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Kavya-Manoharan DUS Family of Distributions with Diverse Applications

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Abstract: In this work, a new family of distributions is proposed without adding any new parameters to any known baseline distribution. This family, the Kavya-Manoharan Dinesh-Umesh-Sanjay (KM-DUS) family, is the mixture of the Kavya-Manoharan (KM) and the Dinesh-Umesh-Sanjay (DUS) families. A new two-parameter distribution, the KM-DUS- Weibull (KM-DUS-W), is built on the foundation of the Weibull distribution to model time-to-event data sets. The KM-DUS-W is a more convenient and computationally tractable alternative to the Weibull distribution and a parsimonious but effective modeling aid for survival and reliability data. The primary statistical properties including the probability density function, cumulative distribution function, quantile function, moments, order statistics and entropy are obtained. Parameter estimation is accomplished with numerous classical and Bayesian estimators. Maximum Likelihood, Least Squares, Weighted Least Squares, Maximum Product Spacing (MPS), Cramér-von Mises, Anderson-Darling, Right-Tailed Anderson-Darling, Percentile estimation, and Bayesian approaches under Squared Error, LINEX, and General Entropy loss functions. Amongst these, the non-Bayesian ones always provide the most efficient estimates for any sample size. Applicability of the KM-DUS-RIW distribution in real life is illustrated through failure times of the 84 Aircraft Windshield, COVID-19 death rate for Angola between 14/06/2020 and 20/2/2022, carbon fibers breaking stress and 30 observations of the March precipitation pattern (in inches) in Minneapolis/St Paul. Comparative goodness-of-fit of log-likelihood, AIC, BIC, HQIC, and Kolmogorov-Smirnov statistics give evidence that KM-DUS-W model is better performing than other alternative models like Weibull, Gumbel, log-normal, new generalized logistic-x transformed exponential and Burr type XII distributions. These results give support to the KM-DUS-W distribution as an alternative option for modeling complex lifetime data.

Keywords: Kavya-Manoharan family; Dinesh-Umesh-Sanjay family; Weibull distribution; Estimation; Goodness of Fit

1 Introduction

One prominent method for generating flexible family of distributions involves using generator functions, as seen in the Beta-Generated Family introduced by [7] and extended by [8]. The Kumaraswamy distribution has been a particularly fruitful generator, leading to the Kumaraswamy-G (K-G) family by [17], and the Kumaraswamy-G Poisson family by [9]. Other Kumaraswamy-based families include the Sine Kumaraswamy-G [10], gamma Kumaraswamy-G [11], new Kumaraswamy Kumaraswamy [12], Kavya-Manoharan Kumaraswamy by [26], unit exponentiated half logistic power

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series class by [27], Kumaraswamy Poisson-G [13], and generalized inverted Kumaraswamy-G [14]. Beyond these, authors have developed distributions based on diverse transformations and generators. For example, the logistic-X family [15], and odd-Perks-G [16] are other examples of general families. Specific applications and transformations have also been used to develop families such as the Odd log-logistic Poisson-G family [39], the Topp-Leone-G [19], and the Dinesh-Umesh-Sanjay (DUS) family [20], with the latter being used in the DUS Topp Leone family [77]. Other families, such as the Gamma-X [33] and Topp-Leone-G [61], use specific distributions as generators, while the Rank Transmutation Maps (RTM) [32] and Type-I Heavy Tailed (TI-HT) [40] employ unique transformation methods to create new models. The Generalized Family of Odd-Lindley Distribution was developed by [6], the Chen-G Class of Distributions by [18], the Type I Half-Logistic Family by [21], and the Sine generalized family of distributions by [76]. [31] introduced the Alpha-log-power transformed-G (ALPT-G), while [36] created the Burr-Hatke-G. The EHL-PGW-G was developed by [29], the extended cosine-G by [43], and the extended odd Frechet by [38]. [37] proposed the generalized odd log-logistic-G and [41] the new flexible Generalized Family (NFGF). The odd Lindley-G was a contribution of [34], the Odd log-logistic Topp-Leone G of [39], and the Power Lindley-G of [35]. The Transmuted-G (Quadratic) was introduced by [32], the Type I half-logistic exponentiated-G by [30], and the Type II log-logistic by [22], sine-exponentiated Weibull-H by [23], truncated Muth by [24], odd inverse power generalized Weibull-G by [25], . Finally, the Weighted cosine-G was developed by [42].

The pursuit of more flexible probability distributions has become a significant driving force in modern statistical research, as traditional models like the exponential, Weibull, and Rayleigh distributions often prove inadequate for accurately modeling real-world phenomena characterized by non-monotonic hazard rates, heavy tails, or varying skewness. To address these limitations, researchers have developed new distribution families by extending or transforming existing ones, as evidenced by the works of [46], [47], [49], and [50]. One notable recent innovation is the Kavya-Manoharan (KM) family, introduced by [2], which gained attention for its simple yet effective exponential transformation of the baseline CDF to reshape tail behavior and improve model fit without adding extra parameters. Concurrently, the Dinesh-Umesh-Sanjay (DUS) generator, proposed by [20], offers a general framework for creating new distribution families by embedding a baseline distribution into a new functional form involving the exponential function, which is particularly effective at modifying baseline hazard functions for applications in survival analysis and reliability engineering. The KM-DUS family was proposed as a novel generalization approach to overcome the rigid functional forms and limitations present even in some existing generalized distributions by combining these two frameworks. This fusion significantly expands the range of shapes that the resulting distributions can capture while preserving parsimony, thereby enhancing modeling power and adaptability beyond what classical and some generalized distributions offer. To illustrate this utility, this study focuses on modifying the Weibull (W) distribution [1], which, despite its inherent flexibility, can still be constrained by its rigid functional form in more complex scenarios. By embedding the W distribution into the KM-DUS framework, the resulting KM-DUS-W distribution inherits and amplifies the desirable features of both families—greater flexibility in the hazard function, enhanced adaptability to various data types, and maintained mathematical simplicity-making it a significant contribution to the literature on generalized distributions and aligning with the trend toward constructing versatile, analytically tractable models for a wide variety of datasets [51,52,53,54,55,56,57]. Ultimately, the KM-DUS-W distribution offers both theoretical richness and practical utility for modeling complex survival, reliability, environmental, and financial data where traditional models often prove insufficient.

The remaining components of this study are arranged as follows: Section 2 details the construction of the KM-DUS family of distributions. Following this, Section 3 presents a special case of this family. Section 4 then outlines the properties, and Section 5 covers the inference methods. The study proceeds with a simulation in Section 6, followed by the application of the model in Section 7, and concludes with a summary in Section 8.

2 Construction of KM-DUS Family of Distributions

Ref. [2] introduced the Kavya-Manoharan (KM) family of distributions with CDF and PDF respectively given as ;

$$G(x;\zeta) = \frac{e}{e-1}\left[1 - e^{-F(x;\zeta)}\right]; \quad x \in \Re, \tag{1}$$

and

$$g(x;\zeta) = \frac{e}{e-1} f(x;\zeta) e^{-F(x;\zeta)}. \tag{2}$$

[20] proposed a generalized generator of distributions known as the Dinesh-Umesh-Sanjay (DUS) with CDF and PDF respectively denoted as

$$F(x;\zeta) = \frac{e^{H(x;\zeta)} - 1}{e-1}, \tag{3}$$

and

$$f(x;\zeta) = \frac{h(x;\zeta)e^{H(x;\zeta)}}{e-1}. \tag{4}$$

If we substitute equations (1) and (2) into equations (3) and (4), a new generalized family of distributions named Kavya-Manoharan Dinesh-Umesh-Sanjay (KM-DUS) family is formulated, with CDF and PDF respectively presented as;

$$G(x;\zeta) = \frac{e}{e-1}\left[1 - e^{-\frac{1}{e-1}\left(e^{H(x;\zeta)}-1\right)}\right]; \quad x > 0, \tag{5}$$

and

$$g(x;\zeta) = \frac{e}{(e-1)^2} h(x;\zeta) \exp\left(H(x;\zeta) - \frac{1}{e-1}\left(e^{H(x;\zeta)} - 1\right)\right). \tag{6}$$

Notice that there is no additional parameter in equations (5) and (6), hence parsimony is guaranteed.

2.1 Linearization of the KM-DUS Family

To present the CDF and PDF of the KM-DUS family in an infinite compact form, we utilize the power series $e^{-x} = \sum_{j=0}^{\infty}(-1)^j\frac{x^j}{j!}$, so that

$$G(x;\zeta) = \frac{e}{e-1}\left[1 - \sum_{j=0}^{\infty}(-1)^j\left(\frac{1}{e-1}\right)^j \frac{\left(e^{H(x;\zeta)}-1\right)^j}{j!}\right].$$

Since $(x-1)^j = \sum_{k=0}^{j}(-1)^k\binom{j}{k}x^{j-k}$, then the final compact form for the CDF is

$$G(x;\zeta) = \frac{e}{e-1}\left[1 - \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{j+k}}{j!}\left(\frac{1}{e-1}\right)^j\binom{j}{k}e^{(j-k)H(x;\zeta)}\right].$$

Similarly, the PDF can be written as

$$g(x;\zeta) = \frac{e}{(e-1)^2}h(x;\zeta)\sum_{i=0}^{\infty}\sum_{h=0}^{i}\sum_{l=0}^{i-h}\frac{(-1)^{h+l}}{i!}\binom{i}{h}\binom{i-h}{l}\left(\frac{1}{e-1}\right)^{i-h}H^i(x;\zeta)e^{(i-h-l)H(x;\zeta)}.$$

2.2 Generic Quantile Function

Let $X \sim G(\zeta)$ and $0 < u < 1$, the quantile function is obtained by inverting equation (5), so that

$$Q(u,\zeta) = H^{-1}\left\{\ln\left(1 + (1-e)\ln\left[1 - \left(1 - \frac{1}{e}\right)u\right]\right)\right\},$$

with $Q(u,\zeta)$, generation random samples that assume the KM-DUS based distribution is first done by generating random samples that follow $H(x;\zeta)$.

3 Special case of the KM-DUS Family

Classical models such as the Weibull are limited in several ways which include failing to model data with non-standard hazard patterns, sensitivity in the presence of influential observations, Consider the Weibull (W) distribution constructed by [1], with CDF and PDF respectively expressed as

$$H(x;\alpha,\beta) = 1 - e^{-\beta x^{\alpha}}; \quad x \geq 0, \tag{7}$$

and

$$h(x;\alpha,\beta) = \alpha\beta x^{\alpha-1} e^{-\beta x^{\alpha}}. \tag{8}$$

In an attempt to enhance the usability of this Weibull distribution, this study utilizes the proposed KM-DUS family to alter the functional form of the RIW distribution. This is achieved by substituting equations (7) and (8) into equations (5) and (6) to obtain the CDF, PDF and Hazard function of the Kavya-Manoharan Dinesh-Umesh-Sanjay Weibull (KM-DUS-W) distribution given respectively as;

$$G(x;\alpha,\beta) = \frac{e}{e-1}\left[1 - \exp\left(-\frac{1}{e-1}\left(e^{1-e^{-\beta x^{\alpha}}} - 1\right)\right)\right]; \quad x > 0, \quad \alpha > 0, \quad \beta > 0, \tag{9}$$

and

$$g(x;\alpha,\beta) = \frac{e}{(e-1)^2}\,\alpha\beta x^{\alpha-1} e^{-\beta x^{\alpha}} \exp\left(1 - e^{-\beta x^{\alpha}} - \frac{1}{e-1}\left(e^{1-e^{-\beta x^{\alpha}}} - 1\right)\right); \quad x > 0,\ \alpha,\beta > 0, \tag{10}$$

and

$$h(x;\alpha,\beta) = \frac{\alpha\beta x^{\alpha-1} e^{-\beta x^{\alpha}} \exp\left(1 - e^{-\beta x^{\alpha}}\right)}{\exp\left(1 - \frac{1}{e-1}\left(e^{1-e^{-\beta x^{\alpha}}} - 1\right)\right) - 1}. \tag{11}$$

*Remark.*The functional form of the Weibull distribution was altered in the KM-DUS-W distribution without introducing additional parameter(s). This improves the flexibility of the Weibull distribution which is observed in the shapes of the hazard function in Figure (2).

