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New Asymptotic and Monotonic Features of Solutions to Neutral Differential Equations and their Applications in Oscillation Theory

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Abstract: This work aims to develop more efficient conditions for evaluating the oscillatory performance of solutions to a class of functional differential equations. We employ the Riccati method and a comparison method with several different approaches. These improved approaches enable us to obtain different criteria that are suitable for different constraints on the parameters. Furthermore, we use some recurrence relations that allow the oscillatory criteria to be applied iteratively. We support the theoretical results and demonstrate their novelty by applying our results to specific cases and comparing the results with previous results in the literature.

Keywords: differential equations; nonlinear equations; oscillatory performance; Riccati and comparison methods; comparison principle.

1 Introduction

This paper focuses on investigating the oscillatory behavior of all solutions of the following canonical half-linear neutral differential equation with multiple delays

$$\left(a(u)F(x(u))\left[X'(u)\right]^{\alpha}\right)' + c_1(u)x^{\alpha}(\rho_1(u)) + c_2(u)x^{\alpha}(\rho_2(u)) + \cdots + c_n(u)x^{\alpha}(\rho_n(u)) = 0, \quad (1)$$

for $u \geq u_0$, $n \in \mathbb{Z}^+$, and

$$X(u) = x(u) + b(u)x(\varpi(u)).$$

We propose the following conditions as necessary,

$(\mathrm{A}_1)\alpha$ is a ratio of any to positive integers;
$(\mathrm{A}_2)a,\ b,\ c_i \in \mathbf{C}([u_0,\infty),\mathbb{R}^+)$ where $i = 1,2,\cdots,n$, $n \in \mathbb{N}$, $a'(u) > 0$, $b(u) \leq b_0$, $b_0 \in \mathbb{R}^+$, and

$$A(u) = \int_{u_0}^{u} a^{-1/\alpha}(\xi)\,\mathrm{d}\xi \to \infty \quad \text{as} \quad u \to \infty;$$

$(\mathrm{A}_3)\varpi,\ \rho_i \in \mathbf{C}^1([u_0,\infty),\mathbb{R})$ with

$$\rho(u) = \min_{1\leq i\leq n}\{\rho_i(u)\};$$

$\rho(u) \leq u$, and $\lim_{u\to\infty}\varpi(u) = \lim_{u\to\infty}\rho(u) = \infty$;

$(\mathrm{A}_4)F \in \mathbf{C}^1(\mathbb{R},[\kappa_1,\kappa_2])$ and $\kappa_1, \kappa_2 \in \mathbb{R}^+$.

However, the following additional conditions would be needed in certain results,

$(\overline{\overline{\mathrm{A}}}_1)\varpi(\rho) = \rho(\varpi)$ and $\varpi' \geq \varpi_0 > 0$;
$(\overline{\overline{\mathrm{A}}}_2)\rho' \geq 0$.

The function $x \in \mathbf{C}^2([u_0,\infty),\mathbb{R})$, is defined as a solution of equation (1) if it satisfies $aF(x)[X']^{\alpha} \in \mathbf{C}^1([u_0,\infty),\mathbb{R})$, x fulfills (1) on the interval $[u_0,\infty)$, and for every $u_1 \geq u_0$, $\sup\{|x(u)| : u \geq u_1\} > 0$. Such a solution is termed oscillatory if it possesses zeros that are arbitrarily large; otherwise, it is referred to as nonoscillatory. The equation (1) itself is called oscillatory if every solution exhibits oscillatory behavior; if not, it is classified as nonoscillatory.

Second-order differential equations (DEs) are frequently used in mathematical modeling in a variety of scientific domains [1, 2]. For the purpose of precisely describing primary and secondary changes in natural processes, these equations are essential. For example, they are used to simulate the rates at which epidemics spread in biology and to describe motion and acceleration in physics. They are also used in environmental studies to comprehend the dispersion of contamination, for a

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thorough analysis of these applications [3, 4], and for results concerning the existence and positivity of solutions in neutral delay models, see [5,6].

The oscillatory behavior of functional differential equations (FDEs) has attracted a lot of attention lately. Sharp oscillation conditions for second-order FDEs have been established in several investigations, see for instance [7,8]. The linkages and inequalities employed in the study of oscillation have been improved by other research, such as [9,10]. Higher-order equations have also been the subject of some study and analysis, see [11,12]. Moreover, By expanding the findings of second order, the research of oscillation for solutions of even order equations has also advanced, this development is seen in [13,14].

The primary motivation behind this study is to address the challenges in understanding the oscillatory behavior of solutions to neutral functional differential equation (1) with delays. The goal is to provide a comprehensive set of conditions for the oscillation of solutions of (1), thereby extending the applicability of existing theorems. By leveraging powerful comparison techniques and the Riccati method, our results offer a broader and more flexible framework for analyzing the oscillatory nature of solutions, which is essential for both theoretical advancements and practical applications in various areas of mathematical modeling.

This paper is organized as follows: The following section discusses the preliminary definitions, notations, and the key lemmas used in deriving the oscillation criteria for these equations. In Sections 3 and 4, we present the main theoretical results, where we establish several oscillation criteria by applying comparison techniques and the Riccati method. Section 5 presents a detailed comparison between the different criteria, highlighting the strengths and conditions for their applicability, as well as how they compare to previous work in the literature. Furthermore, it provides several illustrative examples and discuss the practical implications of our results.

2 Preliminaries and Lemmas

In this section, we set up the necessary notation and establish the main and auxiliary lemmas that describe the relationships between the solution, its derivatives, and the associated neutral terms, providing the foundation for the oscillation criteria developed later. So, let

$$\widehat{\kappa} = \left(\frac{\kappa_1}{\kappa_2}\right)^{-1/\alpha},$$

$$\widehat{c}(u) = \min_{n\in\mathbb{N}} \left\{ \sum_{i=1}^{n} c_i(u), \sum_{i=1}^{n} c_i(\varpi(u)) \right\},$$

and

$$\tau(u) = \begin{cases} \varpi^{-1}(\rho(u)), & u \geq \varpi(u) \geq \rho(u), \\ \rho(u), & u \leq \varpi(u). \end{cases}$$

Moreover, let $\varpi^{[0]}(u) = u$, $\varpi^{[-j]}(u) = \varpi^{-1}\left(\varpi^{[-j+1]}(u)\right)$, and

$$B(u) = \sum_{i=1}^{n} c_i(u) \left(\sum_{j=1}^{k} \left(\prod_{\ell=1}^{2j-1} \frac{1}{b\left(\varpi^{[-\ell]}(\rho(u))\right)} \right) \times \left[1 - \frac{A^{\widehat{\kappa}}\left(\varpi^{[-2j]}(\rho(u))\right)}{b\left(\varpi^{[-2j]}(\rho(u))\right) A^{\widehat{\kappa}}\left(\varpi^{[-2j+1]}(\rho(u))\right)} \right] \right)^{\alpha},$$

for $j = 1, 2, \ldots, k$ and $k \in \mathbb{Z}^+$.

