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Exploring Wave Function Dynamics in Quantum Mechanics with Nonlinear Interaction

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Abstract: The nonlinear Schrödinger-Bopp-Podolsky system has recently attracted a lot of interest since it is useful for modelling a variety of physical phenomena, including optical solitons and Bose-Einstein condensates. In this article, we used an enhanced modified extended tanh-expansion approach to examine the dynamics of wave functions in quantum mechanics with nonlinear interactions and relativistic effects on the Schrödinger-Bopp-Podolsky system. The proposed approach provides precise and effective analytical results. In order to develop a generalized version of the solitary wave solution ansatz, we first formulate the Schrödinger-Bopp-Podolsky system as a coupled set of nonlinear partial differential equations. The enhanced approach enables us to efficiently explore a large parameter space and locate numerous families of solitary wave solutions that have distinctive characteristics. In order to examine the characteristics of solitary waves of the Schrödinger-Bopp-Podolsky system under various conditions, such as variation in the nonlinearity coefficients and external potentials, we employ our enhanced approach. The results of this investigation considerably advance knowledge of the Schrödinger-Bopp-Podolsky system and its solitary wave behavior. The distinctive feature of this article is, to give graphical representations of the obtained solutions using MATLAB and Mathematica simultaneously. Future research into related nonlinear wave equations and their applications in other physical systems will benefit greatly from the enhanced approach described here.

Keywords: Nonlinear Schrödinger Bopp- Podolsky(NLS-BP) system, enhanced modified extended tanh-expansion method, solitary wave solutions, relativistic effects

1 Introduction

In recent years, the study of ultrashort pulse propagation in nonlinear optical fibers has gained significant attention due to its relevance in high-speed optical communication systems, nonlinear optics, and fiber lasers. Ultrashort pulses, characterized by durations on the order of femtoseconds or picoseconds, exhibit complex nonlinear interactions with the optical medium as they travel through the fiber [1–20]. The dynamics of several physical phenomena, such as Bose-Einstein condensates and nonlinear optics, are modelled mathematically by the nonlinear Schrödinger-Bopp-Podolsky (NLS-BP) system. Using the nonlinear Schrödinger equation and the Bopp-Podolsky equation, the NLS-BP system forms a nonlinear partial differential equation. The Bopp-Podolsky equation describes the behavior of a vector field, whereas the nonlinear Schrödinger equation describes the behavior of a complex scalar field.

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The NLS-BP system can be written in the following form:

$$\begin{cases} i\psi_t &= -\nabla^2\psi + \lambda|\psi|^2\psi + \gamma v, \\ v_t &= -\Delta v - \lambda|\psi|^2 v, \end{cases} \tag{1}$$

where ψ is a complex scalar field(complex valued wave function), v is a vector field(real valued wave function), λ and γ are constants, and ∇^2 and Δ are the Laplacian operators in the scalar and vector fields, respectively. The first Eq. in the system 1 describes the evolution of the scalar field under the influence of a nonlinearity, while the second Eq. in the system 1 describes the evolution of the vector field under the influence of the scalar field. The NLS-BP system exhibits a variety of interesting phenomena, including solitons, breathers, and chaos. Solitons are localized wave packets that maintain their shape and speed over long distances, while breathers are localized wave packets that oscillate periodically in time. Chaos refers to a state of disorder and unpredictability in the system behavior.

The study of the nonlinear Schrödinger-Bopp-Podolsky system has applications in various areas of physics. The nonlinear Schrödinger-Bopp-Podolsky equation is used in nonlinear optics to model the propagation of intense laser beams through nonlinear optical media. It aids in the comprehension of phenomena like pulse compression, optical solitons, and self-focusing. This is especially important for the design and improvement of laser systems and photonic devices. When the field operators in a quantum field theory fulfill commutation relations, the nonlinear Schrödinger-Bopp-Podolsky equation can be extended to explain the theory. It has uses in researching particle interactions, quantum dynamics, and the emergence of particle-like excitations in quantum systems. In cold atom physics, cold atom systems like Bose-Einstein condensates, can demonstrate nonlinear behavior as a result of atomic interactions. A theoretical framework for investigating the dynamics and characteristics of these systems, including the development of solitary waves and the behavior of vortices, is provided by the nonlinear Schrödinger-Bopp-Podolsky equation. The nonlinear Schrödinger-Bopp-Podolsky equation can be used to study the transmission of intense electromagnetic waves through plasmas in plasma physics. It aids in the comprehension of phenomena such as self-focusing, interactions between laser and plasma, and the formation of high-energy particle beams in plasma-based accelerators.

In the study of condensed matter systems, such as superconductors and superfluids, where the wave functions of electrons or Cooper pairs can demonstrate nonlinear interactions and relativistic-like behavior, the nonlinear Schrödinger-Bopp-Podolsky equation is important. The nonlinear Schrödinger-Bopp-Podolsky equation has been used to model wave propagation in excitable media, nerve fibers, and protein dynamics in biological systems. It offers understanding of phenomena like wave patterns in cardiac tissue or soliton propagation in nerve impulses. In the study of quantum information and quantum computing, where the dynamics of quantum states and quantum gates can involve nonlinear interactions, the nonlinear Schrödinger-Bopp-Podolsky equation plays a significant role. It applies to topics like quantum computers based on quantum optics or the manipulation of quantum states in nonlinear systems. The nonlinear Schrödinger-Bopp-Podolsky system can be applied in a variety of domains, as shown by these scenarios, and it offers a mathematical framework for comprehending and forecasting complicated wave processes with nonlinear and relativistic effects.

The methods for obtaining exact explicit solutions of nonlinear partial differential equations are the extended generalized Riccati equation mapping method [29, 30], improved F-expansion method [31], ϕ^6 model expansion method [32], modified extended direct algebraic method [33], the exp-function method [34], Hirota's direct method [37, 38], Kudryashov method [35, 36], the extended auxiliary equation method [39, 40], modified method of simplest equation [41, 42], and extended Fan's sub-equation method [43].

The goal of the current article is to extend the enhanced modified extended tanh-expansion method to make further progress. The enhanced modified extended tanh-expansion method [44–51] has a variety of advantages for investigating and analyzing nonlinear partial differential equations. It is appropriate for a wide class of partial differential equations because to its broad application, which enables it to manage many sorts of nonlinearities. The method is flexible enough to allow for the inclusion of parameters, allowing for the investigation of different solution properties. Its systematic and straightforward procedure makes it easy to construct exact solutions. The accuracy of the constructed solutions and their physical interpretability provide illuminating information about the underlying physical system, aiding in the understanding of dynamics and phenomena. Additionally, the method might present novel types of solutions, advancing the discipline. The enhanced Modified extended tanh-expansion method is a potent tool for studying nonlinear partial differential equations as a result, and it can significantly enhance research in this area.

2 Algorithm for Enhanced Modified Extended tanh-expansion method

In this section, the algorithm for the enhanced modified extended tanh-expansion method are introduced. The main steps of our method are outlined below.

Suppose we have the following nonlinear partial differential equation

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \cdots) = 0, \tag{2}$$

where P is a polynomial in $u = u(x,t)$ and u is an unknown wave function, while subscripts represent the partial derivatives.

Step 1. Using the following wave transformation for traveling wave solutions

$$u = U(\zeta), \quad \zeta = x - ct, \tag{3}$$

where c is the wave speed.

Step 2. Substituting the Eq. 3 into Eq. 2 yields a nonlinear ordinary differential equation

$$O(U, U', U'', U''', \cdots) = 0. \tag{4}$$

Step 3. Now let $U(\zeta)$ which can be expressed into a polynomial in $Q(\zeta)$

$$U(\zeta) = A_0 + \sum_{i=1}^{m} A_i Q^i(\zeta) + \sum_{i=1}^{m} B_i Q^{-i}(\zeta), \tag{5}$$

where A_i, B_i are constants to be determined later, and $Q(\zeta)$ satisfying the following Riccati equation

$$Q'(\zeta) = \sigma + Q^2(\zeta) \tag{6}$$

Step 4. In order to find the value of the positive integer m, the homogeneous balance between the highest order derivatives and the nonlinear terms found in Eq. 4 is used.

Step 5. Plugging Eq. 5 with Eq. 6 into Eq. 4 and gathering all coefficients of $Q^s(\zeta)$ where $s = 0, 1, 2, \cdots$, then will yield a system of algebraic equations with respect to $Q(\zeta)$ where $i = \pm 1, \pm 2, \cdots n$.

Step 6. After solving the algebraic system of equations and plugging the results into Eq. 5, then we obtain general form of the solitary wave solutions of Eq. 6, which admits the following solutions.

Type-I. When $\sigma < 0$, we have

$$\begin{cases} Q_1(\zeta) &= -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta), \\ Q_2(\zeta) &= -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\zeta), \\ Q_3(\zeta) &= -\sqrt{-\sigma}\left(\tanh(2\sqrt{-\sigma}\zeta) + i\varepsilon\,\mathrm{sech}(2\sqrt{-\sigma}\zeta)\right), \\ Q_4(\zeta) &= \frac{\sigma - \sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}{1+\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}, \\ Q_5(\zeta) &= \frac{\sqrt{-\sigma}(5-4\cosh(2\sqrt{-\sigma}\zeta))}{3+4\sinh(2\sqrt{-\sigma}\zeta)}, \\ Q_6(\zeta) &= \frac{\varepsilon\sqrt{-\sigma(p^2+s^2)} - p\sqrt{-\sigma}\cosh(2\sqrt{-\sigma}\zeta)}{p\sinh(2\sqrt{-\sigma}\zeta)+s}, \\ Q_7(\zeta) &= \varepsilon\sqrt{-\sigma}\left[1 - \frac{2p}{p+\cosh(2\sqrt{-\sigma}\zeta) - \varepsilon\sinh(2\sqrt{-\sigma}\zeta)}\right], \end{cases} \tag{7}$$

Type-II. When $\sigma > 0$, we have

$$\begin{cases} Q_8(\zeta) &= \sqrt{\sigma}\tan(\sqrt{\sigma}\zeta), \\ Q_9(\zeta) &= -\sqrt{\sigma}\cot(\sqrt{\sigma}\zeta), \\ Q_{10}(\zeta) &= \sqrt{\sigma}\left[\tan(2\sqrt{\sigma}\zeta) + \varepsilon\sec(2\sqrt{\sigma}\zeta)\right], \\ Q_{11}(\zeta) &= -\sqrt{\sigma}\left[\frac{1-\tan(\sqrt{\sigma}\zeta)}{1+\tan(\sqrt{\sigma}\zeta)}\right], \\ Q_{12}(\zeta) &= \sqrt{\sigma}\left[\frac{4-5\cos(2\sqrt{\sigma}\zeta)}{3+5\sin(2\sqrt{\sigma}\zeta)}\right], \\ Q_{13}(\zeta) &= \frac{\varepsilon\sqrt{\sigma(p^2-s^2)} - p\sqrt{\sigma}\cos(2\sqrt{\sigma}\zeta)}{p\sin(2\sqrt{\sigma}\zeta)+s}, \\ Q_{14}(\zeta) &= i\varepsilon\sqrt{\sigma}\left[1 - \frac{2p}{p+\cos(2\sqrt{\sigma}\zeta) - i\varepsilon\sin(2\sqrt{\sigma}\zeta)}\right]. \end{cases} \tag{8}$$

where p and s are arbitrary constants.

Type-III. When $\sigma = 0$, we have

$$Q_{15}(\zeta) = \frac{1}{Q}(\zeta) \tag{9}$$

In the next section, we will apply the enhanced modified extended tanh-expansion method to find the solitary wave solutions to the nonlinear Schrödinger Bopp-Podolsky(NLS-BP)system.

