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Analytical Exploration of the $(4+1)$-Dimensional Fokas Equation: Exact Traveling Wave Solutions

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Abstract: This research paper aims at obtaining closed-form solutions for the traveling wave solutions of the $(4+1)$-dimensional Fokas equation. To solve this difficult system of nonlinear partial differential equations, we will use a new improved direct algebraic method. The given technique can be seen to work well for the purpose of converting the governing equation into a set of algebraic equations from where one can extract various traveling wave solutions. Here our results concern different forms of the solutions as solitons, periodic and rational functions. The obtained solutions are important for understanding the physical processes which are characterized by this equation and may be of use in Fluid mechanics, Nonlinear optics, Plasma physics and other areas.

Keywords: $(4+1)$-dimensional Fokas equation, traveling wave solutions, new extended direct algebraic method, bright soliton, dark soliton.

1 Introduction

Nonlinear evolution equations (NLEEs) as a class of equations are derived from the nonlinear sciences and those have important roles in analyze of the nonlinear phenomenon. Another well preserved by self-reinforcement wave packet is referred to as soliton or solitary wave that moves at a constant velocity. The nonlinearity and dispersion of the medium are eliminated which in a result cause the formation of solitons. The class of weakly nonlinear dispersive PDEs which models physical systems has so-called solitons In nonlinear PDEs, more striking phenomena exist than soliton solutions. A specific kind of a localized traveling wave solution of a nonlinear PDE which is immensely stable is known as a soliton [1–20]. Nonlinear Partial differential equations (NPDEs) are a powerful and indispensable tool of modern mathematical modeling, applied for the description of many sophisticated phenomena in science and engineering. Evaluating these equations is at the heart of reproducing highly complex behaviors and remains instrumental in developing theory and technology from physics and fluid mechanics to biological and financial systems. One of the issues arising with the study of NPDEs is the question of determining exact solutions to these equations, which is essential for determining the physical processes going on and for modeling many systems described by such equations. Traditionally, getting analytical solutions to NPDEs has been quite hard and even to date, some problems are yet to be solved. Since these equations are mutually dependent and quite elaborate at most times, simple solutions are rarely possible, which calls for enhanced methods. Much has been done in this area in the recent past perhaps due to the

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increase in computational power and availability of better algorithms. These have made the solution of NPDEs possible and have created opportunities for applying solutions previously pipe dreams. The sophisticated symbolic computation systems like Maple, Mathematica, MATLAB, etc. have gone further in encouraging the researchers by providing more efficient instruments to analyze broader areas of solution techniques. These tools have enabled one to obtain results that hitherto had been out of the reach of the researcher working on NPDEs. It enriches the existing research on exact solutions of NPDEs by presenting a new method suitable for a certain type of such equations. Thus, using the potential of the current methods of computational mathematics, this work intends to contribute to the development of the existing theoretical material and provide a new look at the behavior of the given complex systems that are characterized by NPDEs. The conclusions drawn in this paper contribute not only to the development of knowledge on these systems but also to the further research done in the fascinating and most diverse field.
The techniques for finding exact analytical solutions of nonlinear partial differential equations are the tanh function technique [41], the exp function technique [42], the F-expansion method [43], Hirota-s direct method [44], Kudryashov technique [45], the modified extended direct algebraic method [64], the extended auxiliary equation technique [47], the modified technique of the simplest equation [48] and the new extended direct algebraic technique [62]. Some of the methods adapted for numerical solution of the nonlinear partial differential equations are finite element method [31, 32], finite volume method [33, 34], generalized finite difference method [35, 36], collocation method [37, 38] ,Galerkin finite element method [39, 40].
In this paper, we are interested in applying the new extended direct algebraic method for solving the presented model trying to reveal their dynamics. The organization of the paper is as follows: in section 2, the algorithm for applying the new extended direct algebraic method is introduced with its main steps revealing different types of solutions. In section 3, the application of the presented method on the $(4+1)$-dimensional Fokas equation is illustrated along with the graphical representations of the obtained solutions. Finally, section 6 present the discussion of the obtained results along with a conclusion of the present and possible future work.

2 Algorithm for New Extended Direct Algebraic Method

In this section, the algorithm for the new extended direct algebraic method is introduced, see [49, 50]. In the following we will outline the main steps of our method.
Consider a general nonlinear PDE in the form

$$Q(u, u_x, u_y, u_z, u_w, u_t, u_{tt}, ...) = 0, \tag{1}$$

where Q is a polynomial function of its argument, and the subscripts denote partial derivatives. We seek its traveling wave solutions by using transformation wave

$$u(x,y,z,w,t) = U(\zeta), \quad \zeta = ax + by + cz + dw + vt. \tag{2}$$

Substituting Eq. 2 into Eq. 1 yields a nonlinear ordinary differential equation

$$Q(u, u^{'}, u^{''}, u^{'''}, ...) = 0, \tag{3}$$

where the prime denotes differentiation with respect to ζ. Let us consider that Eq.3 has a formal solution of the form

$$U(\zeta) = \sum_{j=0}^{n} \eta_j P^j(\zeta), \quad \eta_n \neq 0, \tag{4}$$

where the $\eta_j\,(0 \leq j \leq n)$ are constants coefficients to be determined later, and n is a positive integer which is found by homogenous balancing principle between the highest nonlinear term and the highest derivative in Eq. 3 and $P(\zeta)$ satisfies the NODE Eq. 3 in the form of

$$P^{'}(\zeta) = \ln A\left(\Psi + \Theta P(\zeta) + \Omega P^2(\zeta)\right), \quad A \neq (0,1), \tag{5}$$

where Ψ, Θ and Ω are constants. Some special solutions of the NODE are given in [49, 50].

3 Application of the new extended direct algebraic method

Considering $(4+1)-$ dimensional Fokas equation [65]

$$4u_{tx} - u_{xxxy} + u_{xyyy} + 12u_x u_y + 12u u_{xy} - 6u_{zw} = 0. \tag{6}$$

u : dependent variable representing a physical quantity (e.g., wave amplitude, fluid velocity)
x, y, z, t : independent variables representing spatial coordinates and time, respectively
Making a wave transformation

$$u(x,y,z,w,t) = U(\zeta), \tag{7}$$

with $\zeta = ax + by + cz + dw + vt$.
Plugging Eq.7 into 6, then resulting nonlinear ordinary differential equation is:

$$(4av-6cd)U''(\zeta)+12ab\left(U(\zeta)U'(\zeta)\right)'+\left(-a^3b+ab^3\right)U^{iv}(\zeta)=0. \tag{8}$$

Integrating Eq.8 twice and neglecting constants of integration

$$(4av-6cd)U(\zeta)+6abU^2(\zeta)+\left(-a^3b+ab^3\right)U''(\zeta)=0. \tag{9}$$

Balancing between the highest nonlinear term and the highest derivative U'' with U^2 in Eq. 9 gives $N=2$. Thus, Eq. 9 has the formal solution

$$U(\zeta)=\eta_0+\eta_1P(\zeta)+\eta_2P^2(\zeta), \tag{10}$$

substituting Eq. 10 along with Eq. 5 into Eq. 9 and setting the coefficients of all powers of $P^i, i=0,1\cdots$, to zero, we yield the following system of algebraic equations

$$\begin{cases} 6ab\,\eta_2{}^2+6ab^3\eta_2\ln^2 A\Omega^2-6a^3b\eta_2\ln^2 A\Omega^2=0, \\ -10a^3b\,\eta_2\ln^2 A\Theta\,\Omega+10ab^3\eta_2\ln^2 A\Theta\,\Omega-2a^3b\eta_1\ln^2 A\Omega^2+12ab\,\eta_1\eta_2+2ab^3\eta_1\ln^2 A\Omega^2=0, \\ 8ab^3\eta_2\ln^2 A\Psi\Omega+6ab\,\eta_1{}^2+12ab\,\eta_0\eta_2+4ab^3\eta_2\ln^2 A\Theta^2-8a^3b\eta_2\ln^2 A\Psi\Omega-6cd\,\eta_2+4av\eta_2 \\ +3ab^3\eta_1\ln^2 A\Theta\,\Omega-4a^3b\eta_2\ln^2 A\Theta^2-3a^3b\eta_1\ln^2 A\Theta\,\Omega=0, \\ -6a^3b\eta_2\ln^2 A\Psi\Theta-2a^3b\eta_1\ln^2 A\Omega\,\Psi+4av\eta_1-a^3b\eta_1\ln^2 A\Theta^2-6cd\,\eta_1+6ab^3\eta_2\ln^2 A\Psi\Theta \\ +2ab^3\eta_1\ln^2 A\Omega\,\Psi+12ab\,\eta_0\eta_1+ab^3\eta_1\ln^2 A\Theta^2=0, \\ 4av\,\eta_0+2ab^3\eta_2\ln^2 A\Psi^2+6ab\,\eta_0{}^2-a^3b\eta_1\ln^2 A\Theta\,\Psi-6cd\,\eta_0-2a^3b\eta_2\ln^2 A\Psi^2 \\ +ab^3\eta_1\ln^2 A\Theta\,\Psi=0. \end{cases} \tag{11}$$