Lemma 1. *[15] Consider the functions G_1, $G_2 \in \mathbb{R}^+$. The following inequality holds:*

$$G_1^{\alpha} + G_2^{\alpha} \geq \frac{1}{m_1}(G_1 + G_2)^{\alpha},$$

for α defined as in (A_1) and m_1 is a positive constant defined as

$$m_1 = \begin{cases} 1, & \alpha \leq 1, \\ 2^{\alpha-1}, & \alpha > 1. \end{cases}$$

Lemma 2. *[16] Consider the function*

$$G_3(\xi) = m_2\xi - m_3\xi^{1+1/\alpha},$$

for $\xi \in \mathbb{R}$, $m_2 \in \mathbb{R}$, $m_3 \in \mathbb{R}^+$, and α defined as in (A_1). This function attains its maximum at the critical value

$$G_3(\xi^*) = \max_{\xi\in\mathbb{R}}(G_3) = \left(\frac{m_2}{\alpha+1}\right)^{\alpha+1}\left(\frac{\alpha}{m_3}\right)^{\alpha}$$

at $\xi^* = \left(\frac{\alpha}{\alpha+1}\frac{b_1}{b_2}\right)^{\alpha}$.

Lemma 3. *Under the canonical condition in (A_2), positive decreasing solutions of (1) are excluded. Consequently, only one class of positive solutions to (1) remains—namely, positive increasing solutions, for which x, $x' > 0$, while $\left(aF(x)[X']^{\alpha}\right)' \leq 0$.*

Proof. Assume that x is an eventually positive solution of (1) for $u \in [u_0, \infty)$. The definition of (1) and conditions (A_1)-(A_4) imply that

$$\left(aF(x)\left[X'\right]^{\alpha}\right)' = -\left[c_1 x^{\alpha}(\rho_1) + c_2 x^{\alpha}(\rho_2) + \cdots + c_n x^{\alpha}(\rho_n)\right] \leq 0.$$

Then, x has a fixed sign, i. e., $x' > 0$ or $x' < 0$. But the canonical condition in (A_2) excludes the existence of any positive decreasing solutions. And this completes the proof.

Lemma 4. *Assume that x is an eventually positive solution of (1). Under the satisfaction of (A_1)-(A_4), the following monotonic characteristics arise, X, $X' > 0$ and $\left(X/A^{\widehat{\kappa}}\right)' \leq 0$.*

Proof. Assume that x is an eventually positive solution of (1) for $u \in [u_0,\infty)$. Since X is a positive function and $\left(aF(x)[X']^{\alpha}\right)'$ is nonpositive, then X' has fixed sign. Assume, on contrarily, that $X' \le 0$, then the nonincreasing monotonicity of $aF(x)[X']^{\alpha}$ implies that $(aF(x))^{1/\alpha}X' \le -m_4^{1/\alpha}$ for $m_4 \in \mathbb{R}^+$. As a result, we have

$$X' \le -m_4^{1/\alpha}(aF(x))^{-1/\alpha} \le -\left(\frac{m_4}{\kappa_2}\right)^{1/\alpha} a^{-1/\alpha},$$

for $F^{-1/\alpha}(x) \ge \kappa_2^{-1/\alpha}$ arises from (A_4). By integrating the last inequality over $[u_1,\infty)$, $u_1 \ge u_0$, and taking the limits for both sides as $u \to \infty$, we have

$$X \le m_5 - \left(\frac{m_4}{\kappa_2}\right)^{1/\alpha} A = -\infty \quad \text{as} \quad u \to \infty,$$

for $m_5 = X(u_1)$, which is a contradiction. So, we obtain that $X' > 0$.

Now, again from the nonincreasing monotonicity of $aF(x)[X']^{\alpha}$, one can conclude that

$$\begin{aligned}
& X(u) \\
\ge & \int_{u_1}^{u} \frac{1}{[a(\xi)F(x(\xi))]^{1/\alpha}}[a(\xi)F(x(\xi))]^{1/\alpha}X'(\xi)\,\mathrm{d}\xi \\
\ge & \kappa_2^{-1/\alpha}[a(u)F(x(u))]^{1/\alpha}X'(u)\int_{u_1}^{u}\frac{1}{a^{1/\alpha}(\xi)}\mathrm{d}\xi \\
\ge & \left(\frac{\kappa_1}{\kappa_2}\right)^{1/\alpha} a^{1/\alpha}(u)X'(u)\int_{u_1}^{u}\frac{1}{a^{1/\alpha}(\xi)}\mathrm{d}\xi \\
= & \frac{1}{\widehat{\kappa}}a^{1/\alpha}(u)A(u)X'(u),
\end{aligned}$$

for $F^{1/\alpha}(x) \ge \kappa_1^{1/\alpha}$. And so,

$$\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{X}{A^{\widehat{\kappa}}}\right] = \frac{\widehat{\kappa}}{a^{1/\alpha}A^{\widehat{\kappa}+1}}\left[\frac{1}{\widehat{\kappa}}a^{1/\alpha}AX' - X\right] \le 0.$$

And this completes the proof of this Lemma.

Lemma 5. *Assume that x is an eventually positive solution of (1). Under the satisfaction of (A_1)-(A_4) and ($\overline{\overline{A}}_1$), we obtain*

$$\left[z + \frac{b_0^{\alpha}}{\varpi_0}z(\varpi)\right]' + \frac{1}{m_1}\frac{\kappa_1}{\kappa_2^2}\widehat{c}A^{\alpha}(\rho)z(\rho) \le 0, \tag{2}$$

eventually, where $z = aF(x)[X']^{\alpha}$.

