h is the set size and s is the cycle count.
- To clarify the RSS structure, we utilize the following matrix notation:

$$\begin{array}{ccc}
 \text{Initial Observation} & & \text{Ranked Observation} \\
 \left(\begin{array}{cccc} Z_{(1:1)} & Z_{(1:2)} & \dots & Z_{(1:h)} \\ Z_{(2:1)} & Z_{(2:2)} & \dots & Z_{(2:h)} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{(h:1)} & Z_{(h:2)} & \dots & Z_{(h:h)} \end{array} \right) & \Rightarrow & \left(\begin{array}{cccc} \boxed{Z_{11}} & Z_{21} & \dots & Z_{h1} \\ Z_{12} & \boxed{Z_{22}} & \dots & Z_{h2} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{1h} & Z_{2h} & \dots & \boxed{Z_{hh}} \end{array} \right) \Rightarrow \boxed{\text{RSS } (s=1)} \\
 & & \boxed{Z_{11}, Z_{22}, \dots, Z_{hh}}
 \end{array}$$

4.2 MLEs under RSS Method

Suppose that Z_{aq} be the order statistics (OS) of a -th sample ($a = 1, 2, \dots, h_z$) in the q -th cycle ($q = 1, 2, \dots, s_z$) from the strength random variable follows IPHLD (v_1, ϑ). Similarly, let W_{bl} be the OS of b -th sample ($b = 1, 2, \dots, h_w$) in the l -th cycle ($l = 1, 2, \dots, s_w$) from the stress random variable follows the IPHLD (v_2, ϑ). Here h_z and h_w are the set sizes, s_z and s_w are the cycles numbers, the sample size of strength is $M_1^\bullet = h_z \times s_z$ and the sample size of stress is $M_2^\bullet = h_w \times s_w$.

It should be noted that if the ranking of the observations is perfect, then according to PDF of Z_{aq} and W_{bl} , they are exactly the PDF of a -th and b -th ordered statistics, respectively. For instance, the PDF of Z_{aq} is given by:

$$\begin{aligned}
 f_{Z_{aq}}(z_{aq}) &= \frac{h_z!}{(a-1)!(h_z-a)!} f(z_{aq}) [F(z_{aq})]^{a-1} [1-F(z_{aq})]^{h_z-a} \\
 &= \frac{h_z!}{(a-1)!(h_z-a)!} \frac{2v_1 \vartheta z_{aq}^{-\vartheta-1} e^{v_1 z_{aq}^{-\vartheta}}}{(1+e^{v_1 z_{aq}^{-\vartheta}})^2} \left[\frac{2}{1+e^{v_1 z_{aq}^{-\vartheta}}} \right]^{a-1} \left[1 - \frac{2}{1+e^{v_1 z_{aq}^{-\vartheta}}} \right]^{h_z-a}.
 \end{aligned}$$

It should be mentioned that the PDF of W_{bl} is obtained in a similar way. Therefore, we do not reproduce it here for the sake of brevity. The likelihood function, without constant, based on RSS is:

$$\begin{aligned}
 l^{RSS}(v_1, v_2, \vartheta | z) &= \prod_{q=1}^{s_z} \prod_{a=1}^{h_z} \frac{2v_1 \vartheta z_{aq}^{-\vartheta-1} e^{v_1 z_{aq}^{-\vartheta}}}{(1+e^{v_1 z_{aq}^{-\vartheta}})^2} \left[\frac{2}{1+e^{v_1 z_{aq}^{-\vartheta}}} \right]^{a-1} \left[\frac{e^{v_1 z_{aq}^{-\vartheta}} - 1}{1+e^{v_1 z_{aq}^{-\vartheta}}} \right]^{h_z-a} \\
 &\times \prod_{l=1}^{s_w} \prod_{b=1}^{h_w} \frac{2v_2 \vartheta w_{bl}^{-\vartheta-1} e^{v_2 w_{bl}^{-\vartheta}}}{(1+e^{v_2 w_{bl}^{-\vartheta}})^2} \left[\frac{2}{1+e^{v_2 w_{bl}^{-\vartheta}}} \right]^{b-1} \left[\frac{e^{v_2 w_{bl}^{-\vartheta}} - 1}{1+e^{v_2 w_{bl}^{-\vartheta}}} \right]^{h_w-b}.
 \end{aligned} \tag{18}$$

Then the log-likelihood function, say $l^\bullet$ based on RSS is given by

$$\begin{aligned} \log l^{\bullet} = & M_1^{\bullet} \log(v_1 \vartheta) + \sum_{q=1}^{s_z} \sum_{a=1}^{h_z} \left[v_1 z_{aq}^{-\vartheta} - (\vartheta + 1) \log z_{aq} \right. \\ & \left. - (h_z + 1) \log \left(1 + e^{v_1 z_{aq}^{-\vartheta}} \right) - (h_z - a) \log \left(e^{v_1 z_{aq}^{-\vartheta}} - 1 \right) \right] \\ & + M_2^{\bullet} \log(v_2 \vartheta) + \sum_{l=1}^{s_w} \sum_{b=1}^{h_w} \left[v_2 w_{bl}^{-\vartheta} - (\vartheta + 1) \log w_{bl} \right. \\ & \left. - (h_w + 1) \log \left(1 + e^{v_2 w_{bl}^{-\vartheta}} \right) - (h_w - b) \log \left(e^{v_2 w_{bl}^{-\vartheta}} - 1 \right) \right]. \end{aligned} \quad (19)$$

The first partial derivatives of log-likelihood (19) with respect to ϑ, v_1 , and v_2 respectively, are given by

$$\begin{aligned} \frac{\partial \log l^{\bullet}}{\partial \vartheta} = & \frac{M_1^{\bullet}}{\vartheta} - \sum_{q=1}^{s_z} \sum_{a=1}^{h_z} \left[\log z_{aq} + v_1 z_{aq}^{-\vartheta} \log z_{aq} \right. \\ & \left. - \frac{(h_z + 1) v_1 z_{aq}^{-\vartheta} \log z_{aq}}{1 + e^{-v_1 z_{aq}^{-\vartheta}}} - \frac{(h_z - a) v_1 z_{aq}^{-\vartheta} \log z_{aq}}{1 - e^{-v_1 z_{aq}^{-\vartheta}}} \right] \\ & + \frac{M_2^{\bullet}}{\vartheta} - \sum_{l=1}^{s_w} \sum_{b=1}^{h_w} \left[\log w_{bl} + v_2 w_{bl}^{-\vartheta} \log w_{bl} \right. \\ & \left. - \frac{(h_w + 1) v_2 w_{bl}^{-\vartheta} \log w_{bl}}{1 + e^{-v_2 w_{bl}^{-\vartheta}}} - \frac{(h_w - b) v_2 w_{bl}^{-\vartheta} \log w_{bl}}{1 - e^{-v_2 w_{bl}^{-\vartheta}}} \right], \end{aligned} \quad (20)$$

$$\frac{\partial \log l^{\bullet}}{\partial v_1} = \frac{M_1^{\bullet}}{v_1} + \sum_{q=1}^{s_z} \sum_{a=1}^{h_z} \left[z_{aq}^{-\vartheta} - \frac{(h_z + 1) z_{aq}^{-\vartheta}}{1 + e^{-v_1 z_{aq}^{-\vartheta}}} - \frac{(h_z - a) z_{aq}^{-\vartheta}}{1 - e^{-v_1 z_{aq}^{-\vartheta}}} \right], \quad (21)$$

$$\frac{\partial \log l^{\bullet}}{\partial v_2} = \frac{M_2^{\bullet}}{v_2} + \sum_{l=1}^{s_w} \sum_{b=1}^{h_w} \left[w_{bl}^{-\vartheta} - \frac{(h_w + 1) w_{bl}^{-\vartheta}}{1 + e^{-v_2 w_{bl}^{-\vartheta}}} - \frac{(h_w - b) w_{bl}^{-\vartheta}}{1 - e^{-v_2 w_{bl}^{-\vartheta}}} \right]. \quad (22)$$

Since the likelihood equations (20)–(22) cannot have explicit solutions, we must use iterative techniques to determine the MLE $\hat{v}_1^{RSS}$, $\hat{v}_2^{RSS}$, and $\hat{\vartheta}^{RSS}$ of v_1, v_2 , and ϑ . The likelihood equations were solved using the Newton–Raphson algorithm implemented in R (R Core Team). Numerical derivatives and the Hessian were computed when needed using the `numDeriv` package. Then, by incorporating the MLE $\hat{v}_1^{RSS}$, and $\hat{v}_2^{RSS}$, into (17), the MLE $\hat{\eta}^{RSS}$ of η , based on RSS, is obtained as:

$$\hat{\eta}^{RSS} = \sum_{j_1, j_2=0}^{\infty} \frac{(-1)^{j_1+j_2} (j_1+1) 4 \hat{v}_1^{RSS}}{\hat{v}_1^{RSS} (j_1+1) + \hat{v}_2^{RSS} j_2}.$$

4.2 MLEs under SRS Method

Suppose that $Z_1, Z_2, \dots, Z_{M_1^{\bullet}}$, be an SRS of size $M_1^{\bullet}$ taken from IPHLD (v_1, ϑ) , and $W_1, W_2, \dots, W_{M_2^{\bullet}}$ be an SRS of size $M_2^{\bullet}$ taken from IPHLD (v_2, ϑ) , where we assume that both stress and strength samples are independent. To calculate the MLE $\hat{\eta}^{SRS}$ of η , we obtain at the first the MLE $\hat{v}_1^{SRS}$, $\hat{v}_2^{SRS}$ and $\hat{\vartheta}^{SRS}$ of v_1, v_2 , and ϑ . The likelihood function of the observed sample is as follows:

$$l^{SRS}(v_1, v_2, \vartheta | \underline{z}) = \prod_{j=1}^{M_1^{\bullet}} \frac{2 v_1 \vartheta z_j^{-\vartheta-1} e^{v_1 z_j^{-\vartheta}}}{(1 + e^{v_1 z_j^{-\vartheta}})^2} \prod_{k=1}^{M_2^{\bullet}} \frac{2 v_2 \vartheta w_k^{-\vartheta-1} e^{v_2 w_k^{-\vartheta}}}{(1 + e^{v_2 w_k^{-\vartheta}})^2}. \quad (23)$$

