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Advanced Algebraic Approaches to Soliton Solutions in Shallow Water Wave Equations

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Abstract: In this paper, soliton solutions of the $(1+1)$-dimensional Boussinesq equation are studied with an improved scheme of the direct algebraic method. The Boussinesq equation as concerns the propagation of long waves in shallow water in addition to other physical systems, exhibit both dispersal and nonlinearity. With this new approach we obtain analytical soliton solutions and make a study of their characteristics. The obtained solutions are said in terms of positive forms and this is an improvement of the perception of the dynamics of the equation. The performance of the method is illustrated by calculation, and further use of the method in the related nonlinear systems is also discussed. This work adds to an ongoing research on the analysis of the solution of nonlinear partial differential equations and provides understanding on the solitons in the dispersive media. The obtained solutions as well have been depicted graphically for better understanding.

Keywords: Boussinesq equation, traveling wave solutions, new extended direct algebraic method for Boussinesq equation.

1 Introduction

Nonlinear evolution equations (NLEEs) as a class of equations are derived from the nonlinear sciences and those have important roles in analyze of the nonlinear phenomenon. Another well preserved by self-reinforcement wave packet is referred to as soliton or solitary wave that moves at a constant velocity. The nonlinearity and dispersion of the medium are eliminated which in a result cause the formation of solitons. The class of weakly nonlinear dispersive PDEs which models physical systems has so-called solitons In nonlinear PDEs, more striking phenomena exist than soliton solutions. A specific kind of a localized traveling wave solution of a nonlinear PDE which is immensely stable is known as a soliton [1–20].

It is used for analysis of long waves with small amplitude in shallow water and other physical processes in nonlinear lattice theory and plasma physics. In its $(1+1)$-dimensional form, the equation is expressed as:

$$Q_{tt} - Q_{xx} = \lambda Q_{xxxx} + (Q^2)_{xx}, \tag{1}$$

where $Q(x,t)$ the wave profile and λ is a parameter that determines the scale of dispersion. The coordinates nonlinearity and dispersion in order to form soliton solutions i.e., localized increments that retain their form and propagate at a uniform velocity. The investigation of solitons in the framework of the Boussinesq equation is importance for the analysis of wave processes in different physical systems. Computational procedures of classical

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analysis to solve a system of nonlinear PDEs encounter difficulties in finding exact expressions. Thus, it is critical to develop new analytical approaches that will help to study such systems.

The Boussinesq equation captures the interplay between dispersion and nonlinearity in wave propagation. The term λQ_{xxxx} â represents the dispersive effect, which tends to spread the wave out, while the term $(Q^2)_{xx}$ accounts for the nonlinear interaction, which can lead to wave steepening and the formation of solitons.

From a physical perspective of one and a half-dimensional Boussinesq equation, especially the newly derived solitons, one can easily see the complexities of the interaction between the dispersion and nonlinearity in wave motion. Using the new extended direct algebraic method we have been able to found types of traveling wave solutions which include singular soliton solutions, periodic soliton solutions, rational soliton solutions and both dark and bright soliton solutions. These stand for different wave phenomena which are possible in dispersive medium, for example, shallow water waves or nonlinear lattices. For example, the bright solitons correspond to the localized wave of elevation while the dark solitons describe localized drops of the wave profile. This should give a guarantee to the efficiency of the proposed method and the possibility to analyze the waves behaviors in different physical systems by the help of such opportunities to find soliton solutions with the use of different forms of initial functions. The results obtained in this work have an important implications for the analysis of wave processes in the dispersive media and also in the fields such as mathematical biology, physics, chemistry and fluid dynamics where similar nonlinear partial differential equations are used for modeling of the complex phenomena.

A vast amount of work has been carried out in the last few decades studying the properties of nonlinear partial differential equations for the reason that they provide very ideal models of phenomena in various science and engineering disciplines. In fact, the exact solutions of the above equations help hand and glove in analyzing the various global physical phenomena and dynamism of the process. These kinds of solutions are required for interpreting the qualitative properties of a number of occurrences in various branches of the natural science. In the last few years much attention has been paid to attempt to find exact solutions by using nonlinear partial differential equation models. Since most nonlinear PDEs do not have closed-form solutions, the study of such solutions whether they are travelling wave solutions or other classes of solutions are desirable for developing the understanding of these systems.

The methods for obtaining exact explicit solutions of nonlinear partial differential equations are the tanh-function method [41], the exp-function method [42], the F-expansion method [43], Hirota's direct method [44], Kudryashov method [45], the modified extended direct algebraic method [64], the extended auxiliary equation method [47], modified method of simplest equation [48] and the new extended direct algebraic method [62].Examples of the methods for solving nonlinear partial differential equations numerically are the finite element method [31, 32], finite volume method [33, 34], generalized finite difference method [35, 36], collocation method [37, 38] and Galerkin finite element method [39, 40].

2 Algorithm for New Extended Direct Algebraic Method

In this section, the algorithm for the new extended direct algebraic method is introduced, see [49, 50]. In the following we will outline the main steps of our method.

Consider a general nonlinear PDE in the form

$$Q(u, u_x, u_y, u_z, u_w, u_t, u_{tt}, \ldots.) = 0, \tag{2}$$

where Q is a polynomial function of its argument, and the subscripts denote partial derivatives. We seek its traveling wave solutions by using transformation wave

$$u(x,y,z,w,t) = U(\zeta), \quad \zeta = ax + by + cz + dw + vt. \tag{3}$$

Substituting Eq. 3 into Eq. 2 yields a nonlinear ordinary differential equation

$$Q(u, u^{'}, u^{''}, u^{'''}, \ldots) = 0, \tag{4}$$

where the prime denotes differentiation with respect to ζ.
Let us consider that Eq.4 has a formal solution of the form

$$U(\zeta) = \sum_{j=0}^{n} H_j P^j(\zeta), \; H_n \neq 0, \tag{5}$$

where the $H_j\,(0 \leq j \leq n)$ are constants coefficients to be determined later, and n is a positive integer which is found by homogenous balancing principle between the highest nonlinear term and the highest derivative in Eq. 4 and $P(\zeta)$ satisfies the NODE Eq. 4 in the form of

$$P^{'}(\zeta) = \ln(A)\left(\Theta + \Psi P(\zeta) + \Gamma P^2(\zeta)\right), \;\; A \neq (0,1), \tag{6}$$

where Θ, Ψ and Γ are constants. Some special solutions of the NODE are given in [49, 50].

3 Application of the new extended direct algebraic method

In numerical models, water waves, and coastal engineering, the Boussinesq equation is a long-wavelength and weakly nonlinear approximation that is used to simulate water waves in shallow seas and harbours.

A Scottish engineer named John Scott Russell closely observed solitary waves. John Scott Russell's observation served as the basis for Joseph Boussinesq's approximation. Boussinesq's simulation of one-dimensional water waves in 1872 established that, in addition to the water depth, the vertical velocity is linear and the horizontal velocity is constant [27].

Considering $(1+1)$ dimensional Boussinesq equation

$$Q_{tt} - Q_{xx} = \lambda Q_{xxxx} + \left(Q^2\right)_{xx}. \tag{7}$$

where $Q = Q(x,t)$ represents the wave envelope containing x as a spatial variable and t as a temporal variable. Here λ is an arbitrary constant.