Proof. Assume that x is an eventually positive solution of (1) for $u \in [u_0,\infty)$. It is clear that (1) can be written as

$$\left(aF(x)[X']^{\alpha}\right)' \le -x^{\alpha}(\rho)\sum_{i=1}^{n}c_i. \tag{3}$$

Multiplying both sides by b_0^{α} and using ($\overline{\overline{A}}_1$) implies that

$$\begin{aligned}
& \frac{b_0^{\alpha}}{\varpi_0}\left(a(\varpi)F(x(\varpi))[X'(\varpi)]^{\alpha}\right)' \\
\le & \frac{b_0^{\alpha}}{\varpi'}\left(a(\varpi)F(x(\varpi))[X'(\varpi)]^{\alpha}\right)' \\
= & -b_0^{\alpha}x^{\alpha}(\rho(\varpi))\sum_{i=1}^{n}c_i(\varpi).
\end{aligned} \tag{4}$$

But

$$\begin{aligned}
& x^{\alpha}(\rho)\sum_{i=1}^{n}c_i + b_0^{\alpha}x^{\alpha}(\rho(\varpi))\sum_{i=1}^{n}c_i(\varpi) \\
\ge & \widehat{c}(u)[x^{\alpha}(\rho) + b_0^{\alpha}x^{\alpha}(\rho(\varpi))] \\
\ge & \widehat{c}(u)[x^{\alpha}(\rho) + b^{\alpha}x^{\alpha}(\rho(\varpi))].
\end{aligned}$$

Then applying Lemma 1 gives

$$\begin{aligned}
& x^{\alpha}(\rho)\sum_{i=1}^{n}c_i + b_0^{\alpha}x^{\alpha}(\rho(\varpi))\sum_{i=1}^{n}c_i(\varpi) \\
\ge & \frac{1}{m_1}\widehat{c}(u)[x(\rho) + bx(\rho(\varpi))]^{\alpha} \\
= & \frac{1}{m_1}\widehat{c}X^{\alpha}(\rho).
\end{aligned} \tag{5}$$

Using (3), (4), and (5) we get

$$\begin{aligned}
& \left[aF(x)[X']^{\alpha} + \frac{b_0^{\alpha}}{\varpi_0}a(\varpi)F(x(\varpi))[X'(\varpi)]^{\alpha}\right]' \\
= & \left(aF(x)[X']^{\alpha}\right)' + \frac{b_0^{\alpha}}{\varpi_0}\left(a(\varpi)F(x(\varpi))[X'(\varpi)]^{\alpha}\right)' \\
\le & -x^{\alpha}(\rho)\sum_{i=1}^{n}c_i - b_0^{\alpha}x^{\alpha}(\rho(\varpi))\sum_{i=1}^{n}c_i(\varpi) \\
\le & -\frac{1}{m_1}\widehat{c}X^{\alpha}(\rho).
\end{aligned} \tag{6}$$

From Lemma 4, we obtain that

$$-X^{\alpha}(\rho) \le -\frac{1}{\widehat{\kappa}^{\alpha}}A^{\alpha}(\rho)a(\rho)\left(X'(\rho)\right)^{\alpha}.$$

Leading to

$$\begin{aligned}
& \left[aF(x)[X']^{\alpha} + \frac{b_0^{\alpha}}{\varpi_0}a(\varpi)F(x(\varpi))[X'(\varpi)]^{\alpha}\right]' \\
\le & -\frac{1}{m_1\widehat{\kappa}^{\alpha}}\widehat{c}a(\rho)A^{\alpha}(\rho)\left(X'(\rho)\right)^{\alpha} \\
= & -\frac{1}{m_1\widehat{\kappa}^{\alpha}}\frac{1}{F(x(\rho))}\widehat{c}A^{\alpha}(\rho)a(\rho)F(x(\rho))\left(X'(\rho)\right)^{\alpha} \\
\le & -\frac{1}{m_1\widehat{\kappa}^{\alpha}}\frac{1}{\kappa_2}\widehat{c}A^{\alpha}(\rho)a(\rho)F(x(\rho))\left(X'(\rho)\right)^{\alpha} \\
= & -\frac{1}{m_1}\frac{\kappa_1}{\kappa_2^2}\widehat{c}A^{\alpha}(\rho)a(\rho)F(x(\rho))\left(X'(\rho)\right)^{\alpha},
\end{aligned}$$

i,e

$$\left[z + \frac{b_0^{\alpha}}{\varpi_0}z(\varpi)\right]' \le -\frac{1}{m_1}\frac{\kappa_1}{\kappa_2^2}\widehat{c}A^{\alpha}(\rho)z(\rho),$$

for $z = aF(x)[X']^{\alpha}$. Then, the proof is complete.

Lemma 6. *Assume that x is an eventually positive solution of (1). If $\varpi(u) \le u$, $\varpi' \ge 0$, and $b_0 > A^{\widehat{\kappa}}(u)/A^{\widehat{\kappa}}(\varpi(u))$, then the following inequalities arise*

$$x > \sum_{j=1}^{k}\left(\prod_{\ell=1}^{2j-1}\frac{1}{b\left(\varpi^{[-\ell]}\right)}\right)\left[1-\frac{\frac{A^{\widehat{\kappa}}\left(\varpi^{[-2j]}\right)}{A^{\widehat{\kappa}}\left(\varpi^{[-2j+1]}\right)}}{b\left(\varpi^{[-2j]}\right)}\right]X \tag{7}$$

and

$$\left(a\,F(x)\,\left[X'\right]^{\alpha}\right)'+B\,X^{\alpha}(\rho)\le 0 \tag{8}$$

eventually, for $j=1,2,...,k$ and $k\in\mathbb{Z}^{+}$.

Proof. Assume that x is an eventually positive solution of (1) for $u\in[u_0,\infty)$. Proceeding as in the proof of Lemma 4 in [17], we obtain that

$$x > \sum_{j=1}^{k}\left(\prod_{\ell=1}^{2j-1}\frac{1}{b\left(\varpi^{[-\ell]}\right)}\right)\left[X\left(\varpi^{[-2j+1]}\right)-\frac{X\left(\varpi^{[-2j]}\right)}{b\left(\varpi^{[-2j]}\right)}\right]. \tag{9}$$

But $u\le\varpi^{[-2j+1]}(u)\le\varpi^{[-2j]}(u)$, then, the increasing monotonicity of X in Lemma 4 gives $X\le X\left(\varpi^{[-2j+1]}\right)$ and

$$X\left(\varpi^{[-2j]}\right)\le\frac{A^{\widehat{\kappa}}\left(\varpi^{[-2j]}\right)}{A^{\widehat{\kappa}}\left(\varpi^{[-2j+1]}\right)}X\left(\varpi^{[-2j+1]}\right).$$

And so, (7) holds. From (3) we obtain

$$\begin{aligned}&\left(aF(x)\left[X'\right]^{\alpha}\right)'\\&\le -x^{\alpha}(\rho)\sum_{\ell=1}^{n}c_i\\&\le -\left(\sum_{j=1}^{k}\left(\prod_{\ell=1}^{2j-1}\frac{1}{b\left(\varpi^{[-\ell]}(\rho(u))\right)}\right)\right.\\&\quad\left.\times\left[1-\frac{A^{\widehat{\kappa}}\left(\varpi^{[-2j]}(\rho(u))\right)}{b\left(\varpi^{[-2j]}(\rho(u))\right)A^{\widehat{\kappa}}\left(\varpi^{[-2j+1]}(\rho(u))\right)}\right]\right)^{\alpha}\\&\quad\times X^{\alpha}(\rho)\sum_{\ell=1}^{n}c_i\\&=-B\,X^{\alpha}(\rho).\end{aligned}$$

The proof is complete.