The joint log likelihood function of the observed sample is

$$\begin{aligned}\log \ell_1^* &= M_1^* \log(2v_1 \vartheta) + M_2^* \log(2v_2 \vartheta) \\ &+ \sum_{j=1}^{M_1^*} \left[v_1 z_j^{-\vartheta} - (\vartheta + 1) \log z_j - 2 \log(1 + e^{v_1 z_j^{-\vartheta}}) \right] \\ &+ \sum_{k=1}^{M_2^*} \left[v_2 w_k^{-\vartheta} - (\vartheta + 1) \log w_k - 2 \log(1 + e^{v_2 w_k^{-\vartheta}}) \right].\end{aligned}$$

The first partial derivatives of log-likelihood with respect to ϑ , v_1 , and v_2 , respectively, are given by

$$\frac{\partial \log \ell_1^*}{\partial \vartheta} = \frac{M_1^* + M_2^*}{\vartheta} - \sum_{j=1}^{M_1^*} \left[\log z_j + v_1 z_j^{-\vartheta} \log z_j - \frac{2v_1 z_j^{-\vartheta} \log z_j}{(1 + e^{-v_1 z_j^{-\vartheta}})} \right] - \sum_{k=1}^{M_2^*} \left[\log w_k + v_2 w_k^{-\vartheta} \log w_k - \frac{2v_2 w_k^{-\vartheta} \log w_k}{(1 + e^{-v_2 w_k^{-\vartheta}})} \right], \quad (24)$$

$$\frac{\partial \log \ell_1^*}{\partial v_1} = \frac{M_1^*}{v_1} + \sum_{j=1}^{M_1^*} \left[z_j^{-\vartheta} - \frac{2z_j^{-\vartheta}}{1 + e^{-v_1 z_j^{-\vartheta}}} \right], \quad (25)$$

$$\frac{\partial \log \ell_1^*}{\partial v_2} = \frac{M_2^*}{v_2} + \sum_{k=1}^{M_2^*} \left[w_k^{-\vartheta} - \frac{2w_k^{-\vartheta}}{1 + e^{-v_2 w_k^{-\vartheta}}} \right]. \quad (26)$$

Equations (24)–(26) have no closed form solution. Consequently, numerical techniques are used to determine the MLEs $\hat{\vartheta}^{RSS}$, $\hat{v}_1^{RSS}$, and $\hat{v}_2^{RSS}$, of ϑ , v_1 , and v_2 . The likelihood equations were solved using the Newton–Raphson algorithm implemented in R (R Core Team). Numerical derivatives and the Hessian were computed when needed using the numDeriv package. After that, by including the MLE $\hat{v}_1^{RSS}$, and $\hat{v}_2^{RSS}$ into (17), the MLE $\hat{\eta}^{RSS}$ of η , based on RSS, is obtained.

5 Bayesian Estimator of η

Here, we look at the BE of the SSR under SRS and RSS. In order to derive the BEs, prior distributions are necessary. For a particular Bayesian estimation problem, Arnold and Press [54] mentioned that there is no exact procedure for choosing priors.

5.1 Bayesian Estimators under RSS

Here, we investigate this estimation problem by assuming gamma prior distributions. We selected gamma priors due to their flexible nature; they also include non-informative priors as a particular case. Assuming that the parameters v_1 , v_2 , and ϑ have gamma distributions with the following PDF

$$g_r(v_r) = \frac{\beta_r^{\alpha_r}}{\Gamma(\alpha_r)} v_r^{\alpha_r-1} e^{-\beta_r v_r}, \quad \beta_r, \alpha_r, v_r > 0, \quad r = 1, 2,$$

$$g_3(\vartheta) = \frac{\beta_3^{\alpha_3}}{\Gamma(\alpha_3)} \vartheta^{\alpha_3-1} e^{-\beta_3 \vartheta}, \quad \beta_3, \alpha_3, \vartheta > 0.$$

The joint prior of v_1 , v_2 , and ϑ is given by:

$$g^*(v_1, v_2, \vartheta) \propto v_1^{\alpha_1-1} v_2^{\alpha_2-1} \vartheta^{\alpha_3-1} e^{-(\beta_1 v_1 + \beta_2 v_2 + \beta_3 \vartheta)}. \quad (27)$$

Thus, the joint posterior distribution of v_1 , v_2 and ϑ , based on RSS, is produced as follows by employing the likelihood function (18) and the joint prior distribution (27),

$$\Pi(v_1, v_2, \vartheta | data) = \frac{l^{RSS}(v_1, v_2, \vartheta | \underline{z}) g^*(v_1, v_2, \vartheta)}{\int l^{RSS}(v_1, v_2, \vartheta | \underline{z}) g^*(v_1, v_2, \vartheta) d(v_1 v_2 \vartheta)}. \quad (28)$$

We suggest the MCMC approach to determine the BEs of η , as the posterior PDF in (28) cannot be reduced analytically to a closed form.

5.2 Hyper-parameter determination

We choose informative gamma priors for the Bayesian analysis by matching the first two moments of the priors to empirical moments of MLEs obtained from a collection of k auxiliary samples (as in Dey et al. [55]). Concretely, let

$$\bar{\vartheta} = \frac{1}{k} \sum_{j=1}^k \hat{\vartheta}^{(j)}, \quad \bar{v}_1 = \frac{1}{k} \sum_{j=1}^k \hat{v}_1^{(j)}, \quad \bar{v}_2 = \frac{1}{k} \sum_{j=1}^k \hat{v}_2^{(j)},$$

denote the sample means of the k MLEs, and let the corresponding unbiased sample variances be

$$s_{\vartheta}^2 = \frac{1}{k-1} \sum_{j=1}^k \left(\hat{\vartheta}^{(j)} - \bar{\vartheta} \right)^2, \quad s_{v_1}^2 = \frac{1}{k-1} \sum_{j=1}^k \left(\hat{v}_1^{(j)} - \bar{v}_1 \right)^2, \quad s_{v_2}^2 = \frac{1}{k-1} \sum_{j=1}^k \left(\hat{v}_2^{(j)} - \bar{v}_2 \right)^2.$$

Adopting the $\text{Gamma}(\alpha, \beta)$ parameterization (so that $\mathbb{E}[\text{Gamma}(\alpha, \beta)] = \alpha/\beta$ and $\text{Var}[\text{Gamma}(\alpha, \beta)] = \alpha/\beta^2$), we match the prior mean and variance to the empirical moments. Solving the two moment equations for each parameter yields the closed-form estimates

$$\begin{aligned} \alpha_1 &= \frac{\bar{\vartheta}}{s_{\vartheta}^2}, & \beta_1 &= \frac{\bar{\vartheta}^2}{s_{\vartheta}^2}, \\ \alpha_2 &= \frac{\bar{v}_1}{s_{v_1}^2}, & \beta_2 &= \frac{\bar{v}_1^2}{s_{v_1}^2}, \\ \alpha_3 &= \frac{\bar{v}_2}{s_{v_2}^2}, & \beta_3 &= \frac{\bar{v}_2^2}{s_{v_2}^2}. \end{aligned} \quad (29)$$

5.3 The Method of MCMC

We examine the MCMC approach in this subsection in order to determine the BE of η . Equation (28) provides the joint posterior density, which may be expressed as

$$\begin{aligned} \Pi(v_1, v_2, \vartheta | \text{data}) &\propto v_1^{\alpha_1 + M_1^* - 1} v_2^{\alpha_2 + M_2^* - 1} \vartheta^{\alpha_3 + M_1^* + M_2^* - 1} e^{-(\beta_1 v_1 + \beta_2 v_2 + \beta_3 \vartheta)} \\ &\times e^{\sum_{q=1}^{s_z} \sum_{a=1}^{h_z} [v_1 z_{aq}^{-\vartheta} - (\vartheta + 1) \log z_{aq}] - \sum_{q=1}^{s_z} \sum_{a=1}^{h_z} (1 + h_z) \log(1 + e^{v_1 z_{aq}^{-\vartheta}}) + \sum_{q=1}^{s_z} \sum_{a=1}^{h_z} (a - h_z) \log(e^{v_1 z_{aq}^{-\vartheta}} - 1)} \\ &\times e^{\sum_{l=1}^{s_w} \sum_{b=1}^{h_w} [v_2 w_{bl}^{-\vartheta} - (\vartheta + 1) \log w_{bl}] - \sum_{l=1}^{s_w} \sum_{b=1}^{h_w} (1 + h_w) \log(1 + e^{v_2 w_{bl}^{-\vartheta}}) + \sum_{l=1}^{s_w} \sum_{b=1}^{h_w} (b - h_w) \log(e^{v_2 w_{bl}^{-\vartheta}} - 1)}. \end{aligned} \quad (30)$$

The conditional posterior density of v_1 given ϑ and data may be obtained from (30) as follows:

$$\begin{aligned} \Pi_1(v_1 | \vartheta, \text{data}) &\propto v_1^{\alpha_1 + M_1^* - 1} \exp \left\{ -v_1 \left[\beta_1 + \sum_{q=1}^{s_z} \sum_{a=1}^{h_z} z_{aq}^{-\vartheta} \right] - \sum_{q=1}^{s_z} \sum_{a=1}^{h_z} (1 + h_z) \log(1 + e^{v_1 z_{aq}^{-\vartheta}}) \right. \\ &\quad \left. + \sum_{q=1}^{s_z} \sum_{a=1}^{h_z} (a - h_z) \log(e^{v_1 z_{aq}^{-\vartheta}} - 1) \right\}. \end{aligned} \quad (31)$$