Figure (1) shows the graph of the PDF. The distribution takes the shape of L, bump, leptokurtic, mesokurtic, platikurtic and asymmetric. Figure (2) contains the plots of the hazard function with different unique shapes reflecting the versatility of the distribution.

3.1 Mixture Specification

Theorem 1.*Let $X \sim$ KM-DUS-W (α,β), with PDF in equation (10), then the mixture representation is given as;*

$$\begin{aligned} g(x;\alpha,\beta) = \frac{\alpha\beta e^2}{e-1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{k} \frac{(-1)^{i+j+k+r+s+q}}{i!j!s!q!}\binom{j}{k}\left(\frac{1}{e-1}\right)^{k}\binom{k}{r} e^{(k-r)}(j-k)^{s} \\ \times (k-r)^{q}\beta^{s+i}x^{\alpha(i+s+1)-1}e^{-q\beta x^{\alpha}}. \end{aligned} \tag{12}$$

*Proof.*To derive the mixture representation of the given function $g(x;\alpha,\beta)$, we can express it as a series expansion using the following identities;

$$e^{-x} = \sum_{i=0}^{\infty}(-1)^{i}\frac{x^{i}}{i!}; \quad (x-y)^{j} = \sum_{k=0}^{j}(-1)^{k}\binom{j}{k}x^{j-k}y^{k}; \quad \text{and} \quad (x-1)^{k} = \sum_{r=0}^{k}(-1)^{r}\binom{k}{r}x^{k-r}.$$

The rest is trivial.

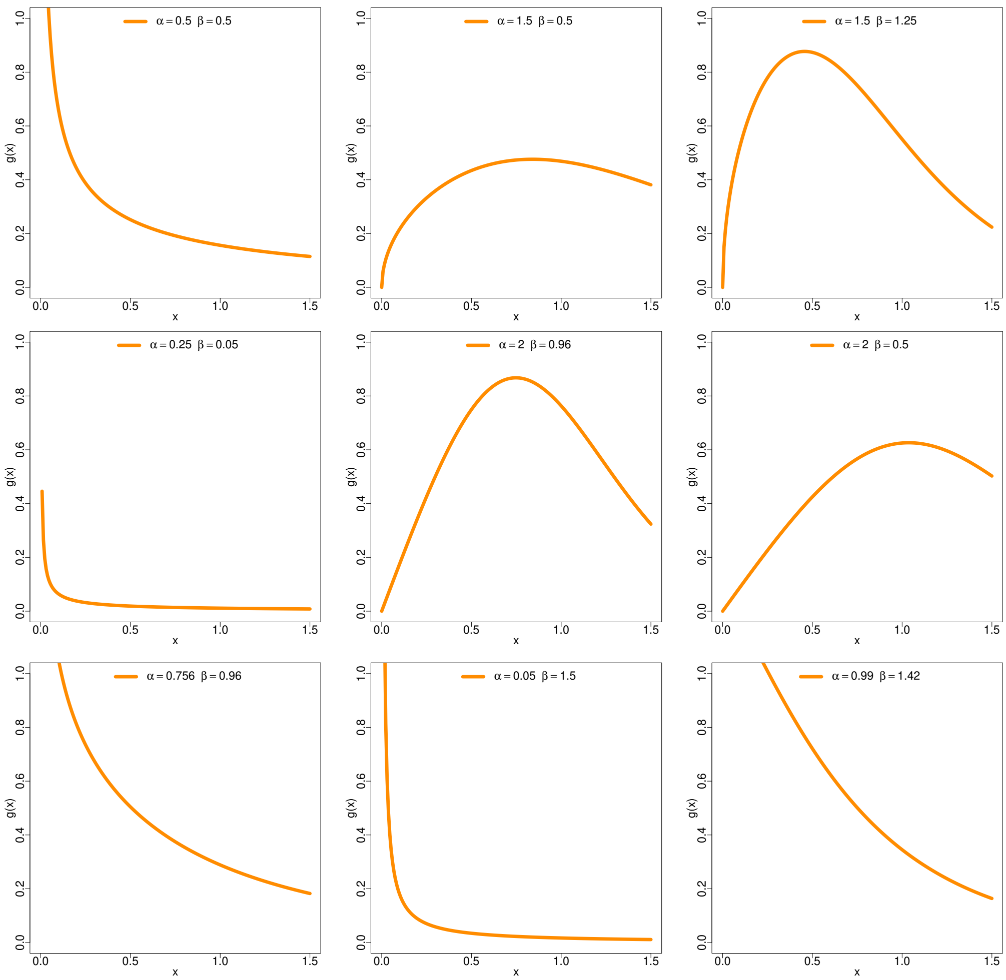


Fig. 1: PDF Plots of KM-DUS-W Distribution

4 Properties

A thorough investigation of properties of the KM-DUS-W distribution, including the quantile function, moments and other related measures, moment-generating function, stress-strength reliability analysis, mean residual life function, distribution of sth order statistic, and entropy, will be dealt with in this section.

4.1 Moment

The r$-th$ moment is defined as $\mu_\omega' = \mathbb{E}X^\omega = \int\limits_{-\infty}^{\infty} x^\omega(x;\alpha,\beta)\,dx$. Substituting equation (12) for $g(x;\alpha,\beta)$, the moment becomes

$$\mu_\omega' = \frac{\alpha\beta e^2}{e-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{k}\frac{(-1)^{i+j+k+r+s+q}}{i!\,j!\,s!\,q!}\binom{j}{k}\left(\frac{1}{e-1}\right)^k\binom{k}{r}e^{(k-r)}(j-k)^s$$
$$\times\,(k-r)^q\beta^{s+i}\int_0^\infty x^{\omega+\alpha(i+s+1)-1}e^{-q\beta x^\alpha}\,dx.$$

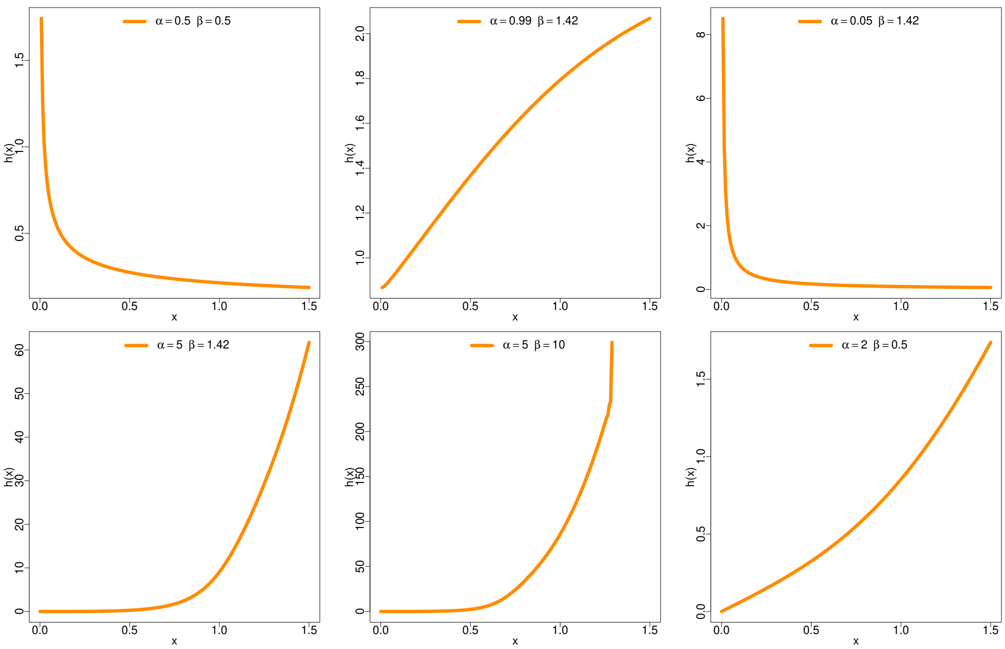


Fig. 2: Hazrad Function Plots of KM-DUS-W Distribution

Let $u = q\beta x^{\alpha} \implies x = \left(\frac{u}{q\beta}\right)^{\frac{1}{\alpha}}$ and $dx = \frac{u^{\frac{1}{\alpha}-1}}{\alpha(q\beta)^{\frac{1}{\alpha}}}\,du$. By change of variable when $x = 0$; $u = 0$ and when $x = \infty$; $u = \infty$, so that

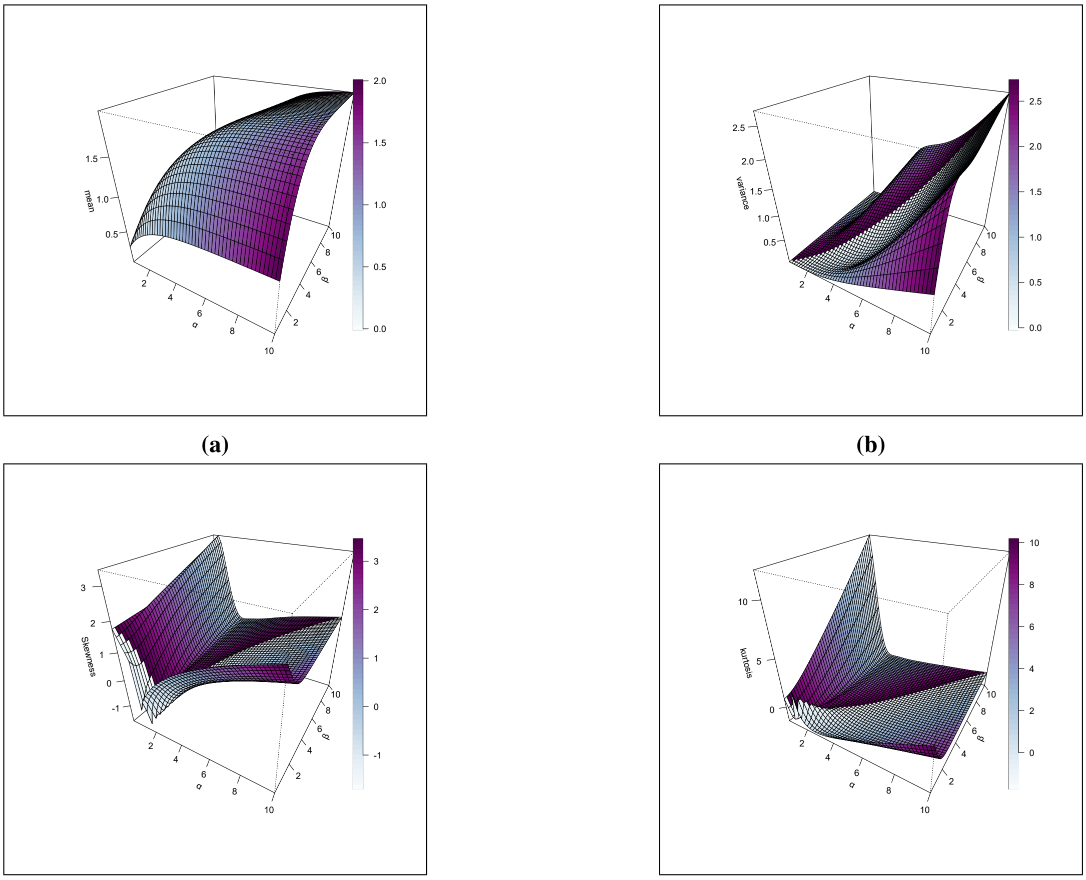
$$\begin{aligned}\mu'_{\omega} = &\frac{\beta e^2}{e-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{k}\frac{(-1)^{i+j+k+r+s+q}}{i!j!s!q!}\binom{j}{k}\left(\frac{1}{e-1}\right)^{k}\binom{k}{r}e^{(k-r)}(j-k)^{s}\\ &\times\frac{(k-r)^{q}}{\beta^{\frac{\omega}{\alpha}+1}q^{\frac{\omega}{\alpha}+i+s+1}}\Gamma\left[\frac{\omega}{\alpha}+i+s+1\right];\quad \text{for}\quad \omega = 1,2,\cdots\end{aligned}\tag{13}$$