3 Application of Enhanced Modified extended tanh- expansion method

Consider the nonlinear Schrödinger Bopp- Podolsky(NLS-BP)system,

$$\begin{aligned} i\frac{\partial \psi}{\partial t} &= -\nabla^2\psi + \lambda|\psi|^2\psi + \gamma v, \\ \frac{\partial v}{\partial t} &= -\Delta v - \lambda|\psi|^2 v. \end{aligned} \tag{10}$$

Making wave transformation

$$\psi(x,t) = U(\zeta)e^{-i\eta} \tag{11}$$

$$v(x,t) = V(\zeta) \tag{12}$$

where $\zeta = x - ct$ and $\eta = -kx + wt + \theta$. Plugging Eq. 11 and Eq. 12 into Eq. 10, then we have obtained following NODEs,

$$U'' - U\kappa^2 + \lambda U^3 + \gamma V + U'c, \tag{13}$$

$$-cV = V'' - \lambda U^2 V. \tag{14}$$

Applying homogeneous balance between the highest order derivatives and the nonlinear terms found in Eqs. 13 and Eq. 14, we have $m = 1$ and $n = 2$. Hence using 5, the formal solutions of Eq. 13 and Eq. 14 are,

$$U(\zeta) = A_0 + A_1 Q(\zeta) + \frac{B_1}{Q(\zeta)}, \tag{15}$$

$$V(\zeta) = F_0 + F_1 Q(\zeta) + F_2 Q^2(\zeta) + \frac{G_1}{Q(\zeta)} + \frac{G_2}{Q^2(\zeta)}. \tag{16}$$

Plugging Eqs.15, 16 into Eqs.13, 14 respectively, then the following systems of algebraic equations obtained,

$$\begin{cases} 2A_1 + \lambda A_1^3 = 0, \\ 3\lambda A_0 A_1^2 - \sigma^2 A_1 + \lambda F_2 + cA_1 = 0, \\ 2A_1\sigma + \lambda F_1 + 3\lambda A_0^2 A_1 + 3\lambda A_1^2 B_1 = 0, \\ -\sigma^2 A_1 \sigma + 6\lambda A_0 A_1 B_1 + cA_1\sigma + \lambda F_0 + \lambda A_0^3 + \sigma^2 B_1 - cB_1 = 0, \\ 2B_1\sigma + \lambda G_1 + 3\lambda A_0^2 B_1 + 3\lambda A_1 B_1^2 = 0, \\ \lambda G_2 + \sigma^2 B_1 \sigma + 3\lambda A_0 B_1^2 - cB_1\sigma = 0, \\ 2B_1\sigma^2 + \lambda B_1^3 = 0, \end{cases} \tag{17}$$

and

$$\begin{cases} \lambda A_1^2 F_1 + 2\lambda A_0 A_1 F_2 - 2F_1 = 0, \\ \lambda A_1^2 F_0 + \lambda A_0^2 F_2 + 2\lambda A_0 A_1 F_1 + 2\lambda A_1 B_1 F_2 - cF_2 - 8F_2\sigma = 0, \\ \lambda A_1^2 G_1 + \lambda A_0^2 F_1 + 2\lambda A_0 B_1 F_2 - 2F_1\sigma + 2\lambda A_1 B_1 F_1 - cF_1 + 2\lambda A_0 A_1 F_0 = 0, \\ 2\lambda A_0 A_1 G_1 + \lambda B_1^2 F_2 + 2\lambda A_0 B_1 F_1 + \lambda A_1^2 G_2 + 2\lambda A_1 B_1 F_0 + \lambda A_0^2 F_0 - 2F_2\sigma^2 - cF_0 - 2G_2 = 0, \\ 2\lambda A_0 B_1 F_0 + 2\lambda A_1 B_1 G_1 + \lambda A_0^2 G_1 + 2\lambda A_0 A_1 G_2 + \lambda B_1^2 F_1 - cG_1 - 2G_1\sigma = 0, \\ -8G_2\sigma + 2\lambda A_1 B_1 G_2 - cG_2 + 2\lambda A_0 B_1 G_1 + \lambda A_0^2 G_2 + \lambda B_1^2 F_0 = 0, \\ \lambda B_1^2 G_1 - 2G_1\sigma^2 + 2\lambda A_0 B_1 G_2 = 0, \\ -6G_2\sigma^2 + \lambda B_1^2 G_2 = 0. \end{cases} \tag{18}$$

Solving Eq. 17 with help of Maple, we have,

$$A_1 = \pm\frac{2}{\sqrt{-2\lambda}},\ A_0 = \pm\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma + \gamma F_1\sqrt{-2\lambda})},\ B_1 = \pm\frac{2\sigma}{\sqrt{-2\lambda}},\ F_1 = F_1. \tag{19}$$

Solving Eq. 18 with help of Maple, and using Eq. 19 we have,

$$F_0 = 0,\ F_2 = \frac{12(40\sigma + 3c)}{\gamma\sqrt{-6\lambda(72\sigma + 6c)}},\ F_1 = \frac{2(40\sigma + 3c)}{\sqrt{-2\lambda}\gamma},\ G_1 = -\frac{2\sigma^2(40\sigma + 3c)}{\sqrt{-2\lambda}\gamma(7\sigma + c)},\ G_2 = -\frac{2\sigma^3(40\sigma + 3c)}{\gamma(7\sigma + c)\sqrt{-\lambda(12\sigma + c)}}. \tag{20}$$

Substituting Eq. 19 and Eq. 20 together with Eq. 15 into Eq. 11, we obtained the following solitary wave solutions of Eq. 10 are:

Type-I. When $\sigma < 0$, we have

$$\psi_1(x,t) = \left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})} + \frac{2}{\sqrt{-2\lambda}}\left(-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)\right) + \frac{2\sigma}{\sqrt{-2\lambda}}\left(-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)\right)^{-1}\right]e^{-i\eta}, \quad (21)$$

$$\begin{aligned} v_1(x,t) &= F_1\left(-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)\right) + \frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)\right)^2 \\ &- \frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)\right)^{-1} - \frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)\right)^{-2}, \end{aligned} \quad (22)$$

$$\psi_2(x,t) = \left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})} + \frac{2}{\sqrt{-2\lambda}}\left(-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\zeta)\right) + + \frac{2\sigma}{\sqrt{-2\lambda}}\left(-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\zeta)\right)^{-1}\right]e^{-i\eta}, \quad (23)$$

$$\begin{aligned} v_2(x,t) &= F_1\left(-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\zeta)\right) + \frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\zeta)\right)^2 \\ &- \frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\zeta)\right)^{-1} - \frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\zeta)\right)^{-2}, \end{aligned} \quad (24)$$

$$\begin{aligned} \psi_3(x,t) = \Bigg[&\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})} + \frac{2}{\sqrt{-2\lambda}}\left(-\sqrt{-\sigma}\left(\tanh(2\sqrt{-\sigma}\zeta)+i\varepsilon\,\mathrm{sech}(2\sqrt{-\sigma}\zeta)\right)\right) \\ &+ \frac{2\sigma}{\sqrt{-2\lambda}}\left(-\sqrt{-\sigma}\left(\tanh(2\sqrt{-\sigma}\zeta)+i\varepsilon\,\mathrm{sech}(2\sqrt{-\sigma}\zeta)\right)\right)^{-1}\Bigg]e^{-i\eta}, \end{aligned} \quad (25)$$

$$\begin{aligned} v_3(x,t) = F_1&\left(-\sqrt{-\sigma}\left(\tanh(2\sqrt{-\sigma}\zeta)+i\varepsilon\,\mathrm{sech}(2\sqrt{-\sigma}\zeta)\right)\right) \\ &+ \frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(-\sqrt{-\sigma}\left(\tanh(2\sqrt{-\sigma}\zeta)+i\varepsilon\,\mathrm{sech}(2\sqrt{-\sigma}\zeta)\right)\right)^2 \\ &- \frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(-\sqrt{-\sigma}\left(\tanh(2\sqrt{-\sigma}\zeta)+i\varepsilon\,\mathrm{sech}(2\sqrt{-\sigma}\zeta)\right)\right)^{-1} \\ &+ - \frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(-\sqrt{-\sigma}\left(\tanh(2\sqrt{-\sigma}\zeta)+i\varepsilon\,\mathrm{sech}(2\sqrt{-\sigma}\zeta)\right)\right)^{-2}, \end{aligned} \quad (26)$$

$$\psi_4(x,t) = \left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})} + \frac{2}{\sqrt{-2\lambda}}\left(\frac{\sigma-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}{1+\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}\right) + \frac{2\sigma}{\sqrt{-2\lambda}}\left(\frac{\sigma-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}{1+\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}\right)^{-1}\right]e^{-i\eta}, \quad (27)$$

$$\begin{aligned} v_4(x,t) &= F_1\left(\frac{\sigma-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}{1+\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}\right) + \frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(\frac{\sigma-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}{1+\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}\right)^2 \\ &- \frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(\frac{\sigma-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}{1+\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}\right)^{-1} + - \frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(\frac{\sigma-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}{1+\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta)}\right)^{-2}, \end{aligned} \quad (28)$$

$$\psi_5(x,t)=\left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})}+\frac{2}{\sqrt{-2\lambda}}\left(\frac{\sqrt{-\sigma}(5-4\cosh(2\sqrt{-\sigma}\zeta))}{3+4\sinh(2\sqrt{-\sigma}\zeta)}\right)+\frac{2\sigma}{\sqrt{-2\lambda}}\left(\frac{\sqrt{-\sigma}(5-4\cosh(2\sqrt{-\sigma}\zeta))}{3+4\sinh(2\sqrt{-\sigma}\zeta)}\right)^{-1}\right]e^{-i\eta}, \quad (29)$$

$$v_5(x,t)=F_1\left(\frac{\sqrt{-\sigma}(5-4\cosh(2\sqrt{-\sigma}\zeta))}{3+4\sinh(2\sqrt{-\sigma}\zeta)}\right)+\frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(\frac{\sqrt{-\sigma}(5-4\cosh(2\sqrt{-\sigma}\zeta))}{3+4\sinh(2\sqrt{-\sigma}\zeta)}\right)^2-\frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(\frac{\sqrt{-\sigma}(5-4\cosh(2\sqrt{-\sigma}\zeta))}{3+4\sinh(2\sqrt{-\sigma}\zeta)}\right)^{-1}+-\frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(\frac{\sqrt{-\sigma}(5-4\cosh(2\sqrt{-\sigma}\zeta))}{3+4\sinh(2\sqrt{-\sigma}\zeta)}\right)^{-2}, \quad (30)$$

$$\psi_6(x,t)=\left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})}+\frac{2}{\sqrt{-2\lambda}}\left(\frac{\varepsilon\sqrt{-\sigma(p^2+s^2)}-p\sqrt{-\sigma}\cosh(2\sqrt{-\sigma}\zeta)}{p\sinh(2\sqrt{-\sigma}\zeta)+s}\right)+\frac{2\sigma}{\sqrt{-2\lambda}}\left(\frac{\varepsilon\sqrt{-\sigma(p^2+s^2)}-p\sqrt{-\sigma}\cosh(2\sqrt{-\sigma}\zeta)}{p\sinh(2\sqrt{-\sigma}\zeta)+s}\right)^{-1}\right]e^{-i\eta}, \quad (31)$$

$$v_6(x,t)=F_1\left(\frac{\varepsilon\sqrt{-\sigma(p^2+s^2)}-p\sqrt{-\sigma}\cosh(2\sqrt{-\sigma}\zeta)}{p\sinh(2\sqrt{-\sigma}\zeta)+s}\right)+\frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(\frac{\varepsilon\sqrt{-\sigma(p^2+s^2)}-p\sqrt{-\sigma}\cosh(2\sqrt{-\sigma}\zeta)}{p\sinh(2\sqrt{-\sigma}\zeta)+s}\right)^2-\frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(\frac{\varepsilon\sqrt{-\sigma(p^2+s^2)}-p\sqrt{-\sigma}\cosh(2\sqrt{-\sigma}\zeta)}{p\sinh(2\sqrt{-\sigma}\zeta)+s}\right)^{-1}+-\frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(\frac{\varepsilon\sqrt{-\sigma(p^2+s^2)}-p\sqrt{-\sigma}\cosh(2\sqrt{-\sigma}\zeta)}{p\sinh(2\sqrt{-\sigma}\zeta)+s}\right)^{-2}, \quad (32)$$

$$\psi_7(x,t)=\left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})}+\frac{2}{\sqrt{-2\lambda}}\left(\varepsilon\sqrt{-\sigma}\left[1-\frac{2p}{p+\cosh(2\sqrt{-\sigma}\zeta)-\varepsilon\sinh(2\sqrt{-\sigma}\zeta)}\right]\right)+\frac{2\sigma}{\sqrt{-2\lambda}}\left(\varepsilon\sqrt{-\sigma}\left[1-\frac{2p}{p+\cosh(2\sqrt{-\sigma}\zeta)-\varepsilon\sinh(2\sqrt{-\sigma}\zeta)}\right]\right)^{-1}\right]e^{-i\eta}, \quad (33)$$

$$v_7(x,t)=F_1\left(\varepsilon\sqrt{-\sigma}\left[1-\frac{2p}{p+\cosh(2\sqrt{-\sigma}\zeta)-\varepsilon\sinh(2\sqrt{-\sigma}\zeta)}\right]\right)+\frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(\varepsilon\sqrt{-\sigma}\left[1-\frac{2p}{p+\cosh(2\sqrt{-\sigma}\zeta)-\varepsilon\sinh(2\sqrt{-\sigma}\zeta)}\right]\right)^2-\frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(\varepsilon\sqrt{-\sigma}\left[1-\frac{2p}{p+\cosh(2\sqrt{-\sigma}\zeta)-\varepsilon\sinh(2\sqrt{-\sigma}\zeta)}\right]\right)^{-1}+-\frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(\varepsilon\sqrt{-\sigma}\left[1-\frac{2p}{p+\cosh(2\sqrt{-\sigma}\zeta)-\varepsilon\sinh(2\sqrt{-\sigma}\zeta)}\right]\right)^{-2}. \quad (34)$$