The conditional posterior density of v_2 given ϑ and data is:

$$\begin{aligned} \Pi_2(v_2 | \vartheta, \text{data}) &\propto v_2^{\alpha_2 + M_2^* - 1} \exp \left\{ -v_2 \left(\beta_2 + \sum_{l=1}^{s_w} \sum_{b=1}^{h_w} z_{bl}^{-\vartheta} \right) \right. \\ &\quad \left. - \sum_{l=1}^{s_w} \sum_{b=1}^{h_w} (1 + h_w) \log(1 + e^{v_2 w_{bl}^{-\vartheta}}) + \sum_{l=1}^{s_w} \sum_{b=1}^{h_w} (b - h_w) \log(e^{v_2 w_{bl}^{-\vartheta}} - 1) \right\}. \end{aligned} \quad (32)$$

The conditional posterior density of ϑ given v_1, v_2 and data is:

$$\begin{aligned} \Pi_3(\vartheta | v_1, v_2, \text{data}) &\propto \vartheta^{\alpha_3 + M_1^* + M_2^* - 1} \exp \left\{ \sum_{q=1}^{s_z} \sum_{a=1}^{h_z} \left[v_1 z_{aq}^{-\vartheta} - \beta_3 \vartheta - (\vartheta + 1) \log z_{aq} \right] - \sum_{q=1}^{s_z} \sum_{a=1}^{h_z} (1 + h_z) \log(1 + e^{v_1 z_{aq}^{-\vartheta}}) \right. \\ &\quad \left. + \sum_{q=1}^{s_z} \sum_{a=1}^{h_z} (a - h_z) \log(e^{v_1 z_{aq}^{-\vartheta}} - 1) \right\} \\ &\times \exp \left\{ \sum_{l=1}^{s_w} \sum_{b=1}^{h_w} \left[v_2 w_{bl}^{-\vartheta} - (\vartheta + 1) \log w_{bl} \right] - \sum_{l=1}^{s_w} \sum_{b=1}^{h_w} (1 + h_w) \log(1 + e^{v_2 w_{bl}^{-\vartheta}}) + \sum_{l=1}^{s_w} \sum_{b=1}^{h_w} (b - h_w) \log(e^{v_2 w_{bl}^{-\vartheta}} - 1) \right\} \end{aligned} \quad (33)$$

To create samples from conditional posterior distributions, we employ the Metropolis-Hastings (M-H) method within the Gibbs sampling process. Let $v_1^{(t)}, v_2^{(t)}$, and $\vartheta^{(t)}$ for $t = 1, 2, \dots, N$ be the observations produced by Equations (31), (32), and (33), respectively, after starting values $v_1^{(0)}, v_2^{(0)}$, and $\vartheta^{(0)}$ are established. Once the first M iterations are taken as the burn-in time, the Bayes estimators of η under SQEL and LLF are provided by:

$$\hat{\eta}_{SQEL} = \frac{1}{N-M} \sum_{j=M+1}^N \hat{\eta}^{(j)}, \quad \hat{\eta}_{LLF} = \frac{-1}{\lambda} \log \left[\frac{1}{N-M} \sum_{j=M+1}^N e^{-\lambda \hat{\eta}^{(j)}} \right],$$

where,

$$\hat{\eta}^{(t)} = \sum_{j_1, j_2=0}^{\infty} \frac{(-1)^{j_1+j_2} (j_1+1) 4 \hat{v}_1^{(t)}}{\hat{v}_1^{(t)} (j_1+1) + \hat{v}_2^{(t)} j_2},$$

5.4 Bayesian Estimators under SRS

In this sub-section, the BEs of η are obtained using SRS. Suppose that $Z_1, Z_2, \dots, Z_{M_1^*}$, be SRS of size M_1^* , where $j=1, 2, \dots, M_1^*$ drawn from IPHLD (v_1, ϑ) . Also, let $W_1, W_2, \dots, W_{M_2^*}$, be SRS of size M_2^* , where $k=1, 2, \dots, M_2^*$ drawn from IPHLD (v_2, ϑ) . Then, the joint posterior distribution of v_1, v_2 and ϑ , based on SRS, is produced by utilizing the likelihood function (23) and the joint prior distribution (27),

$$\begin{aligned} \Pi^*(v_1, v_2, \vartheta | \text{data}) &\propto v_1^{\alpha_1 + M_1^* - 1} v_2^{\alpha_2 + M_2^* - 1} \vartheta^{\alpha_3 + M_1^* + M_2^* - 1} e^{-(\beta_1 v_1 + \beta_2 v_2 + \beta_3 \vartheta)} \\ &\times e^{-\sum_{j=1}^{M_1^*} \left[(\vartheta + 1) \log z_j - v_1 z_j^{-\vartheta} + 2 \log(1 + e^{v_1 z_j^{-\vartheta}}) \right]} \\ &\times e^{-\sum_{k=1}^{M_2^*} \left[(\vartheta + 1) \log w_k - v_2 w_k^{-\vartheta} + 2 \log(1 + e^{v_2 w_k^{-\vartheta}}) \right]}. \end{aligned} \quad (34)$$

The conditional posterior density of v_1 given ϑ and data is as follows:

$$\Pi_1^*(v_1 | \vartheta, \text{data}) \propto v_1^{\alpha_1 + M_1^* - 1} \exp \left\{ -\beta_1 v_1 + \sum_{j=1}^{M_1^*} \left[v_1 z_j^{-\vartheta} - 2 \ln(1 + e^{v_1 z_j^{-\vartheta}}) \right] \right\}$$

The conditional posterior density of v_2 given ϑ and data is as follows:

$$\Pi_2^*(v_2 | \vartheta, \text{data}) \propto v_2^{\alpha_2 + M_2^* - 1} \exp \left\{ -\beta_2 v_2 + \sum_{k=1}^{M_2^*} \left[v_2 w_k^{-\vartheta} - 2 \ln(1 + e^{v_2 w_k^{-\vartheta}}) \right] \right\}$$

The conditional posterior density of ϑ given v_1, v_2 , and data is as follows:

$$\begin{aligned} \Pi_3^*(\vartheta | v_1, v_2, \text{data}) &\propto \vartheta^{\alpha_3 + M_1^* + M_2^* - 1} e^{-\beta_3 \vartheta} \exp \left\{ -\sum_{j=1}^{M_1^*} \left[(\vartheta + 1) \log z_j - v_1 z_j^{-\vartheta} + 2 \log(1 + e^{v_1 z_j^{-\vartheta}}) \right] \right\} \\ &\times \exp \left\{ -\sum_{k=1}^{M_2^*} \left[(\vartheta + 1) \log w_k - v_2 w_k^{-\vartheta} + 2 \log(1 + e^{v_2 w_k^{-\vartheta}}) \right] \right\}. \end{aligned}$$

We use the M-H approach in the Gibbs sampling procedure to generate samples from conditional posterior distributions (as discussed in the previous subsection). Also, the BEs of η under SQEL and LLF are obtained as similar to the procedure defined above.

5.5 The MCMC implementation procedure.

Although the posterior distributions of model parameters do not admit closed forms, Bayesian inference can be carried out using MCMC method. Below we provide a detailed, reproducible procedure used in this paper.

1. Posterior (up to normalizing constant). Derive the joint log-posterior (log likelihood + log prior) for the parameter vector θ . If conditional distributions are not of standard form, sampling requires M-H steps.

2. Sampling algorithm (M-H within Gibbs).

1. Initialize M chains with different starting values $\theta^{(0,m)}$ for $m = 1, \dots, M$ (we used $M = 3$).

2. For iteration $t = 1, \dots, T$ and for each chain m :

(a) For each parameter component θ_j :

– If the full conditional of θ_j is standard, draw $\theta_j^{(t)}$ from it (Gibbs step).

– Otherwise, propose $\theta_j^* \sim q_j(\cdot | \theta_j^{(t-1)})$ (we used a normal random-walk on an appropriate transformed scale, e.g., log scale for positive parameters) and accept/reject with M-H acceptance probability

$$\alpha = \min \left\{ 1, \frac{\pi(\theta_j^* | \text{rest}) q_j(\theta_j^{(t-1)} | \theta_j^*)}{\pi(\theta_j^{(t-1)} | \text{rest}) q_j(\theta_j^* | \theta_j^{(t-1)})} \right\},$$

where $\pi(\cdot | \text{rest})$ denotes the conditional posterior density (up to normalizing constant).

3. Practical tuning and run settings. We recommend (and used) the following practical settings: number of iterations $T = 12,000$, burn-in $B = 2,000$, no thinning by default (unless autocorrelation is very large). Proposal variances were tuned in pilot runs to achieve acceptance rates in the range 20%–40% for random-walk MH steps.

5. Estimation of derived quantity $\eta = P(W < Z)$. For the stress–strength reliability η that is a deterministic function of parameters $\eta = \eta(\theta)$, compute $\eta^{(t)} = \eta(\theta^{(t)})$ at each posterior draw $\theta^{(t)}$. Then obtain:

– Bayes estimator for η under squared-error loss: posterior mean $\hat{\eta}_{\text{Bayes}} = \frac{1}{N} \sum_t \eta^{(t)}$.

– Posterior median for η (if using absolute loss).

6 Simulation Study

This section presents an extensive simulation study to compare the performance of reliability estimates derived from RSS and their counterparts from SRS. The comparison employs Bias, mean squared error (MSE), and relative efficiency as evaluation criteria. In the simulation setup, set sizes (h) and the number of cycles (s) are chosen as $(h_z, h_w) = (2, 2), (3, 3), (4, 4)$, and $(6, 6)$, with $s = 3$ and 6 , respectively. Consequently, sample sizes for RSS samples are calculated as $M_1^* = h_z \times s_z$ and $M_2^* = h_w \times s_w$. Additionally, sample sizes for SRS samples are determined as $(M_1^*, M_2^*) = (6, 6), (9, 9), (12, 12), (18, 18), (12, 12), (18, 18), (24, 24)$, and $(36, 36)$.