3 Comparison-Based Theorems

This section develops new oscillation criteria using the comparison theorem method, which enables us to determine the oscillatory behavior of our equation by comparing it with first-order equations.

Theorem 1. *Let $(\overline{\overline{\mathrm{A}}}_1)$ holds. If the following delay differential equation*

$$y'+\frac{1}{m_1}\frac{\varpi_0}{\varpi_0+b_0^{\alpha}}\frac{\kappa_1}{\kappa_2^2}\widehat{c}A^{\alpha}(\rho)\,y(\tau)=0, \tag{10}$$

oscillates. Then (1) is oscillatory.

Proof. Assume, on the contrary, that x is an eventually positive solution of (1) for $u\in[u_0,\infty)$. Let, $\varpi(u)\ge u$. Since $aF(x)\left[X'\right]^{\alpha}\le 0$, then

$$\begin{aligned}&aF(x)\left[X'\right]^{\alpha}+\frac{b_0^{\alpha}}{\varpi_0}a(\varpi)F(x(\varpi))\left[X'(\varpi)\right]^{\alpha}\\&\le\left(1+\frac{b_0^{\alpha}}{\varpi_0}\right)aF(x)\left[X'\right]^{\alpha}.\end{aligned}$$

Substituting into (2), we have

$$\begin{aligned}y'&\le-\frac{1}{m_1}\frac{\kappa_1}{\kappa_2^2}\widehat{c}A^{\alpha}(\rho)\,a(\rho)\,F(x(\rho))\left[X'(\rho)\right]^{\alpha}\\&\le-\frac{1}{m_1}\frac{\varpi_0}{\varpi_0+b_0^{\alpha}}\frac{\kappa_1}{\kappa_2^2}\widehat{c}A^{\alpha}(\rho)\,y(\rho),\end{aligned} \tag{11}$$

where

$$\begin{aligned}y&=aF(x)\left[X'\right]^{\alpha}+\frac{b_0^{\alpha}}{\varpi_0}a(\varpi)F(x(\varpi))\left[X'(\varpi)\right]^{\alpha}\\&\ge 0.\end{aligned}$$

According to Theorem 1 in [18], the differential inequality (11) admits a positive nonoscillatory solution under the given conditions, which leads to a contradiction with the oscillatory nature of (10) and completes the proof of this part.
For the remaining case, letting $\rho(u)\le\varpi(u)\le u$, then

$$\begin{aligned}&aF(x)\left[X'\right]^{\alpha}+\frac{b_0^{\alpha}}{\varpi_0}a(\varpi)F(x(\varpi))\left[X'(\varpi)\right]^{\alpha}\\&\le\left(1+\frac{b_0^{\alpha}}{\varpi_0}\right)a(\varpi)F(x(\varpi))\left[X'(\varpi)\right]^{\alpha}.\end{aligned}$$

Substituting into (2), yields

$$y'\le-\frac{1}{m_1}\frac{\varpi_0}{\varpi_0+b_0^{\alpha}}\frac{\kappa_1}{\kappa_2^2}\widehat{c}A^{\alpha}(\rho)\,y\left(\varpi^{-1}(\rho)\right),$$

and this inequality also has a positive solution according to Theorem 1 in [18], which leads to a contradiction with the oscillatory nature of (10) and completes the proof.

Corollary 1. *Let $(\overline{\overline{\mathrm{A}}}_1)$ holds. If*

$$\liminf_{u\to\infty}\int_{\tau(u)}^{u}\widehat{c}(\xi)A^{\alpha}(\rho(\xi))\,\mathrm{d}\xi>\frac{m_1\kappa_2^2\left(\varpi_0+b_0^{\alpha}\right)}{\mathrm{e}\kappa_1\varpi_0}, \tag{12}$$

then (1) is oscillatory.

Proof. By Theorem 2 in [19], (10) is oscillatory under condition (12). Thus, applying Theorem 1, we conclude that (1) is oscillatory as well, thereby completing the proof.

Theorem 2. *Let $\varpi(u) \leq u, \varpi' \geq 0$, and $b_0 > A^{\widehat{\kappa}}(u)/A^{\widehat{\kappa}}(\varpi(u))$. If the delay differential equation*

$$z' + \frac{\kappa_1}{\kappa_2^2} B\, A^{\alpha}(\rho)\, z(\rho) = 0 \tag{13}$$

oscillates, then (1) is oscillatory.

Proof. Assume, on the contrary, that x is an eventually positive solution of (1) for $u \in [u_0, \infty)$. Letting $z = aF(x)[X']^{\alpha}$ in (8) and using Lemma 6, then we have

$$\begin{aligned} z' &\leq -BX^{\alpha}(\rho) \\ &\leq -\frac{1}{\widehat{\kappa}^{\alpha}} BA^{\alpha}(\rho)\, a(\rho)\left[X'(\rho)\right]^{\alpha} \\ &\leq -\frac{1}{\widehat{\kappa}^{\alpha}}\frac{1}{\kappa_2} BA^{\alpha}(\rho)\, z(\rho) \\ &= -\frac{\kappa_1}{\kappa_2^2} BA^{\alpha}(\rho)\, z(\rho). \end{aligned}$$

However, this inequality also admits a positive solution according to Theorem 1 in [18], which contradicts the oscillatory behavior of (13) and thereby completes the proof.

Remark. As demonstrated in the previous theorem, by applying the nonincreasing monotonicity of $X/A^{\widehat{\kappa}}$ in Lemma 4, we have derived an oscillation criterion that does not rely on the monotonic constraints $(\overline{\overline{\mathrm{A}}}_1)$, and $(\overline{\overline{\mathrm{A}}}_2)$. This establishes a more general framework for analyzing oscillatory behavior.

Corollary 2. *Let $\varpi(u) \leq u, \varpi' \geq 0$, and $b_0 > A^{\widehat{\kappa}}(u)/A^{\widehat{\kappa}}(\varpi(u))$. If*

$$\liminf_{u\to\infty} \int_{\rho(u)}^{u} B(\xi)\, A^{\alpha}(\rho(\xi))\, \mathrm{d}\xi > \frac{\kappa_2^2}{\mathrm{e}\kappa_1}, \tag{14}$$

then (1) is oscillatory.

Proof. By Theorem 2 in [19], (10) is oscillatory under condition (14). Thus, applying Theorem 2, we conclude that (1) is oscillatory as well, thereby completing the proof.

4 Riccati-Based Theorems

In this section, we apply the Riccati substitution technique to develop additional oscillation criteria for (1). The results obtained here complement those of the previous section and provide a broader framework for the study of oscillatory behavior for (1).