Type-II. When $\sigma > 0$, we have

$$\psi_8(x,t) = \left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})} + \frac{2}{\sqrt{-2\lambda}}\left(\sqrt{\sigma}\tan(\sqrt{\sigma}\zeta)\right) + \frac{2\sigma}{\sqrt{-2\lambda}}\left(\sqrt{\sigma}\tan(\sqrt{\sigma}\zeta)\right)^{-1}\right]e^{-i\eta}, \quad (35)$$

$$v_8(x,t) = F_1\left(\sqrt{\sigma}\tan(\sqrt{\sigma}\zeta)\right) + \frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(\sqrt{\sigma}\tan(\sqrt{\sigma}\zeta)\right)^2 - \frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(\sqrt{\sigma}\tan(\sqrt{\sigma}\zeta)\right)^{-1} + -\frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(\sqrt{\sigma}\tan(\sqrt{\sigma}\zeta)\right)^{-2},, \quad (36)$$

$$\psi_9(x,t) = \left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})} + \frac{2}{\sqrt{-2\lambda}}\left(-\sqrt{\sigma}\cot(\sqrt{\sigma}\zeta)\right) + \frac{2\sigma}{\sqrt{-2\lambda}}\left(-\sqrt{\sigma}\cot(\sqrt{\sigma}\zeta)\right)^{-1}\right]e^{-i\eta}, \quad (37)$$

$$v_9(x,t) = F_1\left(-\sqrt{\sigma}\cot(\sqrt{\sigma}\zeta)\right) + \frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(-\sqrt{\sigma}\cot(\sqrt{\sigma}\zeta)\right)^2 - \frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(-\sqrt{\sigma}\cot(\sqrt{\sigma}\zeta)\right)^{-1} + -\frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(-\sqrt{\sigma}\cot(\sqrt{\sigma}\zeta)\right)^{-2}, \quad (38)$$

$$\psi_{10}(x,t) = \left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})} + \frac{2}{\sqrt{-2\lambda}}\left(-\sqrt{-\sigma}\left(\sqrt{\sigma}\left[\tan(2\sqrt{\sigma}\zeta)+\varepsilon\sec(2\sqrt{\sigma}\zeta)\right]\right)\right) + \frac{2\sigma}{\sqrt{-2\lambda}}\left(\sqrt{\sigma}\left[\tan(2\sqrt{\sigma}\zeta)+\varepsilon\sec(2\sqrt{\sigma}\zeta)\right]\right)^{-1}\right]e^{-i\eta}, \quad (39)$$

$$v_{10}(x,t) = F_1\left(\sqrt{\sigma}\left[\tan(2\sqrt{\sigma}\zeta)+\varepsilon\sec(2\sqrt{\sigma}\zeta)\right]\right) + \frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(\sqrt{\sigma}\left[\tan(2\sqrt{\sigma}\zeta)+\varepsilon\sec(2\sqrt{\sigma}\zeta)\right]\right)^2 - \frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(\sqrt{\sigma}\left[\tan(2\sqrt{\sigma}\zeta)+\varepsilon\sec(2\sqrt{\sigma}\zeta)\right]\right)^{-1} + -\frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(\sqrt{\sigma}\left[\tan(2\sqrt{\sigma}\zeta)+\varepsilon\sec(2\sqrt{\sigma}\zeta)\right]\right)^{-2}, \quad (40)$$

$$\psi_{11}(x,t) = \left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})} + \frac{2}{\sqrt{-2\lambda}}\left(-\sqrt{\sigma}\left[\frac{1-\tan(\sqrt{\sigma}\zeta)}{1+\tan(\sqrt{\sigma}\zeta)}\right]\right) + \frac{2\sigma}{\sqrt{-2\lambda}}\left(-\sqrt{\sigma}\left[\frac{1-\tan(\sqrt{\sigma}\zeta)}{1+\tan(\sqrt{\sigma}\zeta)}\right]\right)^{-1}\right]e^{-i\eta}, \quad (41)$$

$$v_{11}(x,t) = F_1\left(-\sqrt{\sigma}\left[\frac{1-\tan(\sqrt{\sigma}\zeta)}{1+\tan(\sqrt{\sigma}\zeta)}\right]\right) + \frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(-\sqrt{\sigma}\left[\frac{1-\tan(\sqrt{\sigma}\zeta)}{1+\tan(\sqrt{\sigma}\zeta)}\right]\right)^2 - \frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(-\sqrt{\sigma}\left[\frac{1-\tan(\sqrt{\sigma}\zeta)}{1+\tan(\sqrt{\sigma}\zeta)}\right]\right)^{-1} + -\frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(-\sqrt{\sigma}\left[\frac{1-\tan(\sqrt{\sigma}\zeta)}{1+\tan(\sqrt{\sigma}\zeta)}\right]\right)^{-2}, \quad (42)$$

$$\psi_{12}(x,t) = \left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})} + \frac{2}{\sqrt{-2\lambda}}\left(\sqrt{\sigma}\left[\frac{4-5\cos(2\sqrt{\sigma}\zeta)}{3+5\sin(2\sqrt{\sigma}\zeta)}\right]\right) \quad (43)$$

$$+ \frac{2\sigma}{\sqrt{-2\lambda}}\left(\sqrt{\sigma}\left[\frac{4-5\cos(2\sqrt{\sigma}\zeta)}{3+5\sin(2\sqrt{\sigma}\zeta)}\right]\right)^{-1}\right]e^{-i\eta}, \quad (44)$$

$$v_{12}(x,t) = F_1\left(\sqrt{\sigma}\left[\frac{4-5\cos(2\sqrt{\sigma}\zeta)}{3+5\sin(2\sqrt{\sigma}\zeta)}\right]\right) + \frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(\sqrt{\sigma}\left[\frac{4-5\cos(2\sqrt{\sigma}\zeta)}{3+5\sin(2\sqrt{\sigma}\zeta)}\right]\right)^2$$
$$-\frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(\sqrt{\sigma}\left[\frac{4-5\cos(2\sqrt{\sigma}\zeta)}{3+5\sin(2\sqrt{\sigma}\zeta)}\right]\right)^{-1} + -\frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(\sqrt{\sigma}\left[\frac{4-5\cos(2\sqrt{\sigma}\zeta)}{3+5\sin(2\sqrt{\sigma}\zeta)}\right]\right)^{-2}, \tag{45}$$

‘

$$\psi_{13}(x,t) = \left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})} + \frac{2}{\sqrt{-2\lambda}}\left(\frac{\varepsilon\sqrt{\sigma(p^2-s^2)}-p\sqrt{\sigma}\cos(2\sqrt{\sigma}\zeta)}{p\sin(2\sqrt{\sigma}\zeta)+s}\right)\right. \tag{46}$$
$$\left.+\frac{2\sigma}{\sqrt{-2\lambda}}\left(\frac{\varepsilon\sqrt{\sigma(p^2-s^2)}-p\sqrt{\sigma}\cos(2\sqrt{\sigma}\zeta)}{p\sin(2\sqrt{\sigma}\zeta)+s}\right)^{-1}\right]e^{-i\eta}, \tag{47}$$

$$v_{13}(x,t) = F_1\left(\frac{\varepsilon\sqrt{\sigma(p^2-s^2)}-p\sqrt{\sigma}\cos(2\sqrt{\sigma}\zeta)}{p\sin(2\sqrt{\sigma}\zeta)+s}\right) + \frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(\frac{\varepsilon\sqrt{\sigma(p^2-s^2)}-p\sqrt{\sigma}\cos(2\sqrt{\sigma}\zeta)}{p\sin(2\sqrt{\sigma}\zeta)+s}\right)^2$$
$$-\frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(\frac{\varepsilon\sqrt{\sigma(p^2-s^2)}-p\sqrt{\sigma}\cos(2\sqrt{\sigma}\zeta)}{p\sin(2\sqrt{\sigma}\zeta)+s}\right)^{-1} +$$
$$-\frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(\frac{\varepsilon\sqrt{\sigma(p^2-s^2)}-p\sqrt{\sigma}\cos(2\sqrt{\sigma}\zeta)}{p\sin(2\sqrt{\sigma}\zeta)+s}\right)^{-2}, \tag{48}$$

$$\psi_{14}(x,t) = \left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})} + \frac{2}{\sqrt{-2\lambda}}\left(i\varepsilon\sqrt{\sigma}\left[1-\frac{2p}{p+\cos(2\sqrt{\sigma}\zeta)-i\varepsilon\sin(2\sqrt{\sigma}\zeta)}\right]\right)\right.$$
$$\left.+\frac{2\sigma}{\sqrt{-2\lambda}}\left(i\varepsilon\sqrt{\sigma}\left[1-\frac{2p}{p+\cos(2\sqrt{\sigma}\zeta)-i\varepsilon\sin(2\sqrt{\sigma}\zeta)}\right]\right)^{-1}\right]e^{-i\eta}, \tag{49}$$