Solving the above system of algebraic Eq. 11 with the aid of Maple, we have the following results;

Set 1

$$\begin{cases} \eta_0=\Psi\,\Omega\,\left(a^2-b^2\right)\ln^2 A, \\ \eta_1=\Theta\,\Omega\,\left(a^2-b^2\right)\ln^2 A, \\ \eta_2=\Omega^2\left(a^2-b^2\right)\ln^2 A, \\ v=\dfrac{1}{4}b\left(b^2-a^2\right)\left(\Theta^2-4\Psi\,\Omega\right)\ln^2 A-\frac{3cd}{2a}. \end{cases} \tag{12}$$

Set 2

$$\begin{cases} \eta_0=\dfrac{1}{6}\left(a^2-b^2\right)\left(\Theta^2+2\Psi\,\Omega\right)\ln^2 A, \\ \eta_1=\Theta\,\Omega\,\left(a^2-b^2\right)\ln^2 A, \\ \eta_2=\Omega^2\left(a^2-b^2\right)\ln^2 A, \\ v=\dfrac{1}{4}b\left(b^2-a^2\right)\left(\Theta^2-4\Psi\,\Omega\right)\ln^2 A+\frac{3cd}{2a}. \end{cases} \tag{13}$$

substituting Eq. 12 along with Eq. 10 into Eq. 7, we get the following exact solutions of Eq. 6

Case 1

Family 1. When $\Theta^2-4\Psi\Omega<0$ and $\Omega\neq 0$, then the traveling wave solutions are given by

$$u_{1,1}(\zeta)=\Psi\,\Omega\,\left(a^2-b^2\right)\ln^2 A+\Theta\,\Omega\,\left(a^2-b^2\right)\ln^2 A\left[-\frac{\Theta}{2\Omega}+\frac{\sqrt{-(\Theta^2-4\Psi\,\Omega)}}{2\Omega}\tan_A\left(\frac{\sqrt{-(\Theta^2-4\Psi\,\Omega)}\,\zeta}{2}\right)\right] \\ +\Omega^2\left(a^2-b^2\right)\ln^2 A\left[-\frac{\Theta}{2\Omega}+\frac{\sqrt{-(\Theta^2-4\Psi\,\Omega)}}{2\Omega}\tan_A\left(\frac{\sqrt{-(\Theta^2-4\Psi\,\Omega)}\,\zeta}{2}\right)\right]^2 \tag{14}$$

$$u_{2,1}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left[-\frac{\Theta}{2\Omega} - \frac{\sqrt{-(\Theta^2-4\Psi\Omega)}}{2\Omega}\cot_A\left(\frac{\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta}{2}\right)\right] + \Omega^2\left(a^2-b^2\right)\ln^2 A\left[-\frac{\Theta}{2\Omega} - \frac{\sqrt{-(\Theta^2-4\Psi\Omega)}}{2\Omega}\cot_A\left(\frac{\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta}{2}\right)\right]^2 \quad (15)$$

$$u_{3,1}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left[-\frac{\Theta}{2\Omega} + \frac{\sqrt{-(\Theta^2-4\Psi\Omega)}}{2\Omega}\left(\tan_A\left(\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta\right) \pm \sqrt{pq}\sec_A\left(\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta\right)\right)\right] + \Omega^2\left(a^2-b^2\right)\ln^2 A\left[-\frac{\Theta}{2\Omega} + \frac{\sqrt{-(\Theta^2-4\Psi\Omega)}}{2\Omega}\left(\tan_A\left(\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta\right) \pm \sqrt{pq}\sec_A\left(\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta\right)\right)\right]^2 \quad (16)$$

$$u_{4,1}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left[-\frac{\Theta}{2\Omega} + \frac{\sqrt{-(\Theta^2-4\Psi\Omega)}}{2\Omega}\left(-\cot_A\left(\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta\right) \pm \sqrt{pq}\csc_A\left(\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta\right)\right)\right] + \Omega^2\left(a^2-b^2\right)\ln^2 A\left[-\frac{\Theta}{2\Omega} + \frac{\sqrt{-(\Theta^2-4\Psi\Omega)}}{2\Omega}\left(-\cot_A\left(\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta\right) \pm \sqrt{pq}\csc_A\left(\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta\right)\right)\right]^2 \quad (17)$$

$$u_{5,1}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\frac{\Theta}{2\Omega} + \frac{\sqrt{-(\Theta^2-4\Psi\Omega)}}{4\Omega}\left(\tan_A\left(\frac{1}{4}\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta\right) - \cot_A\left(\frac{1}{4}\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta\right)\right)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\frac{\Theta}{2\Omega} + \frac{\sqrt{-(\Theta^2-4\Psi\Omega)}}{4\Omega}\left(\tan_A\left(\frac{1}{4}\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta\right) - \cot_A\left(\frac{1}{4}\sqrt{-(\Theta^2-4\Psi\Omega)}\,\zeta\right)\right)\right\}^2. \quad (18)$$

Family2. When $\Theta^2-4\Psi\Omega>0$ and $\Omega\neq 0$, then the traveling wave solutions are given by

$$u_{6,2}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\frac{\Theta}{2\Omega} - \frac{\sqrt{(\Theta^2-4\Psi\Omega)}}{2\Omega}\tanh_A\left(\frac{\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta}{2}\right)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\frac{\Theta}{2\Omega} - \frac{\sqrt{(\Theta^2-4\Psi\Omega)}}{2\Omega}\tanh_A\left(\frac{\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta}{2}\right)\right\}^2, \quad (19)$$

$$u_{7,2}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\frac{\Theta}{2\Omega} - \frac{\sqrt{(\Theta^2-4\Psi\Omega)}}{2\Omega}\coth_A\left(\frac{\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta}{2}\right)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\frac{\Theta}{2\Omega} - \frac{\sqrt{(\Theta^2-4\Psi\Omega)}}{2\Omega}\coth_A\left(\frac{\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta}{2}\right)\right\}^2, \quad (20)$$

$$u_{8,2}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\frac{\Theta}{2\Omega} + \frac{\sqrt{(\Theta^2-4\Psi\Omega)}}{2\Omega}\left(-\tanh_A\left(\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta\right) \pm i\sqrt{pq}\ \ csch\ _A\left(\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta\right)\right)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\frac{\Theta}{2\Omega} + \frac{\sqrt{(\Theta^2-4\Psi\Omega)}}{2\Omega}\left(-\tanh_A\left(\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta\right) \pm i\sqrt{pq}\ \ csch\ _A\left(\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta\right)\right)\right\}^2 \quad (21)$$

$$u_{9,2}(\zeta) = \Psi\Omega\,(a^2-b^2)\ln^2 A + \Theta\,\Omega\,(a^2-b^2)\ln^2 A \left\{-\frac{\Theta}{2\Omega} + \frac{\sqrt{(\Theta^2-4\Psi\Omega)}}{2\Omega}\left(-\coth_A\left(\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta\right) \pm \sqrt{pq}\quad csch\quad_A\left(\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta\right)\right)\right\} + \Omega^2\,(a^2-b^2)\ln^2 A \left\{-\frac{\Theta}{2\Omega} + \frac{\sqrt{(\Theta^2-4\Psi\Omega)}}{2\Omega}\left(-\coth_A\left(\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta\right) \pm \sqrt{pq}\quad csch\quad_A\left(\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta\right)\right)\right\}^2 \quad (22)$$

