Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/190606

Metric Dimension and Secure Metric Dimension of Some Graphs

Ashraf Elrokh[1,*], *Eman S. Almotairi* [2], *Nabawia Elramly* [1] *and Ebitsam Mostafa* [1]

[1]Mathematics and Computer Since Department, Faculty of Science, Menoufia University, Menoufia, 32511, Egypt
[2]Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

Received: 22 May 2025, Revised: 20 Jun. 2025, Accepted: 22 Aug. 2025
Published online: 1 Nov. 2025

Abstract: The metric dimension of a graph is the smallest number of vertices from which the vector of distances to every vertex in the graph is unique. A resolving set S of G with the minimum cardinality is a metric basis of G, and $|S|$ is the metric dimension of G. A resolving set S is secure if for any $v \in V - S$, there exists $x \in S$ such that $(S - \{x\}) \cup \{v\}$ is a resolving set. For some graphs, the value of the resolving set of the graph is determined and defined, the value of the secure resolving number is determined and defined. The results show that different graph families have different Secure resolving set , we will demonstrate that the resolving set dimension and the secure resolving set dimension of including the total graph , the pendent edge graph , the tadpole graph , the open diagonal ladder graph and the bridge graph. This paper investigates the metric and secure metric dimensions of several graph families, including the total graph, pendent edge graph, tadpole graph, open diagonal ladder graph, and bridge graph. For each graph type, we determine the minimal resolving sets and analyze their resolving sets and analyze their structural characteristics. The results reveal notable differences in resolving behavior across these graph classes, offering insights relevant to applications in network discovery, combinatorial optimization, and pattern recognition.

Keywords: Metric dimension, Secure Metric dimension, Resolving set, Graph

1 Introduction

Graph theory is a branch of mathematics that explores the relationship between vertices and edges. One of the fundamental concepts in this field is the resolving set, which has numerous applications in areas such as network discovery [1,2], connected joins in graphs [3], strategies for the Mastermind game [4,5], pattern recognition [6], combinatorial optimization [7,8], image processing [9], and game theory [10].

Let $G = (V,E)$ be a connected, simple and finite graph. Suppose an ordering $(x_1,x_2,...,x_k)$ is imposed on a subset $\overline{S} = \{x_1,x_2,...,x_k \subseteq V(G)\}$. The metric representation of a vertex $b \in V(G)$ with respect to S is defined as the ordered k-tuples $r(b|\overline{S}) = ((x_1,b),(x_2,b),\ldots,d(x_k,b))$ where $d(x_i,b)$ denotes the shortest path distance between x_i and b. If $r(x|\overline{S}) = r(b|\overline{S})$ implies $x = b$ for all $x,b \in V(G)$, then $\overline{S}$ is called a resolving set of G.

A resolving set with the smallest possible cardinality is known as a metric basis, and its cardinality is referred to as the metric dimension of G, denoted by $dim(G)$ [26].

The concept of resolving sets was introduced nearly four decades ago by Slater, who referred to them as locating sets. Independently, Harary and Melter developed a similar concept and introduced the term ”metric dimension.” Since then, extensive research has been conducted on various types of resolving sets, including resolving dominating sets and independent resolving sets.

In graph theory, the notion of security is also associated with different types of sets. In this paper, our primary objective is to compute both the metric dimension and the secure metric dimension for several families of graphs.

* Corresponding author e-mail: ashraf.hefnawy68@yahoo.com

2 Definitions and basic terminology

Definition 1 [11] . A simple graph G consists of a non-empty finite set of $V(G)$ of elements called vertices (or nodes), and a finite set $E(G)$of distinct unordered pairs of distinct elements of $V(G)$ called edges. We call $V(G)$ the vertex set and $E(G)$ the edge set of . An edge $\{v,w\}$ is said to join the vertices v and , and is usually abbreviated to vw. For example , Fig.1 represents the simple graph G whose vertex set $V(G)$ is $\{v,w,z\}$, and whose edge set $E(G)$ consists of the edges uv, uw, vw and wz.

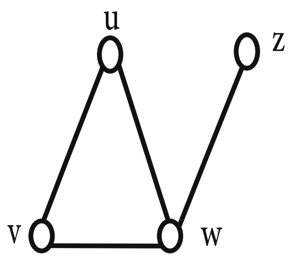

Figure 1. Example of simple graph.

Definition 2 [11] . Resolving Set For a connected graph $G = (V,E)$, a set of vertices $T \subseteq V(G)$ resolves G if every vertex of G is uniquely determined by its vector of distances to the vertices in $r(v|T) = (d(v,x_1), d(v,x_2), ...d(v,x_k))$ is unique for every$v \in V(G)$. For example Figure 2 . $T = (v_1, v_2), (v|T) = ((0,1),(1,0),(1,1),(2,1),(2,2))$, is unique, $T = (v_1, v_2)$, is Resolving Set.

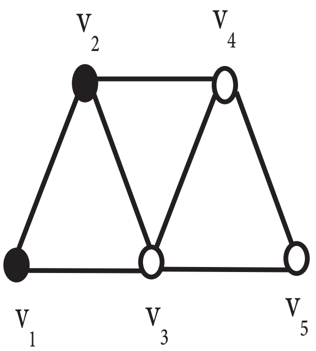

Figure 2. Example of Resolving set.

Definition 3 [12] . Secure Resolving Set a subset T of G is a SR set of G if T is resolving and for any $x \in V - T$, there exists $y \in T$ such that $(T - \{y\}) \cup \{x\}$ is a resolving set of G. The minimum cardinality of a SR set of G is known as the secure resolving dimension of , and is marked by $sdim(G)$. For example, Figure 3.

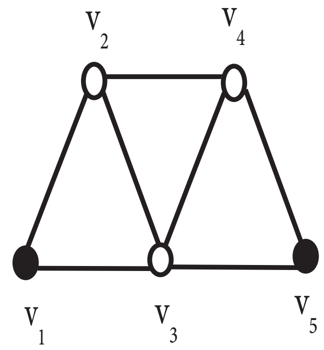

Figure 3. Example of secure resolving set.

$T = (v_1, v_2)$ is Resolving Set.
$y = \{v_2\}$
$x = (v_3, v_4, v_5)$
$H = (T - \{y\}) \bigcup \{x\} = (v_1, v_5)$
$r(v|H) = ((0,2),(1,2),(1,1),(2,1),(2,0))$ is unique,
$H = (v_1, v_5)$ is Resolving Set , T is Secure Resolving Set.