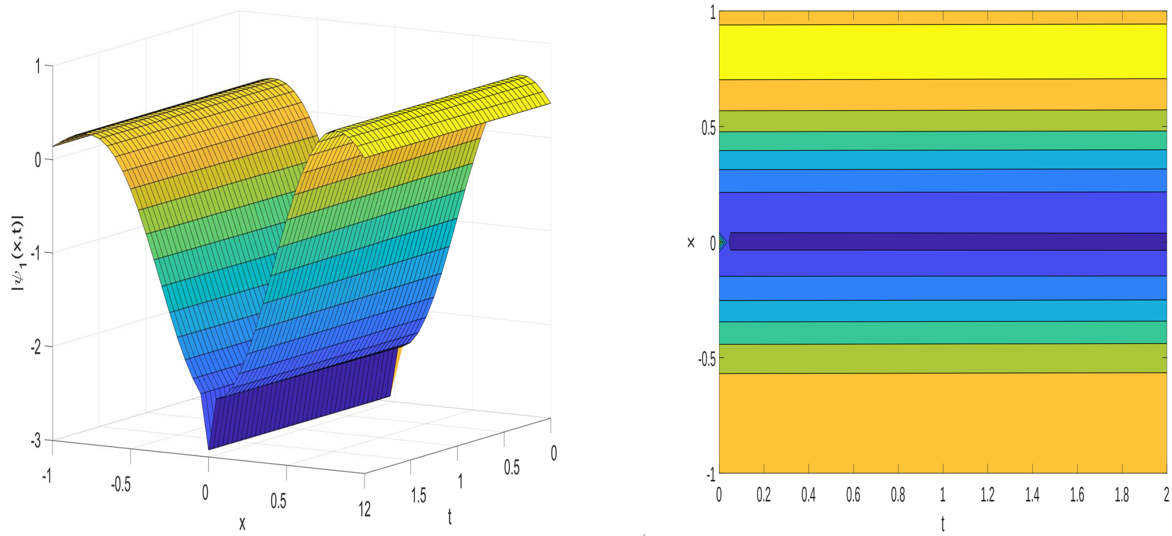
$$v_{14}(x,t) = F_1\left(i\varepsilon\sqrt{\sigma}\left[1-\frac{2p}{p+\cos(2\sqrt{\sigma}\zeta)-i\varepsilon\sin(2\sqrt{\sigma}\zeta)}\right]\right)$$
$$+\frac{12(40\sigma+3c)}{\gamma\sqrt{-6\lambda(72\sigma+6c)}}\left(i\varepsilon\sqrt{\sigma}\left[1-\frac{2p}{p+\cos(2\sqrt{\sigma}\zeta)-i\varepsilon\sin(2\sqrt{\sigma}\zeta)}\right]\right)^2$$
$$-\frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(i\varepsilon\sqrt{\sigma}\left[1-\frac{2p}{p+\cos(2\sqrt{\sigma}\zeta)-i\varepsilon\sin(2\sqrt{\sigma}\zeta)}\right]\right)^{-1}$$
$$+-\frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(i\varepsilon\sqrt{\sigma}\left[1-\frac{2p}{p+\cos(2\sqrt{\sigma}\zeta)-i\varepsilon\sin(2\sqrt{\sigma}\zeta)}\right]\right)^{-2}, \tag{50}$$

Type-III. When $\sigma = 0$, we have

$$\psi_{15}(x,t) = \left[\frac{1}{6\lambda}\sqrt{-6\lambda(-8\sigma+\gamma F_1\sqrt{-2\lambda})} + \frac{2}{\sqrt{-2\lambda}}\left(\frac{1}{\zeta}\right) + \frac{2\sigma}{\sqrt{-2\lambda}}\left(\frac{1}{\zeta}\right)^{-1}\right]e^{-i\eta}, \tag{51}$$

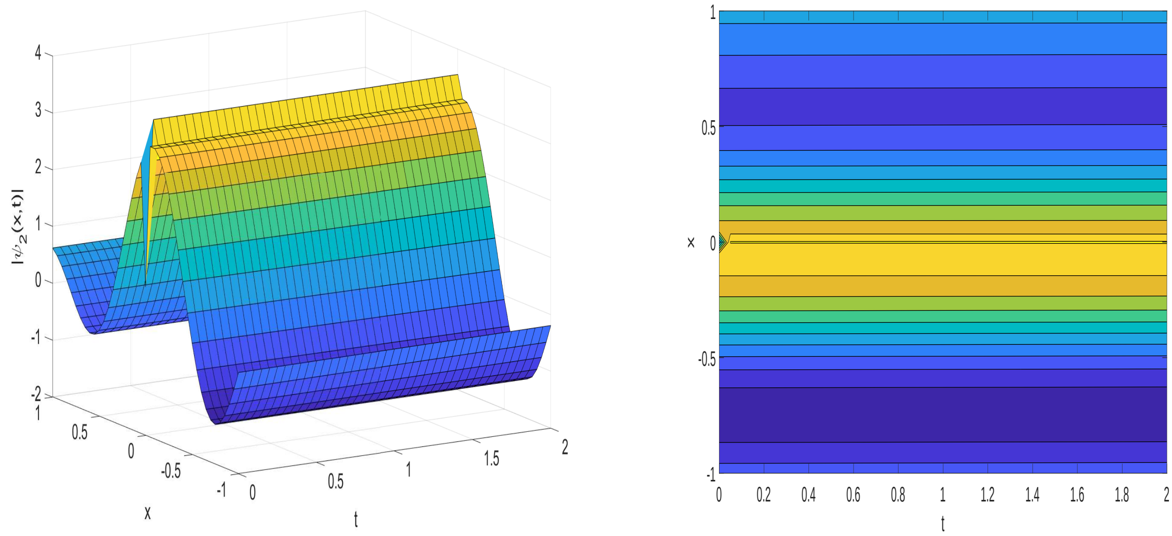
$$v_{15}(x,t) = F_1\left(\frac{1}{\zeta}\right) - \frac{6F_1(-1+\sigma^2)}{c(-1+3\sigma^2)}\left(\frac{1}{\zeta}\right)^2 - \frac{2\sigma^2(40\sigma+3c)}{\sqrt{-2\lambda}\gamma(7\sigma+c)}\left(\frac{1}{\zeta}\right)^{-1} + -\frac{2\sigma^3(40\sigma+3c)}{\gamma(7\sigma+c)\sqrt{-\lambda(12\sigma+c)}}\left(\frac{1}{\zeta}\right)^{-2}. \tag{52}$$

4 The Graphical Representation



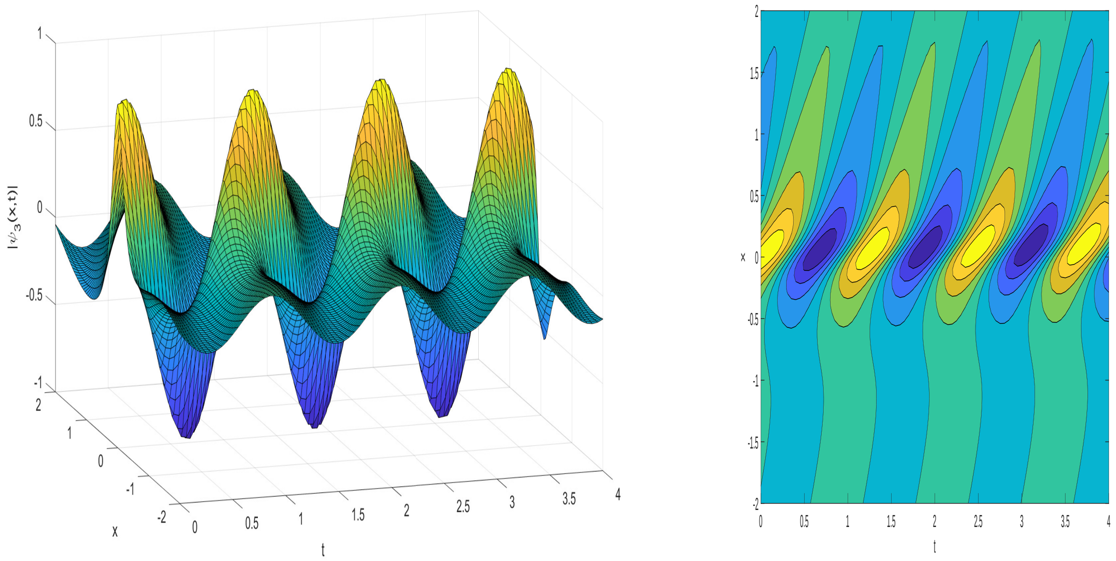
(a) **(b)**

Fig. 1: Select $\sigma = -0.22, \lambda = 11.02, \gamma = 11.60, k = 5.55, w = 0.0102, \theta = 0.012, c = 0.0015, f_1 = 0.002$ for 3D and contour graphs of Eq. 21.



(a) **(b)**

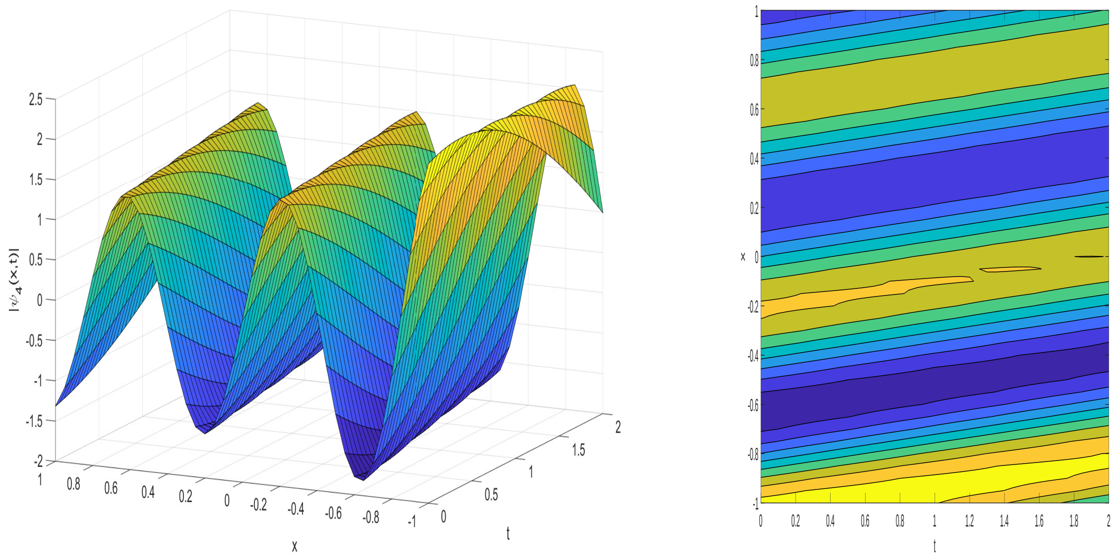
Fig. 2: Select $\sigma = -0.022, \lambda = 10.02, \gamma = 2.60, k = 6.55, w = 0.0102, \theta = 1.012, c = 0.0015$ for 3D and contour graphs of Eq. 23.



(a) **(b)**

Fig. 3: Select $\sigma = -0.22, \lambda = 11.02, \gamma = 11.60, k = 1.55, w = 5.2, \theta = 0.012, c = 0.015, f_1 = 0.002$ for 3D and contour graphs of Eq. 25.

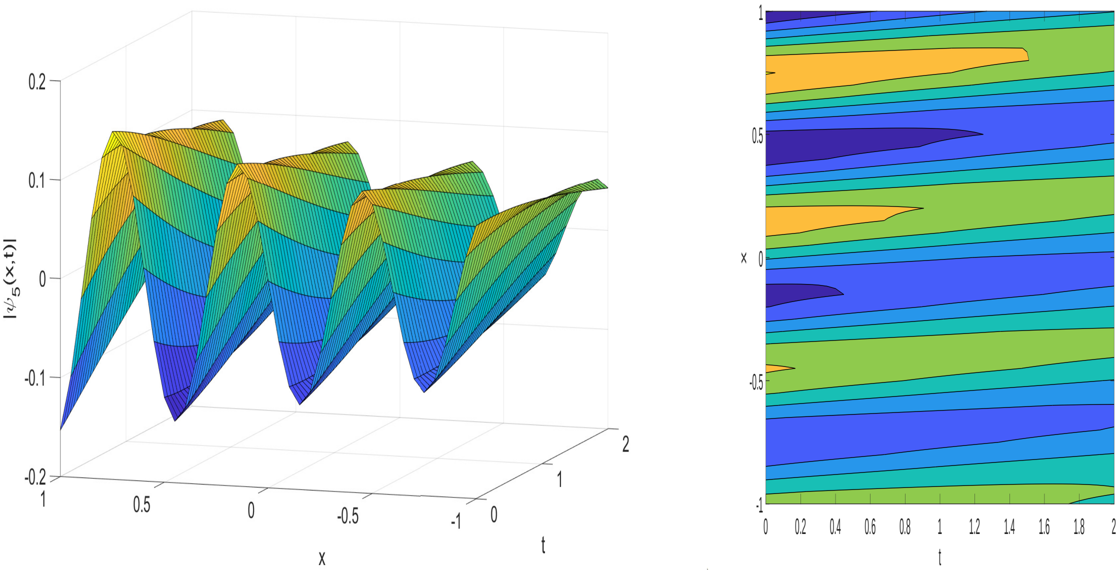
.