$$u_{10,2}(\zeta) = \Psi\Omega\,(a^2-b^2)\ln^2 A + \Theta\,\Omega\,(a^2-b^2)\ln^2 A \left[-\frac{\Theta}{2\Omega} - \frac{\sqrt{(\Theta^2-4\Psi\Omega)}}{4\Omega}\left\{\tanh_A\left(\frac{\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta}{4}\right) + \coth_A\left(\frac{\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta}{4}\right)\right\}\right] + \Omega^2\,(a^2-b^2)\ln^2 A \left[-\frac{\Theta}{2\Omega} - \frac{\sqrt{(\Theta^2-4\Psi\Omega)}}{4\Omega}\left\{\tanh_A\left(\frac{\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta}{4}\right) + \coth_A\left(\frac{\sqrt{(\Theta^2-4\Psi\Omega)}\,\zeta}{4}\right)\right\}\right]^2. \quad (23)$$

Family3. When $\Psi\Omega > 0$ and $\Theta = 0$, then the traveling wave solutions are given by

$$u_{11,3}(\zeta) = \Psi\Omega\,(a^2-b^2)\ln^2 A + \Theta\,\Omega\,(a^2-b^2)\ln^2 A \left\{\sqrt{\frac{\Psi}{\Omega}}\tan_A\left(\sqrt{\Psi\Omega}\zeta\right)\right\} + \Omega^2\,(a^2-b^2)\ln^2 A\left\{\sqrt{\frac{\Psi}{\Omega}}\tan_A\left(\sqrt{\Psi\Omega}\zeta\right)\right\}^2, \quad (24)$$

$$u_{12,3}(\zeta) = \Psi\Omega\,(a^2-b^2)\ln^2 A + \Theta\,\Omega\,(a^2-b^2)\ln^2 A \left\{-\sqrt{\frac{\Psi}{\Omega}}\cot_A\left(\sqrt{\Psi\Omega}\zeta\right)\right\} + \Omega^2\,(a^2-b^2)\ln^2 A\left\{-\sqrt{\frac{\Psi}{\Omega}}\cot_A\left(\sqrt{\Psi\Omega}\zeta\right)\right\}^2, \quad (25)$$

$$u_{13,3}(\zeta) = \Psi\Omega\,(a^2-b^2)\ln^2 A + \Theta\,\Omega\,(a^2-b^2)\ln^2 A \left\{\sqrt{\frac{\Psi}{\Omega}}\left(\tan_A\left(2\sqrt{\Psi\Omega}\zeta\right) \pm \sqrt{pq}\sec_A\left(2\sqrt{\Psi\Omega}\zeta\right)\right)\right\} + \Omega^2\,(a^2-b^2)\ln^2 A\left\{\sqrt{\frac{\Psi}{\Omega}}\left(\tan_A\left(2\sqrt{\Psi\Omega}\zeta\right) \pm \sqrt{pq}\sec_A\left(2\sqrt{\Psi\Omega}\zeta\right)\right)\right\}^2, \quad (26)$$

$$u_{14,3}(\zeta) = \Psi\Omega\,(a^2-b^2)\ln^2 A + \Theta\,\Omega\,(a^2-b^2)\ln^2 A \left\{\sqrt{\frac{\Psi}{\Omega}}\left(-\cot_A\left(2\sqrt{\Psi\Omega}\zeta\right) \pm \sqrt{pq}\csc_A\left(2\sqrt{\Psi\Omega}\zeta\right)\right)\right\} + \Omega^2\,(a^2-b^2)\ln^2 A\left\{\sqrt{\frac{\Psi}{\Omega}}\left(-\cot_A\left(2\sqrt{\Psi\Omega}\zeta\right) \pm \sqrt{pq}\csc_A\left(2\sqrt{\Psi\Omega}\zeta\right)\right)\right\}^2, \quad (27)$$

$$u_{15,3}(\zeta) = \Psi\Omega\,(a^2-b^2)\ln^2 A + \Theta\,\Omega\,(a^2-b^2)\ln^2 A \left\{\frac{1}{2}\sqrt{\frac{\Psi}{\Omega}}\left(\tan_A\left(\frac{1}{2}\sqrt{\Psi\Omega}\zeta\right) - \cot_A\left(\frac{1}{2}\sqrt{\Psi\Omega}\zeta\right)\right)\right\} + \Omega^2\,(a^2-b^2)\ln^2 A\left\{\frac{1}{2}\sqrt{\frac{\Psi}{\Omega}}\left(\tan_A\left(\frac{1}{2}\sqrt{\Psi\Omega}\zeta\right) - \cot_A\left(\frac{1}{2}\sqrt{\Psi\Omega}\zeta\right)\right)\right\}^2. \quad (28)$$

Family4. When $\Psi\Omega < 0$ and $\Theta = 0$, then the traveling wave solutions are given by

$$u_{16,4}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\sqrt{-\frac{\Psi}{\Omega}}\tanh_A\left(\sqrt{-\Psi\Omega}\zeta\right)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\sqrt{-\frac{\Psi}{\Omega}}\tanh_A\left(\sqrt{-\Psi\Omega}\zeta\right)\right\}^2, \quad (29)$$

$$u_{17,4}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\sqrt{-\frac{\Psi}{\Omega}}\coth_A\left(\sqrt{-\Psi\Omega}\zeta\right)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\sqrt{-\frac{\Psi}{\Omega}}\coth_A\left(\sqrt{-\Psi\Omega}\zeta\right)\right\}^2, \quad (30)$$

$$u_{18,4}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{\sqrt{-\frac{\Psi}{\Omega}}\left(-\tanh\left(2\sqrt{-\Psi\Omega}\zeta\right)\right)\pm i\sqrt{pq}\quad csch\quad _A(2\sqrt{-\Psi\Omega}\zeta)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{\sqrt{\frac{-\Psi}{\Omega}}\left(-\tanh\left(2\sqrt{-\Psi\Omega}\zeta\right)\right)\pm i\sqrt{pq}\quad csch\quad _A(2\sqrt{-\Psi\Omega}\zeta)\right\}^2, \quad (31)$$

$$u_{19,4}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\sqrt{\frac{\Psi}{\Omega}}\left(-\coth_A\left(2\sqrt{-\Psi\Omega}\zeta\right)\pm\sqrt{pq}\quad csch\quad _A\left(2\sqrt{-\Psi\Omega}\zeta\right)\right)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{\sqrt{-\frac{\Psi}{\Omega}}\left(-\coth_A\left(2\sqrt{-\Psi\Omega}\zeta\right)\pm\sqrt{pq}\quad csch\quad _A\left(2\sqrt{-\Psi\Omega}\zeta\right)\right)\right\}^2 \quad (32)$$

$$u_{20,4}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\frac{1}{2}\sqrt{-\frac{\Psi}{\Omega}}\left(\tanh_A\left(\frac{1}{2}\sqrt{-\Psi\Omega}\zeta\right)+\coth_A\sqrt{-\Psi\Omega}\zeta\right)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\frac{1}{2}\sqrt{-\frac{\Psi}{\Omega}}\left(\tanh_A\left(\frac{1}{2}\sqrt{-\Psi\Omega}\zeta\right)+\coth_A\sqrt{-\Psi\Omega}\zeta\right)\right\}^2. \quad (33)$$