The first moment obtained when $\omega = 1$ in equation (13) is the mean of the KM-DUS-W distribution given as;

$$\begin{aligned}\mu = &\frac{\beta e^2}{e-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{k}\frac{(-1)^{i+j+k+r+s+q}}{i!j!s!q!}\binom{j}{k}\left(\frac{1}{e-1}\right)^{k}\binom{k}{r}e^{(k-r)}(j-k)^{s}\\ &\times\frac{(k-r)^{q}}{\beta^{\frac{1}{\alpha}+1}q^{\frac{1}{\alpha}+i+s+1}}\Gamma\left[\frac{1}{\alpha}+i+s+1\right].\end{aligned}$$

The second, third and fourth moments are important measures required in determining the variance, standard deviation, skewness and kurtosis are obtained when $r = 2,3$ and 4 respectively.

Figures (3) represents 3D plots of the Mean, Variance, Skewness and Kurtosis of the proposed KM-DUS-RIW distribution. Based on the provided three-dimensional plots, we can infer the behavior of the mean, variance, skewness, and kurtosis of the KM-DUS-RIW distribution as its parameters, α and σ, are varied. The plots collectively demonstrate the distribution's considerable flexibility and its ability to model a diverse range of data characteristics. The plot for the mean (a) shows a clear and consistent relationship with the parameters. As the shape parameter σ increases, the mean of the distribution also increases, particularly when the other parameter, α, is large. This suggests that the distribution shifts to the right as σ grows. The variance plot (b) follows a similar trend, showing that the spread of the distribution increases with larger values of σ. This indicates that the parameters not only shift the location of the distribution but also influence its dispersion. The plots for skewness (c) and kurtosis (d) reveal a more complex and interesting relationship. The skewness is consistently positive for all plotted parameter combinations, but it is at its highest for smaller values of α and larger values of σ. The surface shows a steep drop-off, indicating a rapid change in skewness as the parameters move away from this region. A similar behavior is observed with kurtosis, which also exhibits a positive, heavy-tailed distribution across the parameter space. The plots for both skewness and kurtosis show a unique flat region for high values of α and low values of σ, after which they steeply rise, suggesting that the distribution becomes highly skewed and leptokurtic in specific parameter regions. This flexibility in controlling the tail behavior makes the distribution useful for modeling data with extreme values.



(a) **(b)** **(c)** **(d)**

Fig. 3: Plots of (a) Mean, (b) Variance, (c) Skewness and (d) Kurtosis of KM-DUS-RIW distribution

Table (1) shows the empirical summary statistics—mean (μ), variance (σ^2), standard deviation (σ), skewness (Sk), kurtosis (Ku), and coefficient of variation (CV)—for different combinations of shape parameters (α, β) and sample sizes $n \in \{25, 50, 100, 200, 500, 1000\}$. As would be expected, larger n stabilizes estimates of μ and reduces variability. Larger values of α and β would yield larger mean estimates and lower relative dispersion (CV), while the extreme values of skewness and kurtosis at low n are indicative of large variability and non-normality, particularly for low α values.

4.2 Quantile Function

The u$-th$ quantile function for the KM-DUS-W distribution is derived as the inverse of the CDF in equation (9).

$$x_u = \left(-\frac{1}{\beta}\ln\left(1-\ln\left[1+(1-e)\ln\left\{1-\left(1-\frac{1}{e}\right)u\right\}\right]\right)\right)^{\frac{1}{\alpha}}; \quad u \in (0,1). \tag{14}$$

Table 1: Summary of Basic Statistics

Parameters	n	μ	σ^2	σ	Sk	Ku	CV
$\alpha = 0.05, \beta = 0.75$	25	0.03318	0.00071	0.02662	2.13429	4.98791	0.80236
	50	0.07318	0.03151	0.17750	4.14230	17.48965	2.42543
	100	0.03905	0.00107	0.03269	1.60763	2.25682	0.83710
	200	0.04096	0.00376	0.06133	6.38540	56.10753	1.49720
	500	0.04268	0.00524	0.07236	7.71392	81.25716	1.69566
	1000	0.04920	0.02126	0.14580	13.10721	202.11051	2.96376
$\alpha = 0.5, \beta = 0.75$	25	0.71474	0.32889	0.57349	2.13434	4.98765	0.80237
	50	1.57668	14.62379	3.82411	4.14229	17.48947	2.42541
	100	0.84124	0.49593	0.70422	1.60766	2.25680	0.83712
	200	0.88251	1.74589	1.32132	6.38551	56.10907	1.49723
	500	0.91935	2.43054	1.55902	7.71400	81.25905	1.69578
	1000	1.05986	9.86757	3.14127	13.10724	202.11099	2.96385
$\alpha = 0.05, \beta = 1.25$	25	0.12246	0.00290	0.05388	1.39762	2.12205	0.43995
	50	0.15985	0.03220	0.17944	3.37554	11.54481	1.12251
	100	0.13313	0.00418	0.06464	1.06624	0.47825	0.48555
	200	0.13057	0.00764	0.08740	3.20392	16.80312	0.66937
	500	0.13308	0.00857	0.09260	4.01334	25.02845	0.69577
	1000	0.13629	0.01580	0.12570	7.00607	72.74627	0.92233
$\alpha = 0.5, \beta = 1.5$	25	0.79741	0.08216	0.28664	1.20704	1.54260	0.35946
	50	0.96004	0.66838	0.81754	3.12661	9.94955	0.85157
	100	0.85196	0.11656	0.34141	0.93250	0.14918	0.40073
	200	0.83190	0.19140	0.43750	2.59530	11.27701	0.52590
	500	0.84548	0.20493	0.45269	3.29479	17.21825	0.53542
	1000	0.85595	0.32753	0.57230	5.50241	48.03277	0.66861
$\alpha = 0.05, \beta = 1.75$	25	0.21942	0.00445	0.06672	1.07014	1.17137	0.30409
	50	0.25210	0.02938	0.17141	2.93216	8.79434	0.67994
	100	0.23173	0.00625	0.07903	0.83718	-0.05746	0.34105
	200	0.22608	0.00962	0.09809	2.21727	8.26887	0.43385
	500	0.22934	0.00998	0.09990	2.84354	12.94875	0.43559
	1000	0.23090	0.01466	0.12106	4.55886	34.39667	0.52431
$\alpha = 0.5, \beta = 2.0$	25	0.83493	0.04845	0.22011	0.96800	0.91908	0.26362
	50	0.93126	0.27535	0.52474	2.77772	7.92751	0.56347
	100	0.87451	0.06736	0.25954	0.76604	-0.19621	0.29679
	200	0.85364	0.09956	0.31553	1.96308	6.44925	0.36964
	500	0.86470	0.10096	0.31775	2.53746	10.35448	0.36747
	1000	0.86798	0.14044	0.37475	3.93510	26.30485	0.43175

*Remark.*Simulation of random sample from $G(x;\alpha,\beta)$ is done by first simulating random sample $U_i \sim$ Uniform $(0,1)$; $\quad i = 1,2,\cdots,n$. Then the random sample

$X_i = \left(-\frac{1}{\beta}\ln\left(1-\ln\left[1+(1-e)\ln\left\{1-\left(1-\frac{1}{e}\right)u\right\}\right]\right)\right)^{\frac{1}{\alpha}}; \quad i = 1,2,\cdots,n$ follow $G(x;\alpha,\beta)$.

4.3 Moment Generating Function

Utilizing equation (12), the moment generating function $M_X(t) = \mathbb{E}\left(e^{tX}\right)$ can easily be obtained as;

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} g(x;\alpha,\beta)\,dx,$$

which simplifies to

$$M_X(t) = \frac{\alpha\beta e^2}{e-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{k}\frac{(-1)^{i+j+k+r+s+q}}{i!j!s!q!}\binom{j}{k}\left(\frac{1}{e-1}\right)^k\binom{k}{r}e^{(k-r)}(j-k)^s \times (k-r)^q\beta^{s+i}\int_0^\infty e^{tx}x^{\alpha(i+s+1)-1}e^{-q\beta x^\alpha}\,dx.$$

Recall that $e^{tx} = \sum\limits_{p=0}^{\infty}\frac{t^p x^p}{p!}$, so that

$$M_X(t) = \frac{\alpha\beta e^2}{e-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{k}\frac{(-1)^{i+j+k+r+s+q}}{i!j!s!p!q!}\binom{j}{k}\left(\frac{1}{e-1}\right)^k\binom{k}{r}e^{(k-r)}(j-k)^s \times (k-r)^q\beta^{s+i}t^p\int_0^\infty e^{tx}x^{\alpha(i+s+1)-1}e^{-q\beta x^\alpha}\,dx.$$

Let $u = q\beta x^\alpha \implies x = \left(\frac{u}{q\beta}\right)^{\frac{1}{\alpha}}$ and $dx = \frac{u^{\frac{1}{\alpha}-1}}{\alpha(q\beta)^{\frac{1}{\alpha}}}\,du$, so that

$$M_X(t) = \frac{\beta e^2}{e-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{k}\frac{(-1)^{i+j+k+r+s+q}}{i!j!s!p!q!}\binom{j}{k}\left(\frac{1}{e-1}\right)^k\binom{k}{r}e^{(k-r)}(j-k)^s \times (k-r)^q\beta^{s+i}t^p\int\limits_0^\infty\left(\frac{u}{q\beta}\right)^{\frac{1}{\alpha}[p+\alpha(i+s+1)-1]}\frac{u^{\frac{1}{\alpha}-1}e^{-u}}{\alpha(q\beta)^{\frac{1}{\alpha}}}\,du.$$

After simplification, the final results is

$$M_X(t) = \frac{\beta e^2}{e-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{k}\frac{(-1)^{i+j+k+r+s+q}}{i!j!s!p!q!}\binom{j}{k}\left(\frac{1}{e-1}\right)^k\binom{k}{r}e^{(k-r)}(j-k)^s \times (k-r)^q t^p\frac{\Gamma\left[\frac{p}{\alpha}+i+s+1\right]}{\beta^{\frac{p}{\alpha}+1}q^{\frac{p}{\alpha}+i+s+1}};\quad t\in\Re. \tag{15}$$