Definition 4 [13] . The total graph $T(G)$ of G is the graph with the vertex set $V \bigcup E$ and two vertices are adjacent whenever they are either adjacent or incident in G .

Definition 5 [14] . The pendant number of a graph , denoted by $\prod p(G)$, is the least number of vertices in a graph such that they are the end vertices of a path in a given path decomposition of a graph G .

Definition 6 [15] . bridge of a connected graph is a graph edge whose removal disconnects the graph. More generally, a bridge is an edge of a not-necessarily-connected graph whose removal increases the number of components of G .

Definition 7 [16] . The (m,n)-tadpole graph, also called a dragon graph or kite graph is the graph obtained by joining a cycle graph C_m to a path graph $P_n$with a bridge .

3 Main Results

Here, we display that the metric dimension and secure metric dimension of including the total graph , the pendent edge graph , the tadpole graph , the open diagonal ladder graph and the bridge graph.

Metric dimension. Is the smallest number of vertices from which the vector of distances to every vertex in the graph is unique .

Theorem 1 Let $T(G)$ be total graph with n vertices and k blocks then the metric dimension equal to 2.

Proof. We label total graph as shown in Figure 4 . We select a subset $T = \{v_1, v_{(n-(2k-1))}\}$, and we must show that $dim(T(G)) = 2$ for $n \geq 3$, $n = 2k+1, k \geq 1$.
The proof is as follows:-

Let $T=(v_1,v_{(n-(2k-1))})$ is a metric dimension of total graph where $K=1,2,3,\ldots..n$ and $n=2k+1$

Begin $d(v_1,T)=(0,1)$

for$(i=2;i\leq n-1;i=i+2)$ do

$d(v_i,T)=(\frac{i}{2},\frac{i}{2}-1)$

end

for $(i=3;i\leq n;i=i+2)$ **do**

$d(v_i,T)=(\frac{i-1}{2},\frac{i-1}{2})$

end

end

This completes the proof .

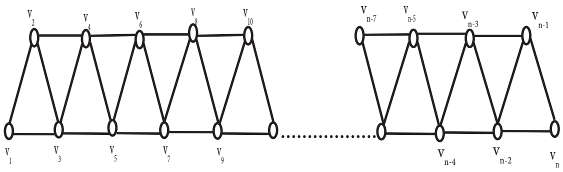

Figure 4. The total graph.

- Ex1:- The example of total graph block 2:-
$T=(v_1,v_2)$ is resolving set Figure 5 :-
$d(v_1,T)=(0,1), d(v_2,T)=(1,0)$,
$d(v_3,T)=(1,1), d(v_4,T)=(2,1)$,
$d(v_5,T)=(2,2)$.

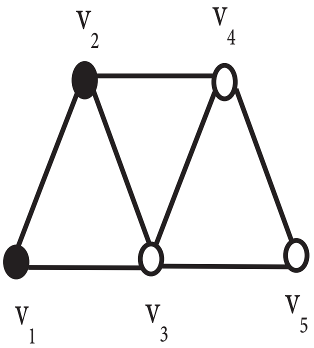

Figure 5. Total graph with block 2.

Theorem 2. Let $\prod p(G)$ be pendent edge graph with n vertices and k blocks then the metric dimension equal to 2.

Proof. We label pendent edge graph as shown in Figure 6 . We select a subset $T=\{v_1,v_{(n-k)}\}$, and we must show that $dim(\prod p(G))=2$ for $n\geq 5$,$n=3k+2$, $k\geq 1$.

The proof is as follows:-

Let $T=(v_1,v_{(n-k)})$ is a metric dimension of pendent edge graph where $K=1,2,3,\ldots..n$ and $n=3k+2$

for $(i=1;i\leq n-k;i=i+1)$ **do**

$d(v_i,T)=(i-1,n-k-i)$

end

for$(i=n-k+1;i\leq n;i=i+1)$**do**

$d(v_i,T)=(2i-4k-4,2n-2i+2)$

end

end

This completes the proof .

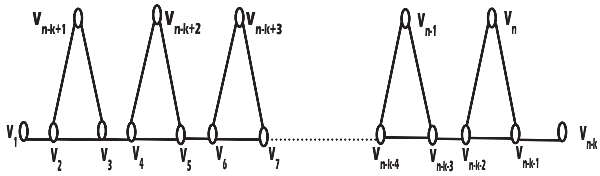

Figure 6. The pendent edge graph.

- Ex2 :-The example of pendent edge graph block 2:-
$T=(v_1,v_6)$ is resolving set Figure 7:-
$d(v_1,T)=(0,5), d(v_2,T)=(1,4)$,
$d(v_3,T)=(2,3), d(v_4,T)=(3,2)$,
$d(v_5,T)=(4,1), d(v_6,T)=(5,0)$,
$d(v_7,T)=(2,4), d(v_8,T)=(4,2)$.

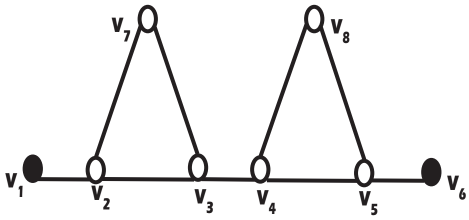

Figure 7. Pendent edge graph with block 2.

Theorem 3. Let G be Open Diagonal ladder graph$O(Dl_n)$ with n vertices and k blocks then the metric dimension equal to $n/2$.

Proof.

We label Open Diagonal ladder graph as shown in Figure 8. We select a subset $T=\{v_1,\ldots,v_{(n/2)}\}$, and we must show that $dim(O(Dl_n)(G))=\frac{n}{2}$ for $n\geq 4$, $n=2k+2$, $k\geq 3$.

The proof is as follows:-

Case 1 $k=1$ then $dim(G)=2$.

Case 2 $k=2$ then $dim(G)=4$.