(a) **(b)**

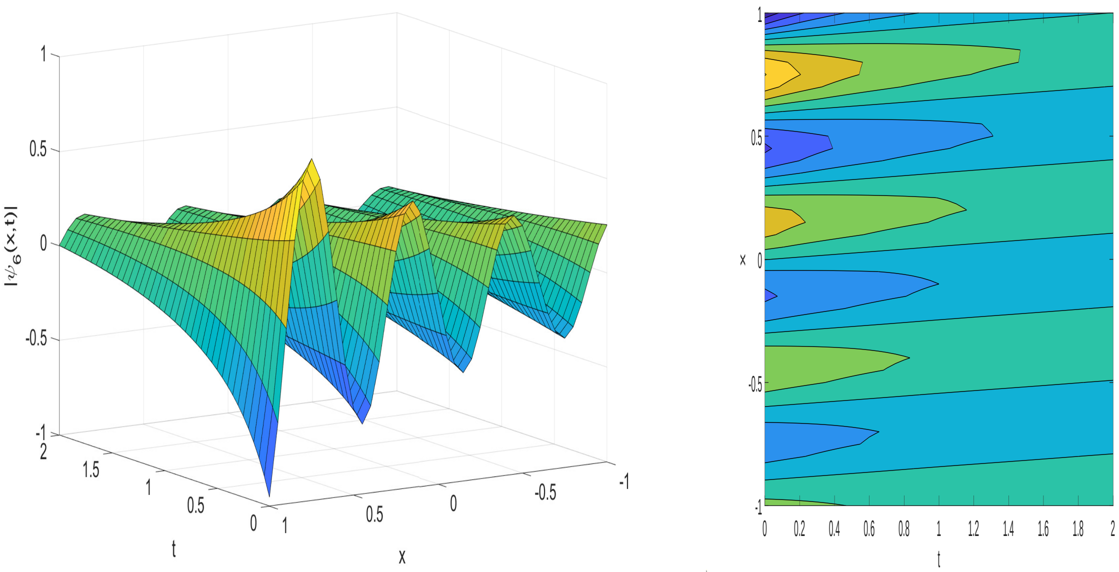
Fig. 4: Select for $\sigma = -0.92, \lambda = 5.02, \gamma = 4.76, k = 7.55, w = \frac{2\pi}{k}, \theta = 1.012, c = 0.0095, f_1 = 0.002$ for 3D and contour graphs of Eq. 27.

.



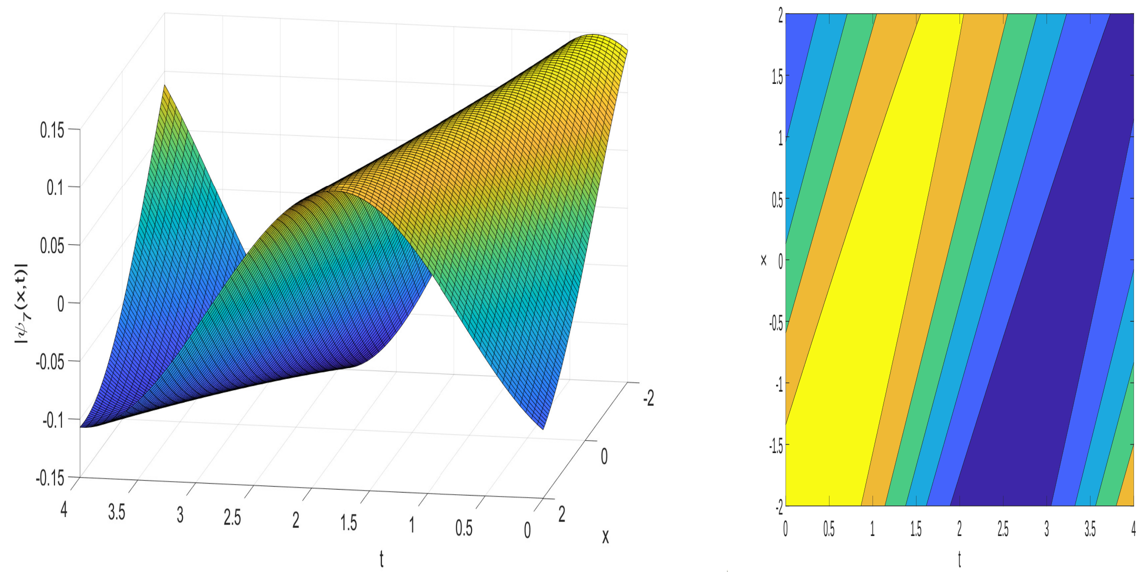
(a) **(b)**

Fig. 5: Select for $\sigma = -0.0042, \lambda = 2.02, \gamma = 2.76, k = 10.55, w = \frac{2\pi}{k}, \theta = 0.12, c = 0.87, f_1 = 0.002$ for 3D and contour and 2D graphs of Eq. 29.



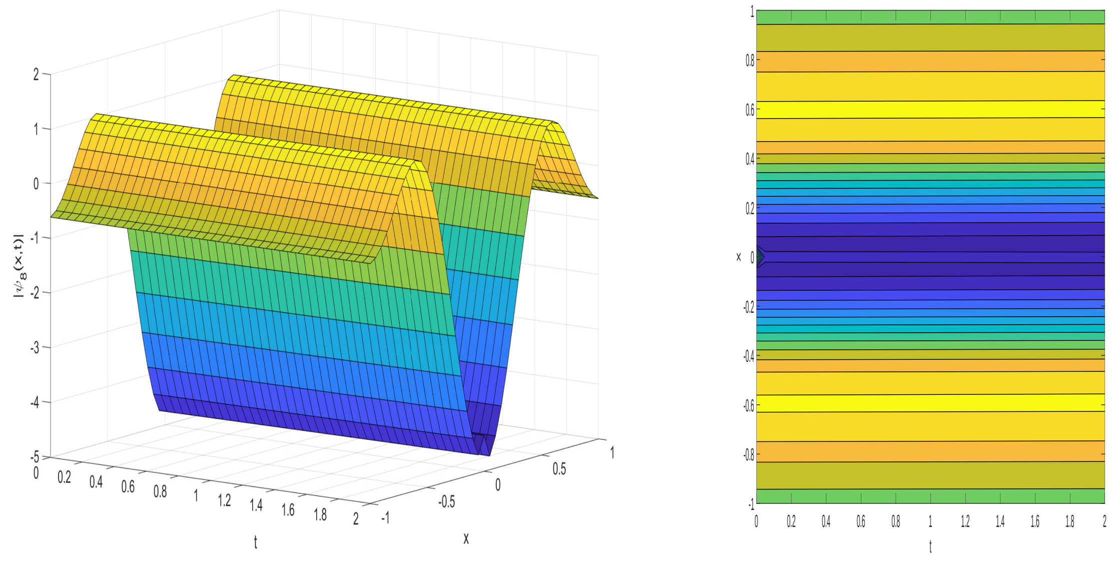
(a) **(b)**

Fig. 6: Select for $\sigma = -0.0042, \lambda = 2.02, \gamma = 2.76, k = 10.55, w = \frac{2\pi}{k}, \theta = 0.12, c = 1.80, p = 4, s = 1, f_1 = 0.002$ for 3D and contour graphs of Eq. 31.



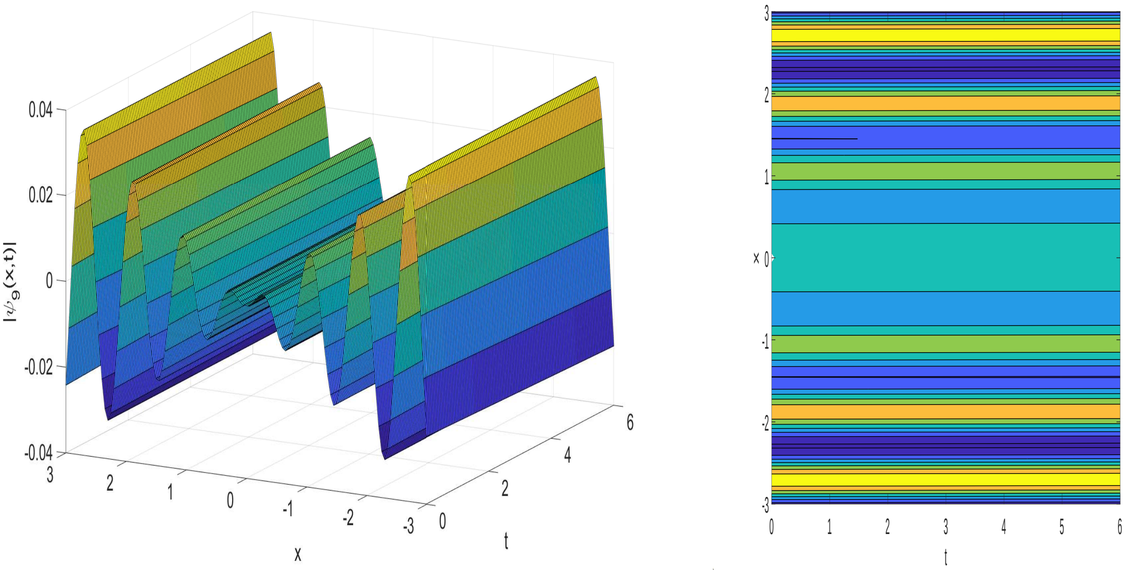
(a) (b)

Fig. 7: Select for $\sigma = -0.0042, \lambda = 2.02, \gamma = 2.76, k = 0.55, w = 1.44, \theta = 0.012, c = 0.008, p = 4, f_1 = 0.002$ for 3D, contour and 2D graphs of Eq. 33.



(a) (b)

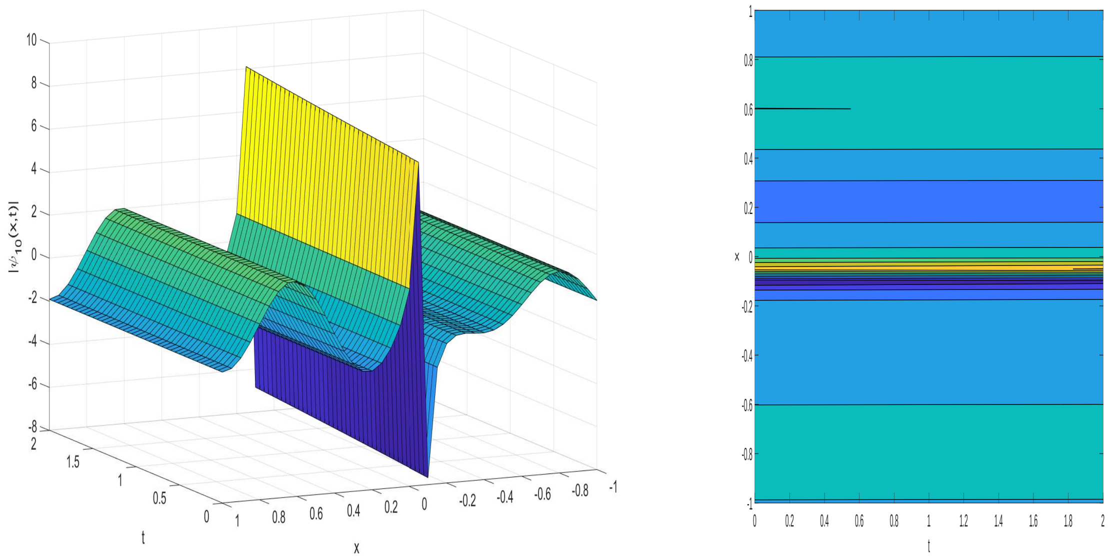
Fig. 8: Select $\sigma = 0.22, \lambda = 5.02, \gamma = 1.6, k = 7.55, w = 0.0102, \theta = 0.012, c = 0.0015, f_1 = 0.002$ for 3D and contour graphs of Eq. 35.



(a) **(b)**

Fig. 9: Select $\sigma = 0.022, \lambda = 5.02, \gamma = 1.60, k = 7.55, w = 0.0102, \theta = 0.012, c = 0.0015, f_1 = 0.002$ for 3D and contour graphs of Eq. 37.