When water waves of various wavelengths are related, this is referred to as a frequency dispersion phenomena; in the case of an infinitesimal wave amplitude, it is also referred to as a linear frequency dispersion. This makes the approximation accurate. Although waves can propagate in multiple directions according to the Boussinesq equation, it is more advantageous to take these into account. The Boussinesq equation utilizing a new extended direct algebraic technique for generating strong and reliable soltons. Now using the traveling wave transformation for this purpose

$$Q(x,t) = q(\zeta), \tag{8}$$

where $\zeta = kx + vt$,

$$Q(x,t) = q(kx + vt), \tag{9}$$

substituting Eq. 9 into Eq. 7. We have the following NODE

$$v^2 q''(\zeta) - k^2 q''(\zeta) - k^2 \left(q^2(\zeta)\right)'' - \lambda k^4 q^{iv}(\zeta) = 0, \tag{10}$$

Integrating Eq.10 twice and neglecting the constants of integration

$$\left(v^2 - k^2\right) q(\zeta) - k^2 q^2(\zeta) - \lambda k^4 q''(\zeta), \tag{11}$$

Balancing q'' with q^2 in Eq. 11 gives $N = 2$. Thus, Eq. 11 has the formal solution

$$q(\zeta) = H_0 + H_1 P(\zeta) + H_2 P^2(\zeta), \tag{12}$$

substituting Eq. 12 along with Eq. 6 into Eq. 11 and setting the coefficients of all powers of $P^i, i = 0, 1\cdots$, to zero, we yield the a system of algebraic equations and solved this system with the aid of Maple or Mathematica, we obtained following values for constants;

Case 1

$$\begin{cases} H_0 = -6\lambda k^2 \Gamma \Theta \ln^2(A), \\ H_1 = -6\lambda k^2 \Psi \Gamma \ln^2(A), \\ H_2 = -6\lambda k^2 \Gamma^2 \ln^2(A), \\ v = \sqrt{1 + \lambda k^2 \Psi^2 - 4\lambda k^2 \Gamma \Theta} \ln(A). \end{cases} \tag{13}$$

Case 2

$$\begin{cases} H_0 = -\lambda k^2 \left(\Psi^2 + 2\Gamma\Theta\right) \ln^2(A), \\ H_1 = -6\lambda k^2 \Psi \Gamma \ln^2(A), \\ H_2 = -6\lambda k^2 \Gamma^2 \ln^2(A), \\ v = k\sqrt{1 - \lambda k^2 \Psi^2 + 4\lambda k^2 \Gamma \Theta} \ln(A). \end{cases} \tag{14}$$

We established following families of exact solutions using Eq.13 along with Eq.12 into Eq. 7

Family1. When $\Psi^2 - 4\Theta\Gamma < 0$ and $\Gamma \neq 0$, then the traveling wave solutions are given by

$$Q_{1,1}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\ln^2(A)\Psi\Gamma\left[-\frac{\Psi}{2\Gamma}+\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\tan_A\left(\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta}{2}\right)\right]$$
$$-6\lambda k^2\ln^2(A)\Gamma^2\left[-\frac{\Psi}{2\Gamma}+\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\tan_A\left(\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta}{2}\right)\right]^2 \quad (15)$$

$$Q_{2,1}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\Psi\Gamma\ln^2(A)\left[-\frac{\Psi}{2\Gamma}-\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\cot_A\left(\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta}{2}\right)\right]$$
$$-6\lambda k^2\ln^2(A)\Gamma^2\left[-\frac{\Psi}{2\Gamma}-\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\cot_A\left(\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta}{2}\right)\right]^2 \quad (16)$$

$$Q_{3,1}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\ln^2(A)\Psi\Gamma$$
$$\left[-\frac{\Psi}{2\Gamma}+\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\left(\tan_A\left(\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\pm\sqrt{pq}\sec_A\left(\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\right)\right]$$
$$-6\lambda k^2\ln^2(A)\Gamma^2$$
$$\left[-\frac{\Psi}{2\Gamma}+\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\left(\tan_A\left(\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\pm\sqrt{pq}\sec_A\left(\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\right)\right]^2 \quad (17)$$

$$Q_{4,1}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\ln^2(A)\Psi\Gamma$$
$$\left[-\frac{\Psi}{2\Gamma}+\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\left(-\cot_A\left(\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\pm\sqrt{pq}\csc_A\left(\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\right)\right]-6\lambda k^2\ln^2(A)\Gamma^2$$
$$\left[-\frac{\Psi}{2\Gamma}+\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\left(-\cot_A\left(\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\pm\sqrt{pq}\csc_A\left(\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\right)\right]^2 \quad (18)$$

$$Q_{5,1}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\ln^2(A)\Psi\Gamma$$
$$\left\{-\frac{\Psi}{2\Gamma}+\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}}{4\Gamma}\left(\tan_A\left(\frac{1}{4}\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta\right)-\cot_A\left(\frac{1}{4}\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\right)\right\}$$
$$-6\lambda k^2\ln^2(A)\Gamma^2\left\{-\frac{\Psi}{2\Gamma}+\frac{\sqrt{-(\Psi^2-4\Theta\Gamma)}}{4\Gamma}\left(\tan_A\left(\frac{1}{4}\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta\right)-\cot_A\left(\frac{1}{4}\sqrt{-(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\right)\right\}^2. \quad (19)$$

Family2. When $\Psi^2-4\Theta\Gamma>0$ and $\Gamma\neq 0$, then the traveling wave solutions are given by

$$Q_{6,2}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\Psi\Gamma\ln^2(A)\left[-\frac{\Psi}{2\Gamma}-\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\tanh_A\left(\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta}{2}\right)\right]$$
$$-6\lambda k^2\ln^2(A)\Gamma^2\left[-\frac{\Psi}{2\Gamma}-\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\tanh_A\left(\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta}{2}\right)\right]^2 \quad (20)$$

$$Q_{7,2}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\Psi\Gamma\ln^2(A)\left[-\frac{\Psi}{2\Gamma}-\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\coth_A\left(\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta}{2}\right)\right]$$
$$-6\lambda k^2\ln^2(A)\Gamma^2\left[-\frac{\Psi}{2\Gamma}-\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\coth_A\left(\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta}{2}\right)\right]^2 \quad (21)$$

$$Q_{8,2}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\Psi\Gamma\ln^2(A)$$
$$\left\{-\frac{\Psi}{2\Gamma}+\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\left(-\tanh_A\left(\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\pm i\sqrt{pq}\,sech_A\left(\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\right)\right\}$$
$$-6\lambda k^2\ln^2(A)\Gamma^2\left\{-\frac{\Psi}{2\Gamma}+\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\left(-\tanh_A\left(\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\pm i\sqrt{pq}\,sech_A\left(\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\right)\right\}^2 \quad (22)$$

$$Q_{9,2}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\ln^2(A)\Psi\Gamma$$
$$\left\{-\frac{\Psi}{2\Gamma}+\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\left(-\coth_A\left(\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\pm\sqrt{pq}\,csch_A\left(\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\right)\right\}$$
$$-6\lambda k^2\ln^2(A)\Gamma^2\left\{-\frac{\Psi}{2\Gamma}+\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}}{2\Gamma}\left(-\coth_A\left(\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\pm\sqrt{pq}\,csch_A\left(\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta\right)\right)\right\}^2 \quad (23)$$

$$Q_{10,2}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\Psi\Gamma\ln^2(A)$$
$$\left[-\frac{\Psi}{2\Gamma}-\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}}{4\Gamma}\left\{\tanh_A\left(\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta}{4}\right)+\coth_A\left(\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta}{4}\right)\right\}\right]-6\lambda k^2\ln^2(A)\Gamma^2$$
$$\left[-\frac{\Psi}{2\Gamma}-\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}}{4\Gamma}\left\{\tanh_A\left(\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta}{4}\right)+\coth_A\left(\frac{\sqrt{(\Psi^2-4\Theta\Gamma)}\,\zeta}{4}\right)\right\}\right]^2 \quad (24)$$