Theorem 3. *Let $\rho(u) \leq \varpi(u) \leq u$, $(\overline{\overline{\mathrm{A}}}_1)$, and $(\overline{\overline{\mathrm{A}}}_2)$ hold. If there exists a positive function $\delta(u) \in \mathbf{C}^1([u_0,\infty), \mathbb{R}^+)$ such that*

$$\limsup_{u\to\infty} \int_{u_1}^{u} \left[\frac{1}{m_1}\delta(\xi)\,\widehat{c}(\xi) - \Psi(\xi)\right] \mathrm{d}\xi = \infty, \tag{15}$$

then (1) is oscillatory, where

$$\Psi(\xi) = \left(1 + \frac{b_0^{\alpha}}{\varpi_0}\right) \frac{\kappa_2}{(\alpha+1)^{\alpha+1}} \frac{(\delta'(\xi))^{\alpha+1}}{\delta^{\alpha}(\xi)} \frac{a(\rho(\xi))}{(\rho'(\xi))^{\alpha}}.$$

Proof. Assume, on the contrary, that x is an eventually positive solution of (1) for $u \in [u_0, \infty)$. Let the following generalized Riccati functions

$$\theta_1 = \delta \frac{aF(x)[X']^{\alpha}}{X^{\alpha}(\rho)}$$

and

$$\theta_2 = \delta \frac{a(\varpi)F(x(\varpi))[X'(\varpi)]^{\alpha}}{X^{\alpha}(\rho)}.$$

It is obvious from Lemmas 3 and 4 that $\theta_1,\ \theta_2 > 0$. Differentiating θ_1 and using (3) and Lemma 3, we get

$$\begin{aligned} \theta_1' &= \delta \frac{\left(aF(x)[X']^{\alpha}\right)'}{X^{\alpha}(\rho)} \\ &\quad + \frac{\delta'}{\delta}\theta_1 - \alpha\delta\rho' \frac{aF(x)[X']^{\alpha}}{X^{\alpha+1}(\rho)} X'(\rho) \\ &\leq \delta \frac{\left(aF(x)[X']^{\alpha}\right)'}{X^{\alpha}(\rho)} + \frac{\delta'}{\delta}\theta_1 \\ &\quad - \alpha\delta\rho' \frac{aF(x)[X']^{\alpha}}{X^{\alpha+1}(\rho)} \frac{\left(aF(x)[X']^{\alpha}\right)^{1/\alpha}}{a^{1/\alpha}(\rho)F^{1/\alpha}(x(\rho))}. \end{aligned}$$

Then,

$$\begin{aligned} \theta_1' &\leq \delta \frac{\left(aF(x)[X']^{\alpha}\right)'}{X^{\alpha}(\rho)} + \frac{\delta'}{\delta}\theta_1 \\ &\quad - \frac{\alpha}{\kappa_2^{1/\alpha}} \frac{\delta\rho'}{a^{1/\alpha}(\rho)} \left(\frac{aF(x)[X']^{\alpha}}{X^{\alpha}(\rho)}\right)^{1+1/\alpha} \\ &= \delta \frac{\left(aF(x)[X']^{\alpha}\right)'}{X^{\alpha}(\rho)} + \frac{\delta'}{\delta}\theta_1 \\ &\quad - \frac{\alpha}{\kappa_2^{1/\alpha}} \frac{\rho'}{(\delta a(\rho))^{1/\alpha}} \theta_1^{1+1/\alpha}. \end{aligned}$$

Applying Lemma 2 with

$$m_2 = \frac{\delta'}{\delta} \quad \text{and} \quad m_3 = \frac{\alpha}{\kappa_2^{1/\alpha}} \frac{\rho'}{(\delta a(\rho))^{1/\alpha}},$$

implies that

$$\theta_1' \leq \delta \frac{\left(aF(x)[X']^{\alpha}\right)'}{X^{\alpha}(\rho)} + \frac{\kappa_2}{(\alpha+1)^{\alpha+1}} \frac{(\delta')^{\alpha+1}}{\delta^{\alpha}} \frac{a(\rho)}{(\rho')^{\alpha}}. \tag{16}$$

Analogously, for θ_2, using $\rho(u) \leq \varpi(u) \leq u$ and Lemma 3, we derive

$$\begin{aligned}\theta_2' &= \delta \frac{\left(a(\varpi) F(x(\varpi)) [X'(\varpi)]^{\alpha}\right)'}{X^{\alpha}(\rho)} + \frac{\delta'}{\delta}\theta_2 \\ &\quad -\alpha\delta\rho' \frac{a(\varpi) F(x(\varpi)) [X'(\varpi)]^{\alpha}}{X^{\alpha+1}(\rho)} X'(\rho) \\ &\leq \delta \frac{\left(a(\varpi) F(x(\varpi)) [X'(\varpi)]^{\alpha}\right)'}{X^{\alpha}(\rho)} + \frac{\delta'}{\delta}\theta_2 \\ &\quad -\frac{\alpha}{\kappa_2^{1/\alpha}} \frac{\delta\rho'}{a^{1/\alpha}(\rho)} \left(\frac{a(\varpi) F(x(\varpi)) [X'(\varpi)]^{\alpha}}{X(\rho)}\right)^{1+1/\alpha}\end{aligned}$$

Thus,

$$\begin{aligned}\theta_2' &\leq \delta \frac{\left(a(\varpi) F(x(\varpi)) [X'(\varpi)]^{\alpha}\right)'}{X^{\alpha}(\rho)} + \frac{\delta'}{\delta}\theta_2 \\ &\quad -\frac{\alpha}{\kappa_2^{1/\alpha}} \frac{\rho'}{(\delta a(\rho))^{1/\alpha}} \theta_2^{1+1/\alpha}.\end{aligned}$$

Applying Lemma 2 again yields

$$\begin{aligned}\theta_2' &\leq \delta \frac{\left(a(\varpi) F(x(\varpi)) [X'(\varpi)]^{\alpha}\right)'}{X^{\alpha}(\rho)} \\ &\quad + \frac{\kappa_2}{(\alpha+1)^{\alpha+1}} \frac{(\delta')^{\alpha+1}}{\delta^{\alpha}} \frac{a(\rho)}{(\rho')^{\alpha}}.\end{aligned} \tag{17}$$