The set size k plays a central role in ranked set sampling. In practice, larger values of k make visual or expert ranking more difficult and therefore increase the probability of ranking errors; Wolfe [53] discusses this empirical limitation and recommends keeping set sizes modest (typically $k \leq 5$) to avoid excessive ranking error. For this reason (and following classical studies such as Dell and Clutter [56], our simulations and theoretical comparisons focus on small set sizes and assume perfect ranking. This assumption allows us to isolate the statistical gains attributable to the RSS design itself.

The parameter values are chosen as $(v_1, v_2, \vartheta) = (1.2, 0.6, 1.5), (3, 1.2, 2), (3, 1.8, 2)$, where the true values of system reliability η are 0.693, 0.745, and 0.646. We generate 5000 random samples from IPHLD (v_1, ϑ) and IPHLD (v_2, ϑ) distributions. The estimated reliability is evaluated using criteria such as average Bias. MSE. The efficiency measures of RSS concerning SRS are defined as:

$$E_1 = \frac{MSE(\hat{\eta}^{RSS})_{MLE}}{MSE(\hat{\eta}^{RSS})_{MLE}}, E_2 = \frac{MSE(\hat{\eta}^{RSS})_{SQLF}}{MSE(\hat{\eta}^{RSS})_{SQLF}}, E_3 = \frac{MSE(\hat{\eta}^{RSS})_{LLF-I}}{MSE(\hat{\eta}^{RSS})_{LLF-I}}, E_4 = \frac{MSE(\hat{\eta}^{RSS})_{LLF-II}}{MSE(\hat{\eta}^{RSS})_{LLF-II}}.$$

For Bayesian estimation, the SQEL and LLF with two weights as -1.25 and 1.25 , which can be denoted as LLF-I, and LLF-II, respectively. The outcomes of the simulation efficiency (E_1, E_2, E_3, E_4) are detailed in Table 1. The outcomes of the simulation investigation (Bias, and MSE) are detailed in Tables 2–4. Based on Tables 1 through 5, we can draw the following conclusions:

1. Tables 1 through 4 indicate that the parameter and reliability estimates using the RSS scheme demonstrate greater efficiency compared to those obtained through SRS in the majority of cases.
2. Tables 2 to 4 suggest that the efficiency of parameter and reliability estimates utilizing Bayesian estimation methods surpasses those obtained through MLEs in most instances.
3. When it comes to Bayesian estimation, utilizing the LLF with positive weights outperforms Bayesian estimation relying on SQEL and LLF-I with negative weights.
4. As the set sizes (h) and the number of cycles (s) increase, Bias and MSE decrease accordingly.
5. In general, there is compelling evidence suggesting that the proposed methods utilizing RSS exhibit superior performance for point estimation of parameters and η .

Tables 1 to 5 collectively summarize the estimation, simulation and goodness-of-fit results. Table 1 reports stable MLEs with reasonably small standard errors indicating that the IPHLD captures key features of the data. Table 2 shows that BEs tend to yield smaller MSEs in small-sample settings while MLE performance becomes comparable as sample size increases. Table 3 indicates interval coverage is close to nominal and interval lengths decrease with larger samples. Table 4 demonstrates that RSS-based estimators improve efficiency over SRS, notably at moderate sample sizes; and Table 5 presents dataset-specific MLEs and Kolmogorov-Smirnov (KS) statistics, where the preferred model for each dataset is identified by the smallest KS value and is consistent with the empirical parameter ranges.

Table 2: Bias, MSE for ML and Bayesian estimation $v_1 = 1.2, v_2 = 0.6, \vartheta = 1.5$

				MLE		SQLF		LLF-I		LLF-II	
s		h		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
3	SRS	2	ϑ	0.1324	0.152	0.0708	0.106	0.1255	0.1304	0.0167	0.0887
			v_1	0.0786	0.222	0.055	0.1468	0.1335	0.1879	-0.0125	0.1168
			v_2	0.0551	0.0728	0.0708	0.0695	0.1106	0.0894	0.0335	0.0544
			η	-0.0025	0.0123	-0.0191	0.0115	-0.0192	0.0119	-0.0196	0.0111
		3	ϑ	0.0722	0.0681	0.042	0.0507	0.0675	0.0567	0.0166	0.0461
			v_1	0.0582	0.1217	0.0468	0.0789	0.0801	0.0884	0.0136	0.0717
			v_2	0.0145	0.0339	0.0334	0.0324	0.0512	0.0365	0.0161	0.0291
			η	0.0008	0.007	-0.0069	0.0064	-0.0072	0.0065	-0.0068	0.0063
		4	ϑ	0.0406	0.0487	0.0113	0.0185	0.0192	0.0189	0.0034	0.0183
			v_1	0.056	0.0836	0.0107	0.0244	0.0203	0.0252	0.001	0.0239
			v_2	0.0123	0.0257	0.0185	0.015	0.0257	0.0159	0.0113	0.0142
			η	-0.0007	0.0053	-0.006	0.0035	-0.0068	0.0035	-0.0051	0.0035
		6	ϑ	0.0317	0.0336	0.0028	0.0126	0.0082	0.0128	-0.0026	0.0126
			v_1	0.0115	0.0522	0.0008	0.0158	0.0078	0.0161	-0.0061	0.0156
			v_2	0.0116	0.0194	0.0109	0.0098	0.0155	0.0101	0.0064	0.0095
			η	-0.0006	0.0042	-0.0046	0.0022	-0.0049	0.0022	-0.0042	0.0022
	SRS	2	ϑ	0.1028	0.1162	0.042	0.0692	0.0751	0.0794	0.1028	0.0619
			v_1	0.0553	0.1022	0.0588	0.0802	0.0952	0.0939	0.0553	0.0696
			v_2	0.0101	0.037	0.0353	0.0313	0.0532	0.0352	0.0101	0.0283
			η	0.0035	0.0075	-0.0056	0.0066	-0.0057	0.0067	0.0035	0.0065
		3	ϑ	0.0353	0.037	0.0226	0.0285	0.0136	0.0303	0.0353	0.0271
			v_1	0.0308	0.0402	0.0256	0.0295	0.0393	0.0314	0.0308	0.028
			v_2	0.0094	0.013	0.0145	0.0118	0.0216	0.0124	0.0094	0.0112
			η	0.0007	0.0028	-0.0019	0.0027	-0.002	0.0027	0.0007	0.0027
		4	ϑ	0.0091	0.0218	-0.0092	0.0096	0.0023	0.0097	0.0081	0.0095
			v_1	0.0037	0.0203	0.0113	0.0093	0.0151	0.0096	0.0294	0.0091
			v_2	0.0081	0.0068	0.0058	0.0049	0.0084	0.005	0.0081	0.0048
			η	0.0006	0.0015	-0.0003	0.0011	-0.0006	0.0011	0.0006	0.0011
		6	ϑ	-0.0082	0.0125	-0.0082	0.0064	-0.0021	0.0063	-0.0072	0.0061
			v_1	0.0028	0.0138	0.009	0.0064	0.0115	0.0065	0.0283	0.0063

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Table 2 – Continued from previous page

				MLE		SQLF		LLF-I		LLF-II	
6	SRS	2	v_2	0.0072	0.0047	0.0049	0.0032	0.0075	0.0033	0.0072	0.0031
			η	-0.0005	0.0011	-0.0002	0.0008	-0.0005	0.0008	-0.0005	0.0008
			ϑ	0.0455	0.0533	0.0325	0.0519	0.0667	0.0593	-0.0018	0.0474
			v_1	0.0522	0.1053	0.0547	0.0887	0.108	0.1109	0.003	0.0745
			v_2	0.0316	0.0344	0.0493	0.0313	0.0732	0.0374	0.027	0.0268
			η	-0.0062	0.0058	-0.0117	0.0057	-0.0107	0.0058	-0.0132	0.0056
		3	ϑ	0.0117	0.0294	0.0087	0.0293	0.0252	0.0314	-0.0013	0.0278
			v_1	0.0239	0.0459	0.0215	0.0413	0.0454	0.046	-0.0021	0.0381
			v_2	0.0097	0.0153	0.0165	0.0149	0.0278	0.0161	0.0057	0.014
			η	-0.0012	0.0034	-0.0042	0.0033	-0.004	0.0034	-0.0046	0.0033
		4	ϑ	0.0041	0.0214	-0.0015	0.014	0.0049	0.0142	-0.0011	0.0138
			v_1	0.0229	0.0321	0.0108	0.0182	0.019	0.0188	0.0021	0.0178
			v_2	0.0081	0.0107	0.0106	0.0078	0.0155	0.0082	0.0051	0.0076
			η	-0.0007	0.0023	-0.0028	0.0018	-0.0031	0.0018	-0.0026	0.0018
		6	ϑ	-0.0032	0.0165	-0.0015	0.0106	-0.0038	0.0105	-0.0009	0.0102
			v_1	0.0227	0.0245	0.0097	0.0139	0.0155	0.0143	0.0015	0.0137
			v_2	0.0074	0.01	0.0103	0.0076	0.0131	0.008	0.0042	0.0073
			η	-0.0006	0.0022	-0.0027	0.0016	-0.003	0.0016	-0.0019	0.0016
	SRS	2	ϑ	0.0192	0.0329	0.0059	0.0302	0.0257	0.032	0.0159	0.0292
			v_1	0.0493	0.0485	0.0476	0.0417	0.072	0.0474	0.0493	0.0376
			v_2	0.0247	0.0162	0.0337	0.0156	0.0457	0.0175	0.0247	0.014
			η	-0.0023	0.0036	-0.0059	0.0034	-0.0058	0.0035	-0.0023	0.0034
		3	ϑ	-0.0178	0.0146	-0.0051	0.0141	-0.0069	0.0148	-0.0148	0.0135
			v_1	0.041	0.0193	0.0358	0.0189	0.0458	0.0203	0.041	0.0177
			v_2	0.023	0.0075	0.025	0.0074	0.0296	0.0079	0.023	0.0071
			η	-0.0017	0.0015	-0.0036	0.0015	-0.0034	0.0015	-0.0017	0.0015
		4	ϑ	-0.0121	0.0095	-0.005	0.0067	-0.0052	0.0066	-0.0108	0.0061
			v_1	0.0318	0.0096	0.0211	0.0069	0.0242	0.0072	0.0318	0.0067
			v_2	0.0183	0.0042	0.0173	0.0034	0.0191	0.0035	0.0183	0.0033
			η	-0.0012	0.0008	-0.0029	0.0007	-0.003	0.0007	-0.0012	0.0007
		6	ϑ	-0.0082	0.0072	-0.0016	0.0047	-0.0041	0.0046	-0.0102	0.0046
			v_1	0.0302	0.006	0.0122	0.0044	0.0236	0.0046	0.0302	0.0043
			v_2	0.0171	0.0028	0.0132	0.0019	0.0144	0.002	0.0171	0.0019
			η	-0.0006	0.0006	-0.0012	0.0005	-0.0013	0.0005	-0.0006	0.0005