Family3. When $\Theta\Gamma>0$ and $\Psi=0$, then the traveling wave solutions are given by

$$Q_{11,3}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\Psi\Gamma\ln^2(A)\left(\sqrt{\frac{\Theta}{\Gamma}}\tan_A\left(\sqrt{\Theta\Gamma}\zeta\right)\right)$$
$$-6\lambda k^2\ln^2(A)\Gamma^2\left[\sqrt{\frac{\Theta}{\Gamma}}\tan_A\left(\sqrt{\Theta\Gamma}\zeta\right)\right]^2 \quad (25)$$

$$Q_{12,3}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\Psi\Gamma\ln^2(A)\left(-\sqrt{\frac{\Theta}{\Gamma}}\cot_A\left(\sqrt{\Theta\Gamma}\zeta\right)\right)$$
$$-6\lambda k^2\ln^2(A)\Gamma^2\left[-\sqrt{\frac{\Theta}{\Gamma}}\cot_A\left(\sqrt{\Theta\Gamma}\zeta\right)\right]^2 \quad (26)$$

$$Q_{13,3}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\ln^2(A)\Psi\Gamma\left[\sqrt{\frac{\Theta}{\Gamma}}\left(\tan_A\left(2\sqrt{\Theta\Gamma}\zeta\right)\pm\sqrt{\frac{pq\Theta}{\Gamma}}\right)\sec_A\left(2\sqrt{\Theta\Gamma}\zeta\right)\right]$$
$$-6\lambda k^2\ln^2(A)\Gamma^2\left[\sqrt{\frac{\Theta}{\Gamma}}\left(\tan_A\left(2\sqrt{\Theta\Gamma}\zeta\right)\pm\sqrt{\frac{pq\Theta}{\Gamma}}\right)\sec_A\left(2\sqrt{\Theta\Gamma}\zeta\right)\right]^2 \quad (27)$$

$$Q_{14,3}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\ln^2(A)\Psi\Gamma\left[\sqrt{\frac{\Theta}{\Gamma}}\left(\cot_A\left(2\sqrt{\Theta\Gamma}\zeta\right)\pm\sqrt{\frac{pq\Theta}{\Gamma}}\right)\csc_A\left(2\sqrt{\Theta\Gamma}\zeta\right)\right]$$
$$-6\lambda k^2\ln^2(A)\Gamma^2\left[\sqrt{\frac{\Theta}{\Gamma}}\left(\cot_A\left(2\sqrt{\Theta\Gamma}\zeta\right)\pm\sqrt{\frac{pq\Theta}{\Gamma}}\right)\csc_A\left(2\sqrt{\Theta\Gamma}\zeta\right)\right]^2 \quad (28)$$

$$Q_{15,3}(\zeta)=\left(-6\lambda k^2\ln^2(A)\Gamma\Theta\right)-6\lambda k^2\ln^2(A)\Psi\Gamma\left[1/2\sqrt{\frac{\Theta}{\Gamma}}\left(\tan_A\left(\frac{1}{2}\sqrt{\Theta\Gamma}\zeta\right)-\cot_A\left(\frac{1}{2}\sqrt{\Theta\Gamma}\zeta\right)\right)\right]$$
$$-6\lambda k^2\ln^2(A)\Gamma^2\left[\frac{1}{2}\sqrt{\frac{\Theta}{\Gamma}}\left(\tan_A\left(\frac{1}{2}\sqrt{\Theta\Gamma}\zeta\right)-\cot_A\left(\frac{1}{2}\sqrt{\Theta\Gamma}\zeta\right)\right)\right]^2 \quad (29)$$

Family4. When $\Theta\Gamma < 0$ and $\Psi = 0$, then the traveling wave solutions are given by

$$Q_{16,4}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left[-\sqrt{-\frac{\Theta}{\Gamma}}\tanh_A\left(\sqrt{-\Theta\Gamma}\zeta\right)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[-\sqrt{-\frac{\Theta}{\Gamma}}\tanh_A\left(\sqrt{-\Theta\Gamma}\zeta\right)\right]^2 \quad (30)$$

$$Q_{17,4}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left[-\sqrt{-\frac{\Theta}{\Gamma}}\coth_A\left(\sqrt{-\Theta\Gamma}\zeta\right)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[-\sqrt{-\frac{\Theta}{\Gamma}}\coth_A\left(\sqrt{-\Theta\Gamma}\zeta\right)\right]^2 \quad (31)$$

$$Q_{18,4}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left[\sqrt{-\frac{\Theta}{\Gamma}}\left(-\tanh\left(2\sqrt{-\Theta\Gamma}\zeta\right)\right) \pm i\sqrt{pq}\,sech_A(2\sqrt{-\Theta\Gamma}\zeta)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[\sqrt{-\frac{\Theta}{\Gamma}}\left(-\tanh\left(2\sqrt{-\Theta\Gamma}\zeta\right)\right) \pm i\sqrt{pq}\,sech_A(2\sqrt{-\Theta\Gamma}\zeta)\right]^2 \quad (32)$$

$$Q_{19,4}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left[\sqrt{-\frac{\Theta}{\Gamma}}\left(-\coth_A\left(2\sqrt{-\Theta\Gamma}\zeta\right) \pm \sqrt{pq}\,csch_A\left(2\sqrt{-\Theta\Gamma}\zeta\right)\right)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[\sqrt{-\frac{\Theta}{\Gamma}}\left(-\coth_A\left(2\sqrt{-\Theta\Gamma}\zeta\right) \pm \sqrt{pq}\,csch_A\left(2\sqrt{-\Theta\Gamma}\zeta\right)\right)\right]^2 \quad (33)$$

$$Q_{20,4}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left[-\frac{1}{2}\sqrt{-\frac{\Theta}{\Gamma}}\left(\tanh_A\left(\frac{1}{2}\sqrt{-\Theta\Gamma}\zeta\right) + \coth_A\sqrt{-\Theta\Gamma}\zeta\right)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[-\frac{1}{2}\sqrt{-\frac{\Theta}{\Gamma}}\left(\tanh_A\left(\frac{1}{2}\sqrt{-\Theta\Gamma}\zeta\right) + \coth_A\sqrt{-\Theta\Gamma}\zeta\right)\right]^2 \quad (34)$$