Combining (16) and (17), we obtain

$$\begin{aligned}\theta_1' + \frac{b_0^{\alpha}}{\varpi_0}\theta_2' &\leq \delta \frac{\left(a F(x) [X']^{\alpha}\right)'}{X^{\alpha}(\rho)} \\ &\quad + \frac{\kappa_2}{(\alpha+1)^{\alpha+1}} \frac{(\delta')^{\alpha+1}}{\delta^{\alpha}} \frac{a(\rho)}{(\rho')^{\alpha}} \\ &\quad + \frac{b_0^{\alpha}}{\varpi_0}\delta \frac{\left(a(\varpi) F(x(\varpi)) [X'(\varpi)]^{\alpha}\right)'}{X^{\alpha}(\rho)} \\ &\quad + \frac{b_0^{\alpha}}{\varpi_0} \frac{\kappa_2}{(\alpha+1)^{\alpha+1}} \frac{(\delta')^{\alpha+1}}{\delta^{\alpha}} \frac{a(\rho)}{(\rho')^{\alpha}},\end{aligned}$$

i.e. with (6), becomes

$$\begin{aligned}&\theta_1' + \frac{b_0^{\alpha}}{\varpi_0}\theta_2' \\ &\leq \left(1 + \frac{b_0^{\alpha}}{\varpi_0}\right) \frac{\kappa_2}{(\alpha+1)^{\alpha+1}} \frac{(\delta')^{\alpha+1}}{\delta^{\alpha}} \frac{a(\rho)}{(\rho')^{\alpha}} \\ &\quad + \frac{\delta}{X^{\alpha}(\rho)} \left(a F(x) [X']^{\alpha}\right)' \\ &\quad + \frac{\delta}{X^{\alpha}(\rho)} \frac{b_0^{\alpha}}{\varpi_0} \left(a(\varpi) F(x(\varpi)) [X'(\varpi)]^{\alpha}\right)' \\ &\leq \left(1 + \frac{b_0^{\alpha}}{\varpi_0}\right) \frac{\kappa_2}{(\alpha+1)^{\alpha+1}} \frac{(\delta')^{\alpha+1}}{\delta^{\alpha}} \frac{a(\rho)}{(\rho')^{\alpha}} \\ &\quad - \frac{1}{m_1} \widehat{c}\,\delta.\end{aligned}$$

By integrating the last inequality over $[u_1, u]$, we get

$$\int_{u_1}^{u} \left[\frac{1}{m_1} \delta(\xi) \widehat{c}(\xi) - \Psi(\xi)\right] \mathrm{d}\xi \leq m_6,$$

with $m_6 = \theta_1(u_1) + \frac{b_0^{\alpha}}{\varpi_0}\theta_2(u_1)$. A contradiction with (15) completes the proof.

Theorem 4. *Let $\varpi(u) \leq u$, $\varpi' \geq 0$, $b_0 > A^{\widehat{\kappa}}(u)/A^{\widehat{\kappa}}(\varpi(u))$, and $(\overline{\overline{\mathrm{A}}}_2)$ hold. If there exists a positive function $\delta(u) \in \mathbf{C}^1([u_0,\infty), \mathbb{R}^+)$ such that*

$$\limsup_{u\to\infty} \int_{u_1}^{u} [\delta(\xi) B(\xi) - \Phi(\xi)] \mathrm{d}\xi = \infty, \tag{18}$$

then (1) is oscillatory, where

$$\Phi(\xi) = \frac{\kappa_2}{(\alpha+1)^{\alpha+1}} \frac{(\delta'(\xi))^{\alpha+1}}{\delta^{\alpha}(\xi)} \frac{a(\rho(\xi))}{(\rho'(\xi))^{\alpha}}.$$

Proof. Assume, on the contrary, that x is an eventually positive solution of (1) for $u \in [u_0, \infty)$. Proceeding as in the proof of Theorem 3 with (8), we arrive at

$$\begin{aligned}\theta_1' &= \delta \frac{\left(a F(x) [X']^{\alpha}\right)'}{X^{\alpha}(\rho)} + \frac{\delta'}{\delta}\theta_1 \\ &\quad -\alpha\delta\rho' \frac{a F(x) [X']^{\alpha}}{X^{\alpha+1}(\rho)} X'(\rho) \\ &\leq -\delta B + \frac{\delta'}{\delta}\theta_1 - \frac{\alpha}{\kappa_2^{1/\alpha}} \frac{\rho'}{(\delta a(\rho))^{1/\alpha}} \theta_1^{1+1/\alpha}.\end{aligned}$$

Applying Lemma 2, yields

$$\theta_1' \leq -\delta B + \frac{\kappa_2}{(\alpha+1)^{\alpha+1}} \frac{(\delta')^{\alpha+1}}{\delta^{\alpha}} \frac{a(\rho)}{(\rho')^{\alpha}}.$$

By integrating the last inequality over $[u_1, u]$, we get

$$\int_{u_1}^{u} [\delta(\xi) B(\xi) - \Phi(\xi)] \mathrm{d}\xi \leq \theta_1(u_1).$$

A contradiction with (18) completes the proof.

Theorem 5. *Let $\varpi(u) \leq u$, $\varpi' \geq 0$, and $b_0 > A^{\widehat{\kappa}}(u)/A^{\widehat{\kappa}}(\varpi(u))$ hold. If there exists a positive function $\delta(u) \in \mathbf{C}^1([u_0,\infty), \mathbb{R}^+)$ such that*

$$\begin{aligned}&\limsup_{u\to\infty} \int_{u_1}^{u} \left[\delta(\xi) B(\xi) \left(\frac{A^{\widehat{\kappa}}(\rho(\xi))}{A^{\widehat{\kappa}}(\xi)}\right)^{\alpha} - \Theta(\xi)\right] \mathrm{d}\xi \\ &= \infty,\end{aligned} \tag{19}$$

then (1) is oscillatory, where

$$\Theta(\xi) = \frac{\kappa_2}{(\alpha+1)^{\alpha+1}} \frac{(\delta'(\xi))^{\alpha+1}}{\delta^{\alpha}(\xi)} a(\xi).$$

*Proof.*Assume, on the contrary, that x is an eventually positive solution of (1) for $u \in [u_0, \infty)$. Let the following Riccati function

$$\theta_3 = \delta \frac{aF(x)[X']^{\alpha}}{X^{\alpha}},$$

which is clearly positive, and then with (8), we obtain

$$\begin{aligned}\theta_3' &= \delta \frac{\left(aF(x)[X']^{\alpha}\right)'}{X^{\alpha}} + \frac{\delta'}{\delta}\theta_3 - \alpha\delta \frac{aF(x)[X']^{\alpha+1}}{X^{\alpha+1}(\rho)} \\ &\leq -\delta B \frac{X^{\alpha}(\rho)}{X^{\alpha}} + \frac{\delta'}{\delta}\theta_1 - \alpha \frac{1}{(\delta aF(x))^{1/\alpha}}\theta_3^{1+1/\alpha} \\ &\leq -\delta B \frac{X^{\alpha}(\rho)}{X^{\alpha}} + \frac{\delta'}{\delta}\theta_1 - \frac{\alpha}{\kappa_2^{1/\alpha}} \frac{1}{(\delta a)^{1/\alpha}}\theta_3^{1+1/\alpha}.\end{aligned}$$