4.4 Distribution of k-th Order Statistics

The distribution of the k$-th$ order statistics for a random variable X is mathematically defined thus;

$$f_{X_{(k)}}(x;\alpha,\beta) = \frac{n!}{(k-1)!(n-k)!}\left[G(x;\alpha,\beta)\right]^{k-1}\left[1-G(x;\alpha,\beta)\right]^{n-k}g(x;\alpha,\beta),$$

For a random variable $X \sim$ KM-DUS-W (α,β), with $G(x;\alpha,\beta)$ and $g(x;\alpha,\beta)$ as the CDF and PDF defined in equations (9) and (10) respectively, the $f_{X_{(k)}}(x;\alpha,\beta)$, can be constructed as;

Expanding $[G(x;\alpha,\beta)]^{k-1}$, to obtain

$$
\begin{aligned}
[G(x;\alpha,\beta)]^{k-1} &= \left(\frac{e}{e-1}\right)^{k-1}\left[1-\exp\left(-\frac{1}{e-1}\left(e^{1-e^{-\beta x^{\alpha}}}-1\right)\right)\right]^{k-1}\\
&= \left(\frac{e}{e-1}\right)^{k-1}\sum_{i=0}^{\infty}(-1)^{i}\binom{k-1}{i}\exp\left(-\frac{i}{e-1}\left(e^{1-e^{-\beta x^{\alpha}}}-1\right)\right);\quad |i|\leq 1\\
&= \left(\frac{e}{e-1}\right)^{k-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{j!}\binom{k-1}{i}\left(\frac{i}{e-1}\right)^{j}\left(e^{1-e^{-\beta x^{\alpha}}}-1\right)^{j}\\
&= \left(\frac{e}{e-1}\right)^{k-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{j}\frac{(-1)^{i+j+r}}{j!}\binom{k-1}{i}\binom{j}{r}\left(\frac{i}{e-1}\right)e^{(j-r)\left(1-e^{-\beta x^{\alpha}}\right)};\quad r\in\mathbb{Z}\\
&= \left(\frac{e}{e-1}\right)^{k-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{j}\frac{(-1)^{i+j+r+p}}{j!p!}\binom{k-1}{i}\binom{j}{r}\left(\frac{i}{e-1}\right)^{j}e^{j-r}\left(1-e^{-\beta x^{\alpha}}\right)^{p}\\
&= \left(\frac{1}{e-1}\right)^{k-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{j}\frac{(-1)^{i+j+r+p+q}}{j!p!}\binom{k-1}{i}\binom{j}{r}\binom{p}{q}\left(\frac{i}{e-1}\right)^{j}\underbrace{e^{j+k-r-1}e^{-q\beta x^{\alpha}}}_{\text{Not Merged}}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
[1-G(x;\alpha,\beta)]^{n-k} &= \left\{1-\frac{e}{e-1}\left[1-\exp\left(-\frac{1}{e-1}\left(e^{1-e^{-\beta x^{\alpha}}}-1\right)\right)\right]\right\}^{n-k}\\
&= \sum_{d=0}^{\infty}(-1)^{d}\binom{n-k}{d}\left(\frac{e}{e-1}\right)^{d}\left[1-\exp\left(-\frac{1}{e-1}\left(e^{1-e^{-\beta x^{\alpha}}}-1\right)\right)\right]^{d};\quad |d|\leq 1\\
&= \sum_{d=0}^{\infty}\sum_{g=0}^{\infty}(-1)^{d+g}\binom{n-k}{d}\binom{d}{g}\left(\frac{e}{e-1}\right)^{d}\exp\left(-\frac{g}{e-1}\left(e^{1-e^{-\beta x^{\alpha}}}-1\right)\right);\quad |g|\leq 1\\
&= \sum_{d=0}^{\infty}\sum_{g=0}^{\infty}\sum_{f=0}^{\infty}\frac{(-1)^{d+g+f}}{f!}\binom{n-k}{d}\binom{d}{g}\left(\frac{e}{e-1}\right)^{d}\left(\frac{g}{e-1}\right)^{f}\left(e^{1-e^{-\beta x^{\alpha}}}-1\right)^{f}\\
&= \sum_{d=0}^{\infty}\sum_{g=0}^{\infty}\sum_{f=0}^{\infty}\sum_{h=0}^{f}\frac{(-1)^{d+g+f+h}}{f!}\binom{n-k}{d}\binom{d}{g}\binom{f}{h}\left(\frac{e}{e-1}\right)^{d}\left(\frac{g}{e-1}\right)^{f}e^{(f-h)\left(1-e^{-\beta x^{\alpha}}\right)};\quad h\in\mathbb{Z}\\
&= \sum_{d=0}^{\infty}\sum_{g=0}^{\infty}\sum_{f=0}^{\infty}\sum_{l=0}^{\infty}\sum_{h=0}^{f}\frac{(-1)^{d+g+f+h+l}}{f!l!}\binom{n-k}{d}\binom{d}{g}\binom{f}{h}\left(\frac{e}{e-1}\right)^{d}\left(\frac{g}{e-1}\right)^{f}e^{(f-h)}\left(1-e^{-\beta x^{\alpha}}\right)^{l}\\
&= \sum_{d=0}^{\infty}\sum_{g=0}^{\infty}\sum_{f=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{h=0}^{f}\frac{(-1)^{d+g+f+h+l+m}}{f!l!}\binom{n-k}{d}\binom{d}{g}\binom{f}{h}\binom{l}{m}\left(\frac{e}{e-1}\right)^{d}\\
&\times\left(\frac{g}{e-1}\right)^{f}\underbrace{e^{(f-h)}e^{-m\beta x^{\alpha}}}_{\text{Not Merged}};\quad |m|\leq 1,
\end{aligned}
$$

and to conflict in the series expand, replace i,j,k,q and r with a,b,t,c and w respectively in equation (13), so that

$$
\begin{aligned}
g(x;\alpha,\beta) = \frac{\alpha\beta e^{2}}{e-1}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{s=0}^{\infty}\sum_{c=0}^{\infty}\sum_{t=0}^{b}\sum_{w=0}^{t}\frac{(-1)^{a+b+t+w+s+c}}{a!b!s!c!}\binom{b}{t}\left(\frac{1}{e-1}\right)^{t}\binom{t}{r}e^{(t-c)}(b-t)^{s}\\
\times(t-w)^{c}\beta^{s+a}x^{\alpha(a+s+1)-1}e^{-c\beta x^{\alpha}}.
\end{aligned}
$$

Therefore;

$$
\begin{aligned}
f_{X_{(k)}}(x;\alpha,\beta) &= \frac{n!}{(k-1)!(n-k)!}\left[G(x;\alpha,\beta)\right]^{k-1}\left[1-G(x;\alpha,\beta)\right]^{n-k} g(x;\alpha,\beta)\\
&= \frac{n!}{(k-1)!(n-k)!}\cdot\left(\frac{e}{e-1}\right)^{k-1}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{p=0}^{\infty}\sum_{c=0}^{\infty}\sum_{w=0}^{b}\sum_{r=0}^{p}\frac{(-1)^{a+b+p+c+w+r}}{b!p!}\binom{k-1}{a}\binom{b}{w}\binom{p}{r}\\
&\times\left(\frac{a}{e-1}\right)^{b}\underbrace{e^{b+k-w-1}e^{-r\beta x^{\alpha}}}_{\text{Not Merged}}\\
&\times\sum_{d=0}^{\infty}\sum_{g=0}^{\infty}\sum_{f=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{h=0}^{f}\frac{(-1)^{d+g+f+h+l+m}}{f!l!}\binom{n-k}{d}\binom{d}{g}\binom{f}{h}\binom{l}{m}\\
&\times\left(\frac{e}{e-1}\right)^{d}\left(\frac{g}{e-1}\right)^{f}\underbrace{e^{f-h}e^{-m\beta x^{\alpha}}}_{\text{Not Merged}}\\
&\times\frac{\alpha\beta e^{2}}{e-1}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{s=0}^{\infty}\sum_{c=0}^{\infty}\sum_{t=0}^{b}\sum_{w=0}^{t}\frac{(-1)^{a+b+t+w+s+c}}{a!b!s!c!}\binom{b}{t}\left(\frac{1}{e-1}\right)^{t}\binom{t}{r}\\
&\times e^{(t-c)}(b-t)^{s}(t-w)^{c}\beta^{s+a}x^{\alpha(a+s+1)-1}e^{-c\beta x^{\alpha}}.
\end{aligned}
$$

4.5 Measure of Uncertainty

The Rény entropy is a common measure of information loss or gained [28,63,64]. The Rény entropy of order γ, $(\gamma > 0, \gamma \neq 1)$ for a r.v X with a PDF defined in equation (10) is

$$
\begin{aligned}
H_{\gamma}(g) &= \frac{1}{1-\gamma}\log\int_{-\infty}^{\infty} g(x;\alpha,\beta)^{\gamma}\,dx\\
&= \frac{1}{1-\gamma}\log\left[\alpha^{\gamma-1}\beta^{\frac{\gamma-1}{\alpha}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{i+j+k+l}}{i!j!l!}\binom{j}{k}\frac{\gamma^{i+j}e^{2\gamma+j-k}}{(e-1)^{2\gamma+j}}\frac{\Gamma\left[\gamma-\frac{\gamma+1}{\alpha}\right]}{(\gamma+i+l)^{\gamma-\frac{\gamma+1}{\alpha}}}\right].
\end{aligned}
\tag{16}
$$

Proof. Substituting the expression for $g(x;\alpha,\beta)$, we get:

$$
\begin{aligned}
H_{\gamma}(g) &= \frac{1}{1-\gamma}\log\left[\int_{0}^{\infty}\left(\frac{e}{(e-1)^{2}}\alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}}\exp\left(1-e^{-\beta x^{\alpha}}-\frac{1}{e-1}(e^{1-e^{-\beta x^{\alpha}}}-1)\right)\right)^{\gamma}dx\right]\\
&= \frac{1}{1-\gamma}\log\left[\frac{e^{\gamma}\alpha^{\gamma}\beta^{\gamma}}{(e-1)^{2\gamma}}\int_{0}^{\infty}x^{\gamma(\alpha-1)}e^{-\beta\gamma x^{\alpha}}\exp\left\{\gamma\left(1-e^{-\beta x^{\alpha}}-\frac{1}{e-1}(e^{1-e^{-\beta x^{\alpha}}}-1)\right)\right\}dx\right]\\
&= \frac{1}{1-\gamma}\log\left[\frac{e^{2\gamma}\alpha^{\gamma}\beta^{\gamma}}{(e-1)^{2\gamma}}\int_{0}^{\infty}x^{\gamma(\alpha-1)}e^{-\beta\gamma x^{\alpha}}e^{-\gamma e^{-\beta x^{\alpha}}}\exp\left(-\frac{\gamma}{e-1}e^{1-e^{-\beta x^{\alpha}}}\right)dx\right].
\end{aligned}
$$