Family5: when $\Psi = 0$ and $\Gamma = \Theta$, then the singular periodic solutions are given by

$$Q_{21,5}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left[\tan_A(\Theta\zeta)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[\tan_A(\Theta\zeta)\right]^2 \quad (35)$$

$$Q_{22,5}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left[-\cot_A(\Theta\zeta)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left(\ln^2 A\right)\left[-\cot_A(\Theta\zeta)\right]^2 \quad (36)$$

$$Q_{23,5}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left[(\tan_A(2\Theta\zeta) \pm \sqrt{pq})\sec_A(2\Theta\zeta)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[(\tan_A(2\Theta\zeta) \pm \sqrt{pq})\sec_A(2\Theta\zeta)\right]^2 \quad (37)$$

$$Q_{24,5}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left[-(\cot_A(2\Theta\zeta) \pm \sqrt{pq})\csc_A(2\Theta\zeta)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[-(\cot_A(2\Theta\zeta) \pm \sqrt{pq})\csc_A(2\Theta\zeta)\right] \quad (38)$$

$$Q_{25,5}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left[\frac{1}{2}\left(\tan\left(\frac{1}{2}\Theta\zeta\right) - \cot\left(\frac{1}{2}\Theta\zeta\right)\right)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[\frac{1}{2}\left(\tan\left(\frac{1}{2}\Theta\zeta\right) - \cot\left(\frac{1}{2}\Theta\zeta\right)\right)\right]^2 \quad (39)$$

Family6. When $\Psi = 0$ and $\Gamma = -\Theta$, then the traveling wave solutions are given by

$$Q_{26,6}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left[-\tanh_A(\Theta\zeta)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[-\tanh_A(\Theta\zeta)\right]^2 \quad (40)$$

$$Q_{27,6}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left[-\coth_A(\Theta\zeta)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[-\coth_A(\Theta\zeta)\right]^2 \quad (41)$$

$$Q_{28,6}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left[-\tanh_A(2\Theta\zeta) \pm i\sqrt{pq}sech_A(2\Theta\zeta)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[-\tanh_A(2\Theta\zeta) \pm i\sqrt{pq}sech_A(2\Theta\zeta)\right]^2 \quad (42)$$

$$Q_{29,6}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left[-\coth_A(2\Theta\zeta) \pm \sqrt{pq}csch_A(2\Theta\zeta)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[-\coth_A(2\Theta\zeta) \pm \sqrt{pq}csch_A(2\Theta\zeta)\right]^2 \quad (43)$$

$$Q_{30,6}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left[-\frac{1}{2}\left(\tanh_A\left(\frac{1}{2}\Theta\zeta\right) + \coth_A\left(\frac{1}{2}\Theta\zeta\right)\right)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[-\frac{1}{2}\left(\tanh_A\left(\frac{1}{2}\Theta\zeta\right) + \coth_A\left(\frac{1}{2}\Theta\zeta\right)\right)\right]^2 \quad (44)$$

Family7. When $\Psi^2 = 4\Theta\Gamma$ and $\Gamma = -\Theta$, then the traveling wave solutions are given by

$$Q_{31,7}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left[-2\frac{\Theta(\Psi\zeta LnA + 2)}{\Psi^2\zeta LnA}\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[-2\frac{\Theta(\Psi\zeta LnA + 2)}{\Psi^2\zeta LnA}\right]^2 \quad (45)$$

Family 8: When $\Psi = 0, \Theta = mk, (m \neq 0)$ and $\Gamma = k$, then

$$Q_{32,8}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left[A^{k\zeta} - m\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[A^{k\zeta} - m\right]^2 \quad (46)$$

Family 9 . When $\Psi = \Gamma = 0$, then

$$Q_{33,9}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left[\Theta\zeta In(A)\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[\Theta\zeta In(A)\right]^2 \quad (47)$$

Family 10 . When $\Psi = \Theta = 0$ then

$$Q_{34,10}(\zeta) = \left(6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left[\frac{-1}{\Gamma\zeta ln(A)}\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[\frac{-1}{\Gamma\zeta ln(A)}\right]^2 \quad (48)$$

Family 11 . When $\Psi \neq 0, \Theta = 0$, then the traveling wave solutions are given by

$$Q_{35,11}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left[-\frac{p\Psi}{\Gamma(\cosh_A(\Psi\zeta) - \sinh_A(\Psi\zeta) + p)}\right] - 6\lambda k^2 \ln^2(A)\Gamma^2\left[-\frac{p\Psi}{\Gamma(\cosh_A(\Psi\zeta) - \sinh_A(\Psi\zeta) + p)}\right]^2 \quad (49)$$

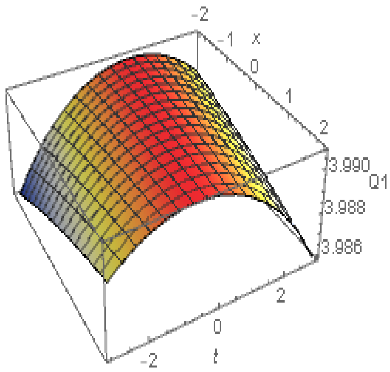
$$Q_{36,11}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6\lambda k^2 \ln^2(A)\Psi\Gamma\left(\ln^2 A\right)\left\{-\frac{\Psi(\sinh_A(\Psi\zeta) + \cosh_A(\Psi\zeta))}{\Gamma(\sinh_A(\Psi\zeta) + \cosh_A(\Psi\zeta) + q)}\right\} - 6\lambda k^2 \ln^2(A)\Gamma^2\left\{-\frac{\Psi(\sinh_A(\Psi\zeta) + \cosh_A(\Psi\zeta))}{\Gamma(\sinh_A(\Psi\zeta) + \cosh_A(\Psi\zeta) + q)}\right\}^2 \quad (50)$$

Family 12 : When $\Psi = l, \Gamma = m\,(m \neq 0)$, and $\Theta = 0$ then the rational solution is given by

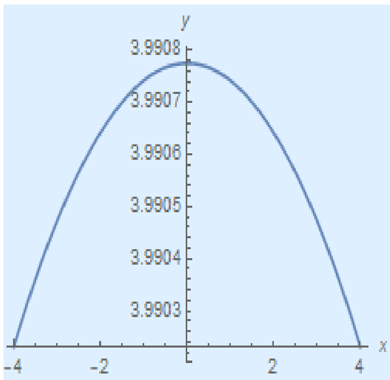
$$Q_{37,12}(\zeta) = \left(-6\lambda k^2 \ln^2(A)\Gamma\Theta\right) - 6lk^2\Psi\Gamma\ln^2(A)\left(\ln^2 A\right)\left\{-\frac{pA^{l\zeta}}{q - mpA^{l\zeta}}\right\} - 6lk^2\Gamma^2\ln^2(A)\left\{-\frac{pA^{l\zeta}}{q - mpA^{l\zeta}}\right\}^2. \quad (51)$$

where ζ is an independent variable, p and q are arbitrary constants greater than zero and called deformation parameters.