The the nonincreasing monotonicity of $X/A^{\widehat{\kappa}}$ implies

$$\begin{aligned}\theta_3' \leq{}& -\delta B \left(\frac{A^{\widehat{\kappa}}(\rho)}{A^{\widehat{\kappa}}}\right)^{\alpha} + \frac{\delta'}{\delta}\theta_1 \\ &- \frac{\alpha}{\kappa_2^{1/\alpha}} \frac{1}{(\delta a)^{1/\alpha}}\theta_3^{1+1/\alpha}.\end{aligned}$$

Now, applying Lemma 2, gives

$$\theta_3' \leq -\delta B \left(\frac{A^{\widehat{\kappa}}(\rho)}{A^{\widehat{\kappa}}}\right)^{\alpha} + \frac{\kappa_2}{(\alpha+1)^{\alpha+1}} \frac{(\delta')^{\alpha+1}}{\delta^{\alpha}} a.$$

By integrating the last inequality over $[u_1, u]$, we get

$$\int_{u_1}^{u} \left[\delta(\xi) B(\xi) \left(\frac{A^{\widehat{\kappa}}(\rho(\xi))}{A^{\widehat{\kappa}}(\xi)}\right)^{\alpha} - \Theta(\xi)\right] d\xi \leq \theta_3(u_1).$$

A contradiction with (19) completes the proof.

5 Comparisons and conclusion

In the previous sections, we derived different criteria in Theorems 1–5 to investigate the oscillatory behavior of solutions to the neutral differential equation (1) with multiple delays ($\rho_i(u)$). Each theorem presents a different lens through which we can examine the solutions, offering valuable insights based on varying conditions and methodologies. Now, we shift our focus to synthesizing the contributions of each theorem and comparing them on a deeper level.

To fully appreciate the utility of each theorem, it is crucial to examine how they diverge in their assumptions, and where they converge in their conclusions. This allows us to not only differentiate their practical applications but also to highlight the unique strengths of each.

1. A key distinguishing factor among the theorems is the use of monotonicity conditions $(\overline{\overline{\mathrm{A}}}_1)$, and $(\overline{\overline{\mathrm{A}}}_2)$. Theorems 1, 3, and 4 incorporate one or both of these conditions, making them more suited for specific scenarios where the solutions display certain monotonic behavior. However, this requirement is relaxed in Theorems 2 and 5, which allows us to apply these theorems in a wider range of non-monotonic cases. The ability to accommodate both scenarios is one of the strengths of our approach, offering flexibility for dealing with more complex or irregular solutions.
2. The next major difference lies in the methodological approach used by the theorems. Theorems 1 and 2 follow a first-order comparison method, which is more straightforward and intuitive for solving simpler cases. On the other hand, Theorems 3, 4, and 5 employ the Riccati technique, a more advanced approach that is particularly useful when dealing with higher-order relationships or more complex interactions between variables. The choice between these methods depends on the specific nature of the NDE being studied, Riccati provides deeper insights, but may be more challenging to apply in straightforward cases.
3. Another layer of distinction lies in the temporal relationships between the terms in the equations. In Theorem 1, we assume that $\varpi(u) \geq \rho(u)$ and $\varpi(u) \geq u$, a conditions that introduces an ordering between time-dependent functions. In other theorems, this constraint is relaxed, allowing more freedom to address cases where the relationship between $\varpi(u)$ and $\rho(u)$ is less strict. This flexibility in handling time-dependent relationships is crucial for covering a broader spectrum of possible behaviors in NDE solutions.

*Example 1.*Consider the NDE

$$\begin{aligned}&\left(\frac{1}{1+\kappa_0 \sin^2(x(u))}\left[(x(u)+b_0 x(\varpi_0 u))'\right]^{\alpha}\right)' \\ &\qquad + \frac{c_0}{u^{\alpha+1}} \sum_{i=1}^{n} x^{\alpha}(\nu_i u) = 0, \qquad (20)\end{aligned}$$

for $u > 0$, ϖ_0, $\nu_i \in (0,1]$, $\kappa_0 \in [0,\infty)$, $c_0 > (0,\infty)$, and

$$\nu = \min_{i \in m} \{\nu_i\},$$

for $i = 1,2,\ldots,n$, $n \in \mathbb{N}$. Moreover, it is obvious that $\kappa_2 = 1$, $\kappa_1 = 1/(1+\kappa_0)$, and $\varpi^{[\ell]}(u) = \varpi_0^{\ell} u$ for $\ell \in \mathbb{Z}$. Some calculations gives

$$\begin{aligned}\widehat{\kappa} &= \sqrt[\alpha]{\kappa_0 + 1}, \\ \widehat{c}(u) &= c_0 u^{-\alpha-1}, \\ \tau(u) &= \nu u,\end{aligned}$$

and

$$B(u) = \left(\left[b_0 - \varpi_0^{-\sqrt[\alpha]{\kappa_0+1}}\right] \sum_{j=1}^{k} \frac{1}{b_0^{2j}}\right)^{\alpha} \frac{c_0}{u^{\alpha+1}}$$
$$= B_0 \frac{c_0}{u^{\alpha+1}}.$$

Applying Theorems 1–5, we obtain

$$c_0 > \left(1 + \frac{b_0^{\alpha}}{\varpi_0}\right) \frac{m_1}{\mathrm{e}\ln(1/\nu)\,\nu^{\alpha}} \quad \text{for} \quad \varpi_0 \geq \nu, \tag{21}$$

$$c_0 > \frac{1}{B_0} \frac{1}{\mathrm{e}\ln(1/\nu)\,\nu^{\alpha}} \quad \text{for} \quad b_0 > \varpi_0^{-\sqrt[\alpha]{\kappa_0+1}}, \tag{22}$$

$$c_0 > \left(1 + \frac{b_0^{\alpha}}{\varpi_0}\right) \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{m_1}{\nu^{\alpha}} \quad \text{for} \quad \varpi_0 \geq \nu, \tag{23}$$

$$c_0 > \frac{1}{B_0 \nu^{\alpha}} \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \quad \text{for} \quad b_0 > \varpi_0^{-\sqrt[\alpha]{\kappa_0+1}}, \tag{24}$$

and

$$c_0 > \frac{1}{B_0 \nu^{\alpha \sqrt[\alpha]{\kappa_0+1}}} \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \quad \text{for} \quad b_0 > \varpi_0^{-\sqrt[\alpha]{\kappa_0+1}}, \tag{25}$$

respectively.