Table 3: Bias, MSE for ML and Bayesian estimation $v_1 = 3, v_2 = 1.2, \vartheta = 2$

				MLE		SQLF		LLF-I		LLF-II	
s		h		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
3	SRS	2	ϑ	0.1705	0.2475	0.087	0.1588	0.1711	0.2054	0.0036	0.1304
			v_1	0.5533	2.4573	0.1174	0.4216	0.2506	0.5386	-0.0191	0.3458
			v_2	0.1047	0.2258	0.0875	0.1524	0.1612	0.1981	0.0166	0.1209
			η	-0.0051	0.0111	-0.0109	0.0064	-0.0138	0.0066	-0.0086	0.0064
		3	ϑ	0.099	0.1321	0.0394	0.08	0.0753	0.0891	0.0034	0.0739
			v_1	0.3017	1.016	0.0671	0.1674	0.1292	0.1966	0.0054	0.1489
			v_2	0.0842	0.1198	0.0707	0.084	0.1057	0.0959	0.0359	0.0744
			η	-0.005	0.0071	-0.0089	0.0035	-0.0106	0.0036	-0.0073	0.0034
		4	ϑ	0.0541	0.0784	0.0032	0.0219	0.0128	0.0222	-0.0063	0.0218
			v_1	0.2139	0.6101	0.0332	0.037	0.0466	0.0387	0.0198	0.0357
			v_2	0.0543	0.0626	0.0156	0.024	0.0257	0.025	0.0054	0.0233
			η	-0.0021	0.0041	-0.0005	0.001	-0.0013	0.001	0.0004	0.001
		6	ϑ	0.0108	0.0591	-0.0025	0.0166	0.004	0.0166	-0.009	0.0163
			v_1	0.1195	0.3703	0.0159	0.0212	0.0237	0.0218	0.0081	0.0208
			v_2	0.0461	0.0584	0.0093	0.0163	0.0157	0.0166	0.0029	0.0161
			η	-0.0018	0.0038	-0.0006	0.0007	-0.0011	0.0007	0	0.0007

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				MLE		SQLF		LLF-I		LLF-II	
	SRS	2	ϑ	0.1478	0.1984	0.0163	0.0537	0.0398	0.0569	0.1478	0.0519
			v_1	0.3971	1.4486	0.019	0.0851	0.0476	0.0881	0.3971	0.0845
			v_2	0.0528	0.1097	0.0191	0.0449	0.0407	0.0483	0.0533	0.0427
			η	0.0036	0.007	-0.0038	0.002	-0.0044	0.002	0.0036	0.002
		3	ϑ	0.0407	0.0816	0.0033	0.0245	0.013	0.0251	0.0407	0.0241
			v_1	0.1807	0.4009	0.0188	0.0307	0.0307	0.0318	0.1807	0.0301
			v_2	0.0513	0.0448	0.0173	0.0192	0.0353	0.0201	0.0534	0.0185
			η	-0.0021	0.0031	-0.0037	0.0008	-0.0043	0.0008	-0.0021	0.0008
		4	ϑ	-0.0211	0.0355	-0.0022	0.0053	-0.0034	0.0053	-0.0211	0.0054
			v_1	0.0779	0.1662	0.0059	0.0054	0.0079	0.0054	0.0779	0.0053
			v_2	0.0279	0.0175	0.0029	0.0045	0.0047	0.0045	0.0279	0.0045
			η	-0.0019	0.0013	-0.0032	0.0002	-0.0025	0.0002	-0.0019	0.0002
		6	ϑ	-0.0179	0.0222	-0.0007	0.0032	0.0005	0.0032	-0.0179	0.0032
			v_1	0.0685	0.108	0.0021	0.0035	0.0033	0.003	0.0768	0.003
			v_2	0.0235	0.0113	-0.0023	0.0027	-0.0023	0.0027	0.0235	0.0027
			η	0.0012	0.0009	0.0008	0.0001	0.0007	0.0001	0.0002	0.0001
6	SRS	2	ϑ	0.08	0.103	0.0534	0.0893	0.1085	0.1066	-0.0029	0.08
			v_1	0.2444	0.8404	0.0763	0.2673	0.1907	0.3288	-0.0378	0.2404
			v_2	0.0537	0.0727	0.0636	0.0703	0.1147	0.0881	0.0148	0.0593
			η	-0.0018	0.0048	-0.0093	0.0037	-0.0099	0.0037	-0.0091	0.0036
		3	ϑ	0.0787	0.0489	-0.0084	0.0433	0.0206	0.0454	-0.0028	0.0426
			v_1	0.1014	0.3088	0.0254	0.1162	0.0736	0.1263	-0.0231	0.1118
			v_2	0.0385	0.0481	0.0393	0.0436	0.0641	0.0491	0.0135	0.0395
			η	-0.0017	0.0031	-0.0065	0.002	-0.0073	0.0021	-0.0057	0.002
		4	ϑ	-0.07	0.0359	-0.0032	0.0208	-0.011	0.0204	-0.0027	0.0201
			v_1	0.0243	0.1732	-0.0026	0.0303	0.0103	0.0303	-0.0114	0.0301
			v_2	0.0377	0.032	0.0158	0.017	0.0238	0.0179	0.0077	0.0163
			η	-0.0016	0.0021	-0.0031	0.0008	-0.0037	0.0008	-0.0025	0.0008
		6	ϑ	-0.0134	0.0284	-0.0021	0.0141	-0.006	0.0141	-0.0022	0.0138
			v_1	0.0235	0.1442	0.0024	0.0192	0.0091	0.0194	-0.0032	0.0192
			v_2	0.0355	0.0227	0.0146	0.0113	0.0237	0.0117	0.0051	0.011
			η	-0.0011	0.0015	-0.0028	0.0005	-0.0035	0.0005	-0.0021	0.0005
	SRS	2	ϑ	0.0328	0.0651	-0.0043	0.0391	0.0147	0.0401	0.0328	0.0389
			v_1	0.1651	0.4058	0.0041	0.0683	0.0313	0.0699	0.1651	0.0671
			v_2	0.0418	0.0457	0.0261	0.0328	0.0429	0.0355	0.0418	0.0308
			η	-0.0015	0.0034	-0.005	0.0015	-0.0059	0.0015	-0.0015	0.0015
		3	ϑ	-0.0143	0.0301	-0.0041	0.0176	-0.0144	0.0173	-0.0143	0.0176
			v_1	0.0637	0.1266	0.0041	0.0268	0.0277	0.0276	0.0637	0.0263
			v_2	0.0174	0.0156	0.0121	0.0131	0.0182	0.0135	0.0174	0.0127
			η	-0.0006	0.0012	-0.0011	0.0006	-0.0014	0.0006	-0.0006	0.0005
		4	ϑ	-0.0137	0.019	-0.0039	0.0048	-0.0104	0.0048	-0.0137	0.0047
			v_1	0.0376	0.0741	0.004	0.0053	0.0119	0.0054	0.0376	0.0053
			v_2	0.0163	0.0086	0.0088	0.0037	0.0103	0.0037	0.0153	0.0036
			η	-0.0005	0.0007	-0.0009	0.0002	-0.001	0.0002	-0.0005	0.0001
		6	ϑ	-0.0124	0.0126	-0.0035	0.003	-0.0091	0.003	-0.0124	0.0029
			v_1	0.0289	0.0468	0.001	0.0031	0.0023	0.0031	0.0289	0.0028
			v_2	0.0122	0.0062	0.0078	0.0026	0.0088	0.0026	0.0122	0.0025
			η	-0.0004	0.0005	-0.0008	0.0001	-0.0008	0.00009	-0.0003	0.0001

Table 4: Bias, MSE, LACI, CP, and LCCI for MLE and Bayesian estimation: $v_1 = 3$, $v_2 = 1.8$, $\vartheta = 2$

				MLE		SQLF		LLF-I		LLF-II	
s		h		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
		2	ϑ	0.1959	0.3288	0.0922	0.1783	0.1767	0.2239	0.0094	0.1502
			v_1	0.5115	2.1954	0.0883	0.4022	0.2262	0.4991	-0.0495	0.3503