4 The graphical representation

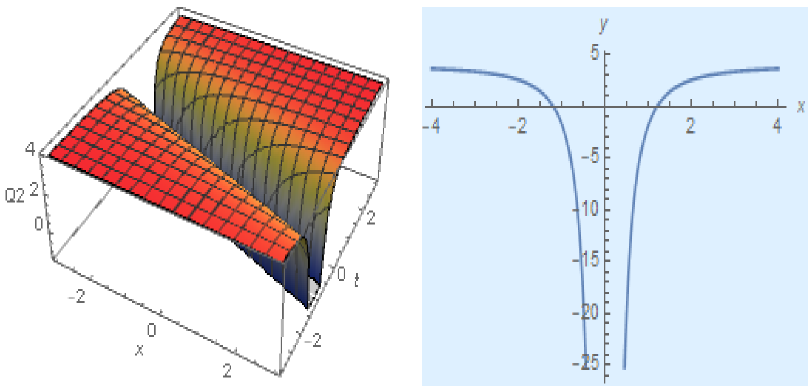


(a)



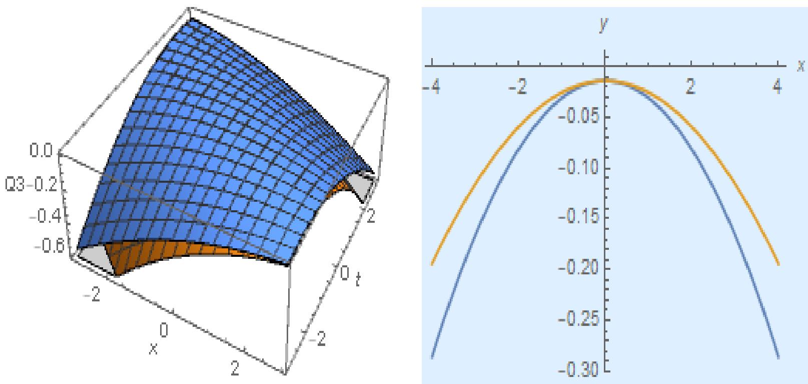
(b)

Fig. 1: The 3-D and 2-D graphs of $q_1(\zeta)$ given by Eq. 15.



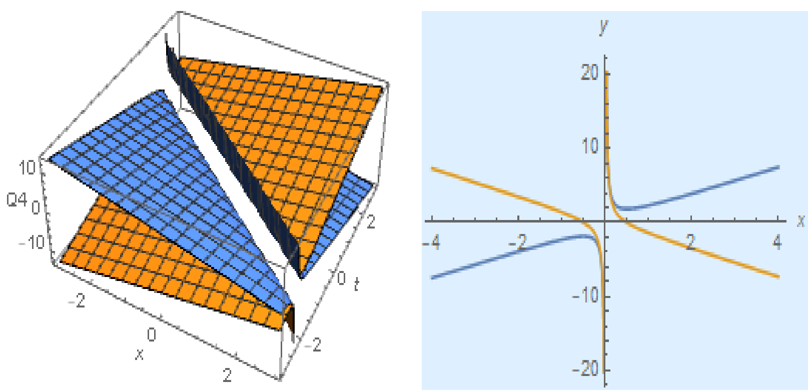
(a) (b)

Fig. 2: The 3-D and 2-D graphs of $q_2(\zeta)$ given by Eq. 16.



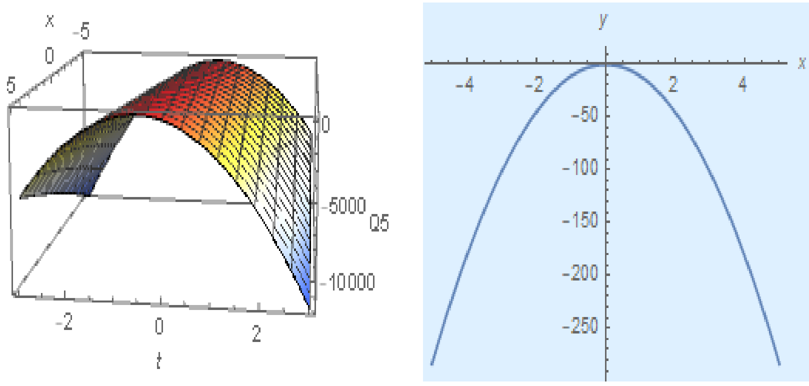
(a) (b)

Fig. 3: The 3-D and 2-D graphs of $q_3(\zeta)$ given by Eq. 17.



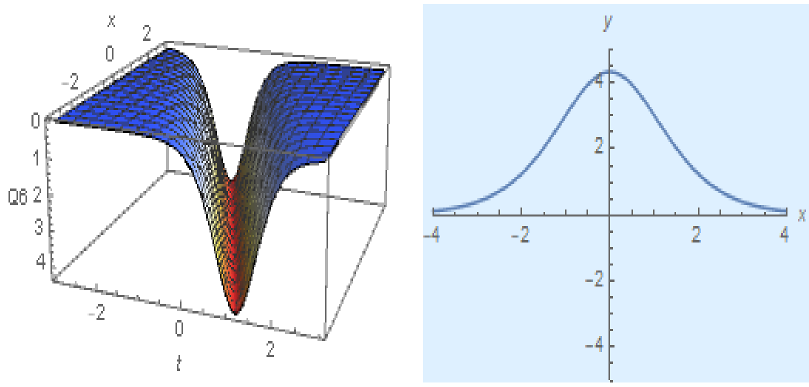
(a) (b)

Fig. 4: The 3-D and 2-D graphs of $q_4(\zeta)$ given by Eq. 18.



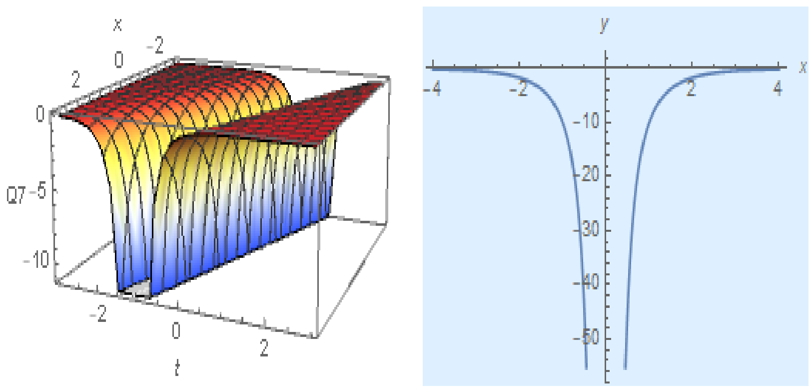
(a) **(b)**

Fig. 5: The 3-D and 2-D graphs of $q_5(\zeta)$ given by Eq. 19.



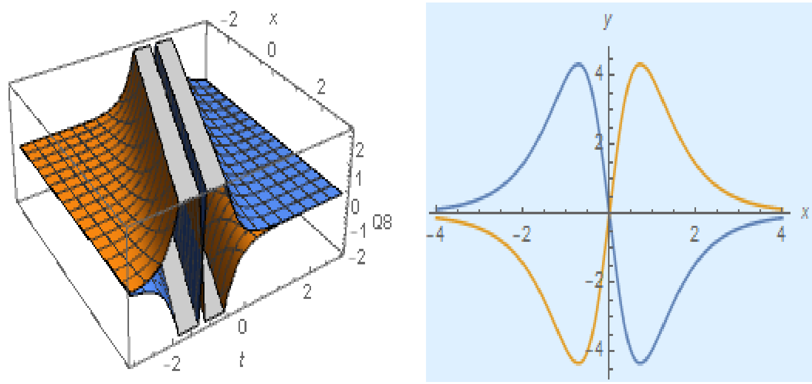
(a) **(b)**

Fig. 6: The 3-D and 2-D graphs of $q_6(\zeta)$ given by Eq. 20.