*Remark.*For $\alpha = 1$, $\kappa_0 = 0$, $\nu = 1$, and $b_0 = 0$, equation (20) reduces to the ordinary Euler-type

$$x''(u) + \frac{c_0}{u^2} x(u) = 0.$$

In this case, condition (23) provides the well-known sharp criterion which is $c_0 > 1/4$.

*Remark.*For further comparison of our results with existing works in the literature, we apply (20) with $\alpha = 1$, $b_0 = 0.5$ and $\varpi_0 = 0.9$ to Theorem 3 in [20], which gives

$$c_0 > \frac{2}{\mathrm{e}\ln(1/\nu)\,\nu\left(1 + \frac{1}{2} c_0 \nu\right)}. \tag{26}$$

*Example 2.*Take the following special case of (20)

$$\left(\frac{1}{1+\sin^2(x(u))}\left[\left(x(u) + 2x\left(\frac{9}{10}u\right)\right)'\right]\right)'$$
$$+ \frac{c_0}{u^2} x(\nu_1 u) = 0, \tag{27}$$

where $\kappa_0 = 1$, $b_0 = 2$, $\varpi_0 = 0.9$, $\alpha = 1$, and $n = 1$. Table 1 shows the minimum values of c_0 for different values of ν_1.

Table 1. Comparison of the oscillation criteria of (27).

	(21)	(22)	(23)	(24)	(26)
$\nu_1 = 0.2$	3.6826	3.8572	4.0278	4.2187	4.336
$\nu_1 = 0.5$	3.4203	3.5824	1.6111	1.6875	3.206

By analyzing the comparisons presented in the previous table, we observe that our proposed criteria outperform each other under specific conditions. For instance, when the $\nu_1 = 0.2$, Criterion (21) yields better results compared to the others. However, when $\nu_1 = 0.5$, Criteria (23) and (24) provide stronger and more effective outcomes. In all cases, it is evident that our criteria consistently surpass those found in previous works (26), demonstrating the overall improvement and broader applicability of our results.

In conclusion, this study establishes comprehensive and sufficient conditions to determine the oscillatory behavior of all solutions to a general class of neutral delay differential equations. Our results extend previous works by covering both cases where $b_0 < 1$ and $b_0 > 1$, which allows for a more inclusive range of scenarios. Unlike prior studies that imposed monotonicity constraints on delay functions, our criteria relax these restrictions, offering a more versatile framework for analyzing NDEs. Additionally, by reducing our results to the ordinary case, we derived sharp oscillation conditions for the Euler differential equation. Another key advancement is the consideration of the delay function $\varpi(u)$, which was often overlooked in previous research, but plays a crucial role in our analysis. The combination of the comparison and Riccati techniques in our approach ensures robust and practical criteria for oscillation, which are applicable to both functional and ordinary differential equations. These contributions expand the theoretical landscape of oscillation criteria and open avenues for future research, particularly in exploring the effects of more complex delay functions and their applications in diverse scientific fields.

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Conflicts of Interest

The author declares no conflict of interest.

References

[1] G. Kristensson, *Second order differential equations: Special Functions and Their Classification* (Springer Science & Business Media, August 2010).

[2] S. J. Farlow, *An introduction to differential equations and their applications* (Courier Corporation, March 2006).

[3] S. Graham Kelly, *Mechanical vibrations* (Sieditions/Theuniversityof Akron, 2012).

[4] L. Perko, *Differential equations and dynamical systems* (Springer Science & Business Media, February 2008).

[5] C. Huang, B. Liu, H. Yang and J. Cao, Positive almost periodicity on sicnns incorporating mixed delays and d operator, *Nonlinear Analysis Modelling and Control* **27** (2022) p. 1–21.

[6] C. Huang, D. Xiaodan and W. Qian, Periodic stability to a neutral nicholson's blowflies model, *Qualitative Theory of Dynamical Systems* **24** (2025) p. 140.

[7] G. E. Chatzarakis, S. R. Grace and I. Jadlovská, On the sharp oscillation criteria for half-linear second-order differential equations with several delay arguments, *Applied Mathematics and Computation* **397** (2021) p. 125915.

[8] O. Moaaz, H. Ramos and J. Awrejcewicz, Second-order emden–fowler neutral differential equations: A new precise criterion for oscillation, *Applied Mathematics Letters* **118** (2021) p. 107172.

[9] O. Moaaz, W. Albalawi and R. Alotaibi, Functional differential equations with distributed deviating arguments: Oscillatory features of solutions, *Mediterranean Journal of Mathematics* **21**(7) (2024) p. 201.

[10] O. Moaaz and H. Ramos, On the oscillation of second-order functional differential equations with a delayed damping term, *Applied Mathematics Letters* (2025) p. 109464.

[11] I. Jadlovská, J. Dzurina, J. R. Graef and S. R. Grace, Sharp oscillation theorem for fourth-order linear delay differential equations, *Journal of Inequalities and Applications* **2022**(1) (2022) p. 122.

[12] O. Moaaz, C. Cesarano and B. Almarri, An improved relationship between the solution and its corresponding function in fourth-order neutral differential equations and its applications, *Mathematics* **11**(7) (2023) p. 1708.

[13] J. Alzabut, S. R. Grace and G. N. Chhatria, New oscillation results for higher order nonlinear differential equations with a nonlinear neutral terms, *Journal of Mathematics and Computer Science* **28**(03) (2022) p. 294–305.

[14] S. Janaki and V. Ganesan, On the oscillatory behavior of a class of even order nonlinear damped delay differential equations with distributed deviating arguments, *International Journal of Apllied Mathematics* **36**(3) (2023) 391–403.

[15] E. Thandapani and T. Li, On the oscillation of third-order quasi-linear neutral functional differential equations, *Archivum Mathematicum* **47**(3) (2011) p. 181–199.

[16] S.-Y. Zhang and Q.-R. Wang, Oscillation of second-order nonlinear neutral dynamic equations on time scales, *Applied Mathematics and Computation* **216**(10) (2010) p. 2837–2848.

[17] A. Essam, O. Moaaz, M. Ramadan, G. AlNemer and I. M. Hanafy, On the monotonic and asymptotic properties of positive solutions to third-order neutral differential equations and their effect on oscillation criteria, *Axioms* **12**(12) (2023) p. 1086.

[18] C. G. Philos, On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, *Archiv Der Mathematik* **36**(1) (1981) p. 168–178.

[19] Y. Kitamura and T. Kusano, Oscillation of first-order nonlinear differential equations with deviating arguments, *Proceedings of the American Mathematical Society* **78**(1) (1980) p. 64–68.

[20] S. R. Grace, J. Dzurina, I. Jadlovská and T. Li, An improved approach for studying oscillation of second-order neutral delay differential equations, *Journal of Inequalities and Applications* **2018**(1) (2018) p. 193.