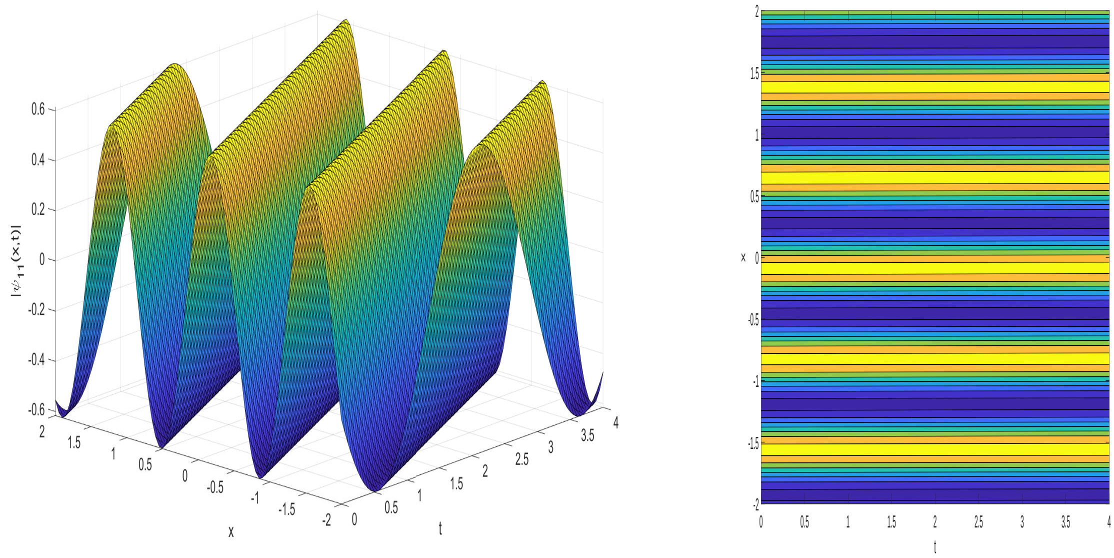
.



(a) **(b)**

Fig. 10: Select $\sigma = 0.22, \lambda = 5.02, \gamma = 11.60, k = 8.55, w = 0.0102, \theta = 0.012, c = 0.0015, f_1 = 2.002$ for 3D and contour graphs of Eq. 39.

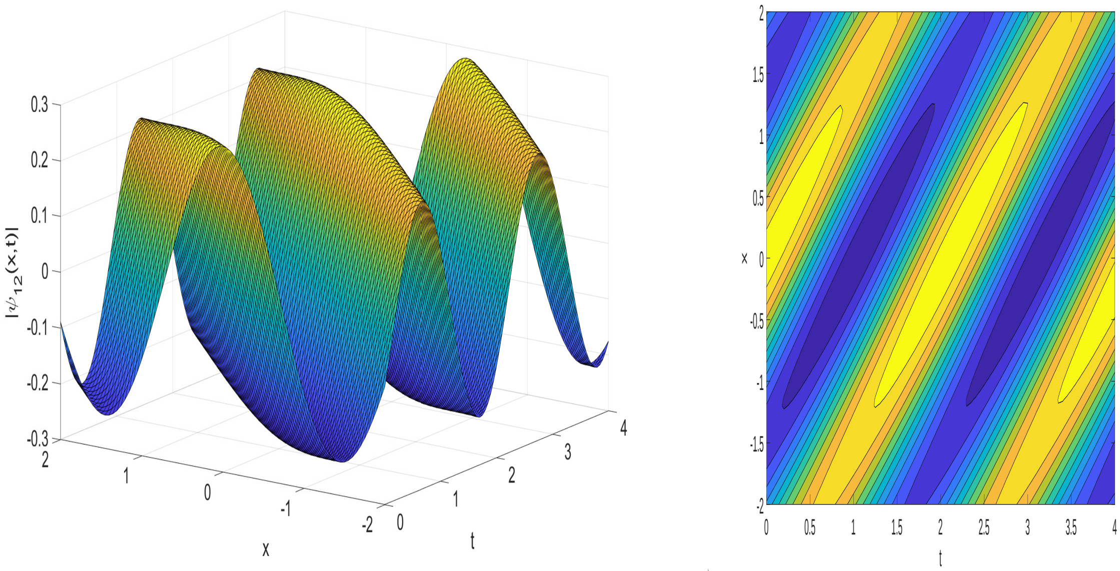
.



(a) **(b)**

Fig. 11: Select $\sigma = 0.022, \lambda = 5.02, \gamma = 1.60, k = 4.55, w = 2.02, \theta = 0.012, c = 0.0015, f_1 = 2.002$ for 3D and contour graphs of Eq. 41.

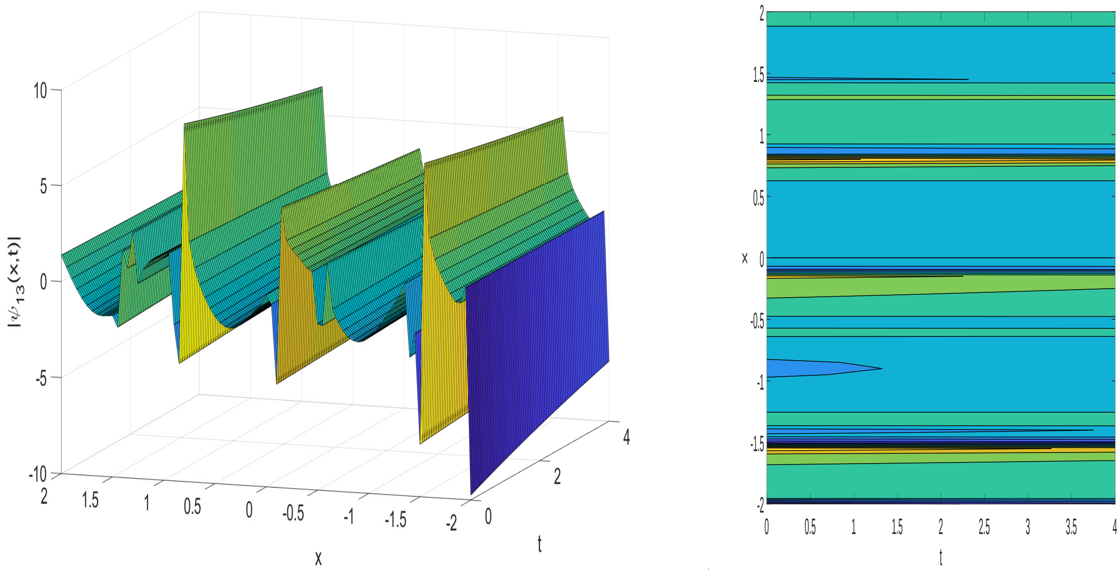
.



(a) **(b)**

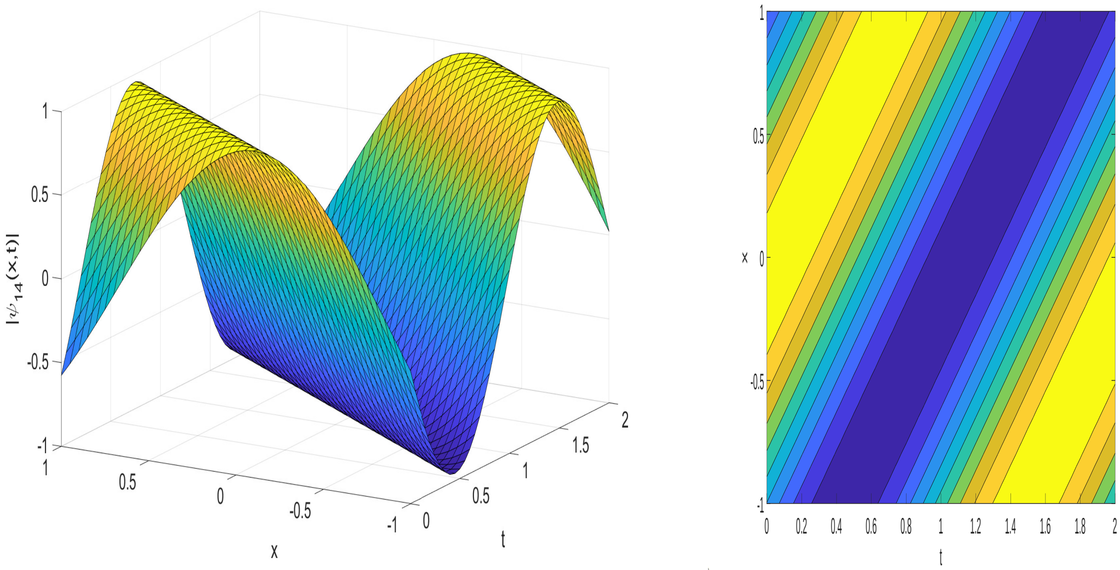
Fig. 12: Select $\sigma = 0.022, \lambda = 5.02, \gamma = 11.60, k = 2.55, w = 3.0102, \theta = 0.012, c = 0.0015, f_1 = 2.002$ for 3D and contour graphs of Eq. 43.

.



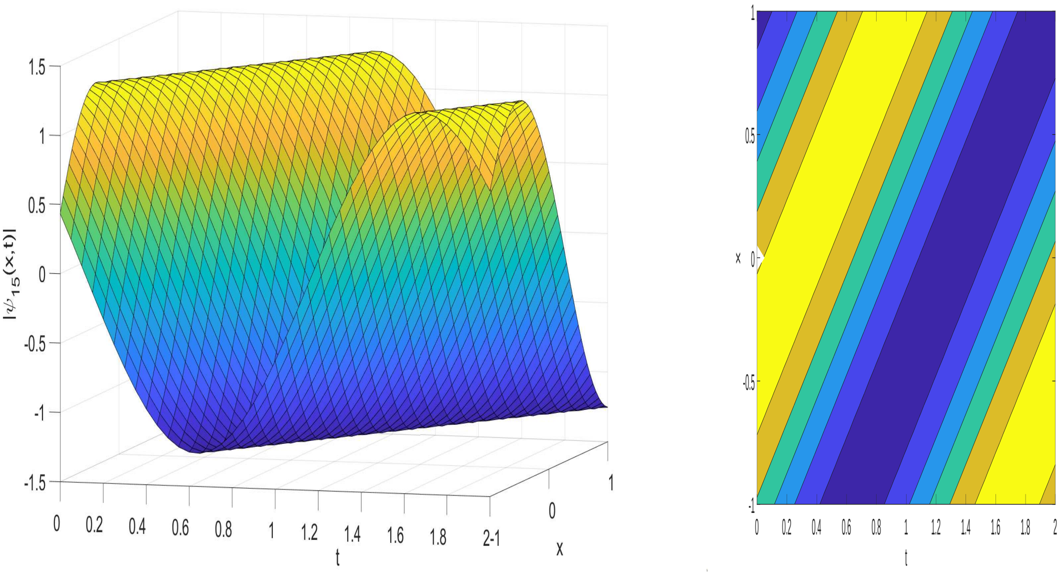
(a) **(b)**

Fig. 13: Select $\sigma = 0.022, \lambda = 0.02, \gamma = 0.0060, k = 0.0055, w = 0.0102, \theta = 0.012, c = 0.0015, p = 2, s = 1,$ $f_1 = 2.002$ for 3D and contour graphs of Eq. 46.