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				MLE		SQLF		LLF-I		LLF-II		
6	SRS	3	v_2	0.2257	0.6355	0.1023	0.25	0.2076	0.3247	-0.044	0.2017	
			η	-0.0017	0.0138	-0.0288	0.0072	-0.0113	0.0073	-0.0064	0.0072	
			ϑ	0.082	0.1288	0.03	0.0835	0.0677	0.0934	-0.0075	0.0772	
			v_1	0.2979	1.0182	0.0776	0.186	0.1375	0.2146	0.0172	0.1657	
			v_2	0.1229	0.2055	0.0678	0.0969	0.1114	0.1117	0.0247	0.0876	
		η	-0.0017	0.008	-0.0204	0.0033	-0.0054	0.0033	-0.0034	0.0033		
		4	ϑ	0.0338	0.0738	-0.0042	0.0254	0.0602	0.0256	-0.007	0.0255	
			v_1	0.1913	0.5848	0.0157	0.035	0.0273	0.0366	0.004	0.0338	
			v_2	0.0955	0.1301	0.0199	0.0254	0.0303	0.0264	0.0095	0.0247	
			η	-0.0013	0.0051	-0.0163	0.0008	-0.0021	0.0008	-0.0011	0.0008	
			ϑ	0.0181	0.0585	-0.0037	0.0169	-0.0102	0.0165	-0.0065	0.0157	
		6	v_1	0.0851	0.3629	0.0067	0.0234	0.0087	0.0234	-0.0037	0.0214	
			v_2	0.0768	0.1069	0.0103	0.0198	0.0181	0.0205	0.0024	0.0193	
			η	-0.0011	0.0048	-0.0148	0.0006	-0.0019	0.0006	-0.001	0.0006	
			2	ϑ	0.1271	0.1667	0.0245	0.0552	0.048	0.0595	0.1271	0.0523
				v_1	0.4012	1.3552	0.0231	0.0747	0.053	0.0759	0.4012	0.0681
		v_2		0.1152	0.2443	0.028	0.0593	0.0453	0.0632	0.1152	0.0572	
		η		0.0057	0.0093	-0.0083	0.0018	-0.0019	0.0017	0.0057	0.0018	
		3		ϑ	0.0535	0.0715	0.0125	0.0247	0.0102	0.0251	0.0535	0.0246
			v_1	0.2045	0.5085	0.018	0.0348	0.0307	0.0366	0.2045	0.0334	
			v_2	0.0693	0.08	0.0226	0.0236	0.0327	0.0249	0.0693	0.0227	
			η	0.0017	0.0041	-0.0019	0.0008	-0.0018	0.0008	0.0017	0.0008	
			ϑ	-0.0216	0.0337	-0.0016	0.0041	0.0011	0.0041	-0.0216	0.0042	
		4	v_1	0.0924	0.1674	0.0091	0.0053	0.0031	0.0054	0.0924	0.0053	
	v_2		0.0514	0.0395	0.0056	0.0049	0.0074	0.0049	0.0514	0.0048		
	η		-0.0015	0.0018	-0.0007	0.0003	-0.0008	0.0002	-0.0015	0.0001		
	6		ϑ	-0.0136	0.0245	-0.0014	0.0032	-0.001	0.0032	-0.0136	0.0032	
			v_1	0.0779	0.1002	0.0009	0.0036	0.0026	0.0034	0.0779	0.0034	
		v_2	0.0304	0.025	0.0026	0.0033	0.0038	0.0033	0.0304	0.0033		
		η	0.0012	0.0011	0.0004	0.0001	-0.0003	0.0001	0.0012	0.0001		
		SRS	2	ϑ	0.0498	0.0886	0.0289	0.0733	0.0849	0.0868	-0.0254	0.0676
	v_1			0.2258	0.8515	0.0752	0.3323	0.1988	0.4204	-0.0467	0.2844	
	v_2			0.0586	0.159	0.0611	0.1312	0.1362	0.1604	-0.0128	0.1153	
	η			0.0017	0.0074	-0.0056	0.0051	-0.0057	0.0051	-0.0055	0.0051	
	ϑ			-0.0092	0.0502	-0.0125	0.041	0.0128	0.0425	-0.0238	0.0409	
	3		v_1	0.1034	0.3494	0.0227	0.1441	0.0747	0.1591	-0.0302	0.1351	
			v_2	0.0479	0.1003	0.0524	0.0761	0.0888	0.0871	0.0116	0.0682	
			η	-0.0016	0.0041	-0.0047	0.0026	-0.0047	0.0027	-0.0046	0.0026	
			ϑ	-0.0021	0.0327	-0.0102	0.0188	-0.0124	0.0185	-0.021	0.0179	
			v_1	0.0299	0.2249	0.0027	0.0304	0.0143	0.0308	-0.009	0.0303	
	4		v_2	0.0398	0.0599	0.0112	0.0228	0.0206	0.0235	0.0016	0.0224	
			η	-0.0015	0.0029	-0.0015	0.0008	-0.0018	0.0008	-0.0011	0.0008	
			ϑ	-0.0019	0.026	-0.0101	0.0132	-0.0123	0.0131	-0.0204	0.013	
			v_1	0.0235	0.1584	-0.0019	0.0196	0.0056	0.0196	-0.0089	0.0196	
			v_2	0.0358	0.0501	0.0092	0.0165	0.0154	0.0167	0.0013	0.0164	
	6		η	-0.0012	0.0022	-0.0015	0.0006	-0.0018	0.0006	-0.0009	0.0005	
			2	ϑ	0.0381	0.0704	0.0215	0.0407	0.0198	0.0419	0.0381	0.0405
				v_1	0.1576	0.4308	0.0129	0.0723	0.04	0.0755	0.1576	0.0708
				v_2	0.0565	0.0887	0.0138	0.0439	0.0336	0.0454	0.0565	0.0434
				η	0.0062	0.004	-0.0013	0.0016	-0.0018	0.0016	0.0062	0.0015
	3			ϑ	-0.0231	0.0322	-0.0162	0.0163	-0.0084	0.0162	-0.0092	0.0157
			v_1	0.0757	0.1435	0.0124	0.0284	0.025	0.0294	0.0757	0.0278	
			v_2	0.0252	0.0332	0.0015	0.0179	0.0102	0.0184	0.0252	0.0177	
			η	0.0012	0.0015	0.0009	0.0006	0.0006	0.0006	0.0012	0.0006	
		ϑ	-0.0203	0.0174	-0.0156	0.0041	-0.0082	0.004	-0.001	0.0039		
	4	v_1	0.0396	0.0781	0.0015	0.0052	0.0034	0.0052	0.0396	0.0052		

Continue on next page

Table 4 – Continued from previous page

			MLE		SQLF		LLF-I		LLF-II		
			v_2	0.024	0.018	0.0013	0.0049	0.0074	0.0049	0.024	0.0048
			η	-0.0011	0.0009	-0.0007	0.0002	-0.0005	0.0002	-0.0011	0.0002
		6	ϑ	-0.0194	0.0126	-0.012	0.0031	-0.0071	0.0031	-0.0009	0.0029
			v_1	0.0342	0.0442	0.0011	0.0034	0.003	0.0035	0.0342	0.0034
			v_2	0.0205	0.012	0.0009	0.0029	0.0059	0.0029	0.0205	0.0028
			η	-0.0005	0.0005	-0.0005	0.0001	-0.0005	0.0001	-0.0005	0.0001

7 Data Application

In this segment, we examine eight authentic datasets to demonstrate the application of the proposed techniques.

1. First Dataset

The breaking strengths of jute fiber at two distinct gauge lengths are displayed as follows:

Z (length 10 mm) is “693.73, 704.66, 323.83, 778.17, 123.06, 637.66, 383.43, 151.48, 108.94, 50.16, 671.49, 183.16, 257.44, 727.23, 291.27, 101.15, 376.42, 163.40, 141.38, 700.74, 262.90, 353.24, 422.11, 43.93, 590.48, 212.13, 303.90, 506.60, 530.55, and 177.25”, and

W (length 20 mm) is 71.46, 419.02, 284.64, 585.57, 456.60, 113.85, 187.85, 688.16, 662.66, 45.58, 578.62, 756.70, 594.29, 166.49, 99.72, 707.36, 765.14, 187.13, 145.96, 350.70, 547.44, 116.99, 375.81, 581.60, 119.86, 48.01, 200.16, 36.75, 244.53, 83.55”. These data sets have been used by Xia et al. [57] and Saraçoğlu et al. [58].

Second Dataset

This set of data involves rainfall recordings documented at the Los Angeles Civic Center spanning from 1943 to 2018, as detailed by Nadar and Kizilaslan [59]. This dataset is structured as follows:

Z (rainfall in February) is “3.07, 8.65, 3.24, 1.52, 0.86, 1.29, 1.41, 1.67, 1.48, 0.63, 0.33, 2.98, 0.68, 0.59, 1.47, 6.46, 3.32, 2.26, 0.15, 11.57, 2.88, 0.23, 1.51, 0.11, 0.49, 8.03, 2.58, 0.67, 0.13, 7.98, 0.14, 3.54, 3.71, 0.17, 8.91, 3.06, 12.75, 1.48, 0.70, 4.37, 2.84, 6.10, 1.22, 1.72, 1.90, 3.12, 4.13, 7.16, 6.61, 3.21, 1.3, 4.94, 0.08, 13.68, 0.56, 5.54, 8.87, 0.29, 4.64, 4.89, 11.02, 2.37, 0.92, 1.64, 3.57, 4.27, 3.29, 0.16, 0.20, 3.58, 0.83, 0.79, 4.17, and 0.03”, and

W (rainfall in March) is “4.55, 2.47, 3.43, 3.66, 0.79, 3.07, 1.40, 0.87, 0.44, 6.14, 0.48, 2.99, 0.56, 1.02, 5.30, 0.31, 0.57, 1.10, 2.78, 1.79, 2.49, 0.53, 2.5, 3.34, 1.49, 2.36, 0.53, 2.70, 3.78, 4.83, 1.81, 1.89, 8.02, 5.85, 4.79, 4.10, 3.54, 8.37, 0.28, 1.29, 5.27, 0.95, 0.26, 0.81, 0.17, 5.92, 7.12, 2.74, 1.86, 6.98, 2.16, 4.06, 1.24, 2.82, 1.17, 0.32, 4.32, 1.47, 2.14, 2.87, 0.05, 0.01, 0.35, 0.48, 3.96, 1.75, 0.54, 1.18, 0.87, 1.60, 0.09, and 2.69”. This data set has been used by Pak et al. [60].