Let $T=(v_1,\ldots,v_{(n/2)})$ is a metric dimension of open diagonal ladder graph where $n=2k+2$ and $dim(G)=k+1=n/2$ and $K=3,4,\ldots..10$

for$(i=1;i\leq n/2;i=i+1)$ **do**

$d(v_i,T)=(|i-1|,|i-2|,|i-3|,\ldots,|i-(n/2-1)|,|n/2-i|)$.

end

$d(v_{(n/2+1)},T)=((n-i),(n-i-1),\ldots,(n-i-(n/2-2)),(i-k))$.

for $(i=n/2+2;i\leq n/2+3;i=i+1)$**do**

$d(v_i,T)=((n-i),(n-i-1),\ldots,(n-i-(k-3)),1,1,(i-k-2))$.

end

for $(i=n/2+4;i\leq n/2+5;i=i+1)$ **do**

$d(v_i,T)=((n-i),(n-i-1),\ldots,(n-i-(k-5)),1,1,(i-k-4),(i-k-3),(i-k-2)$.

end
$for(i = n/2+6; i \leq n/2+7; i = i+1)$ **do**
$d(v_i,T) = ((n-i),(n-i-1),\dots,(n-i-(k-7)),1,1,(i-k-6),\dots,(i-k-3),(i-k-2)$.
end
$for(i = n/2+8; i \leq n/2+9; i = i+1)$ **do**
$d(v_i,T) = ((n-i),(n-i-1),1,1,(i-k-8),\dots,(i-k-3),(i-k-2)$.
end
for$(i = n-1; i \leq n; i = i+1)$ **do**
$d(v_i,T) = ((i-2k),1,(i-k-10),(i-k-9),\dots,(i-k-3),(i-k-2))$.
end
end
This completes the proof .

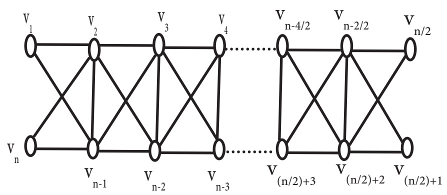

Figure 8. The Open Diagonal ladder graph.

- Ex3 :-The example of The Open Diagonal ladder graph block 3:-
$T = (v_1,v_2,v_3,v_4)$ is resolving set Figure 9 :-
$d(v_1,T) = (0,1,2,3), d(v_2,T) = (1,0,1,2)$,
$d(v_3,T) = (2,1,0,1), d(v_4,T) = (3,2,1,0)$,
$d(v_5,T) = (3,2,1,2), d(v_6,T) = (2,1,1,1)$,
$d(v_7,T) = (1,1,1,2), d(v_8,T) = (2,1,2,3)$.

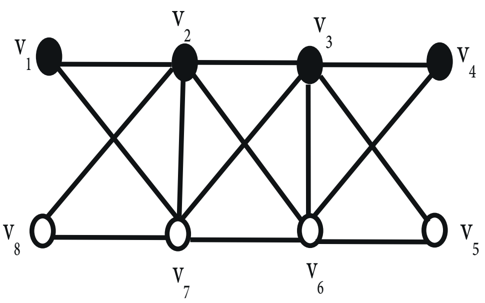

Figure 9. Open Diagonal ladder graph with block 3.

Theorem 4. Let$G$ be Bridge graph with n vertices and k blocks then the metric dimension equal to$n/2+1$.

Proof.
We label Bridge graph as shown in Figure 10 . We select a subset $T = \{v_1,v_2,\dots,v_{(n/2+1)}\}$, and we must show that $dim(G) = \frac{n}{2}+1$ for $n \geq 6$, $n = 3k+2$, $k \geq 1$.

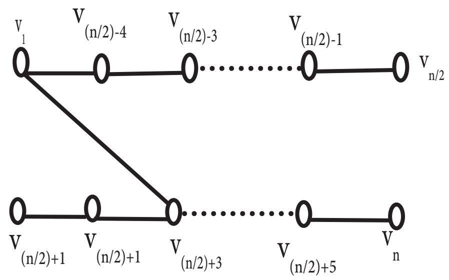

Figure 10. The bridge graph.

The proof is as follows:-
Case 1: k is even
Let $T = (v_1,v_2,\dots,v_{(n/2+1)})$ is a metric dimension of bridge graph with even blocks where $K = 2,4,6,8\dots n$ and $n = 6k$ and $m = 1,2,3,4,5,6,\dots\dots$
for$(i = 1; i \leq n/2; i = i+1)$**do**
$d(v_i,T) = (|i-1|,|i-2|,|i-3|,\dots|i-10|,|i-11|,|i-12|,\dots|i-m|,(i+2))$.
end

$for(i = n/2+1; i \leq n/2+2; i = i+1)$ **do**
$d(v_i,T) = (((n/2+2)-i)+2,(((n/2+2)-i)+2)+1,(((n/2+2)-i)+2)+2,\dots,(((n/2+2)-i)+2)+m,(i-(n/2+1)))$
end

for $(i = n/2+3; i \leq n; i = i+1)$ **do**
$d(v_i,T) = ((i-(n/2+2)),(i-(n/2+2))+1,(i-(n/2+2))+2,\dots((i-(n/2+2)))+m,(i-(n/2+1)))$
end
end
This completes the proof .

- Ex4 :-The example of The bridge graph block 2:-
$T = (v_1,v_2,v_3,v_4,v_5,v_6,v_7)$ is resolving set Figure 11 :-
$d(v_1,T) = (0,1,2,3,4,5,3), d(v_2,T) = (1,0,1,2,3,4,4)$,
$d(v_3,T) = (2,1,0,1,2,3,5), d(v_4,T) = (3,2,1,0,1,2,6)$,
$d(v_5,T) = (4,3,2,1,0,1,7), d(v_6,T) = (5,4,3,2,1,0,8)$.
$d(v_7,T) = (3,4,5,6,7,8,0), d(v_8,T) = (2,3,4,5,6,7,1)$,
$d(v_9,T) = (1,2,3,4,5,6,2), d(v_{10},T) = (2,3,4,5,6,7,3)$,
$d(v_{11},T) = (3,4,5,6,7,8,4), d(v_{12},T) = (4,5,6,7,8,9,5)$.

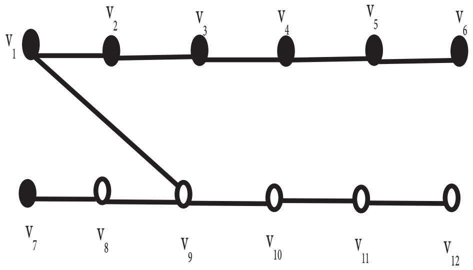

Figure 11. Bridge graph with block 2.