Using the following power series identities;

$$
e^{-\gamma e^{-\beta x^{\alpha}}} = \sum_{i=0}^{\infty}\frac{(-1)^{i}\gamma^{i}}{i!}e^{-i\beta x^{\alpha}},\quad \exp\left(-\frac{\gamma}{e-1}e^{1-e^{-\beta x^{\alpha}}}\right) = \sum_{j=0}^{\infty}\frac{(-1)^{j}}{j!}\left(\frac{\gamma}{e-1}\right)^{j}e^{j(1-e^{-\beta x^{\alpha}})}\quad\text{and}
$$

$$
e^{-je^{-\beta x^{\alpha}}} = \sum_{l=0}^{\infty}\frac{(-1)^{l}j^{l}}{l!}e^{-l\beta x^{\alpha}}.
$$

and making appropriate substitutions

$$
\begin{aligned}
&\int_{0}^{\infty}x^{\gamma(\alpha-1)}e^{-\beta\gamma x^{\alpha}}e^{-\gamma e^{-\beta x^{\alpha}}}\exp\left(-\frac{\gamma}{e-1}e^{1-e^{-\beta x^{\alpha}}}\right)dx\\
&\quad= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+j+l}\gamma^{i+j}}{i!j!l!(e-1)^{j}}e^{j}\int_{0}^{\infty}x^{\gamma(\alpha-1)}e^{-\beta(\gamma+i+l)x^{\alpha}}\sum_{k=0}^{j}\binom{j}{k}(-1)^{k}e^{-k\beta x^{\alpha}}dx.
\end{aligned}
$$

Now, collect all exponents in $e^{-\beta x^{\alpha}}$:

$$\int_0^{\infty} x^{\gamma(\alpha-1)} e^{-\beta(\gamma+i+l+k)x^{\alpha}} dx.$$

Apply the change of variable $u = \beta(\gamma+i+l+k)x^{\alpha} \Rightarrow x = \left(\frac{u}{\beta(\gamma+i+l+k)}\right)^{1/\alpha}$, the integral becomes:

$$\int_0^{\infty} x^{\gamma(\alpha-1)} e^{-\beta(\gamma+i+l+k)x^{\alpha}} dx = \frac{1}{\alpha} [\beta(\gamma+i+l+k)]^{-\frac{\gamma(\alpha-1)+1}{\alpha}} \Gamma\left(\frac{\gamma(\alpha-1)+1}{\alpha}\right).$$

Thus, let $s = \frac{\gamma(\alpha-1)+1}{\alpha} = \gamma - \frac{\gamma-1}{\alpha}$, then:

$$H_{\gamma}(g) = \frac{1}{1-\gamma} \log\left[\frac{e^{2\gamma}\alpha^{\gamma}\beta^{\gamma}}{(e-1)^{2\gamma}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{j} \frac{(-1)^{i+j+k+l}}{i!j!l!} \binom{j}{k} \frac{\gamma^{i+j} e^{2\gamma+j-k}}{(e-1)^{2\gamma+j}} \frac{1}{\alpha} [\beta(\gamma+i+l+k)]^{-s} \Gamma(s)\right].$$

Extracting constants and simplifying:

$$H_{\gamma}(g) = \frac{1}{1-\gamma} \log\left[\alpha^{\gamma-1}\beta^{\frac{\gamma-1}{\alpha}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{j} \frac{(-1)^{i+j+k+l}}{i!j!l!} \binom{j}{k} \frac{\gamma^{i+j} e^{2\gamma+j-k}}{(e-1)^{2\gamma+j}} (\gamma+i+l+k)^{-s} \Gamma(s)\right].$$

Hence, the closed-form of the Rényi entropy is:

$$H_{\gamma}(g) = \frac{1}{1-\gamma} \log\left[\alpha^{\gamma-1}\beta^{\frac{\gamma-1}{\alpha}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{j} \frac{(-1)^{i+j+k+l}}{i!j!l!} \binom{j}{k} \frac{\gamma^{i+j} e^{2\gamma+j-k}}{(e-1)^{2\gamma+j}} \frac{\Gamma\left(\gamma-\frac{\gamma-1}{\alpha}\right)}{(\gamma+i+l+k)^{\gamma-\frac{\gamma-1}{\alpha}}}\right].$$

5 Inference

This section focuses on the estimation of parameters for the KM-DUS-W distribution using both non-Bayesian and Bayesian methods. The purpose of employing multiple approaches is to evaluate their effectiveness in both small and large sample contexts.

5.1 Maximum Likelihood Estimation

Consider the random sample of observations given as $x_1, x_2, \ldots, x_n$, which are independent observations drawn from the KM-DUS-W distribution with parameters α and β unknown. The log-likelihood function [59,60,45] for the distribution in Equation (10) based on a random sample $x_1, x_2, \ldots, x_n$ is given by:

$$\begin{aligned}\ell(\alpha,\beta) = n\log\left(\frac{e}{(e-1)^2}\right) + n\log\alpha + n\log\beta + (\alpha-1)\sum_{i=1}^{n}\log x_i \\ -\beta\sum_{i=1}^{n} x_i^{\alpha} + \sum_{i=1}^{n}\left(1 - e^{-\beta x_i^{\alpha}} - \frac{1}{e-1}\left(e^{1-e^{-\beta x_i^{\alpha}}} - 1\right)\right).\end{aligned} \tag{17}$$

The score functions are:

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log x_i \left(1 - \beta x_i^{\alpha} + \beta x_i^{\alpha} e^{-\beta x_i^{\alpha}} - \frac{\beta x_i^{\alpha}}{e-1} e^{1-e^{-\beta x_i^{\alpha}}} e^{-\beta x_i^{\alpha}}\right), \tag{18}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x_i^{\alpha} + \sum_{i=1}^{n} x_i^{\alpha} e^{-\beta x_i^{\alpha}} - \frac{1}{e-1}\sum_{i=1}^{n} x_i^{\alpha} e^{1-e^{-\beta x_i^{\alpha}}} e^{-\beta x_i^{\alpha}}. \tag{19}$$

These equations (18) and (19) are solved numerically to obtain the maximum likelihood estimates of α and β.

5.1.1 Confidence Interval for MLE

Once the MLEs $\hat{\alpha}$ and $\hat{\beta}$ are obtained by numerically solving the score equations (18) and (19), the asymptotic distribution of the MLEs is approximately:

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} \sim \mathscr{N}\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \mathscr{I}^{-1}(\alpha,\beta)\right),$$

where $\mathscr{I}(\alpha,\beta)$ is the Fisher Information Matrix (FIM), ([68]) given by

$$\mathscr{I}(\alpha,\beta) = -\mathbb{E}\left[\begin{pmatrix}\frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\beta}\\ \frac{\partial^2\ell}{\partial\beta\partial\alpha} & \frac{\partial^2\ell}{\partial\beta^2}\end{pmatrix}\right].$$

Since the expectations are often analytically intractable, we typically use the observed Fisher information by evaluating the second derivatives of the log-likelihood at the MLEs:

$$\hat{\mathscr{I}}(\hat{\alpha},\hat{\beta}) = -\left.\begin{pmatrix}\frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\beta}\\ \frac{\partial^2\ell}{\partial\beta\partial\alpha} & \frac{\partial^2\ell}{\partial\beta^2}\end{pmatrix}\right|_{(\alpha=\hat{\alpha},\beta=\hat{\beta})}.$$

The estimated variance-covariance matrix is

$$\hat{\mathscr{V}} = \hat{\mathscr{I}}^{-1}(\hat{\alpha},\hat{\beta}) = \begin{pmatrix}\mathrm{Var}(\hat{\alpha}) & \mathrm{Cov}(\hat{\alpha},\hat{\beta})\\ \mathrm{Cov}(\hat{\alpha},\hat{\beta}) & \mathrm{Var}(\hat{\beta})\end{pmatrix}.$$

We compute the second-order derivatives $\frac{\partial^2\ell}{\partial\alpha^2}$, $\frac{\partial^2\ell}{\partial\beta^2}$, and $\frac{\partial^2\ell}{\partial\alpha\partial\beta}$ from equations (17), (18), and (19) as follows.

$$\frac{\partial^2\ell}{\partial\alpha^2} = -\frac{n}{\alpha^2} + \sum_{i=1}^{n}(\ln x_i)^2\cdot x_i^{\alpha}\beta\left[(1-\beta x_i^{\alpha})\,e^{-\beta x_i^{\alpha}} + \frac{e^{1-\beta x_i^{\alpha}-e^{-\beta x_i^{\alpha}}}}{e-1}(1-\beta x_i^{\alpha})-1\right], \tag{20}$$

$$\frac{\partial^2\ell}{\partial\beta^2} = -\frac{n}{\beta^2} + \sum_{i=1}^{n}x_i^{2\alpha}e^{-\beta x_i^{\alpha}} - \frac{1}{e-1}\sum_{i=1}^{n}x_i^{2\alpha}e^{1-\beta x_i^{\alpha}-e^{-\beta x_i^{\alpha}}}(1-\beta x_i^{\alpha}), \tag{21}$$

and

$$\frac{\partial^2\ell}{\partial\alpha\partial\beta} = \sum_{i=1}^{n}\ln x_i\cdot x_i^{\alpha}\left[(1-\beta x_i^{\alpha})e^{-\beta x_i^{\alpha}} + \frac{e^{1-\beta x_i^{\alpha}-e^{-\beta x_i^{\alpha}}}}{e-1}(1-\beta x_i^{\alpha})-1\right]. \tag{22}$$

This is symmetric with $\frac{\partial^2\ell}{\partial\beta\partial\alpha}$, as expected from Schwarz's theorem under regularity conditions, see [62]. Then, the approximate $100(1-\gamma)\%$ confidence intervals for α and β are:

$$\hat{\alpha} \pm Z_{\gamma/2}\sqrt{\mathrm{Var}(\hat{\alpha})}, \quad \hat{\beta} \pm Z_{\gamma/2}\sqrt{\mathrm{Var}(\hat{\beta})},$$

where $Z_{\gamma/2}$ is the upper $\gamma/2$ quantile of the standard normal distribution.

5.2 Least Squares Estimation (LSE)

The Least Squares Estimation (LSE) method, which was introduced in [48,58] to estimate the parameters of Beta distribution, provides a basis for parameter estimation for α,β parameters of KM-DUS-W distribution. The method minimizes the total discrepancy between the observed and expected values ensuring the best fit of the distribution model to the data.

The expected value and variance of the CDF for the KM-DUS-W distribution can be expressed as follows:

$$E\left[G\left(x_{j:n}|\alpha,\beta\right)\right] = \frac{j}{n+1}, \quad V\left[G\left(x_{j:n}|\alpha,\beta\right)\right] = \frac{j(n-j+1)}{(n+1)^2(n+2)}.$$

These parameters are estimated by minimizing the following function $L(\alpha,\beta)$:

$$L(\alpha,\beta) = \arg\min_{(\alpha,\beta)} \sum_{j=1}^{n} \left[G(x_{j:n}|\alpha,\beta) - \frac{j}{n+1} \right]^2$$

The estimates of parameters, $\hat{\alpha}_{LSE}$ and $\hat{\beta}_{LSE}$, are determined solving the corresponding system of nonlinear equations defined as follows:

$$\sum_{j=1}^{n} \left[G(x_{j:n}|\alpha,\beta) - \frac{j}{n+1} \right]^2 \Delta_1(x_{j:n}|\alpha,\beta) = 0, \tag{23}$$

and

$$\sum_{j=1}^{n} \left[G(x_{j:n}|\alpha,\beta) - \frac{j}{n+1} \right]^2 \Delta_2(x_{j:n}|\alpha,\beta) = 0. \tag{24}$$

The terms $\Delta_1(x_{j:n}|\alpha,\beta)$ and $\Delta_2(x_{j:n}|\alpha,\beta)$ in equations (18) and (19) can be expressed by supposing $z = e^{1-e^{-\beta x^{\alpha}}}$ and $A = \frac{1}{e-1}$, so that;

$$\Delta_1(x_{j:n}|\alpha,\beta) = \frac{eA}{e-1} e^{-A(z-1)} z e^{-\beta x^{\alpha}} \beta x^{\alpha} \ln x, \tag{25}$$

and

$$\Delta_2(x_{j:n}|\alpha,\beta) = \frac{eA}{e-1} e^{-A(z-1)} z e^{-\beta x^{\alpha}} x^{\alpha}. \tag{26}$$

The expressions in equations (25) and (26) above are derived by partial differentiation of the CDF of the KM-DUS-W distribution as given in equation (9) with respect to α and β.