Family5. When $\Theta = 0$ and $\Omega = \Psi$, then the traveling wave solutions are given by

$$u_{21,5}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{\tan_A(\Psi\zeta)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{\tan_A(\Psi\zeta)\right\}^2, \quad (34)$$

$$u_{22,5}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\cot_A(\Psi\zeta)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\cot_A(\Psi\zeta)\right\}^2, \quad (35)$$

$$u_{23,5}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{(\tan_A(2\Psi\zeta)\pm\sqrt{pq})\sec_A(2\Psi\zeta)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{(\tan_A(2\Psi\zeta)\pm\sqrt{pq})\sec_A(2\Psi\zeta)\right\}^2, \quad (36)$$

$$u_{24,5}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\cot_A(2\Psi\zeta)\pm\sqrt{pq}\csc_A(2\Psi\zeta)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\cot_A(2\Psi\zeta)\pm\sqrt{pq}\csc_A(2\Psi\zeta)\right\}^2, \quad (37)$$

$$u_{25,5}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{\frac{1}{2}\left(\tan\left(\frac{1}{2}\Psi\zeta\right)-\cot\left(\frac{1}{2}\Psi\zeta\right)\right)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{\frac{1}{2}\left(\tan\left(\frac{1}{2}\Psi\zeta\right)-\cot\left(\frac{1}{2}\Psi\zeta\right)\right)\right\}^2. \quad (38)$$

Family6. When $\Theta = 0$ and $\Omega = -\Psi$, then the traveling wave solutions are given by

$$u_{26,6}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\tanh_A(\Psi\zeta)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\tanh_A(\Psi\zeta)\right\}^2, \tag{39}$$

$$u_{27,6}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\coth_A(\Psi\zeta)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\coth_A(\Psi\zeta)\right\}^2, \tag{40}$$

$$u_{28,6}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\tanh_A(2\Psi\zeta) \pm i\sqrt{pq}sech_A(2\Psi\zeta)\right\}$$
$$+\Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\tanh_A(2\Psi\zeta) \pm i\sqrt{pq}sech_A(2\Psi\zeta)\right\}^2, \tag{41}$$

$$u_{29,6}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\coth_A(2\Psi\zeta) \pm \sqrt{pq}csch_A(2\Psi\zeta)\right\}$$
$$+\Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\coth_A(2\Psi\zeta) \pm \sqrt{pq}csch_A(2\Psi\zeta)\right\}^2, \tag{42}$$

$$u_{30,6}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\frac{1}{2}\left(\tanh_A(\frac{1}{2}\Psi\zeta) + \coth_A(\frac{1}{2}\Psi\zeta)\right)\right\}$$
$$+\Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\frac{1}{2}\left(\tanh_A(\frac{1}{2}\Psi\zeta) + \coth_A\frac{1}{2}\Psi\zeta\right)\right\}^2. \tag{43}$$

Family7 When $\Theta^2 = 4\Psi\Omega$, then the traveling wave solutions are given by

$$u_{31,7}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-2\frac{\Psi(\Theta\zeta\ln A + 2)}{\Theta^2\zeta\ln A}\right\}$$
$$+\Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-2\frac{\Psi(\Theta\zeta\ln A + 2)}{\Theta^2\zeta\ln A}\right\}^2 \tag{44}$$

Family 8. When $\Theta = 0, \Psi = mk, (m \neq 0)$ and $\Omega = k$, then the traveling wave solutions are given by

$$u_{32,8}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{A^{k\zeta} - m\right\} \tag{45}$$
$$+\Omega^2\left(a^2-b^2\right)\ln^2 A\left\{A^{k\zeta} - m\right\}^2$$

Family 9. When $\Theta = \Omega = 0$,, then the traveling wave solutions are given by

$$u_{33,9}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{\Psi\zeta ln(A)\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{\Psi\zeta ln(A)\right\}^2 \tag{46}$$

Family 10 : when $\Theta = \Psi = 0$,, then

$$u_{34,10}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{\frac{-1}{\Omega\zeta ln(A)}\right\} \tag{47}$$
$$+\Omega^2\left(a^2-b^2\right)\ln^2 A\left\{\frac{-1}{\Omega\zeta ln(A)}\right\}^2$$

Family 11 : When $\Theta \neq 0, \Psi = 0$, then the traveling wave solutions are given by

$$u_{35,11}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\frac{p\Theta}{\Omega(\cosh_A(\Theta\zeta) - \sinh_A(\Theta\zeta)) + p}\right\} +$$
$$\Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\frac{p\Theta}{\Omega(\cosh_A(\Theta\zeta) - \sinh_A(\Theta\zeta) + p)}\right\}^2, \tag{48}$$

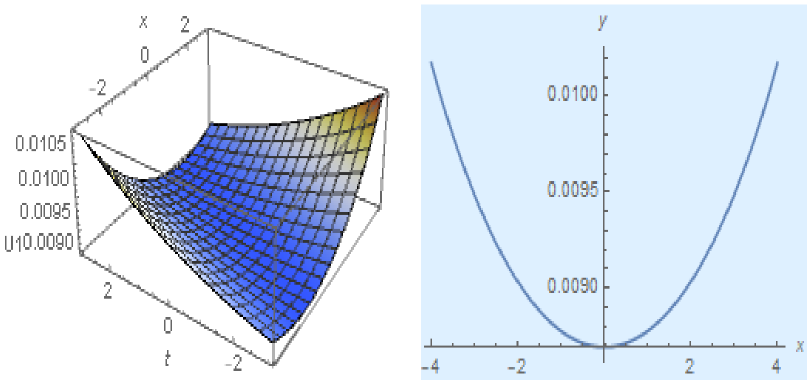
$$u_{36,11}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\frac{\Theta\left(\sinh_A(\Theta\zeta)+\cosh_A(\Theta\zeta)\right)}{\Omega\left(\sinh_A(\Theta\zeta)+\cosh_A(\Theta\zeta)+q\right)}\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\frac{\Theta\left(\sinh_A(\Theta\zeta)+\cosh_A(\Theta\zeta)\right)}{\Omega\left(\sinh_A(\Theta\zeta)+\cosh_A(\Theta\zeta)+q\right)}\right\}^2. \quad (49)$$

Family 12 : When $\Theta = \lambda, \Omega = m\,(m \neq 0)$, and $\Psi = 0$ then the rational solution is given by

$$u_{37,12}(\zeta) = \Psi\Omega\left(a^2-b^2\right)\ln^2 A + \Theta\Omega\left(a^2-b^2\right)\ln^2 A\left\{-\frac{pA^{\lambda\zeta}}{q-m\,pA^{\lambda\zeta}}\right\} + \Omega^2\left(a^2-b^2\right)\ln^2 A\left\{-\frac{pA^{\lambda\zeta}}{q-m\,pA^{\lambda\zeta}}\right\}^2 \quad (50)$$

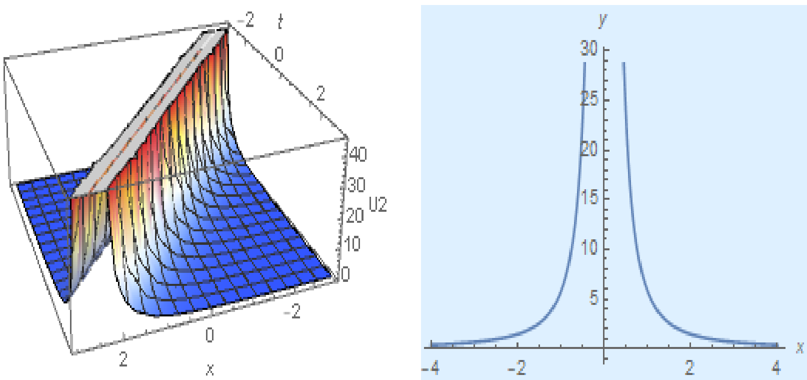
where ζ is an independent variable, p and q are arbitrary constants greater than zero and called deformation parameters.

4 The graphical representation



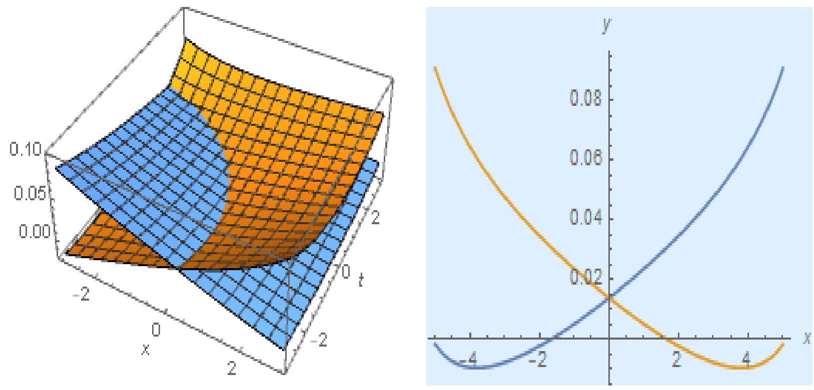
(a) **(b)**

Fig. 1: The 3-D and 2-D graphs of $u_1(\zeta)$ given by Eq. 14.