Case 2:- k is odd.

Let $T = (v_1, v_2, \ldots, v_{(n/2+1)})$ is a metric dimension of bridge graph with odd blocks where $K = 1,3,5,7,9\ldots n$ and $n = 6k$ and $m = 1,2,3,4,5,6,\ldots\ldots$

for $(i = 1; i \leq n/2; i = i+1)$ **do**
$d(v_i,T) = (|i-1|, |i-2|, |i-3|, \ldots |i-10|, |i-11|, |i-12|, \ldots |i-m|, (i+2))$
end

for $(i = n/2+1; i \leq n/2+2; i = i+1)$ **do**
$d(v_i,T) = (((n/2+2)-i)+2, (((n/2+2)-i)+2)+1, (((n/2+2)-i)+2)+2, \ldots, (((n/2+2)-i)+2)+m, (i-(n/2+1)))$
end
for $(i = n/2+3; i \leq n; i = i+1)$ do
$d(v_i,T) = ((i-(n/2+2)), (i-(n/2+2))+1, (i-(n/2+2))+2, \ldots ((i-(n/2+2)))+m, (i-(n/2+1)))$
end
end

This completes the proof .

- Ex5:- The example of The bridge graph block 3:-

-$T = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10})$ is resolving set Figure 12 :-

$d(v_1,T) = (0,1,2,3,4,5,6,7,8,3), d(v_2,T) = (1,0,1,2,3,4,5,6,7,4),$
$d(v_3,T) = (2,1,0,1,2,3,4,5,6,5), d(v_4,T) = (3,2,1,0,1,2,3,4,5,6),$
$d(v_5,T) = (4,3,2,1,0,1,2,3,4,7), d(v_6,T) = (5,4,3,2,1,0,1,2,3,8),$
$d(v_7,T) = (6,5,4,3,2,1,0,1,2,9), d(v_8,T) = (7,6,5,4,3,2,1,0,1,10),$
$d(v_9,T) = (8,7,6,5,4,3,2,1,0,11), d(v_{10},T) = (3,4,5,6,7,8,9,10,11,0),$
$d(v_{11},T) = (2,3,4,5,6,7,8,9,10,1), d(v_{12},T) = (1,2,3,4,5,6,7,8,9,2),$
$d(v_{13},T) = (2,3,4,5,6,7,8,9,10,3), d(v_{14},T) = (3,4,5,6,7,8,9,10,11,4),$
$d(v_{15},T) = (4,5,6,7,8,9,10,11,12,5), d(v_{16},T) = (5,6,7,8,9,10,11,12,13,6),$
$d(v_{17},T) = (6,7,8,9,10,11,12,13,14,7) d(v_{18},T) = (7,8,9,10,11,12,13,14,15,8)$

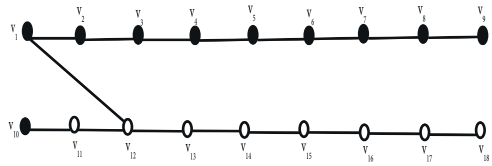

Figure 12. Bridge graph with block 3.

Theorem 5. Let $T(m,n)$be tadpole graph with n vertices and k blocks then the metric dimension equal to 2.

Proof.

We label tadpole graph as shown in Figure 13 . We select a subset $T = \{v_1, v_{(n-k-1)}\}$, and we must show that $dim(G) = 2$ for $n \geq 4$, $n = k+3$, $k \geq 1$.

The proof is as follows:-

Let $T = (v_1, v_{(n-k-1)})$ is a metric dimension of tadpole graph where $K = 1,2,3,\ldots..n$ and $n = k+3$

Begin $d(v_1,T) = (0,1), d(v_2,T) = (1,0)$
$for(i = 3; i \leq n; i = i+1)$ **do**
$d(v_i,T) = (i-2, i-2)$
end
end

This completes the proof .

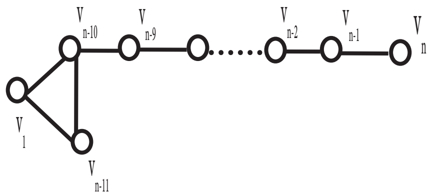

Figure 13. The tadpole graph.

- Ex6:- The example of tadpole graph block 1:-

$T = (v_1, v_2)$ is resolving set Figure 14 :-
$d(v_1,T) = (0,1), d(v_2,T) = (1,0),$
$d(v_3,T) = (1,1), d(v_4,T) = (2,2).$

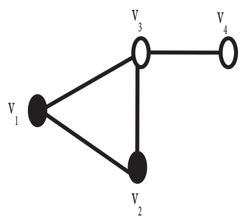

Figure 14. Tadpole graph with block 1.

3.1 Secure metric dimension

Theorem 6. Let $T(G)$ be total graph with n vertices and k blocks then the Secure metric dimension equal to 2.

Proof. We label total graph as shown in Figure 15 . We select a subset $H = \{v_1, v_n\}$, and we must show that $Sdim(T(G)) = 2$ for $n \geq 3$,$n = 2k+1$, $k \geq 1$.

- The proof is as follows:-

Let $H = (v_1, v_n)$ is a Secure metric dimension of total graph where $K = 1,2,3,\ldots..n$ and $n = 2k+1$

Begin $d(v_1,H) = (0,K), d(v_2,H) = (1,K)$

for $(i = 3; i \leq n-2; i = i+2)$ **do**
$d(v_i,H) = (\frac{i-1}{2}, \frac{n-i}{2})$
end
for $(i = 4; i \leq n-3; i = i+2)$ **do**
$d(v_i,H) = (\frac{i}{2}, n-k-\frac{i}{2})$
end
$d(v_{n-1},H) = (K,1), d(v_n,H) = (K,0)$.

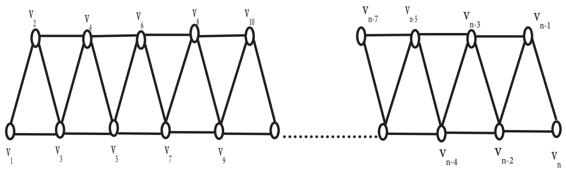

Figure 15. The total graph.