5.3 Weighted Least Squares Estimation (WLSE)

The parameters of the KM-DUS-W distribution, α and β, are estimated by the weighted least squares method [65,66,44]. The estimates $\hat{\alpha}_{WLSE}$ and $\hat{\beta}_{WLSE}$ are obtained from the minimization of the function $W(\alpha,\beta)$ in both α and β:

$$W(\alpha,\beta) = \arg\min_{(\alpha,\beta)} \sum_{j=1}^{n} w_j \left[G(x_{j:n}|\alpha,\beta) - \frac{j}{n+1} \right]^2 \tag{27}$$

The weights w_j are given by

$$w_j = \frac{(n+1)^2(n+2)}{j(n-j+1)}$$

The parameter estimates are found by solving the following set of nonlinear equations:

$$\sum_{j=1}^{n} w_j \left[G(x_{j:n}|\alpha,\beta) - \frac{j}{n+1} \right]^2 \Delta_1(x_{j:n}|\alpha,\beta) = 0 \tag{28}$$

$$\sum_{j=1}^{n} w_j \left[G(x_{j:n}|\alpha,\beta) - \frac{j}{n+1} \right]^2 \Delta_2(x_{j:n}|\alpha,\beta) = 0 \tag{29}$$

Here, $\Delta_1(x_{j:n}|\alpha,\beta)$ and $\Delta_2(x_{j:n}|\alpha,\beta)$ are defined in Equations (25) and (26), respectively.

5.4 Maximum Product of Spacing Estimation (MPS)

Maximum product spacing method, introduced in [5], is a method for estimating in addition to the estimation of maximum likelihood. The method is instead an approximation of Kullback-Leibler information criterion rather than taking the traditional maximum likelihood path. Under this arrangement, it will assume the data are arranged in ascending order and go ahead with:

$$I_s\left(data|\alpha,\beta\right)=\left[\prod_{j=1}^{n+1}D_k\left(x_{j:n}|\alpha,\beta\right)\right]^{\frac{1}{n+1}} \tag{30}$$

Whence $D_k\left(x_{j:n}|\alpha,\beta\right)$ is thus defined as follows: $G\left(x_j|\alpha,\beta\right)-G\left(x_{j-1}|\alpha,\beta\right)$, with $j=1,2,3,\ldots,n$.
Likewise, we could maximize the following function with:

$$N\left(\alpha,\beta\right)=\frac{1}{n+1}\sum_{j=1}^{n+1}\ln\left(D_k\left(x_{j:n}|\alpha,\beta\right)\right) \tag{31}$$

Differentiate $N(\alpha,\beta)$ with respect to α and β to form a nonlinear system of equations: $\frac{\partial N(\alpha,\beta)}{\partial\alpha}=0$, $\frac{\partial N(\alpha,\beta)}{\partial\beta}=0$; solving this system results in parameter estimates.

5.5 Cramér-von Mises Estimation (CvM)

The estimates for the parameters α and β of the KM-DUS-W distribution from the Cramér-von Mises [67] are denoted by $\hat{\alpha}_{\text{CvM}}$ and $\hat{\beta}_{\text{CvM}}$ and were found by minimization with respect to α and β of the objective function $C(\alpha,\beta)$, given as:

$$C\left(\alpha,\beta\right)=\arg\min_{(\alpha,\beta)}\left\{\frac{1}{12n}+\sum_{j=1}^{n}\left[G\left(x_{j:n}|\alpha,\beta\right)-\frac{2j-1}{2n}\right]^2\right\} \tag{32}$$

To get the estimates, we solve the following system of non-linear equations:

$$\sum_{j=1}^{n}\left[G\left(x_{j:n}|\alpha,\beta\right)-\frac{2j-1}{2n}\right]\Delta_1\left(x_{j:n}|\alpha,\beta\right)=0 \tag{33}$$

$$\sum_{j=1}^{n}\left[G\left(x_{j:n}|\alpha,\beta\right)-\frac{2j-1}{2n}\right]\Delta_2\left(x_{j:n}|\alpha,\beta\right)=0 \tag{34}$$

Here, the $\Delta_1(x_{j:n}|\alpha,\beta)$ and $\Delta_2(x_{j:n}|\alpha,\beta)$ functions are defined according to equations (25) and (26).

5.6 Anderson-Darling Estimation (AD)

The first Anderson-Darling estimators for parameters α and β of the KM-DUS-W distribution, $\hat{\alpha}_{\text{AD}}$ and $\hat{\beta}_{\text{AD}}$, correspondingly, are determined by minimizing the function $AD(\alpha,\beta)$, concerning these estimands. The minimization is done over:

$$AD\left(\alpha,\beta\right)=\arg\min_{(\alpha,\beta)}\sum_{j=1}^{n}\left(2j-1\right)\left[\ln\left(G\left(x_{j:n}|\alpha,\beta\right)\right)+\ln\left(1-G\left(x_{n+1-j:n}|\alpha,\beta\right)\right)\right] \tag{35}$$

From solving the following system of non-linear equations, estimates were made:

$$\sum_{j=1}^{n}\left(2j-1\right)\left[\frac{\Delta_1\left(x_{j:n}|\alpha,\beta\right)}{G\left(x_{j:n}|\alpha,\beta\right)}-\frac{\Delta_1\left(x_{n+1-j:n}|\alpha,\beta\right)}{1-G\left(x_{n+1-j:n}|\alpha,\beta\right)}\right]=0 \tag{36}$$

$$\sum_{j=1}^{n}\left(2j-1\right)\left[\frac{\Delta_2\left(x_{j:n}|\alpha,\beta\right)}{G\left(x_{j:n}|\alpha,\beta\right)}-\frac{\Delta_2\left(x_{n+1-j:n}|\alpha,\beta\right)}{1-G\left(x_{n+1-j:n}|\alpha,\beta\right)}\right]=0 \tag{37}$$

In the equations above, the quantities $\Delta_1\left(x_{j:n}|\alpha,\beta\right)$ and $\Delta_2\left(x_{j:n}|\alpha,\beta\right)$ are defined by the equations appearing in (23) and (24), respectively.

5.7 Right-Tailed Anderson-Darling Estimation (RTAD)

The parameter estimates of $\hat{\alpha}_{\text{RTAD}}$ and $\hat{\beta}_{\text{RTAD}}$ for α and β corresponding to the KM-DUS-W distribution are determined by minimizing the function $RA(\alpha,\beta)$ with respect to both parameters using the Right-Tailed Anderson-Darling method.

$$RA(\alpha,\beta) = \arg\min_{(\alpha,\beta)} \left\{ \frac{n}{2} - 2\sum_{j=1}^{n} G(x_{j:n}|\alpha,\beta) - \frac{1}{n}\sum_{j=1}^{n}(2j-1)\ln\left[1-G\left(x_{n+1-j:n}|\alpha,\beta\right)\right] \right\} \tag{38}$$

The estimates are derived by solving the following set of non-linear equations.

$$-2\sum_{j=1}^{n}\frac{\Delta_1(x_{j:n}|\alpha,\beta)}{G(x_{j:n}|\alpha,\beta)} + \frac{1}{n}\sum_{j=1}^{n}(2j-1)\left[\frac{\Delta_1\left(x_{n+1-j:n}|\alpha,\beta\right)}{1-G\left(x_{n+1-j:n}|\alpha,\beta\right)}\right] = 0 \tag{39}$$

$$-2\sum_{j=1}^{n}\frac{\Delta_2(x_{j:n}|\alpha,\beta)}{G(x_{j:n}|\alpha,\beta)} + \frac{1}{n}\sum_{j=1}^{n}(2j-1)\left[\frac{\Delta_2\left(x_{n+1-j:n}|\alpha,\beta\right)}{1-G\left(x_{n+1-j:n}|\alpha,\beta\right)}\right] = 0 \tag{40}$$

The above two functions, namely $\Delta_1(x_{j:n}|\alpha,\beta)$ and $\Delta_2(x_{j:n}|\alpha,\beta)$, are separately defined in Equations (23) and (24).

5.8 Percentile Matching Estimation or Percentile Estimation (PE)

Given a sample $x_1, x_2, \cdots, x_n$ of size n ordered such that $x_1 \le x_2 \le \cdots \le x_n$. Also, given m distinct cumulative probabilities $0 < p_1 < p_2 < \cdots < p_m < 1$ such that the typical choices for large or moderate n is $p_j = \frac{j}{n+1}$, $\quad j = 1, 2, \cdots, m$. For the proposed KM-DUS-W model, the implicit definition of p-th percentile x_p, means it is a function of α and β that is $p_j = G(x_p, \alpha, \beta)$. From the quantile function in equation (14), the p-th percentile is expressed as

$$x_p = \left(-\frac{1}{\beta}\ln\left(1-\ln\left[1+(1-e)\ln\left\{1-\left(1-\frac{1}{e}\right)p\right\}\right]\right)\right)^{\frac{1}{\alpha}}; \quad p \in (0,1). \tag{41}$$

To obtain the optimal function for the PE, we match order statistics to the corresponding order statistics from the sample. Hence, the PE is minimized by the function

$$Q(\alpha,\beta) = \sum_{j=1}^{m}\left\{x_{(j)} - \left(-\frac{1}{\beta}\ln\left(1-\ln\left[1+(1-e)\ln\left\{1-\left(1-\frac{1}{e}\right)p_j\right\}\right]\right)\right)^{\frac{1}{\alpha}}\right\}^2; \quad m \le n. \tag{42}$$

The PE method has undergone stages of development in the literature, see [5] for details. The parameter estimates in equations (23), (24), (27), (28), (30), (35), (36), (38), (39), (40) and (42) were all acquired through the use of the *optim()* function in the statistical programming language R, employing the Newton-Raphson method in order to search for the maximum likelihood iteratively.