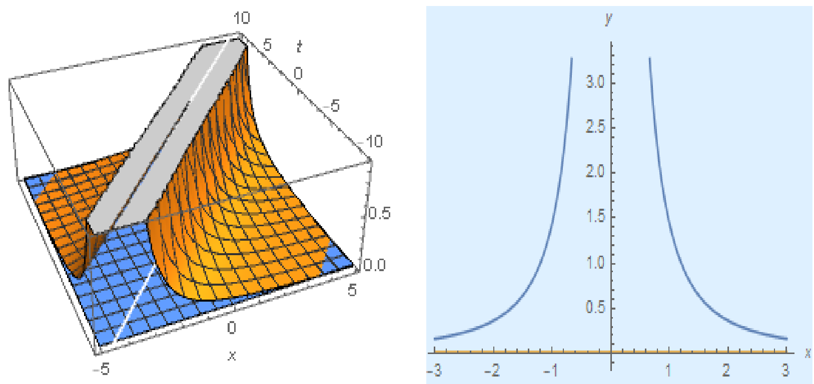
(a) **(b)**

Fig. 2: The 3-D and 2-D graphs of $u_2(\zeta)$ given by Eq. 15.



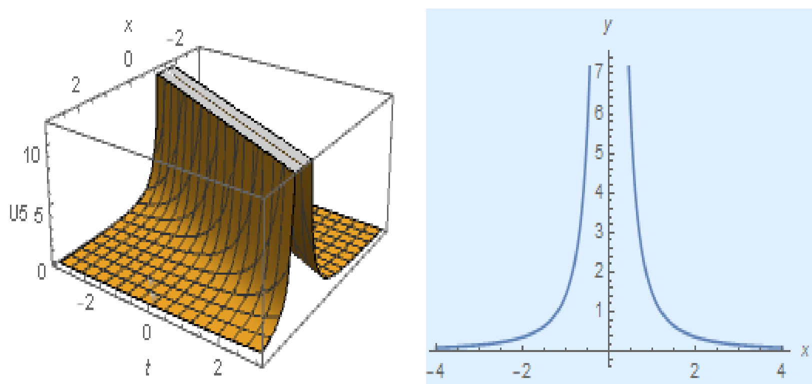
(a) **(b)**

Fig. 3: The 3-D and 2-D graphs of $u_3(\zeta)$ given by Eq.16.



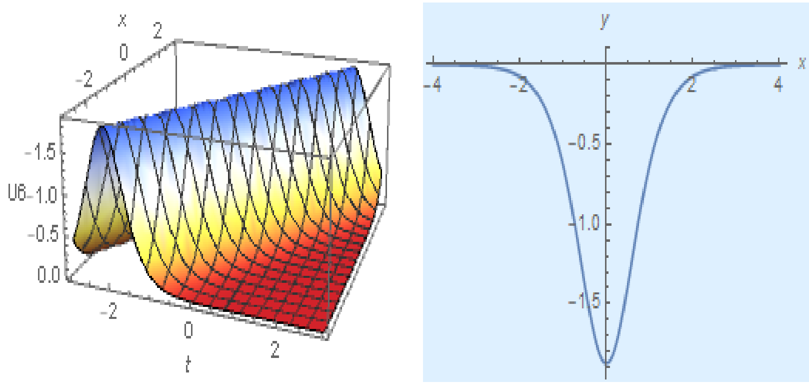
(a) **(b)**

Fig. 4: The 3-D and 2-D graphs of $u_4(\zeta)$ given by Eq. 17.



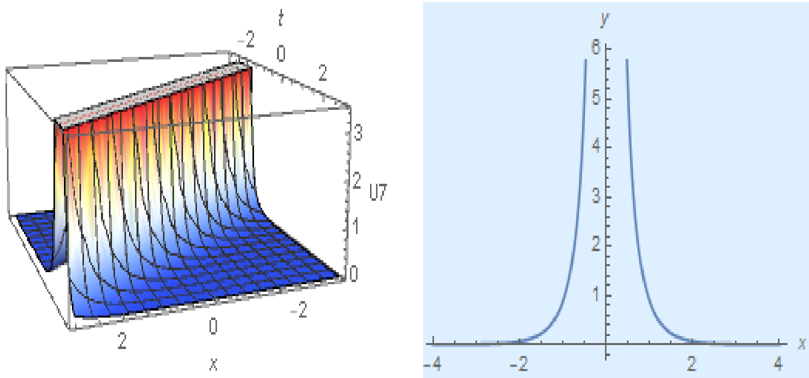
(a) **(b)**

Fig. 5: The 3-D and 2-D graphs of $u_5(\zeta)$ given by Eq. 18.



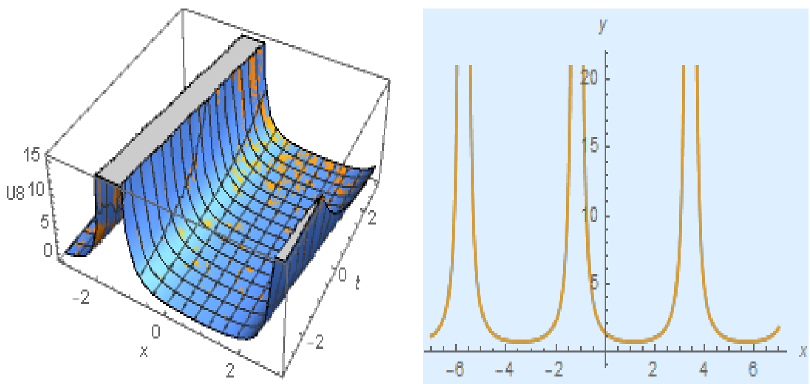
(a) **(b)**

Fig. 6: The 3-D and 2-D graphs of $u_6(\zeta)$ given by Eq. 19.



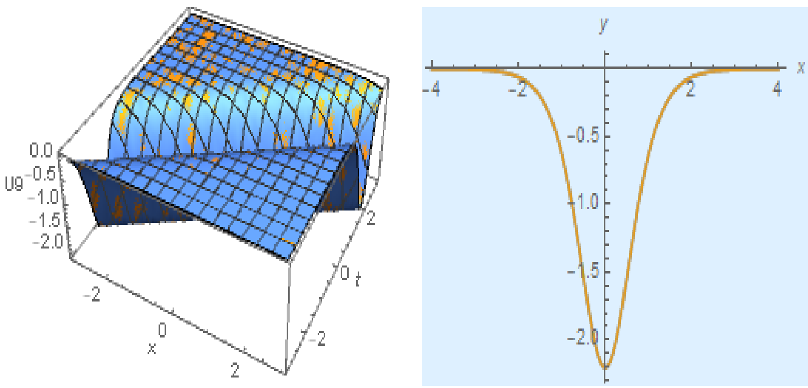
(a) **(b)**

Fig. 7: The 3-D and 2-D graphs of $u_7(\zeta)$ given by Eq. 20.



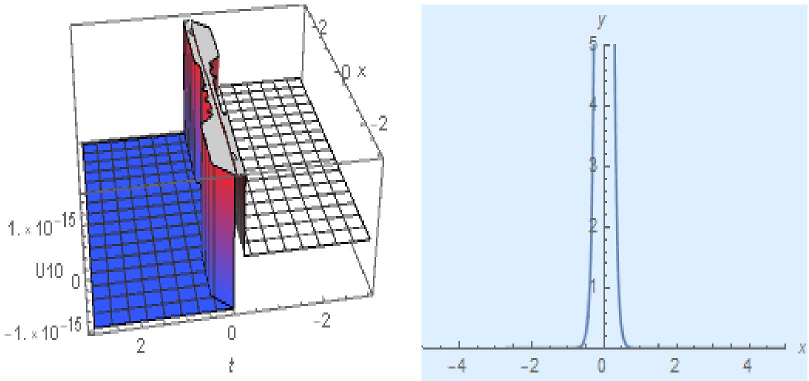
(a) **(b)**

Fig. 8: The 3-D and 2-D graphs of $u_8(\zeta)$ given by Eq. 21.