- Ex7:- The example of total graph block 2:
$H = (v_1, v_5)$ is secure resolving set Figure 16 :
$d(v_1,H) = (0,2), d(v_2,H) = (1,2)$,
$d(v_3,H) = (1,1), d(v_4,H) = (2,1)$,
$d(v_5,H) = (2,0)$.

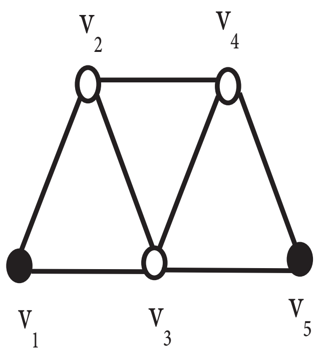

Figure 16. Secure metric dimension of total graph with block 2.

Theorem 7. Let $\prod p(G)$ be pendent edge graph with n vertices and k blocks then the Secure metric dimension equal to 2.

Proof.

We label pendent edge graph as shown in Figure 17 . We select a subset $H = \{v_2, v_{(n-k)}\}$, and we must show that $Sdim\prod(p(G)) = 2$ for $n \geq 5$, $n = 3k+2$, $k \geq 1$.

The proof is as follows:

Let $H = (v_2, v_{(n-k)})$ is a Secure metric dimension of pendent edge graph where $K = 1,2,3,\ldots..n$ and $n = 3k+2$

Begin $d(v_1,H) = (1, n-k-1)$

$for(i = 2; i \leq n-k; i = i+1)$ **do**
$d(vi,H) = (i-2, n-i-k)$
end
$for(i = n-k+1; i \leq n; i = i+1)$ **do**
$d(v_i,H) = (2i-4k-5, 2n-2i+2)$
end
end

This completes the proof .

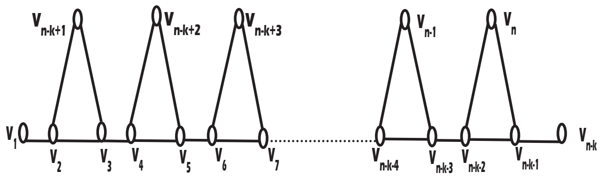

Figure 17. The pendent edge graph.

- Ex8:- The example of pendent edge graph block 2:
$H = (v_2, v_6)$is secure resolving set Figure 18 :-
$d(v_1,H) = (1,5), d(v_2,H) = (0,4)$,
$d(v_3,H) = (1,3), d(v_4,H) = (2,2)$,
$d(v_5,H) = (3,1), d(v_6,H) = (4,0)$,
$d(v_7,H) = (1,4), d(v_8,H) = (3,2)$.

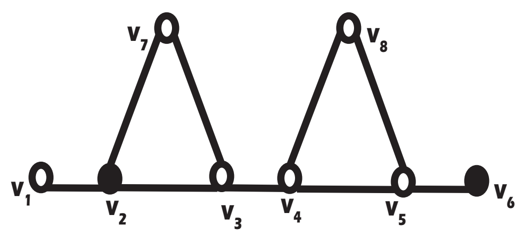

Figure 18. Secure metric dimension of pendent edge graph with block 2.

Theorem 8. Let G be Open Diagonal ladder graph $O(Dl_n)$ with n vertices and k blocks then the Secure metric dimension equal to $\frac{n}{2}$.

Proof. We label Open Diagonal ladder graph as shown in Figure 19. We select a subset $H = \{v_1, \ldots, v_{(n/2+1)}\}$, and we must show that $Sdim\prod(O(Dl\_n)(G)) = \frac{n}{2}$ for $n \geq 4$, $n = 2k+2$, $k \geq 3$.

The proof is as follows:-

Case 1. $k = 1$ then $Sdim(G) = 2$

Case 2. $k = 2$ then $Sdim(G) = 4$

Let $H = (v_1, \ldots, v_{(n/2+1)})$ is a Secure metric dimension of open diagonal ladder graph where

$n = 2k + 2$ and $K = 3, 4, \ldots 10$
$\ni Sdim(G) = k + 1 = n/2$

for$(i = 1; i \leq n/2 - 1; i = i + 1)$**do**

$d(v_i, H) = (|i-1|, |i-2|, |i-3|, \ldots, |i - (\frac{n}{2} - 1)|, |\frac{n}{2} - i|)$.

end

$d(v_{(\frac{n}{2})}, H) = ((n-i), (n-i-1), \ldots, (n-i-(\frac{n}{2}-2)), ((\frac{n}{2}-i)+2))$.

$d(v_{(\frac{n}{2}+1)}, H) = ((n-i), (n-i-1), \ldots, (n-i-(\frac{n}{2}-2)), (i-k-2))$.

for $(i = \frac{n}{2}+2; i \leq \frac{n}{2}+3; i = i+1)$ **do**

$d(v_i, H) = ((n-i), (n-i-1), \ldots, (n-i-(k-3)), 1, 1, (i-k-2))$.

end

for $(i = \frac{n}{2}+4; i \leq \frac{n}{2}+5; i = i+1)$ **do**

$d(v_i, H) = ((n-i), (n-i-1), \ldots, (n-i-(k-5)), 1, 1, (i-k-4), (i-k-3), (i-k-2))$.

end

for $(i = \frac{n}{2}+6; i \leq \frac{n}{2}+7; i = i+1)$ **do**

$d(v_i, H) = ((n-i), (n-i-1), \ldots, (n-i-(k-7)), 1, 1, (i-k-6), \ldots, (i-k-3), (i-k-2))$.

end

for$(i = \frac{n}{2}+8; i \leq \frac{n}{2}+9; i = i+1)$ **do**

$d(v_i, H) = ((n-i), (n-i-1), 1, 1, (i-k-8), \ldots, (i-k-3), (i-k-2))$.

end

for $(i = n-1; i \leq n; i = i+1)$ **do**

$d(v_i, H) = ((i-2k), 1, (i-k-10), (i-k-9), \ldots, (i-k-3), (i-k-2))$.

end

end

This completes the proof .

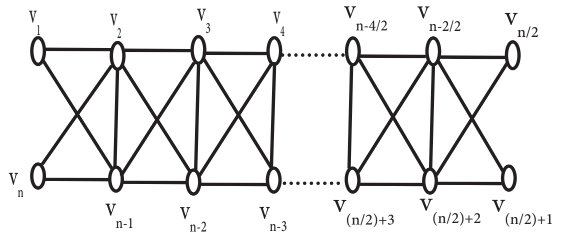

Figure 19. The Open Diagonal ladder graph.