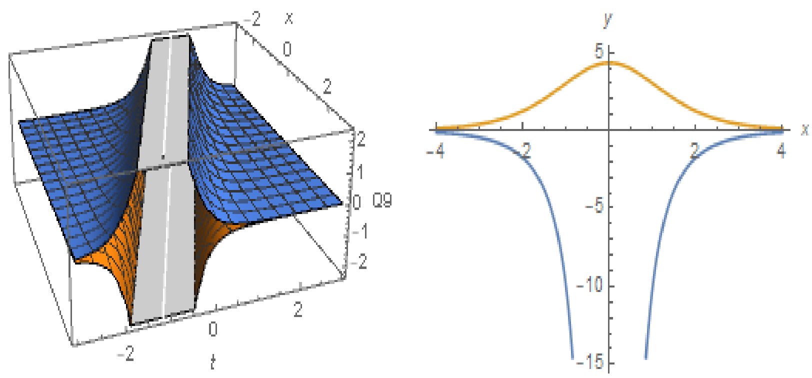
(a) **(b)**

Fig. 7: The 3-D and 2-D graphs of $q_7(\zeta)$ given by Eq. 21.



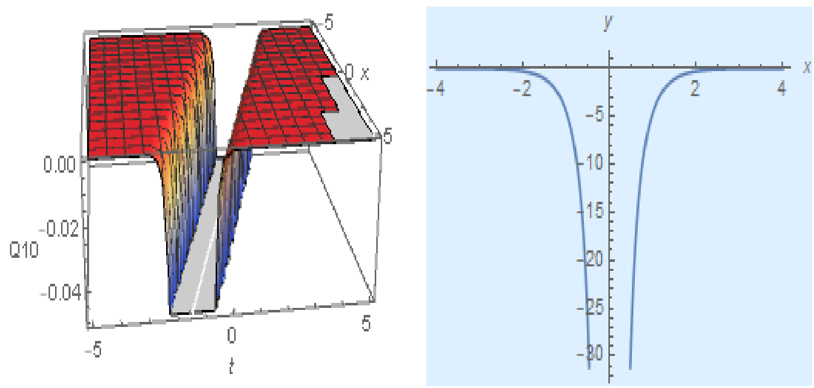
(a) **(b)**

Fig. 8: The 3-D and 2-D graphs of $q_8(\zeta)$ given by Eq. 22.



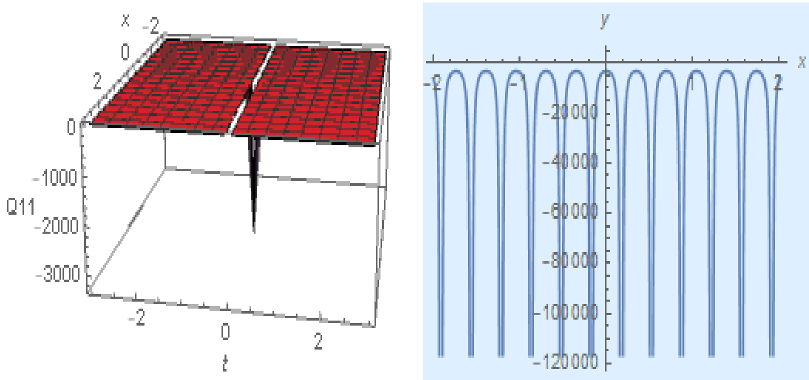
(a) **(b)**

Fig. 9: The 3-D and 2-D graphs of $q_9(\zeta)$ given by Eq. 23.



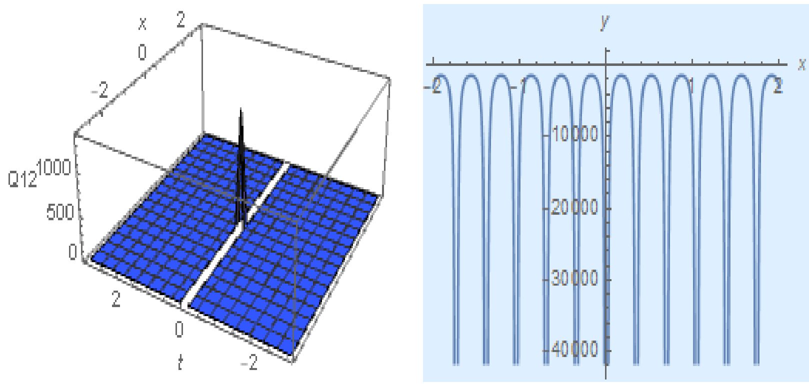
(a) **(b)**

Fig. 10: The 3-D and 2-D graphs of $q_{10}(\zeta)$ given by Eq. 24.



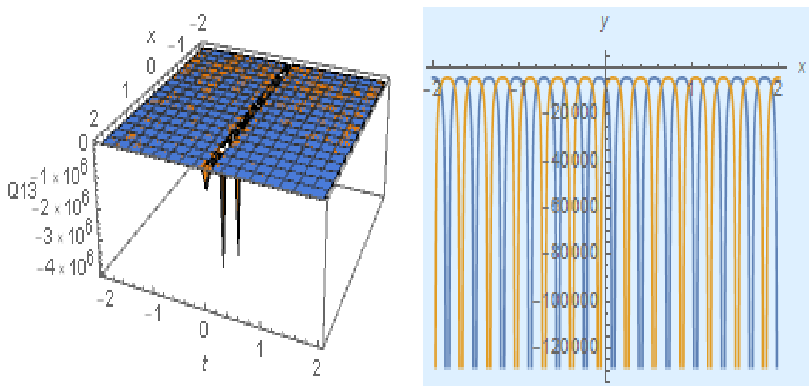
(a) **(b)**

Fig. 11: The 3-D and 2-D graphs of $q_{11}(\zeta)$ given by Eq. 25.



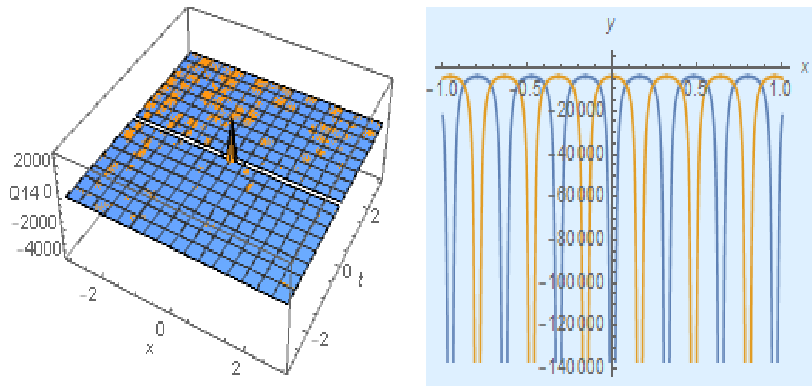
(a) **(b)**

Fig. 12: The 3-D and 2-D graphs of $q_{12}(\zeta)$ given by Eq. 26.