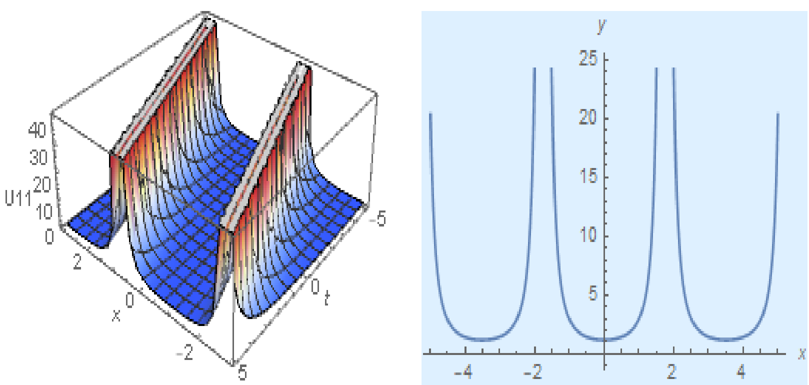
(a) **(b)**

Fig. 9: The 3-D and 2-D graphs of $u_9(\zeta)$ given by Eq. 22.



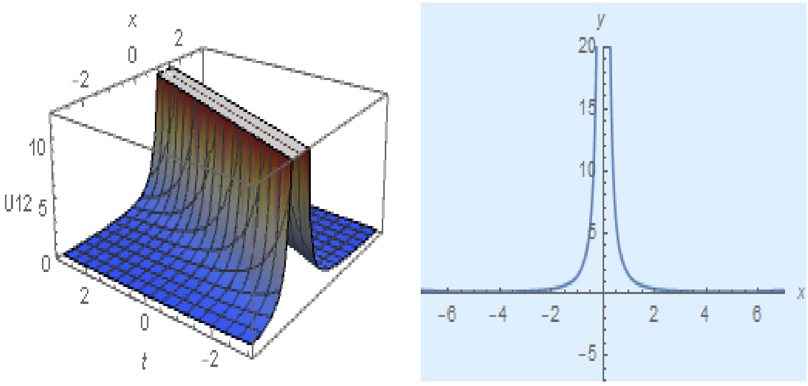
(a) **(b)**

Fig. 10: The 3-D and 2-D graphs of $u_{10}(\zeta)$ given by Eq. 23.



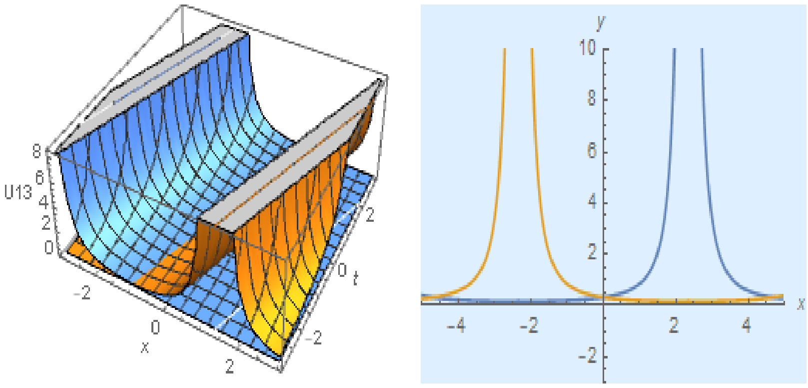
(a) **(b)**

Fig. 11: The 3-D and 2-D graphs of $u_{11}(\zeta)$ given by Eq. 24.



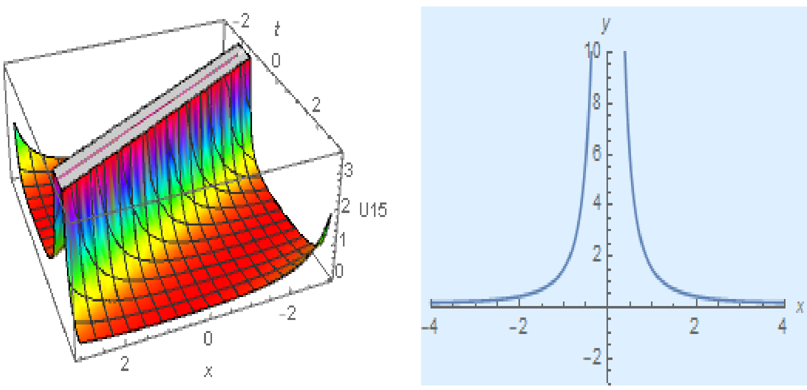
(a) **(b)**

Fig. 12: The 3-D and 2-D graphs of $u_{12}(\zeta)$ given by Eq. 25.



(a) **(b)**

Fig. 13: The 3-D and 2-D graphs of $u_{13}(\zeta)$ given by Eq. 26.



(a) **(b)**

Fig. 14: The 3-D and 2-D graphs of $u_{15}(\zeta)$ given by Eq. 28.

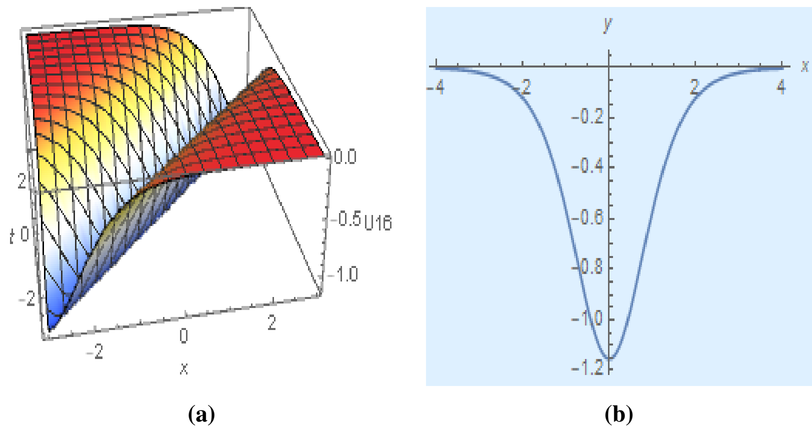


Fig. 15: The 3-D and 2-D graphs of $u_{16}(\zeta)$ given by Eq. 29.

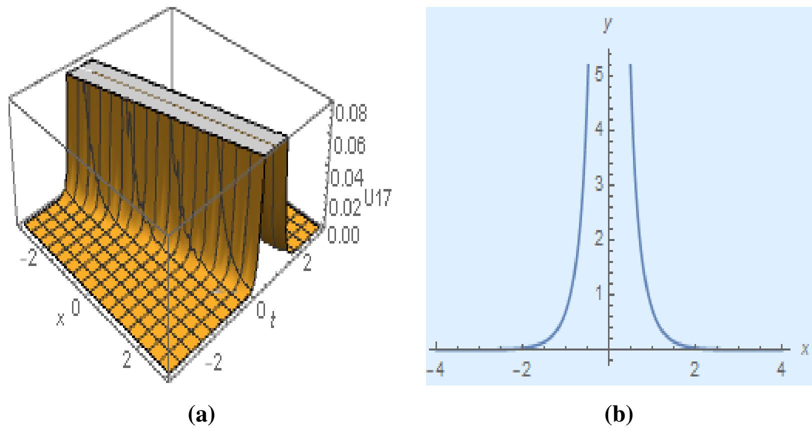


Fig. 16: The 3-D and 2-D graphs of $u_{17}(\zeta)$ given by Eq. 30.