Ex9:- The example of The Open Diagonal ladder graph block 3:

$H = (v_1, v_2, v_3, v_5)$ is secure resolving set Figure 20.
$d(v_1, H) = (0, 1, 2, 3), d(v_2, H) = (1, 0, 1, 2)$,
$d(v_3, H) = (2, 1, 0, 1), d(v_4, H) = (3, 2, 1, 2)$,
$d(v_5, H) = (3, 2, 1, 0), d(v_6, H) = (2, 1, 1, 1)$,
$d(v_7, H) = (1, 1, 1, 2), d(v_8, H) = (2, 1, 2, 3)$.

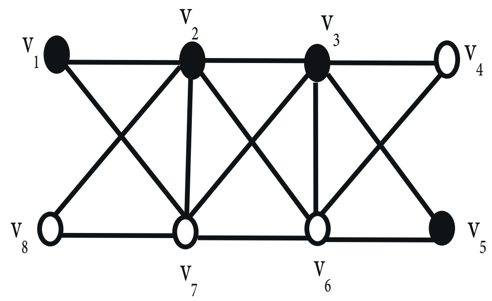

Figure 20. Secure metric dimension of Open Diagonal ladder graph with block 3.

Theorem 9. Let G be Bridge graph with n vertices and k blocks then the Secure metric dimension equal to $\frac{n}{2} + 1$.

Proof. We label Bridge graph as shown in Figure 21 . We select a subset $H = \{v_1, v_2, \ldots, v_{(n/2)}, v_{(n-1)}\}$, and we must show that $Sdim(G) = \frac{n}{2} + 1$ for$n \geq 6$, $n = 3k + 2$, $k \geq 1$.

The proof is as follows:-

Case 1: k is even:-

Let $H = (v_1, v_2, \ldots,, v_{(\frac{n}{2})}, v_{(n-1)})$ is a secure metric dimension of bridge graph with even blocks where $K = 2, 4, 6, 8 \ldots n$ and $n = 6k$ $\ni$
$m = 1, 2, 3, 4, 5, 6, \ldots\ldots$

for $(i = 1; i \leq \frac{n}{2}; i = i+1)$ **do**

$d(v_i, H) = (|i-1|, |i-2|, |i-3|, \ldots |i-10|, |i-11|, |i-12|, \ldots |i-m|, (i + (n/2 - 4)))$.

end

for $(i = \frac{n}{2}+1; i \leq \frac{n}{2}+2; i = i+1)$ **do**

$d(v_i, H) = (((n/2+2)-i)+2), ((n/2+2)-i)+2)+1, ((n/2+2)-i)+2)+2, \ldots, ((n/2+2)-i)+2)+m, (((9/2k-1)-i)+n/4))$.

end

for $(i = n/2+3; i \leq n; i = i+1)$ **do**

$d(v_i, H) = ((i-(n/2+2)), (i-(n/2+2))+1, (i-(n/2+2))+2, \ldots, ((i-(n/2+2)))+m, |((9/2k-1)-i)+n/4|)$

end

end

This completes the proof .

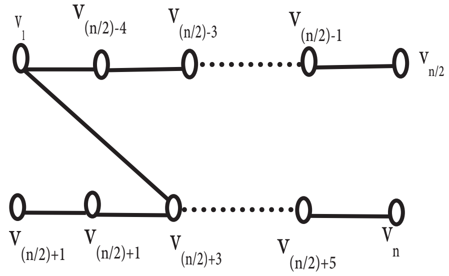

Figure 21. The bridge graph.

- Ex9:- The example of The bridge graph block 2:-

$H = (v_1, v_2, v_3, v_4, v_5, v_6, v_{11})$ is secure resolving set Figure 22 :-

$d(v_1, H) = (0,1,2,3,4,5,3), d(v_2, H) = (1,0,1,2,3,4,4)$,
$d(v_3, H) = (2,1,0,1,2,3,5), d(v_4, H) = (3,2,1,0,1,2,6)$,
$d(v_5, H) = (4,3,2,1,0,1,7), d(v_6, H) = (5,4,3,2,1,0,8)$.
$d(v_7, H) = (3,4,5,6,7,8,4), d(v_8, H) = (2,3,4,5,6,7,3)$,
$d(v_9, H) = (1,2,3,4,5,6,2), d(v_{10}, H) = (2,3,4,5,6,7,1)$,
$d(v_{11}, H) = (3,4,5,6,7,8,0), d(v_{12}, H) = (4,5,6,7,8,9,1)$.

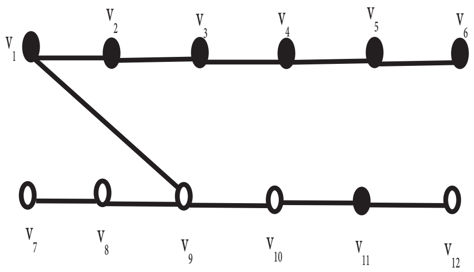

Figure 22. Secure metric dimension of bridge graph with block 2.

Case 2:- k is odd.

Let $H = (v_1 \ldots, v_{(n/2)}, v_{(n-1)})$ is a secure metric dimension of bridge graph with oddd blocks where $K = 1,3,5,7,9\ldots n$ and $n = 6k \ni m = 1,2,3,4,5,6,\ldots\ldots$

for$(i = 1; i \leq n/2; i = i+1)$ **do**
$d(v_i, H) = (|i-1|, |i-2|, |i-3|, \ldots |i-10|, |i-11|, |i-12|, \ldots |i-m|, (i+(n/2-4))$.
end
for $(i = n/2+1; i \leq n/2+2; i = i+1)$ **do**
$d(v_i, H) = (((n/2+2)-i)+2, (((n/2+2)-i)+2)+1, (((n/2+2)-i)+2)+2, \ldots, (((n/2+2)-i)+2)+m, ((n-k-i)+2))$.
end
for $(i = n/2+3; i \leq n; i = i+1)$ do
$d(v_i, H) = ((i-(n/2+2)), (i-(n/2+2))+1, (i-(n/2+2))+2, \ldots, ((i-(n/2+2)))+m, (|n-i-1|))$.
end
end

This completes the proof .