1. Third Dataset

This data set showcases empirical data sourced from Lawless [61]. These data sets entail failure times, measured in minutes, from an experiment conducted on two varieties of electrical insulation. The experiment involved subjecting the insulation to an incrementally rising voltage stress. Each type of insulation was tested twelve times, with the corresponding failure times recorded. For the first type, denoted as “**W**,” the failure times were observed as follows: “21.8, 70.7, 24.4, 138.6, 151.9, 75.3, 12.3, 95.5, 98.1, 43.2, 28.6, and 46.9”. Conversely, for the second type, denoted as “**Z**” the failure times were: “219.3, 79.4, 86.0, 150.2, 21.7, 18.5, 121.9, 40.5, 147.1, 35.1, 42.3, and 48.7”. This data set has been used by Mokhlis et al. [62].

1. Fourth dataset

In this real-world data, we explore the monthly levels of sulphur dioxide concentration in Long Beach, California, spanning from 1956 to 1974. This dataset was documented by Roberts [63] and scrutinized by Wang and Ye [64] to contrast two different models. In this context, let w_1 through w_{20} represent the concentrations of sulphur dioxide in March, and z_1 through z_{20} represent those in August. The specific data are presented below: **W** is “21, 16, 20, 15, 9, 10, 10, 4, 25, 18, 18, 26, 25, 17, 40, 55, 19, 16, 9, and 19.6”, and **Z** is “44, 20, 20, 20, 23, 20, 15, 27, 3, 9, 25, 32, 18, 55, 10, 20, 18, 8, 9, and 20.8”, respectively. This data set has been used by Pak et al. [60].

Initially, an assessment was made to ascertain whether the IPHLD is suitable for analyzing these datasets. Table 5 presents the MLEs alongside standard errors (StEr) and KS distances with associated P-Values, which were utilized to evaluate goodness-of-fit. Based on the obtained P-values, there is no evidence to reject the hypothesis that the data adhere to IPHLD. All data sets have fitting of IPHLD where P-Value is more than 5%.

The model comparison between IPHLD variants is summarized in Table 5 (“MLEs with StEr for parameters of IPHLD and KS test for each data set”). The KS test statistics reported there indicate the preferred model for each dataset; smaller KS values correspond to better empirical fit.

Table 1: Efficiencies between RSS and SRS by using different estimation method

(v_1, v_2, ϑ)			(1.2, 0.6, 1.5)				(3, 1.2, 2)				(3, 1.8, 2)			
s	h		$E1$	$E2$	$E3$	$E4$	$E1$	$E2$	$E3$	$E4$	$E1$	$E2$	$E3$	$E4$
3	2	ϑ	1.3082	1.5319	1.6429	1.434	1.2473	2.9593	3.6105	2.5124	1.9717	3.2277	3.7623	2.8739
		v_1	2.172	1.8302	2.0005	1.679	1.6963	4.951	6.1099	4.0906	1.6199	5.3877	6.5756	5.1472
		v_2	1.9649	2.2181	2.5411	1.9218	2.0587	3.396	4.106	2.8322	2.6014	4.2153	5.1393	3.5287
		η	1.6402	1.7321	1.7661	1.6924	1.5796	3.2564	3.2964	3.2393	1.4855	4.1218	4.1828	4.0959
	3	ϑ	1.8405	1.7806	1.8691	1.7018	1.6182	3.2652	3.5533	3.0642	1.8008	3.3737	3.7297	3.1313
		v_1	3.0314	2.6764	2.8108	2.5643	2.534	5.4453	6.1864	4.9391	2.0024	5.3474	5.8581	4.9644
		v_2	2.6011	2.7536	2.9351	2.591	2.6754	4.3728	4.7703	4.0318	2.5666	4.1019	4.4938	3.8598
		η	2.4549	2.3847	2.405	2.3567	2.3245	4.2813	4.3583	4.2024	1.9569	4.2046	4.1312	4.2078
	4	ϑ	2.2382	1.9288	1.9423	1.9231	2.2106	4.0893	4.1523	4.074	2.1934	6.1556	6.223	6.1421
		v_1	4.1145	2.6193	2.6275	2.6217	3.67	6.8771	7.1436	6.6939	3.4926	6.5665	6.7915	6.3922
		v_2	3.7828	3.0604	3.181	2.9568	3.5708	5.3621	5.5414	5.2387	3.2966	5.1848	5.3266	5.0921
		η	3.5598	3.0693	3.0821	3.0601	3.0424	5.6206	5.6671	5.5959	2.7627	2.6994	5.3763	5.3511
	6	ϑ	2.6957	1.964	2.0202	2.0769	2.6602	5.1693	5.1851	5.0452	2.3857	5.2668	5.16	4.8963
		v_1	3.7785	2.4632	2.4642	2.4784	3.428	6.0245	7.1891	6.9033	3.6214	6.4389	6.8547	6.3003
		v_2	4.1553	3.0511	3.0903	3.0213	5.161	6.1283	6.2531	6.0383	4.2751	5.9589	6.1406	5.8315
		η	3.849	2.858	2.8468	2.8732	4.3454	6.3952	6.4036	6.3999	4.5446	6.5607	6.6209	6.5248
6	2	ϑ	1.6196	1.7217	1.8545	1.6205	1.5819	2.2836	2.6597	2.0561	1.2572	1.7988	2.0694	1.6682
		v_1	2.1694	2.1246	2.3399	1.9826	2.0711	3.9115	4.7051	3.5818	1.9766	4.599	5.5705	4.0171
		v_2	2.1257	2.011	2.1424	1.911	1.5917	2.142	2.4808	1.9216	1.7925	2.99	3.5353	2.66
		η	1.6158	1.6684	1.6761	1.666	1.3934	2.4206	2.4232	2.3657	1.8312	3.2342	3.2893	3.3699
	3	ϑ	2.0116	2.0805	2.128	2.0636	1.626	2.4604	2.615	2.4255	1.5599	2.5163	2.625	2.6147
		v_1	2.3798	2.186	2.2611	2.1586	2.4388	4.3352	4.5826	4.2446	2.4345	5.0637	5.4067	4.8502
		v_2	2.0226	1.9981	2.05	1.9731	3.0914	3.3375	3.6409	3.1098	3.025	4.2477	4.7461	3.8558
		η	2.2324	2.2476	2.2451	2.2556	2.563	3.656	3.7101	3.6233	2.7646	4.2465	4.3678	4.1305
	4	ϑ	2.2414	2.0742	2.1576	2.2721	1.8864	4.2995	4.2705	4.2927	1.8833	4.5383	4.5909	4.5469
		v_1	3.3476	2.6204	2.6181	2.6347	2.3373	5.6756	5.6044	5.7214	2.8818	5.84	5.8987	5.8324
		v_2	2.5288	2.2965	2.3065	2.2995	3.7168	4.6438	4.8117	4.5158	3.3292	4.6909	4.7521	4.6658
		η	2.9841	2.6846	2.6642	2.7096	2.8607	5.0853	5.1157	5.0669	3.3282	4.9503	4.9188	4.9878
	6	ϑ	2.308	2.2446	2.2685	2.2203	2.2436	4.6485	4.7535	4.7226	2.0618	4.2693	4.269	4.4511
		v_1	4.0744	3.1638	3.1312	3.2089	3.0807	6.1405	6.1827	6.7916	3.5821	5.7144	5.6252	5.7801
		v_2	3.6141	3.9804	4.0677	3.8992	3.6376	4.4249	4.5157	4.3541	4.1896	5.7549	5.764	5.7668
		η	3.6877	3.3048	3.3035	3.3097	3.1498	4.831	5.2803	4.816	4.1215	6.3223	6.2247	5.671

Table 5: MLEs with StEr for parameters of IPHLD and KS test for each data set

Data Application		W			Z		
		Estimate	StEr	KS test	Estimate	StEr	KS test
First	ϑ	0.8714	0.12	0.1789	0.9856	0.1355	0.1853
	v	117.8202	66.333	0.2599	265.092	177.5901	0.2248
Second	ϑ	0.5473	0.0469	0.1602	0.5163	0.0404	0.2217
	v	1.3531	0.1488	0.0447	1.3503	0.1466	0.0017
Third	ϑ	1.1183	0.2562	0.1716	1.1993	0.2832	0.1567
	v	79.798	67.7276	0.8147	135.8565	135.5228	0.8862
Fourth	ϑ	1.3554	0.2167	0.2033	1.1214	0.1774	0.226
	v	48.1963	25.0902	0.3804	26.3423	11.26	0.2585

Next in Table 6, we calculate the MLEs and BEs using the SQEL approach for the parameters and η based on the complete sample for each dataset. As shown in Table 6, the estimated value of the SSR $P(W < Z)$ for the first dataset is approximately 0.5661 and 0.5969 for MLEs and BEs, respectively. This suggests that the probability of the coverage percentage (0.5661 and 0.5969) of the breaking strengths of jute fiber at a length of 20 mm being less than the breaking strengths of jute fiber at a length of 10 mm. As shown in Table 6, the estimated value of the SSR $P(W < Z)$ for the second dataset is approximately 0.5050 and 0.5088 for MLEs and BEs, respectively. This suggests that the probability of the coverage percentage of the involves rainfall recordings in February more than involves rainfall recordings in March. As depicted in Table 6, the estimated SSR value $P(W < Z)$ for the third dataset is approximately 0.5783 and 0.5905 for MLEs and BEs, respectively. This indicates that the probability of the coverage percentage of failure in electrical insulation in the

first type of experiment being less than that in the second type of experiment. As indicated in Table 6, the calculated SSR estimate $P(W < Z)$ for the third dataset is around 0.5267 for MLEs and 0.5302 for BEs. This implies that the probability of the coverage percentage of monthly concentration of sulphur dioxide in Long Beach for March is lower than that of the August.