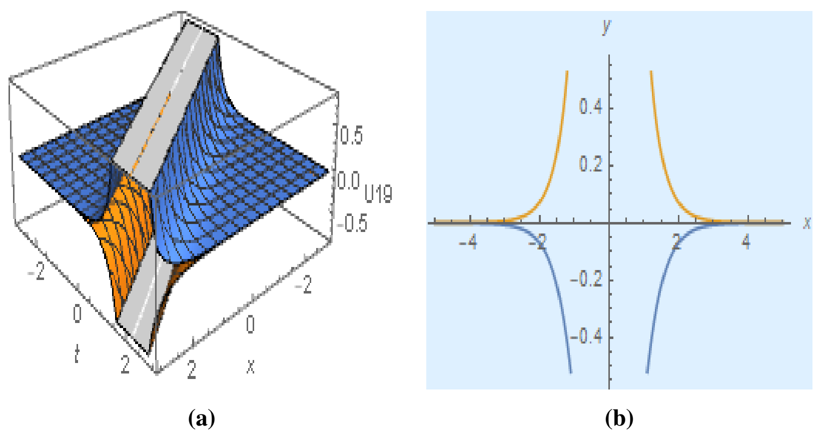
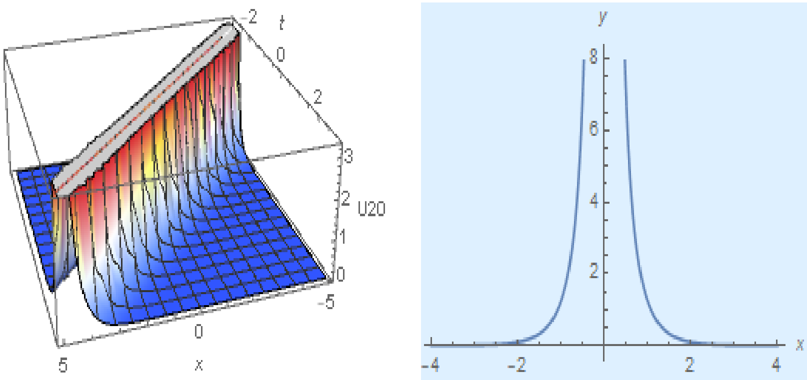
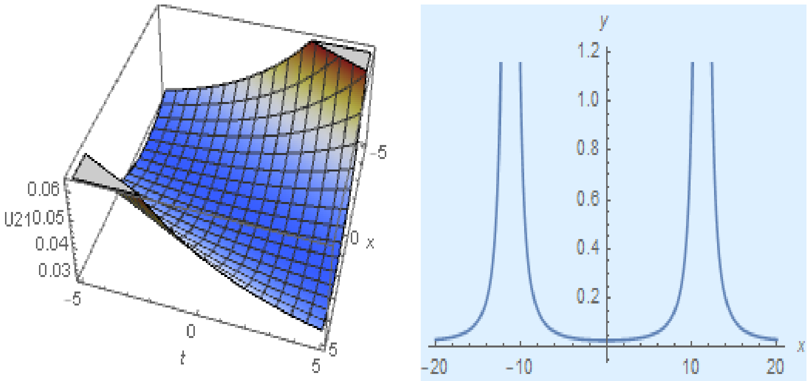


Fig. 17: The 3-D and 2-D graphs of $u_{19}(\zeta)$ given by Eq. 32.



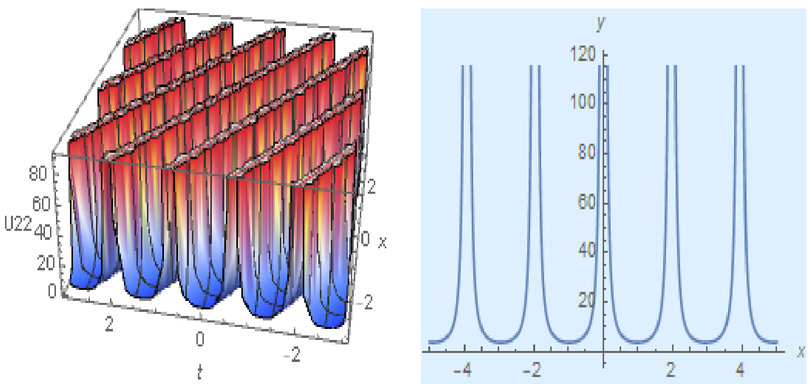
(a) **(b)**

Fig. 18: The 3-D and 2-D graphs of $u_{20}(\zeta)$ given by Eq. 33.



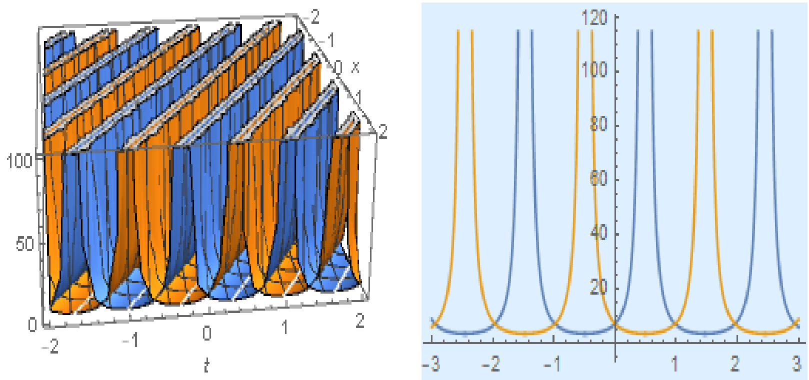
(a) **(b)**

Fig. 19: The 3-D and 2-D graphs of $u_{21}(\zeta)$ given by Eq. 34.



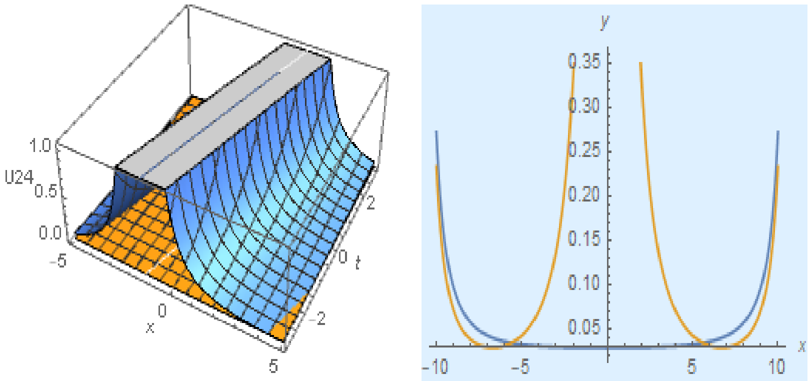
(a) **(b)**

Fig. 20: The 3-D and 2-D graphs of $u_{22}(\zeta)$ given by Eq. 35.



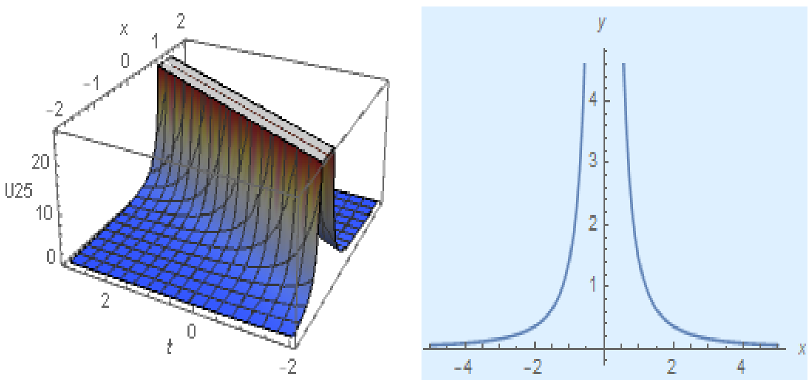
(a) **(b)**

Fig. 21: The 3-D and 2-D graphs of $u_{23}(\zeta)$ given by Eq. 36.