- Ex10:- The example of The bridge graph block 3:-

-$H = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{17})$ is secure resolving set Figure 23:-

$d(v_1, H) = (0,1,2,3,4,5,6,7,8,6), d(v_2, H) = (1,0,1,2,3,4,5,6,7,7)$,
$d(v_3, H) = (2,1,0,1,2,3,4,5,6,8), d(v_4, H) = (3,2,1,0,1,2,3,4,5,9)$,
$d(v_5, H) = (4,3,2,1,0,1,2,3,4,10), d(v_6, H) = (5,4,3,2,1,0,1,2,3,11)$,
$d(v_7, H) = (6,5,4,3,2,1,0,1,2,12), d(v_8, H) = (7,6,5,4,3,2,1,0,1,13)$,
$d(v_9, H) = (8,7,6,5,4,3,2,1,0,14), d(v_{10}, H) = (3,4,5,6,7,8,9,10,11,7)$,
$d(v_{11}, H) = (2,3,4,5,6,7,8,9,10,6), d(v_{12}, H) = (1,2,3,4,5,6,7,8,9,5)$,
$d(v_{13}, H) = (2,3,4,5,6,7,8,9,10,4), d(v_{14}, H) = (3,4,5,6,7,8,9,10,11,3)$,
$d(v_{15}, H) = (4,5,6,7,8,9,10,11,12,2), d(v_{16}, H) = (5,6,7,8,9,10,11,12,13,1)$,
$d(v_{17}, H) = (6,7,8,9,10,11,12,13,14,0) d(v_{18}, H) = (7,8,9,10,11,12,13,14,15,1)$.

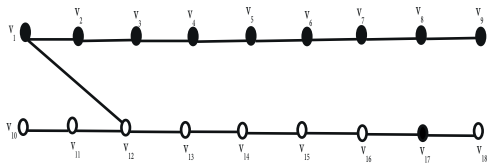

Figure 23. Secure metric dimension of bridge graph with block 3.

Theorem 10. Let$T(m,n)$ be tadpole graph with n vertices and k blocks then the secure metric dimension equal to 2.

Proof.

- We label tadpole graph as shown in Figure 24 . We select a subset $H = \{v_1, v_{(n-k)}\}$, and we must show that $Sdim(G) = 2$ for $n \geq 4$, $n = k+3$, $k \geq 1$.

-The proof is as follows:-

Let $H = (v_1, v_{(n-K)})$ is a secure metric dimension of tadpole graph where $K = 1,2,3,\ldots..n$ and $n = k+3$

Begin $d(v_1, H) = (0,1), d(v_2, H) = (1,1)$
$for(i = 3; i \leq n; i = i+1)$ do
$d(v_i, H) = (i-2, i-(n-k))$
end
end

This completes the proof .

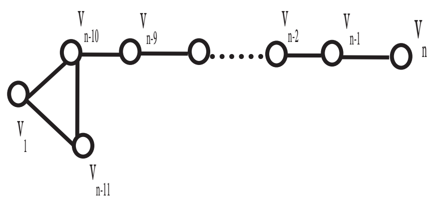

Figure 24. The tadpole graph .

- Ex11:- The example of tadpole graph block 1:
- $H = (v_1, v_3)$ is secure resolving set Figure 25:-

$$d(v_1,T) = (0,1), d(v_2,T) = (1,1),$$
$$d(v_3,T) = (1,0), d(v_4,T) = (2,1).$$

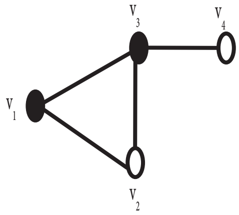

Figure 25. Secure metric dimension of tadpole graph with block 1.

4 Conclusion

In this paper we have calculated and explained the resolving set dimension and the secure resolving set dimension for some graphs like the total graph , the pendent edge graph , the tadpole graph , the open diagonal ladder graph and the bridge graph, as shown in the following table:-

Table 1: Summary of results

Graph type	Resolving set dimension	Secure resolving set dimension
The total graph	$dim(T(G)) = 2$ for $n \geq 3$	$Sdim(T(G)) = 2$ for $n \geq 3$
The pendent edge graph	$dim(\prod p(G)) = 2$ for $n \geq 5$	$Sdim(\prod p(G)) = 2$ for $n \geq 5$
The open diagonal ladder graph	at $k = 1 \rightarrow dim(G) = 2$, at $k = 2 \rightarrow dim(G) = 4$, at $k = 10 \rightarrow dim(O(Dl\_n)(G)) = k+1 = n/2$	at $k = 1 \rightarrow Sdim(G) = 2$, at $k = 2 \rightarrow Sdim(G) = 4$, at $k = 10 \rightarrow Sdim(O(Dl\_n)(G)) = k+1 = n/2$
The bridge graph	$dim(G) = n/2+1$ for $n \geq 6$	$Sdim(G) = n/2+1$ for $n \geq 6$
The tadpole graph	$dim(G) = 2$ for $n \geq 4$	$Sdim(G) = 2$ for $n \geq 4$

Acknowledgement

The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2025).

References

[1] Iqbal M. Batiha , Basma Mohamed and Iqbal H. Jebril ," Secure metric dimension of new classes of graphs " , Received 21 April 2024; accepted 31 May 2024; published online 15 July 2024 DOI , https://doi.org/10.21595/mme.2024.24168.

[2] N. Vesselinova, R. Steinert, D. F. Perez-Ramirez and M. Boman, " Learning combinatorial optimization on graphs: A survey with applications to networking, IEEE Access 8 (2020) ", 120388-120416.

[3] A. Sebő and E. Tannier, On metric generators of graphs, Mathematics of Operations Research 29(2) (2004), 383-393.

[4] C. Glazik, G. Jäger, J. Schiemann and A. Srivastav, Bounds for the static permutation mastermind game, Discrete Mathematics 344(3) (2021), 112253.

[5] A. Berger, C. Chute and M. Stone, Query complexity of mastermind variants, Discrete Mathematics 341(3) (2018), 665-671.

[6] Y. Chen, M. Rohrbach, Z. Yan, Y. Shuicheng, J. Feng and Y. Kalantidis, Graph-based global reasoning networks, In Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition, 2019, pp. 433-442.

[7] Z. Li, Q. Chen and V. Koltun, Combinatorial optimization with graph convolutional networks and guided tree search, Advances in Neural Information Processing Systems (2018), 31.