(a) **(b)**

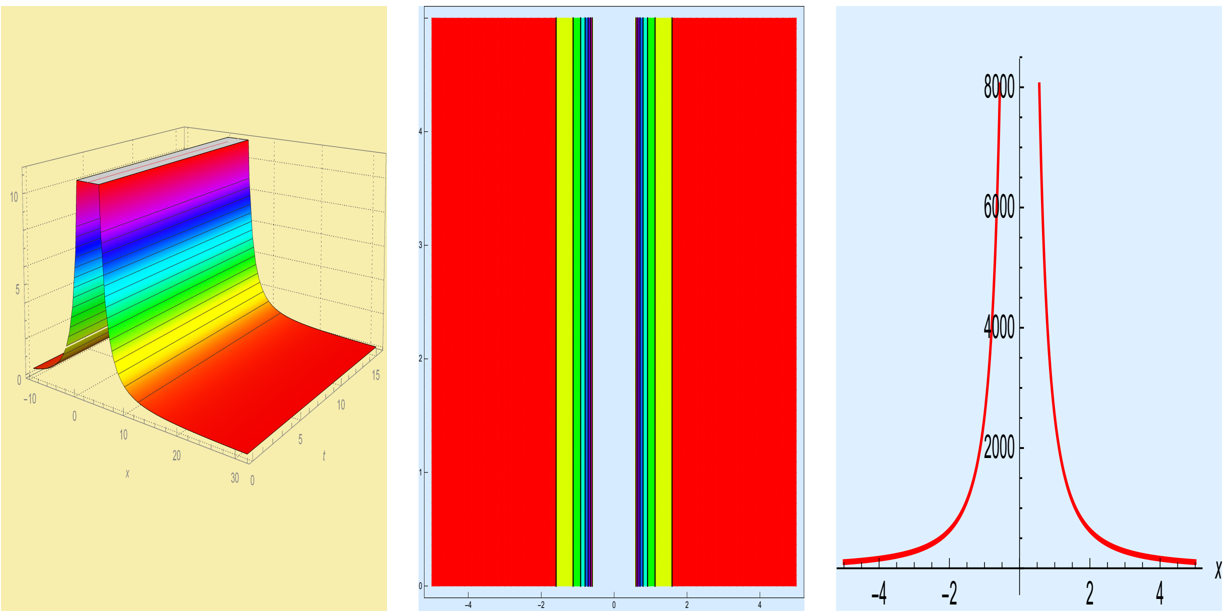
Fig. 14: Select $\sigma = 3.0048, \lambda = 5.02, \gamma = 1.60, k = 2.0055, w = 3.02, \theta = 0.0012, c = 0.0015, p = 2, f_1 = 2.2$ for 3D and contour graphs of Eq. 49.



(a) **(b)**

Fig. 15: Select $\sigma = 0.0048, \lambda = 15.02, \gamma = 11.60, k = 2.0055, w = 3.02, \theta = 0.0012, c = 0.0015, f_1 = 2.002$ for 3D and contour graphs of Eq. 51.

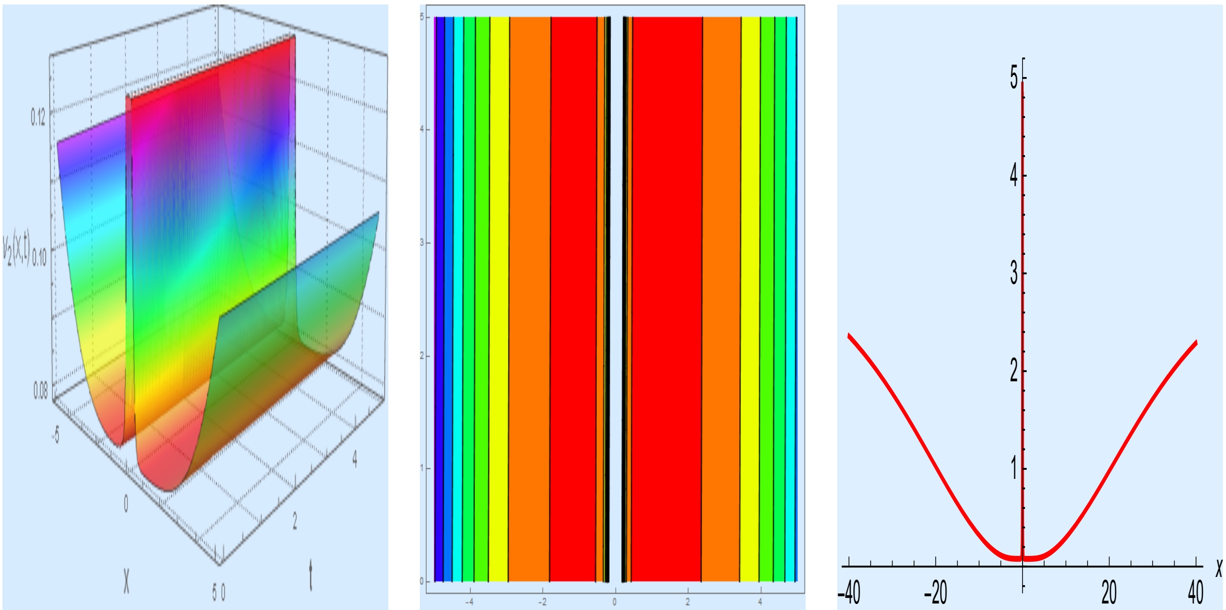
.



(a) **(b)** **(c)**

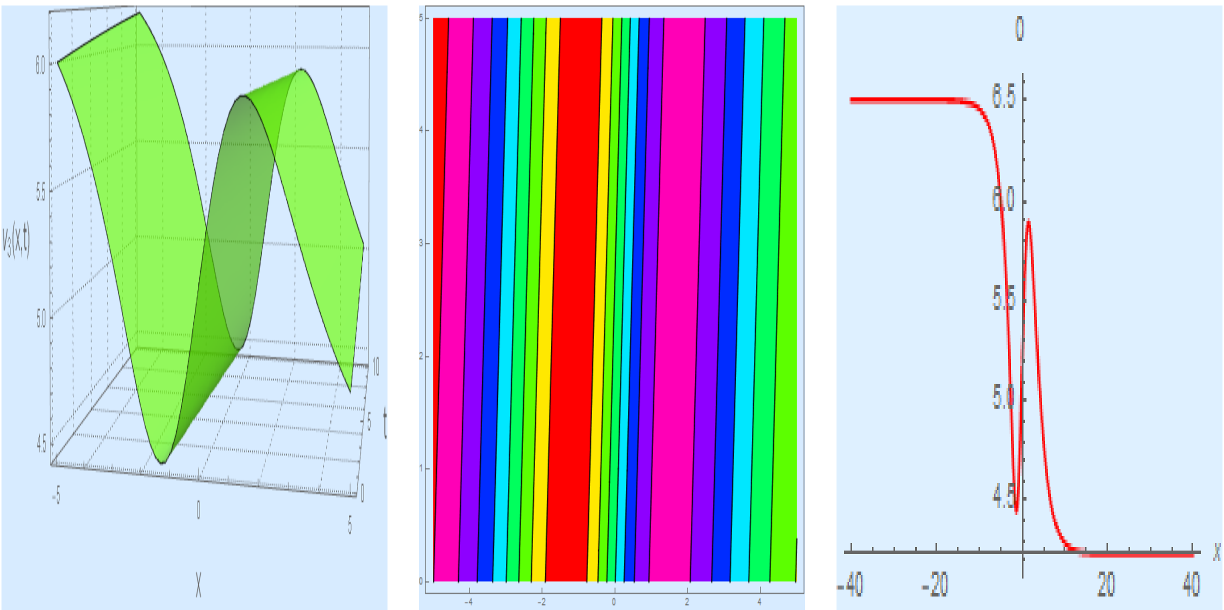
Fig. 16: Select $\sigma = -0.009, \lambda = 0.0025, \gamma = 15.60, c = 0.0125$ for 3D, contour and 2D graphs of Eq. 22.

.



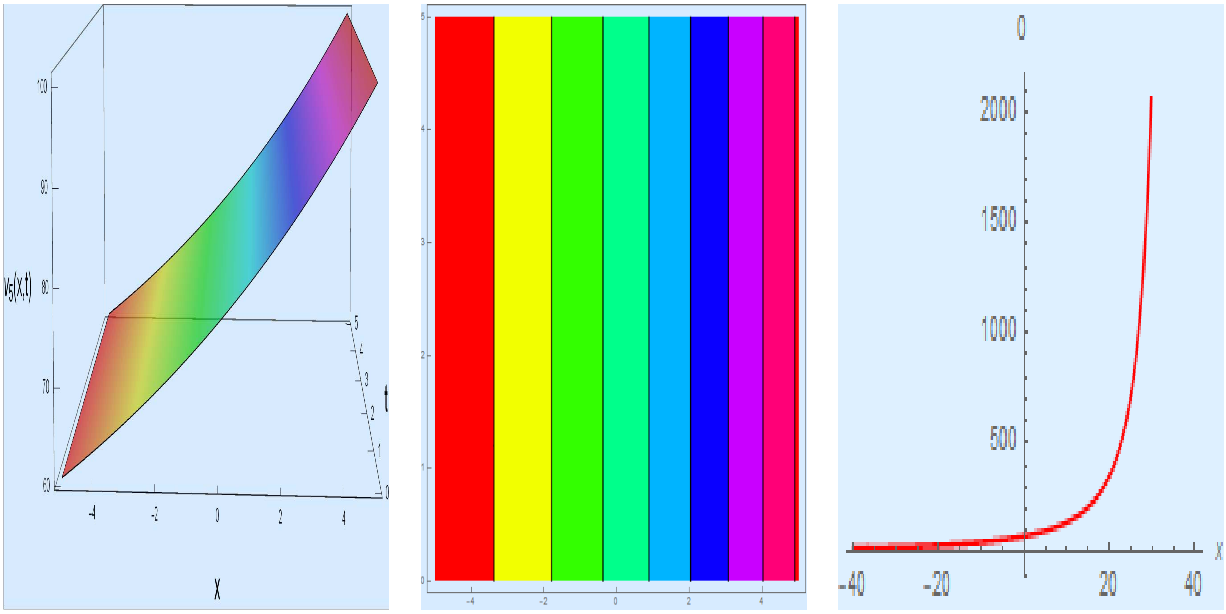
(a) **(b)** **(c)**

Fig. 17: Select $\sigma = -0.00099, \lambda = 0.00095, \gamma = 15.60, c = 0.00825$ for 3D, contour and 2D graphs of Eq. 24.
.



(a) **(b)** **(c)**

Fig. 18: Select $\sigma = -0.0099, \lambda = 0.0095, c = 0.0825$ for 3D, contour and 2D graphs of Eq. 26.
.



(a) **(b)** **(c)**

Fig. 19: Select $\sigma = -0.00009, \lambda = 0.0025, c = 0.225$ for 3D, contour and 2D graphs of Eq. 30.