Table 6: ML and Bayesian estimation of $\eta = P[W < Z]$ for complete sample

Data Application		ML		Bayesian	
		Estimates	StEr	Estimates	StEr
First	ϑ	0.9575	0.0982	1.079	0.0856
	v_1	232.9864	116.969	457.7096	190.8058
	v_2	175.2578	81.9205	315.6849	102.0549
	η	0.5661		0.5969	
Second	ϑ	0.5297	0.0305	0.5293	0.0425
	v_1	1.3767	0.1419	1.4087	0.2006
	v_2	1.33	0.1403	1.3428	0.204
	η	0.505		0.5088	
Third	ϑ	1.1703	0.1966	1.5396	0.144
	v_1	123.6762	89.0012	461.2735	204.2323
	v_2	94.2152	62.8634	338.1461	132.7952
	η	0.5783		0.5905	
Fourth	ϑ	1.2239	0.1382	1.3747	0.2129
	v_1	35.866	13.2656	56.5999	31.6601
	v_2	32.7342	11.317	51.0811	28.4148
	η	0.5267		0.5302	

Figure 3 displays the profile log-likelihood for each parameter (panels are ordered left to right). The blue points indicate the MLEs at the modal peaks. The curvature of each profile near its maximum reflects estimator precision: narrower, more sharply peaked profiles correspond to smaller standard errors (higher precision), whereas broader, flatter profiles indicate greater uncertainty.

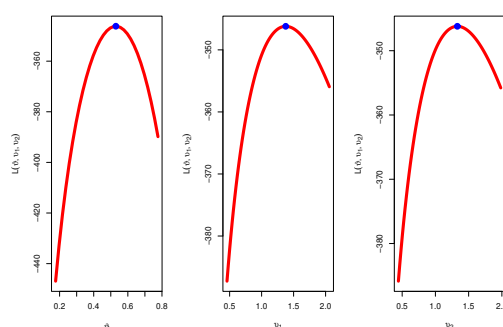


Fig. 3: Profile likelihood for data 2

Figure 4 presents MCMC diagnostics for the three model parameters (left panels: trace plots; right panels: posterior histograms). The trace plots fluctuate around stable central values with no visible drift, indicating stationarity and adequate mixing; the posterior histograms are unimodal and approximately symmetric, supporting the use of posterior means and credible intervals.

Table 7 discussed sample data for SRS and RSS for each data set. Table 8 obtained MLE and BE based on SRS and RSS. The results confirmed the suggest comment of the reliability of the strength of the stress in Table 8.

Table 7: SRS and RSS data

SRS				RSS					
				Data 2					
W		Z				h = 1	h = 2	h = 3	h = 4
36.75	350.7	43.93	262.9	W	s = 1	36.75	99.72	284.64	419.02
36.75	375.81	108.94	303.9		s = 2	145.96	113.85	166.49	419.02
48.01	375.81	108.94	303.9		s = 3	244.53	83.55	578.62	756.7
71.46	419.02	108.94	303.9		s = 4	71.46	578.62	83.55	756.7
83.55	419.02	123.06	353.24		s = 5	187.13	187.13	456.6	707.36
99.72	688.16	123.06	383.43	Z	s = 1	183.16	323.83	383.43	530.55
113.85	688.16	163.4	506.6		s = 2	123.06	506.6	590.48	704.66
119.86	707.36	212.13	530.55		s = 3	50.16	323.83	212.13	671.49
145.96	756.7	212.13	727.23		s = 4	291.27	183.16	700.74	671.49
187.13	765.14	257.44	727.23		s = 5	101.15	141.38	700.74	778.17
Data 3									
Z		W				h = 1	h = 2	h = 3	h = 4
3.29	2.98	3.96	8.37	Z	s = 1	1.41	3.06	4.17	4.17
0.17	3.12	0.48	2.49		s = 2	0.49	0.7	3.21	4.64
0.11	3.06	0.87	2.47		s = 3	0.13	3.32	2.26	13.68
11.02	5.54	4.83	4.32		s = 4	0.17	3.07	4.27	6.61
3.54	3.71	3.43	0.87		s = 5	0.86	4.17	1.41	13.68
0.7	1.64	0.57	5.27	W	s = 1	0.17	0.32	4.79	5.3
6.1	0.11	0.32	0.05		s = 2	0.44	0.28	0.57	5.85
11.02	3.54	1.4	5.27		s = 3	1.86	0.28	2.47	5.85
0.67	8.03	6.98	1.47		s = 4	0.09	1.47	1.81	2.78
4.64	11.57	8.02	0.17		s = 5	1.1	1.4	0.53	6.14
Data 1									
W		Z				h = 1	h = 2	h = 3	
12.3	95.5	121.9	219.3	W	s = 1	12.3	43.2	75.3	
46.9	43.2	150.2	48.7		s = 2	28.6	46.9	95.5	
75.3	70.7	35.1	150.2	Z	s = 1	121.9	48.7	219.3	
					s = 2	40.5	150.2	150.2	
Data 4									
W		Z				h = 1	h = 2	h = 3	h = 4
25	10	32	20	W	s = 1	9	17	18	40
20	18	10	15		s = 2	9	15	17	26
16	17	9	44	Z	s = 1	3	18	20	23
16	25	55	18		s = 2	8	18	20	55

Table 8: MLE and BE based on different loss functions for SRS and RSS

Data		SRS				RSS			
		MLE	SQEL	LLF-I	LLF-II	MLE	SQEL	LLF-I	LLF-II
2	ϑ	0.9823	1.0461	1.0482	1.0472	0.9756	1.0031	1.004	1.0004
	v_1	235.7846	360.2839	798.7612	784.618	297.8361	356.9983	777.9867	111.7332
	v_2	154.71	207.4927	473.2082	455.1502	189.0029	224.379	395.6428	68.265
	η	0.6079	0.6452	0.6241	0.6299	0.6124	0.6163	0.6518	0.6198
3	ϑ	0.5212	0.5210	0.5238	0.5215	0.6421	0.6398	0.6423	0.6372
	v_1	1.5418	1.5681	1.6493	1.5809	1.7338	1.7454	1.7837	1.7098
	v_2	1.3766	1.3962	1.4654	1.4067	1.0761	1.0941	1.1100	1.0787
	η	0.5271	0.5278	0.5283	0.5280	0.6346	0.6319	0.6339	0.6301
1	ϑ	1.1384	1.2385	1.2453	1.2419	1.2629	1.2857	1.2900	1.2881
	v_1	196.9542	349.3407	1006.096	991.909	339.2356	404.0409	1514.547	1498.219
	v_2	80.3071	108.3084	261.6068	248.3473	114.196	134.4084	368.6643	353.9674
	η	0.7382	0.7945	0.8215	0.8274	0.7794	0.7817	0.8319	0.8363
4	ϑ	2.0584	1.6242	1.6590	1.6300	1.2845	1.3350	1.3505	1.3376
	v_1	457.1219	154.7478	428.6469	406.4528	44.6493	54.7926	120.514	100.240
	v_2	353.6045	101.4375	236.4255	215.4917	31.8456	36.9067	84.2469	62.5412
	η	0.5749	0.6203	0.6682	0.6784	0.5777	0.6289	0.6638	0.6835

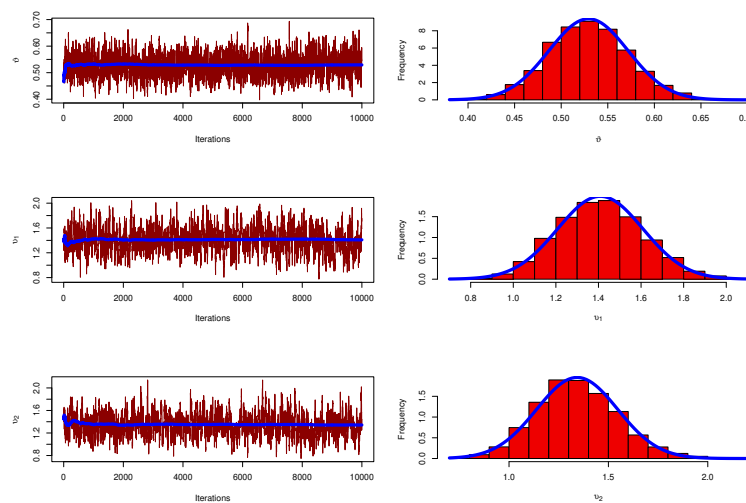


Fig. 4: MCMC plots for data 2

8 Concluding Remarks

A vital statistical technique for modeling different properties is the lifespan distribution. In order to analyze these sorts of data sets, extremely complicated distributions are available in the statistical literature. However, the estimating procedure is challenging due to the enormous number of factors in these distributions. A novel inverted model called the IPHLD is introduced to provide additional avenues for modeling these types of data sets. Statistical properties, such as moments, PWMs, entropy measures, quantile function, incomplete moments, Bonferroni and Lorenz curves are acquired. The Bayesian and non-Bayesian estimation of $\eta = P[W < Z]$ assuming stress (W) and strength (Z) random variables follow the IPHLD with different scale parameters are regarded. The estimators of $\eta = P[W < Z]$ are determined based on SRS and RSS. The Bayes estimates of η under symmetric and asymmetric loss functions are achieved using gamma priors. It is clear that the Bayesian estimators do not exist in their explicit form. Therefore, the MCMC method is used to get the Bayesian estimate. The Monte Carlo simulation is used to assess how well the suggested estimates perform. Finally, eight data sets are analyzed to illustrate the proposed estimation methods. Future study involves estimating the SSR, assuming that stresses and strength random variables THAT follow the IPHLD under more advanced RSS scheme.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

Data Availability: The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

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