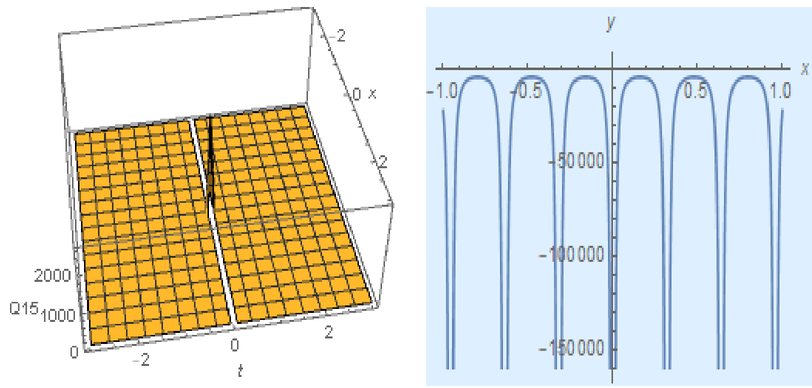
(a) **(b)**

Fig. 13: The 3-D and 2-D graphs of $q_{13}(\zeta)$ given by Eq. 27.



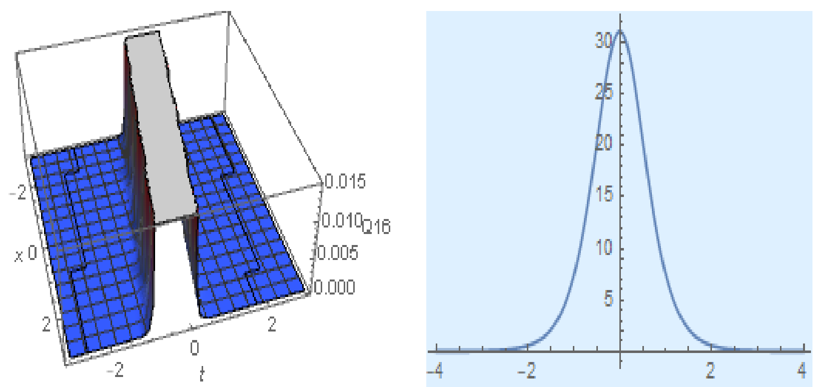
(a) **(b)**

Fig. 14: The 3-D and 2-D graphs of $q_{14}(\zeta)$ given by Eq. 28.



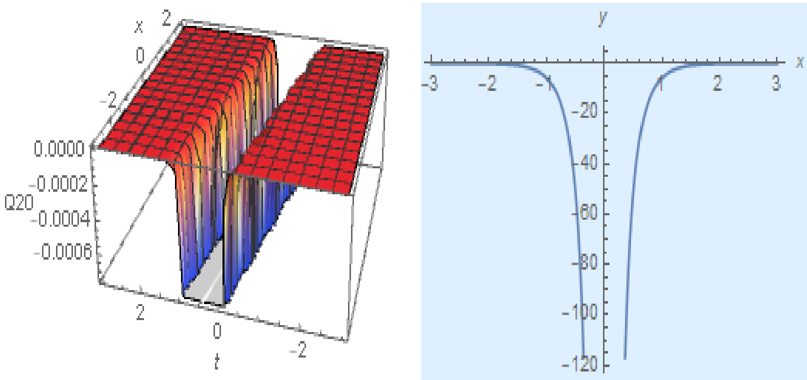
(a) **(b)**

Fig. 15: The 3-D and 2-D graphs of $q_{15}(\zeta)$ given by Eq. 29.



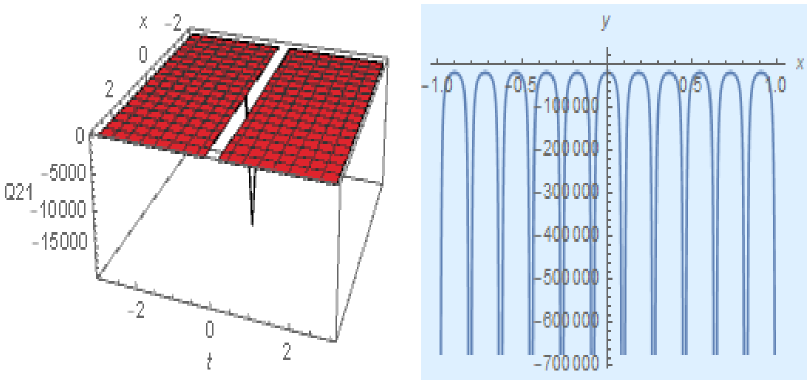
(a) **(b)**

Fig. 16: The 3-D and 2-D graphs of $q_{16}(\zeta)$ given by Eq. 30.



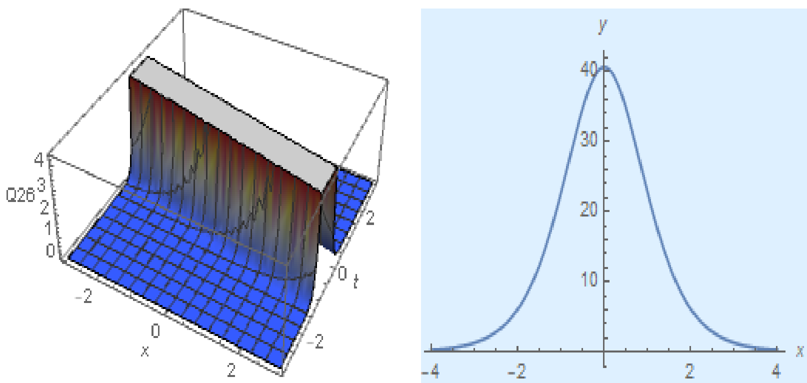
(a) **(b)**

Fig. 17: The 3-D and 2-D graphs of $q_{20}(\zeta)$ given by Eq. 34.



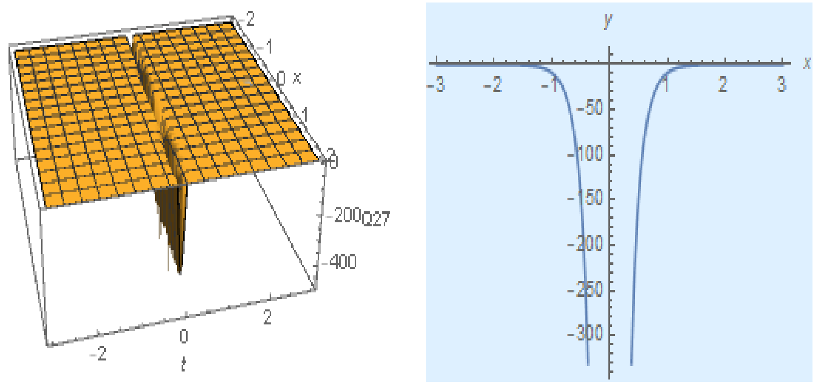
(a) **(b)**

Fig. 18: The 3-D and 2-D graphs of $q_{21}(\zeta)$ given by Eq. 35.



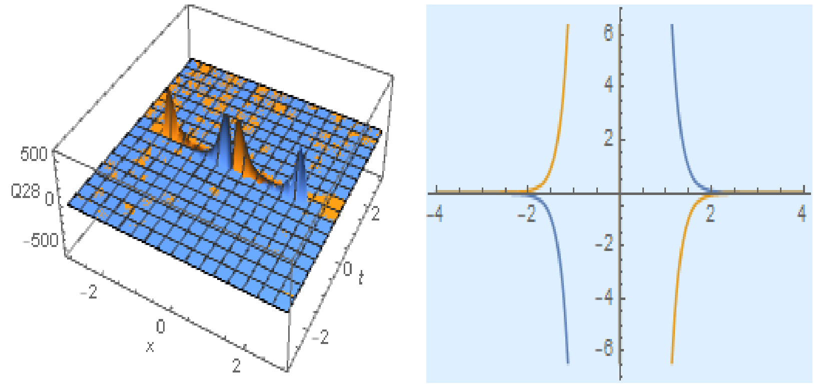
(a) **(b)**

Fig. 19: The 3-D and 2-D graphs of $q_{26}(\zeta)$ given by Eq. 40.