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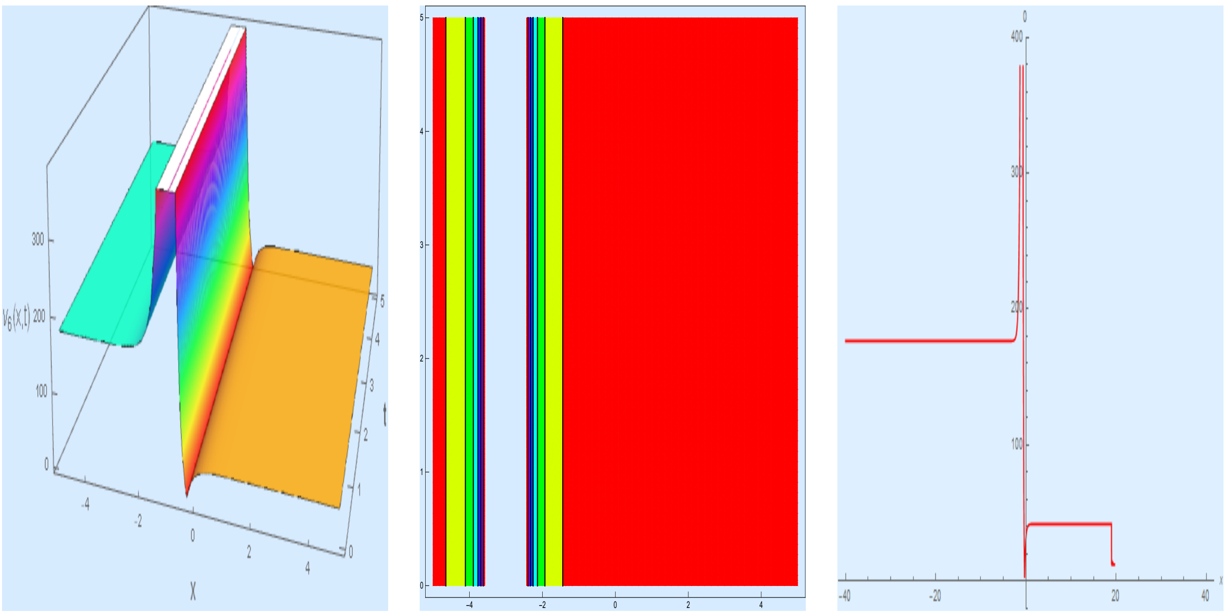
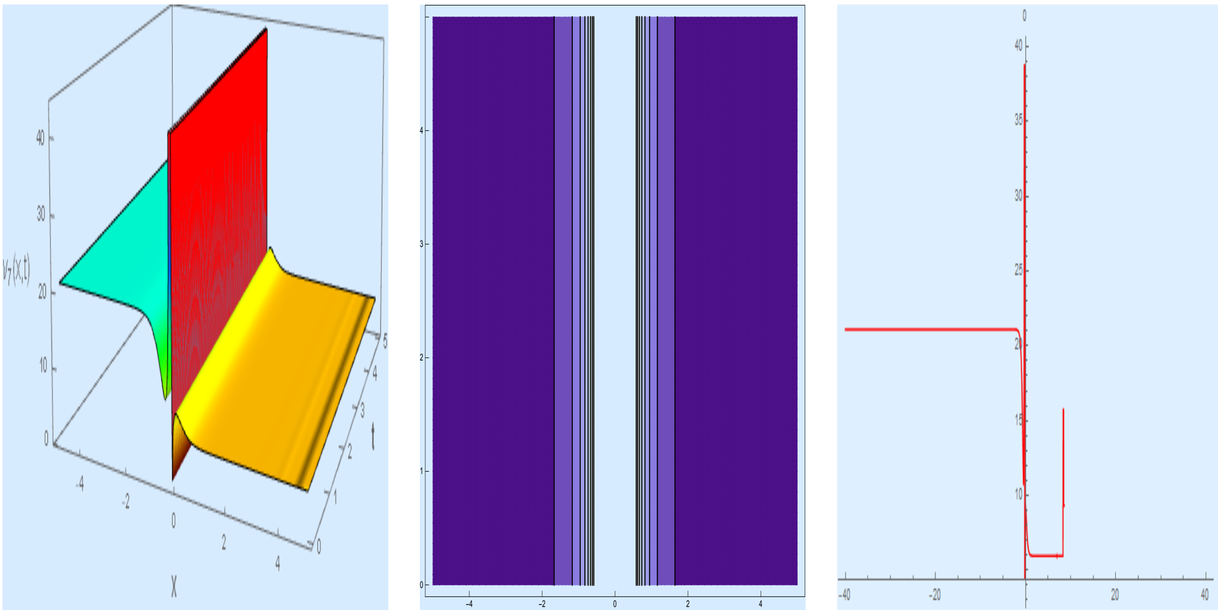


(a) **(b)** **(c)**

Fig. 20: Select $\sigma = -1.0099, \lambda = 0.2095, c = 0.00825, p = 6$ for 3D, contour and 2D graphs of Eq. 32.

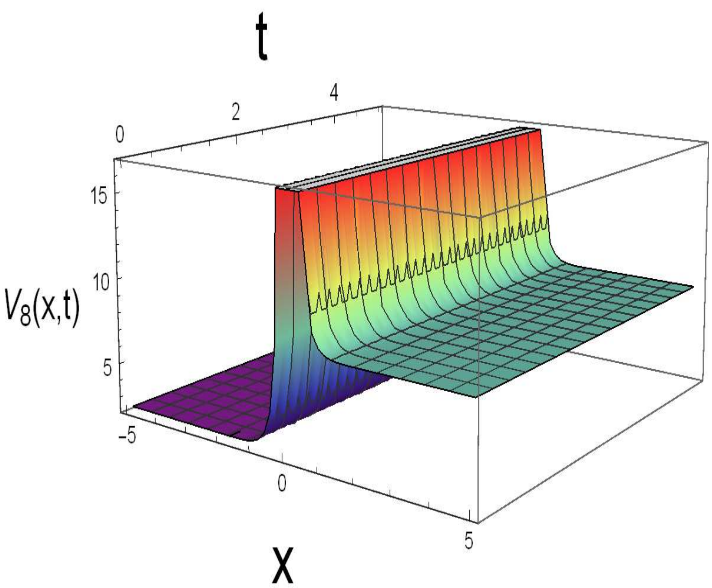
.



(a) **(b)** **(c)**

Fig. 21: Select $\sigma = -5.09, \lambda = 12.2095, c = 1.25, p = 3$. for 3D, contour and 2D graphs of Eq. 34.

.



(a) **(b)**

Fig. 22: Select for $\sigma = 2.09, \lambda = 1.25, c = 0.125$ for 3D, contour and 2D graphs of Eq. 36.

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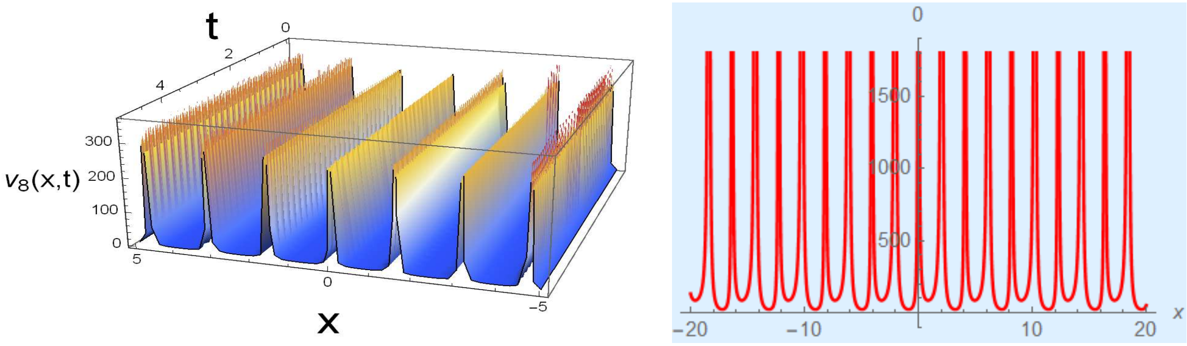
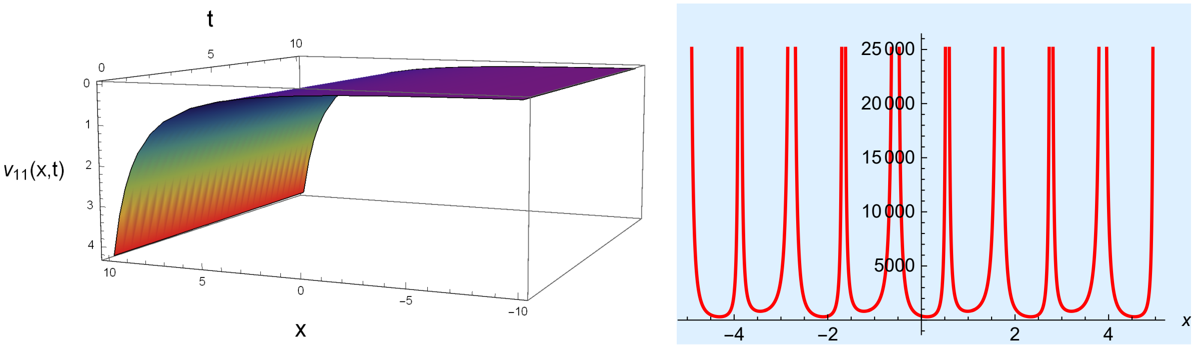


(a) (b)

Fig. 23: Select $\sigma = 0.59, \lambda = 0.0025, c = 0.125$ for 3D and 2D graphs of Eq. 38 .

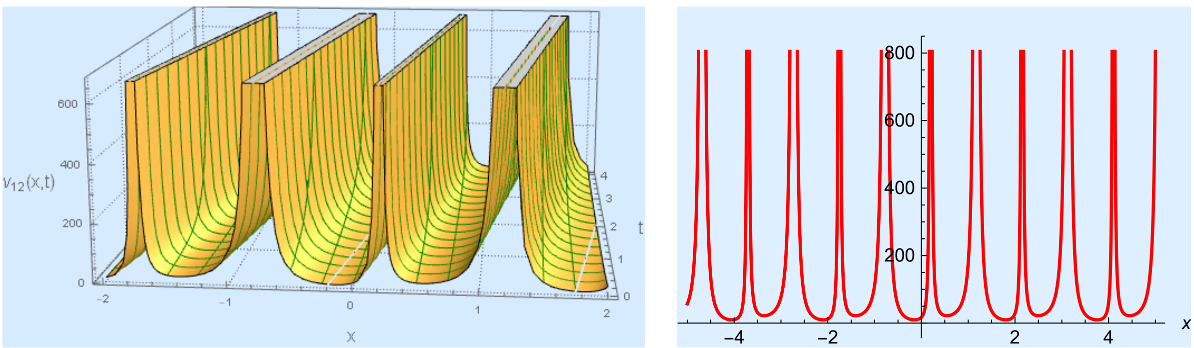
.



(a) (b)

Fig. 24: Select $\sigma = 2.59, \lambda = 1.25, c = 0.125$ for 3D and 2D graphs of Eq. 42.

.



(a) (b)

Fig. 25: Select $\sigma = 2.59, \lambda = 1.25, c = 0.125$ for 3D and 2D graphs of Eq. 45.

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5 Results and Discussion

In order to demonstrate the physical Interpretation of the solutions and to define the nature of solitary waves are illustrated through constructed 3D, contour, and 2D(line) graphs in this section. We can see the wave as it varies through time and space, these type of plots, which are a useful tools for visualizing the solitary waves behavior. The nature of solitary wave for different scenario is given below.
The figures 1, 16, 4, display dark solitary wave solution, and the figures 2, 17, display singular solitary wave solution. The figures 5, 19, 6, 20, 7, 21, display combination of singular solitary wave solutions, and the figures 3, 18, display combination of dark-bright solitary wave solution. The figures 8, 22, 9, 23, 11, 24, display singular periodic solitary wave solutions, and the figures 10, 12, 25, 13, 14, display combination of singular periodic solitary wave solution. The figure 15, displays rational solitary wave solution.

6 Conclusion

The nonlinear Schrödinger-Bopp-Podolsky system, which combines parts of the nonlinear Schrödinger equation and the Bopp-Podolsky equation, is used in this study to describe the dynamics of wave functions in quantum mechanics with nonlinear interactions and relativistic effects. Applications of this research can be found in condensed matter physics, quantum optics, nonlinear wave phenomena, and other areas. Additionally, utilizing this system and an improved modified tanh-expansion method, we have examined and identified solitary wave solutions in this study. By combining this technique with powerful analytical methods, we have achieved accurate and efficient solutions for this complex system. We have successfully obtained a variety of solitary wave solutions using our enhanced modified tanh-expansion method, each of which can be recognized by specific features and behaviors. We have been able to investigate several parameter regimes using the derived ansatz and analytical simulations, revealing delightful phenomena as soliton stability, wave interactions, and the effects of external perturbations on the dynamics of solitary waves. The results of this study strengthen our understanding of the NLS-BP system's solitary wave behavior. The modified method presented in this study will be very helpful for future studies on related nonlinear wave equations and their applications in many physical systems. Our analysis provides novel directions of research. Future research can concentrate on adapting the enhanced method to related nonlinear wave equations or broadening it to more intricate NLS-BP system versions. Additional study on the dynamics and stability of the resulting solitary wave solutions in the presence of external perturbations or consideration of higher-dimensional systems could offer insightful information about the behavior of nonlinear waves. In conclusion, our improved technique has successfully obtained solitary wave solutions for the NLS-BP system. The results of this study make a substantial contribution to the subject of nonlinear wave dynamics and open fresh opportunities for the study, comprehension, and application of solitary waves in many physical environments.

References

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