5.9 Bayesian estimation (BE) under different loss functions

This section discusses the Bayesian estimation of the parameters of the KM-DUS-W distribution. Bayesian estimation links prior knowledge with observed data, employing loss functions such as squared error, LINEX, and generalized entropy to estimate the parameters. We assume independent gamma priors for the parameters α and β, expressed as:

$$\left.\begin{array}{ll} \pi_1(\alpha) \propto \alpha^{s_1-1}e^{-k_1\alpha}, & \alpha > 0,\ s_1 > 0,\ k_1 > 0, \\ \pi_2(\beta) \propto \beta^{s_2-1}e^{-k_2\beta}, & \beta > 0,\ s_2 > 0,\ k_2 > 0 \end{array}\right\} \tag{43}$$

where s_j and k_j for $j = 1, 2$ are hyperparameters. The joint prior distribution for $\phi = (\alpha, \beta)$ is given as:

$$\pi(\phi) \propto \alpha^{s_1-1}\beta^{s_2-1}e^{-k_1\alpha-k_2\beta}. \tag{44}$$

Using observed data $X = (x_1, x_2, \ldots, x_n)$, the posterior distribution is expressed as:

$$\pi(\phi \mid X) \propto \pi(\phi)L(\phi), \tag{45}$$

where $L(\phi)$ is the likelihood function. For the KM-DUS-W distribution, the posterior density becomes:

$$\pi(\phi \mid X) \propto \frac{\alpha^{n-s_1-1}\beta^{n+s_2-1}}{(e-1)^{2n}} \exp\left[2n - \beta\sum_{i=1}^{n} x_i^{\alpha} - \sum_{i=1}^{n}\left(e^{-\beta x_i^{\alpha}} - \frac{1}{e-1}\left(e^{1-e^{-\beta x_i^{\alpha}}} - 1\right)\right)\right]\prod_{i=1}^{n} x_i^{\alpha-1} \tag{46}$$

Bayesian parameter estimates are derived under various loss functions. For the squared error loss (SEL), the Bayes estimator is given by:

$$\hat{\phi}_{\text{BE\_SEL}} = \mathbb{E}[\phi \mid X] = \int \phi\pi(\phi \mid X)d\phi. \tag{47}$$

Alternative loss functions, such as LINEX and generalized entropy loss (GEL), address asymmetric estimation scenarios. The Bayes estimator under LINEX loss is defined as:

$$\hat{\phi}_{\text{BE\_LINEX}} = -\frac{1}{\eta}\log\left(\int e^{-\eta\phi}\pi(\phi \mid X)d\phi\right), \tag{48}$$

where $\eta \neq 0$ reflects the asymmetry in estimation. For GEL, the Bayes estimator becomes:

$$\hat{\phi}_{\text{BE\_GEL}} = \left(\int \phi^{-l}\pi(\phi \mid X)d\phi\right)^{-1/l}, \tag{49}$$

where $l \neq 0$ is the asymmetry parameter.

Since these estimators often lack closed-form solutions, numerical methods like Markov chain Monte Carlo (MCMC) are employed. The MCMC procedure for approximating Bayesian estimates involves the following steps:

1. Initialize the parameters $\phi^{(0)}$ and set the number of iterations M.
2. Generate samples $\phi^{(j)}$ from the posterior distribution $\pi(\phi \mid X)$ using algorithms like the Metropolis-Hastings or Gibbs sampler.
3. Discard the initial τ_D samples as the burn-in period to ensure convergence.
4. Use the remaining $M-\tau_D$ samples to compute Bayesian estimates as follows:

$$\hat{\phi}_{\text{BE\_SEL}} = \frac{1}{M-\tau_D}\sum_{j=\tau_D+1}^{M}\phi^{(j)}, \tag{50}$$

$$\hat{\phi}_{\text{BE\_LINEX}} = -\frac{1}{\eta}\log\left(\frac{1}{M-\tau_D}\sum_{j=\tau_D+1}^{M} e^{-\eta\phi^{(j)}}\right), \tag{51}$$

$$\hat{\phi}_{\text{BE\_GEL}} = \left(\frac{1}{M-\tau_D}\sum_{j=\tau_D+1}^{M}(\phi^{(j)})^{-l}\right)^{-1/l}. \tag{52}$$

This algorithm enables the computation of Bayesian estimates under SEL, LINEX, and GEL loss functions, providing robust parameter estimates tailored to specific applications. For further reading, refer to [3] and [4].

5.9.1 Credible Interval for BE

To calculate a $100(1-\gamma)\%$ credible interval (CI) for $\phi = (\alpha, \beta)$ based on the three loss functions studied, equation (53) is used.

$$\begin{aligned}
&\hat{\phi}_{\text{BE_SEL}} \pm Z_{\gamma/2}\sqrt{\operatorname{var}\left\{\frac{1}{M-\tau_D}\sum_{i=\tau_D}^{M}\phi^{(i)}\right\}},\\
&\hat{\phi}_{\text{BE_LINEX}} \pm Z_{\gamma/2}\sqrt{\operatorname{var}\left\{-\frac{1}{\eta}\log\left[\frac{1}{M-\tau_D}\sum_{i=\tau_D}^{M}\exp\left(-\eta\phi^{(i)}\right)\right]\right\}},\\
&\hat{\phi}_{\text{BE_GEL}} \pm Z_{\gamma/2}\sqrt{\operatorname{var}\left\{\left[\frac{1}{M-\tau_D}\sum_{i=\tau_M}^{N}\left(\phi^{(i)}\right)^{-\tau}\right]^{-\frac{1}{\tau}}\right\}},
\end{aligned} \tag{53}$$

where $Z_{\gamma/2}$ represents the critical value from the standard normal distribution corresponding to the upper $\gamma/2$ percentile (i.e., for the right-tailed probability).

6 Simulation

The aim here is to compare the performance of non-Bayesian and Bayesian estimation methods for the KM-DUS-W distribution parameters in a 1,000-replication study for each sample size (n = 25, 75, 150, 200) using different sets of initial parameter values. The bias and root mean square error (RMSE) were averaged for each replication to assess the accuracy and reliability of the proposed estimators. In the Bayesian situation, asymmetry is accounted for using $\eta = -0.5$ and $\eta = 0.5$ for the LINEX function and $l = -0.5$ and $l = 0.5$ for the GEL loss functions. Hence, the LINEX1, LINEX2, GEL1, and GEL2 respectively. Different scenarios with different sets of initial parameter values were examined, including:

Case I:Table 2 ($\alpha = 1.75$ and $\beta = 2.75$)
Case II:Table 3 ($\alpha = 1.5$ and $\beta = 2.5$)
Case III:Table 4 ($\alpha = 1.75$ and $\beta = 1.25$)
Case IV:Table 5 ($\alpha = 1.8$ and $\beta = 2.8$).

Table 2: Bias and RMSE of the Estimators under Simulation Case I

Class	Method	Estimator	$n=25$ Bias	$n=25$ RMSE	$n=75$ Bias	$n=75$ RMSE	$n=150$ Bias	$n=150$ RMSE	$n=200$ Bias	$n=200$ RMSE
Non-Bayesian	MLE	$\hat{\alpha}$	0.09400	0.09733	0.03859	0.02834	0.01221	0.01354	0.01129	0.01050
		$\hat{\beta}$	0.24274	0.67622	0.07197	0.14483	0.03874	0.07721	0.02839	0.05173
	MPS	$\hat{\alpha}$	0.11310	0.08253	0.05258	0.02697	0.04137	0.01434	0.03131	0.01088
		$\hat{\beta}$	0.16581	0.38172	0.09627	0.12061	0.05945	0.07010	0.04912	0.04823
	LS	$\hat{\alpha}$	0.02064	0.12039	0.00058	0.03904	0.00706	0.01999	0.00399	0.01487
		$\hat{\beta}$	0.03854	0.80794	0.00529	0.19698	0.00286	0.10099	0.00375	0.07015
	WLS	$\hat{\alpha}$	0.00160	0.10501	0.01222	0.03245	0.00030	0.01625	0.00325	0.01223
		$\hat{\beta}$	0.05741	0.64070	0.02402	0.16415	0.01606	0.08394	0.01487	0.05866
	CvM	$\hat{\alpha}$	0.09008	0.14875	0.03538	0.04234	0.01069	0.02056	0.00929	0.01522
		$\hat{\beta}$	0.30112	1.29699	0.08186	0.22668	0.04014	0.10810	0.03152	0.07389
	AD	$\hat{\alpha}$	0.02021	0.09116	0.01366	0.03078	0.00022	0.01554	0.00173	0.01193
		$\hat{\beta}$	0.09653	0.55811	0.02723	0.15378	0.01552	0.08103	0.01240	0.05733
	RTAD	$\hat{\alpha}$	0.05313	0.11930	0.02274	0.03529	0.00721	0.01791	0.00336	0.01344
		$\hat{\beta}$	0.14006	0.63580	0.03610	0.15098	0.02463	0.08261	0.01322	0.05577
	PE	$\hat{\alpha}$	0.07597	0.09044	0.03566	0.03131	0.03245	0.01681	0.02599	0.01314
		$\hat{\beta}$	0.10966	0.42351	0.07098	0.12584	0.04413	0.07432	0.03935	0.05108
Bayesian	SEL	$\hat{\alpha}$	0.24289	0.14198	0.36574	0.16125	0.41723	0.18815	0.42989	0.19629
		$\hat{\beta}$	0.37473	0.61566	0.73934	0.71903	0.90166	0.90782	0.94382	0.96568
	LINEX1	$\hat{\alpha}$	0.26091	0.15361	0.37405	0.16782	0.42189	0.19218	0.43350	0.19948
		$\hat{\beta}$	0.46165	0.76363	0.78947	0.80969	0.93265	0.96934	0.96848	1.01589
	LINEX2	$\hat{\alpha}$	0.22533	0.13150	0.35751	0.15489	0.41259	0.18419	0.42630	0.19313
		$\hat{\beta}$	0.29583	0.50936	0.69143	0.63895	0.87149	0.85022	0.91972	0.91800
	GEL1	$\hat{\alpha}$	0.23410	0.13737	0.36186	0.15832	0.41509	0.18635	0.42825	0.19485
		$\hat{\beta}$	0.34998	0.58782	0.72564	0.69686	0.89340	0.89228	0.93731	0.95293
	GEL2	$\hat{\alpha}$	0.21655	0.12876	0.35408	0.15258	0.41080	0.18276	0.42495	0.19201
		$\hat{\beta}$	0.30121	0.53847	0.69832	0.65406	0.87691	0.86171	0.92429	0.92776

Table 2 gives the simulation results for the bias and root mean square error (RMSE) of the various estimators of the parameters α and β under the different estimation methods and sample sizes $n = 25, 75, 150$, and 200.

Let $\hat{\phi}$ be an estimator of a parameter ϕ. The *bias* of $\hat{\phi}$ is:

$$\mathrm{Bias}(\hat{\phi}) = \mathbb{E}[\hat{\phi}] - \phi,$$

while the *root mean square error* (RMSE) is:

$$\mathrm{RMSE}(\hat{\phi}) = \sqrt{\mathbb{E}[(\hat{\phi} - \phi)^2]}.$$

It is observed from the table that for all non-Bayesian estimators (MLE, MPS, LS, WLS, CvM, AD, RTAD, PE), bias and RMSE decrease with the increasing sample size, in conformity with the asymptotic properties of consistent estimators. Especially, the least squares (LS) and weighted least squares (WLS) methods possess very low bias and RMSE for $\hat{\alpha}$ and $\hat{\beta}$ for all sample sizes, indicating high efficiency.

In contrast to this, Bayesian estimators (SEL, LINEX1, LINEX2, GEL1, GEL2) consistently possess greater bias and RMSE, especially for the parameter β. Among them, LINEX2 and GEL2 estimators possess comparatively less bias and RMSE compared to SEL and LINEX1, which means these loss functions can provide more stable estimates for the specified simulation scheme.

Overall, non-Bayesian methods perform better in this simulation example, particularly when $n \geq 75$. Bayesian methods exhibit more variability, perhaps due to the influence of prior specification and loss function used.