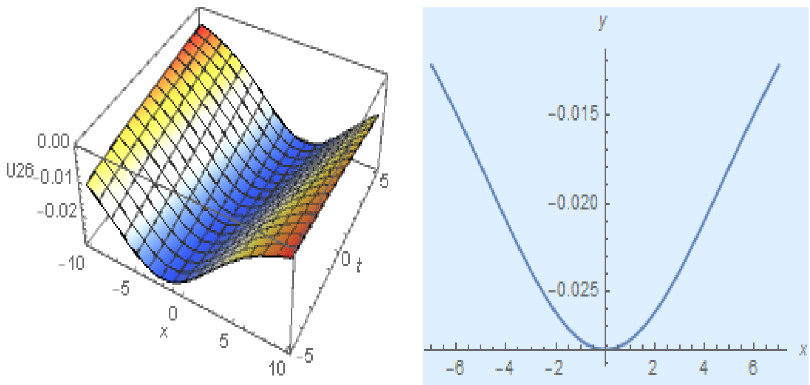
(a) **(b)**

Fig. 22: The 3-D and 2-D graphs of $u_{24}(\zeta)$ given by Eq. 37.



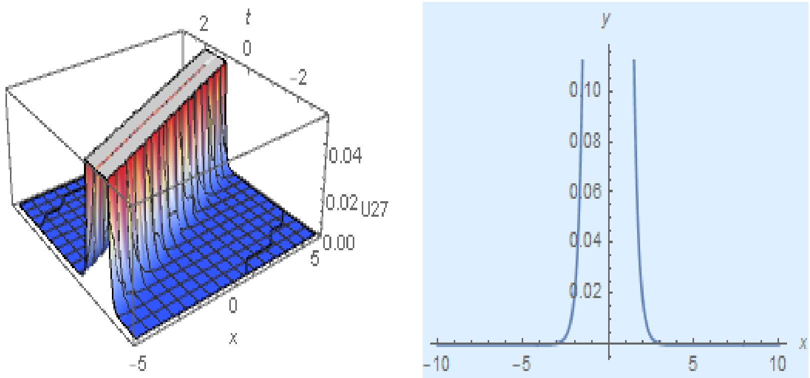
(a) **(b)**

Fig. 23: The 3-D and 2-D graphs of $u_{25}(\zeta)$ given by Eq. 38.



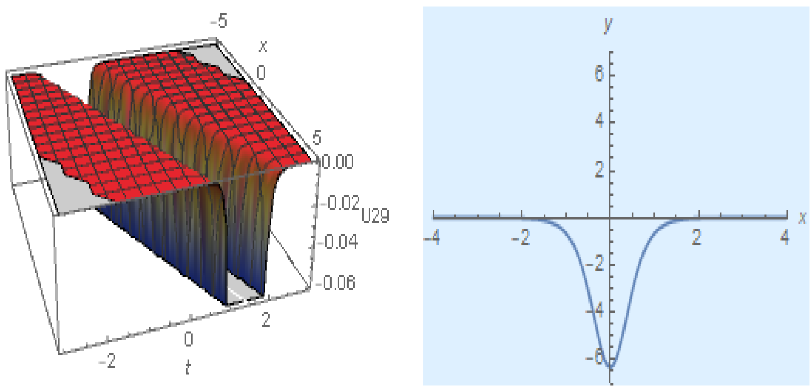
(a) **(b)**

Fig. 24: The 3-D and 2-D graphs of $u_{26}(\zeta)$ given by Eq. 39.



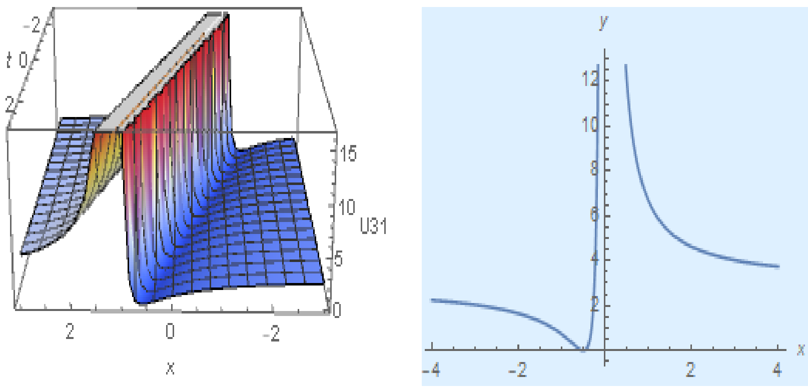
(a) **(b)**

Fig. 25: The 3-D and 2-D graphs of $u_{27}(\zeta)$ given by Eq. 40.



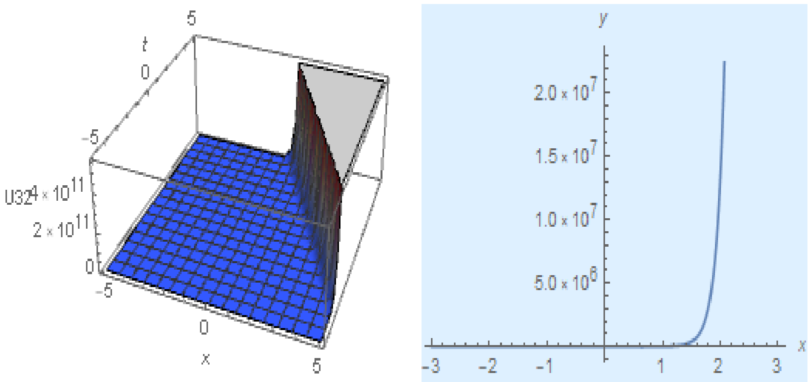
(a) **(b)**

Fig. 26: The 3-D and 2-D graphs of $u_{29}(\zeta)$ given by Eq. 42.



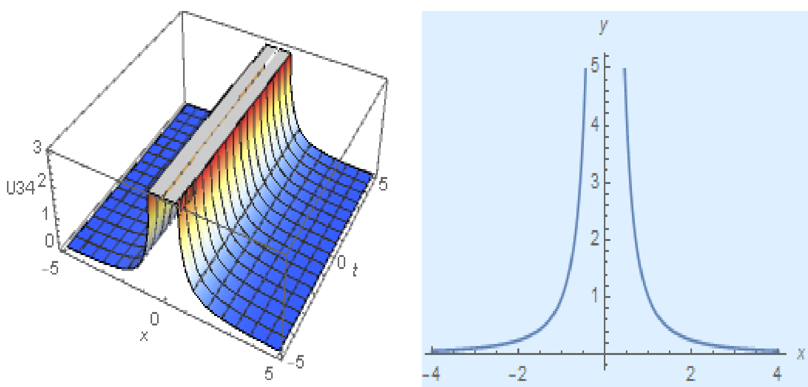
(a) **(b)**

Fig. 27: The 3-D and 2-D graphs of $u_{31}(\zeta)$ given by Eq. 44.



(a) **(b)**

Fig. 28: The 3-D and 2-D graphs of $u_{32}(\zeta)$ given by Eq. 45.



(a) **(b)**

Fig. 29: The 3-D and 2-D graphs of $u_{34}(\zeta)$ given by Eq. 47.

5 Physical Interpretation

The (4+1)-dimensional Fokas equation is a higher-dimensional extension of the well-known Kadomtsev-Petviashvili equation, which models nonlinear wave propagation in dispersive media. It finds applications in various fields, including fluid mechanics, plasma physics, and nonlinear optics. The traveling wave solutions obtained in this study represent different types of wave patterns that can emerge in the systems governed by this equation. Soliton solutions, for example, are associated with temporary, real, localized wave patterns which do not deform or change their velocity while propagating. Periodic solutions are those of wave like form with some form of repetition whereas rational solutions correspond to localized disturbances which decay algebraically. Such solutions are highly useful for understanding the behavior of physical systems that underlie individual waves and can be applied for the computation of many wave processes.

6 Discussion and Conclusion

In the present study, we were able to implement the recent proposed new extended direct algebraic method to obtain a general solution for the traveling wave solutions of the $(4+1)$-dimensional Fokas equation. The acquires solutions contain solitons, periodical functions, rational function and so on. In doing so, the current work makes a contribution to knowledge of the various solutions of this largely nonlinear partial differential equation and lest a potential application in the corresponding fields. The effectiveness of the proposed method is shown in terms of solving high-dimensional nonlinear PDEs while it can be used for other nonlinear systems as well. further investigations can be the stability analysis of the solutions, the physical interpretation of the solutions and last but not the least, numerical studies for the validation of the theory.

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