[8] M. Gasse, D. Chételat, N. Ferroni, L. Charlin and A. Lodi, Exact combinatorial optimization with graph convolutional neural networks, Advances in Neural Information Processing Systems (2019), 32.

[9] A. Ortega, P. Frossard, J. Kovačević, J. M. Moura and P. Vandergheynst, Graph signal processing: Overview, challenges, and applications, Proceedings of the IEEE 106(5) (2018), 808-828.

[10] M. K. Sohrabi and H. Azgomi, A survey on the combined use of optimization methods and game theory, Archives of Computational Methods in Engineering 27(1) (2020), 59-80.

[11] Robin J. Wilson," Introduction to Graph Theory "1972, 1996 chapter 2 , p.17 .

[12] Hemalathaa Subramanian and Subramanian Arasappan "Secure Resolving Sets in a Graph "5 September 2018; Accepted: 19 September 2018; Published: 27 September2018.

[13] N. Inayah, I. W. Sudarsana, S. Musdalifah and N. Daeng Mangesa, On super mean labeling for total graph of path and cycle, International Journal of Mathematics and Mathematical Sciences (2018), 1-5.

[14] Jomon, K, Sebastian., Joseph, Varghese, Kureethara., Sudev, Naduvath., Charles, Dominic. "A study on the pendant number of graph products." Acta Universitatis Sapientiae: Informatica, 11 (2019).:24-40. doi: 10.2478/AUSI-2019-0002 .

[15] Harary, F. " Graph Theory " . Reading, MA: Addison-Wesley, 1994 , P.26 . West, D. B. " Introduction to Graph Theory ", 2nd ed. Englewood Cliffs, NJ: Prentice-Hall,pp. 155-158,2000,P.23.

[16] Kim, S.-R. and Park, J. Y. "On Super Edge-Magic Graphs." Ars Combin. 81, 113-127, 2006. Truszczyński, M. "Graceful Unicyclic Graphs." Demonstatio Math. 17,377-387,1984.

[17] ELrokh, A., Al-Shamiri, M. M. A., Almazah, M. M. A., & El-hay, A. A. (2023). A Novel Problem for Solving Permuted Cordial Labeling of Graphs. Symmetry, 15(4), 825. https://doi.org/10.3390/sym15040825

[18] Abd El-hay, A., & Rabie, A. (2022). Signed Product Cordial labeling of Corona Product between Paths and Second Power of Fan Graphs. Ital. J. Pure Appl. Math, 48, 287-294.

[19] ELrokh, Ashraf, Mohammed M. Ali Al-Shamiri, and Atef Abd El-hay. 2023. "A Novel Radio Geometric Mean Algorithm for a Graph" Symmetry 15, no. 3:570. https://doi.org/10.3390/sym15030570.

[20] Elrokh, A., Al-shamiri, M. A., & Abd El-hay, A. (2022). A NOVEL PROBLEM FOR SOLVING CORDIAL LABELING OF CORONA PRODUCT BETWEEN PATH AND THIRD ORDER OF CONE GRAPHS. Turkish Journal of Computer and Mathematics Education (TURCOMAT), 13(03), 970-986.
[21] Abd El-hay, A., Elmshtaye, Y., & Elrokh, A. (2023). Solving Singed Product Cordial Labeling of Corona Products of Paths and the Third Power of Lemniscate Graphs. Turkish Journal of Computer and Mathematics Education (TURCOMAT), 14(2), 806-823.
[22] K. A. Alsatami, Y. Algrawani, and A. Abd El-hay, A novel problem and algorithm for solving permuted cordial labeling of corona product between two graphs, Mathematical Models in Engineering, Vol. 11, No. 1, "11, Jan. 2025, https://doi.org/10.21595/mme.2024.24609
[23] Atef Abd El-hay, Khalid A. Alsatami, Ashraf ELrokh and Aya Rabie. "Cordial Labeling of Corona Product of Paths and Fourth Order of Lemniscate Graphs". (2025). European Journal of Pure and Applied Mathematics, 18(1), 5470. https://doi.org/10.29020/nybg.ejpam.v18i1.5470
[24] Atef Abd El-hay, Khalid A. Alsatami and Ashraf ELrokh. "A Novel Problem and Algorithm for Solving Cordial Labeling of Some Fifth Power of Graphs". (2025). European Journal of Pure and Applied Mathematics, 18(1), 5812. https://doi.org/10.29020/nybg.ejpam.v18i1.5812
[25] Elmshtaye, Y., Elrokh, A., & Abd El-hay, A. (2025). TOTAL CORDIAL FOR THE CORONA PRODUCT OF PATHS AND THE THIRD POWER OF DOUBLE FANS-GENERALIZED FANS. Advances and Applications in Discrete Mathematics, 42(4), 303-334.
[26] Elrokh, A., Almotairi, E. S., & Mostafa, H. (2025). Computing Metric and Connected Metric Dimension of Some Graphs. European Journal of Pure and Applied Mathematics, 18(3), 6438-6438.

Ashraf Elrokh was born in Melig, Shebin El-Kom, Menoufia, Egypt in 1968. He received the B.S. and M.S. degrees in pure mathematics from faculty of science, Menoufia University, in 1994 and the Ph.D. degree Pure Mathematics (Functional Analysis) from the Faculty of Mathematics and Information, Grethwald University,Germany, in 2004. In addition to over 26 years of teaching and academic experiences. His research interest includes knot theory, graph theory, functional analysis and operation research.

Eman S. Almotairi received the B.Sc. degree in Mathematics from Qassim University, in 2009, the M.Sc. degree in Mathematics from Qassim University in 2020, and the Ph.D. degree in Pure Mathematics from Umm Al-Qura University, in 2024. In addition, she has 14 years of teaching and academic experience. Her research interests include linear algebra, rings, discrete mathematics, graph theory and algebraic graph theory.

Nabawia Elramly earned her Ph.D. in Operations Research in 1977 from Humboldt University, Berlin. She was appointed as a Professor of Operations Research in 1993. Her specialized research area is Parametric Optimization, where she has contributed significantly to the development of mathematical models and solution techniques.

Ebitsam Mostafa received the B.S. in Mathematics and computer science Department from the Faculty of Science, Menoufia University, in 2015. In addition, she has one year of academic experience. Her research interests include graph theory and, discrete mathematics.