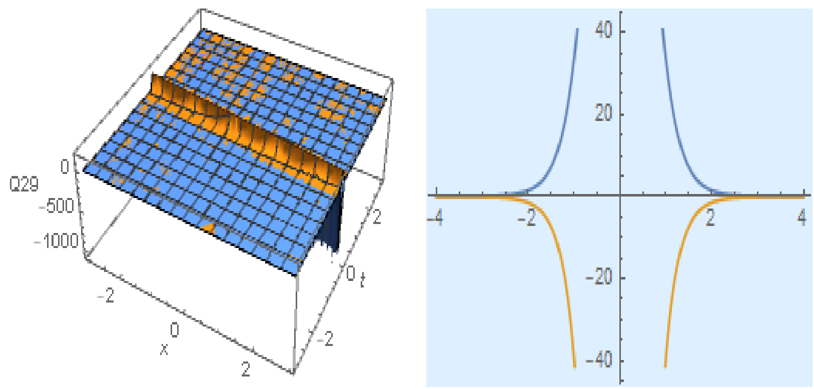
(a) **(b)**

Fig. 20: The 3-D and 2-D graphs of $q_{27}(\zeta)$ given by Eq. 41.



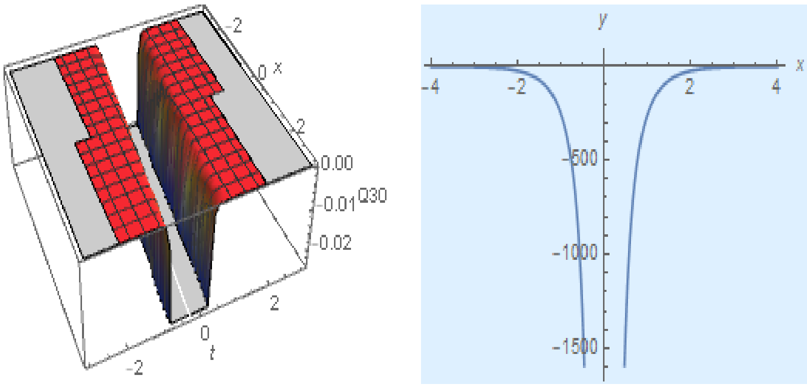
(a) **(b)**

Fig. 21: The 3-D and 2-D graphs of $q_{28}(\zeta)$ given by Eq. 42.



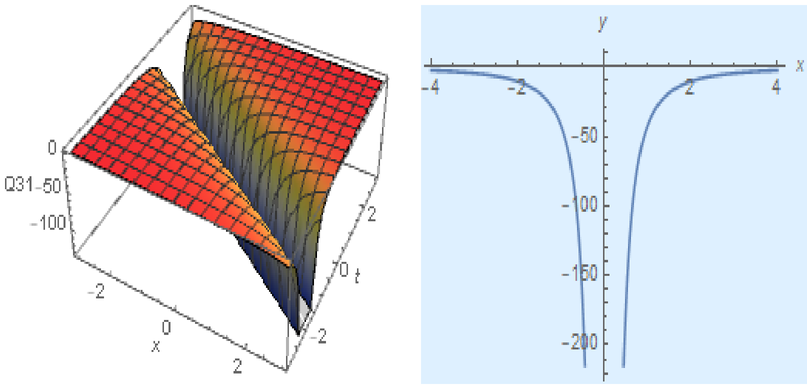
(a) **(b)**

Fig. 22: The 3-D and 2-D graphs of $q_{29}(\zeta)$ given by Eq. 43.



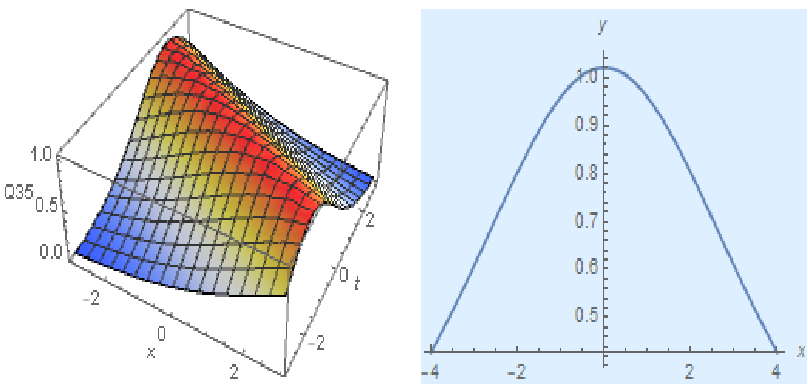
(a) **(b)**

Fig. 23: The 3-D and 2-D graphs of $q_{30}(\zeta)$ given by Eq. 44.



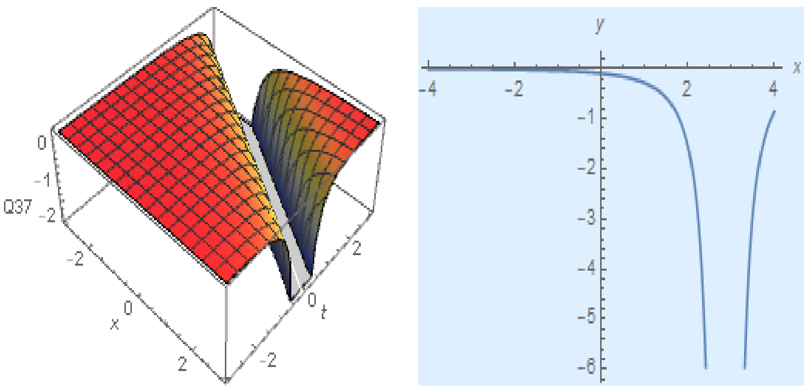
(a) **(b)**

Fig. 24: The 3-D and 2-D graphs of $q_{31}(\zeta)$ given by Eq. 45.



(a) **(b)**

Fig. 25: The 3-D and 2-D graphs of $q_{35}(\zeta)$ given by Eq. 41.



(a) **(b)**

Fig. 26: The 3-D and 2-D graphs of $q_{37}(\zeta)$ given by Eq. 51.

5 Discussion and Conclusion

In this study, we have applied a new extended direct algebraic method to obtain soliton solutions of the $(1+1)$-dimensional Boussinesq equation. The derived solutions are presented in explicit forms, showcasing the method's capability to handle complex nonlinear PDEs. Our results contribute to the understanding of soliton dynamics in dispersive media and provide a foundation for future research in related areas. The success of this method in finding exact solutions highlights its potential as a valuable tool in the study of nonlinear wave equations, with implications across various physical and engineering disciplines. By the application of new extended direct algebraic method we have obtained a series of traveling wave solutions namely: singular solitons, periodic solitons, rational solitons, dark and bright soliton solutions for $(1+1)$-Boussinesq equation. The proposed method is straight forward and more powerful in constructing exact traveling wave solutions of NLPDE's. It can also be applied to other nonlinear partial differential appearing in mathematical biology, physics, chemistry, fluid mechanics and many other fields.

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