Table 3: Bias and RMSE of the Estimators under Simulation Case II

Class	Method	Estimator	$n = 25$		$n = 75$		$n = 150$		$n = 200$	
			Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
Non-Bayesian	MLE	$\hat{\alpha}$	0.08904	0.08488	0.02757	0.02163	0.01266	0.01051	0.01340	0.00770
		$\hat{\beta}$	0.25331	0.72921	0.06993	0.12676	0.03704	0.05722	0.02661	0.03802
	MPS	$\hat{\alpha}$	0.08934	0.06837	0.05057	0.02144	0.03329	0.01084	0.02344	0.00770
		$\hat{\beta}$	0.10135	0.38430	0.07234	0.10283	0.04518	0.05119	0.03866	0.03511
	LS	$\hat{\alpha}$	0.00230	0.10325	0.00678	0.02941	0.00444	0.01553	0.00181	0.01150
		$\hat{\beta}$	0.11213	1.28118	0.00950	0.15753	0.00806	0.07564	0.00534	0.05427
	WLS	$\hat{\alpha}$	0.01491	0.09121	0.00470	0.02481	0.00362	0.01275	0.00719	0.00936
		$\hat{\beta}$	0.12214	0.94073	0.02897	0.13727	0.02171	0.06451	0.01560	0.04493
	CvM	$\hat{\alpha}$	0.10058	0.13218	0.02386	0.03142	0.01075	0.01601	0.01325	0.01189
		$\hat{\beta}$	0.35641	2.23673	0.07397	0.18003	0.03948	0.08101	0.02878	0.05715
	AD	$\hat{\alpha}$	0.02689	0.07700	0.00612	0.02337	0.00266	0.01221	0.00568	0.00898
		$\hat{\beta}$	0.12907	0.60357	0.03128	0.12823	0.02024	0.06173	0.01299	0.04313
	RTAD	$\hat{\alpha}$	0.05527	0.10459	0.01567	0.02646	0.00654	0.01352	0.01128	0.01014
		$\hat{\beta}$	0.16845	0.93568	0.04176	0.13181	0.02326	0.05922	0.01931	0.04271
	PE	$\hat{\alpha}$	0.06448	0.08994	0.03975	0.02778	0.03164	0.01440	0.02071	0.01052
		$\hat{\beta}$	0.05574	0.45952	0.05200	0.11359	0.03713	0.05433	0.03012	0.03823
Bayesian	SEL	$\hat{\alpha}$	0.25087	0.12953	0.33546	0.13353	0.36992	0.14745	0.37807	0.15156
		$\hat{\beta}$	0.38597	0.55825	0.65983	0.57374	0.77392	0.67177	0.80284	0.70137
	LINEX1	$\hat{\alpha}$	0.26501	0.13868	0.34183	0.13813	0.37342	0.15013	0.38076	0.15365
		$\hat{\beta}$	0.45658	0.67398	0.69861	0.63581	0.79706	0.71103	0.82106	0.73285
	LINEX2	$\hat{\alpha}$	0.23705	0.12111	0.32916	0.12906	0.36643	0.14480	0.37539	0.14948
		$\hat{\beta}$	0.32102	0.47059	0.62247	0.51786	0.75130	0.63464	0.78494	0.67121
	GEL1	$\hat{\alpha}$	0.24301	0.12529	0.33203	0.13117	0.36805	0.14605	0.37664	0.15047
		$\hat{\beta}$	0.36406	0.53369	0.64806	0.55678	0.76702	0.66064	0.79743	0.69238
	GEL2	$\hat{\alpha}$	0.22734	0.11730	0.32517	0.12652	0.36432	0.14328	0.37379	0.14829
		$\hat{\beta}$	0.32080	0.48949	0.62460	0.52398	0.75325	0.63874	0.78663	0.67462

Table 3 gives the bias and root mean squared error (RMSE) of the estimators $\hat{\alpha}$ and $\hat{\beta}$ for various estimation procedures under Simulation Case II at varying sample sizes ($n = 25, 75, 150, 200$). For non-Bayesian estimators, those obtained through MLE, MPS, LS, WLS, CvM, AD, RTAD, and PE display decreasing bias and RMSE with an increase in the sample size, indicating consistency. Among these, PE and MLE have relatively small RMSE for large samples. For the Bayesian estimators (SEL, LINEX1, LINEX2, GEL1, GEL2), the estimators are more biased and larger RMSE than their non-Bayesian analogs and the performance degrades with growing sample size, particularly for $\hat{\beta}$. LINEX2 and GEL2 loss functions work fairly better among the Bayesian estimators.

Table 4 results show that, for Simulation Case III, the bias and RMSE of the non-Bayesian estimators decrease as the sample size increases. Among the non-Bayesian techniques, MLE and MPS have low bias and RMSE, particularly in large sample sizes. Conversely, Bayesian estimators exhibit significantly higher bias and RMSE across all sample sizes, with LINEX1 and SEL providing the highest corresponding figures for $\hat{\alpha}$ and $\hat{\beta}$. This indicates that, in the present case,

Table 4: Bias and RMSE of the Estimators under Simulation Case III

Class	Method	Estimator	$n=25$ Bias	$n=25$ RMSE	$n=75$ Bias	$n=75$ RMSE	$n=150$ Bias	$n=150$ RMSE	$n=200$ Bias	$n=200$ RMSE
Non-Bayesian	MLE	$\hat{\alpha}$	0.09611	0.09858	0.03417	0.02777	0.02488	0.01506	0.01329	0.00941
		$\hat{\beta}$	0.05367	0.08744	0.01305	0.02171	0.00358	0.01075	0.00286	0.00783
	MPS	$\hat{\alpha}$	0.11105	0.08322	0.05694	0.02724	0.02905	0.01441	0.02979	0.00975
		$\hat{\beta}$	0.01375	0.05959	0.01257	0.01882	0.01065	0.00998	0.00844	0.00739
	LS	$\hat{\alpha}$	0.00021	0.13184	0.00163	0.03799	0.00751	0.02076	0.00252	0.01282
		$\hat{\beta}$	0.02976	0.13144	0.00500	0.02553	0.00277	0.01188	0.00317	0.00912
	WLS	$\hat{\alpha}$	0.01370	0.11257	0.01091	0.03159	0.01417	0.01747	0.00454	0.01039
		$\hat{\beta}$	0.03053	0.11016	0.00721	0.02319	0.00070	0.01116	0.00044	0.00825
	CvM	$\hat{\alpha}$	0.11415	0.16925	0.03782	0.04138	0.02540	0.02185	0.01077	0.01317
		$\hat{\beta}$	0.08077	0.18964	0.01900	0.02774	0.00392	0.01229	0.00179	0.00935
	AD	$\hat{\alpha}$	0.02776	0.09094	0.01173	0.02994	0.01373	0.01684	0.00380	0.01023
		$\hat{\beta}$	0.03343	0.07923	0.00760	0.02223	0.00059	0.01096	0.00035	0.00819
	RTAD	$\hat{\alpha}$	0.04968	0.11079	0.02110	0.03484	0.01836	0.01840	0.01057	0.01211
		$\hat{\beta}$	0.03227	0.08029	0.00674	0.02218	0.00070	0.01092	0.00064	0.00822
	PE	$\hat{\alpha}$	0.08047	0.09188	0.04307	0.03279	0.01744	0.01692	0.02145	0.01170
		$\hat{\beta}$	0.00890	0.06306	0.01032	0.01921	0.00957	0.01012	0.00726	0.00750
Bayesian	SEL	$\hat{\alpha}$	0.39838	0.26410	0.44763	0.23111	0.46193	0.22823	0.46406	0.22731
		$\hat{\beta}$	0.13529	0.11488	0.14942	0.04892	0.15350	0.03575	0.15441	0.03277
	LINEX1	$\hat{\alpha}$	0.42163	0.28713	0.45718	0.24026	0.46692	0.23299	0.46786	0.23093
		$\hat{\beta}$	0.14850	0.12274	0.15510	0.05109	0.15653	0.03679	0.15672	0.03354
	LINEX2	$\hat{\alpha}$	0.37581	0.24314	0.43819	0.22225	0.45698	0.22356	0.46028	0.22374
		$\hat{\beta}$	0.12246	0.10788	0.14380	0.04685	0.15049	0.03473	0.15212	0.03200
	GEL1	$\hat{\alpha}$	0.38790	0.25521	0.44334	0.22717	0.45969	0.22615	0.46236	0.22571
		$\hat{\beta}$	0.12640	0.11132	0.14544	0.04760	0.15136	0.03506	0.15279	0.03225
	GEL2	$\hat{\alpha}$	0.36699	0.23829	0.43474	0.21943	0.45521	0.22200	0.45894	0.22253
		$\hat{\beta}$	0.10873	0.10488	0.13750	0.04505	0.14708	0.03371	0.14953	0.03122

non-Bayesian estimators provide more stable and trustworthy estimates compared to Bayesian ones, especially as the sample size increases.

The simulation in Table 5 shows that, for increasing sample sizes $n = 25, 75, 150, 200$, the bias and RMSE of the estimators $\hat{\alpha}$ and $\hat{\beta}$ have the tendency to reduce for all methods. Among non-Bayesian methods, MLE and AD estimators have relatively low bias and RMSE, especially for large n. The LS and WLS methods provide better estimates for $\hat{\alpha}$ than $\hat{\beta}$, where the latter is more variable. The CvM method has relatively high RMSE, especially for small n, indicating inefficiency.

Bayesian estimators are more biased and have greater RMSE than non-Bayesian estimators for all sample sizes. Specifically, LINEX1 and LINEX2 estimators have the highest RMSE for $\hat{\beta}$ when n is small and remain inferior even for larger n. The GEL-type estimators also perform in like manner, with modest reductions in RMSE for increasing n, yet remain behind non-Bayesian competitors. SEL method consistently has the highest estimation errors, particularly for $\hat{\beta}$.

Table 6 provides the credible intervals for the Maximum Likelihood Estimator (MLE) and the various Bayesian Estimators (BE) for four sets of parameters α and β. For each case, the non-Bayesian approach provides intervals based on the classical MLE, while the Bayesian approach includes estimators based on the squared error loss (SEL), LINEX loss functions (LINEX1 and LINEX2), and general entropy loss functions (GEL1 and GEL2). It is observed that Bayesian estimators, particularly under asymmetric loss functions (LINEX and GEL), yield interval bounds that are different, indicating the influence of the choice of loss function on interval estimation. Bayesian intervals tend to give narrower bounds for α compared to MLE, while for β, they give wider coverage, depending on the case and the loss function used.

Table 5: Bias and RMSE of the Estimators under Simulation Case IV

Class	Method	Estimator	$n=25$ Bias	$n=25$ RMSE	$n=75$ Bias	$n=75$ RMSE	$n=150$ Bias	$n=150$ RMSE	$n=200$ Bias	$n=200$ RMSE
Non-Bayesian	MLE	$\hat{\alpha}$	0.10414	0.10925	0.03859	0.02916	0.01498	0.01412	0.01178	0.01075
		$\hat{\beta}$	0.29404	0.83436	0.09213	0.16721	0.04303	0.07296	0.01925	0.05347
	MPS	$\hat{\alpha}$	0.10986	0.08976	0.05560	0.02821	0.04035	0.01470	0.03237	0.01119
		$\hat{\beta}$	0.14032	0.44268	0.08470	0.13325	0.05884	0.06565	0.06088	0.05161
	LS	$\hat{\alpha}$	0.00587	0.13750	0.00982	0.04180	0.00220	0.02027	0.00318	0.01501
		$\hat{\beta}$	0.09910	0.98251	0.04766	0.25224	0.01345	0.10284	0.00719	0.07437
	WLS	$\hat{\alpha}$	0.01366	0.12050	0.01815	0.03428	0.00462	0.01667	0.00369	0.01228
		$\hat{\beta}$	0.12384	0.84445	0.05779	0.20325	0.02481	0.08395	0.00452	0.06006
	CvM	$\hat{\alpha}$	0.10992	0.17457	0.04728	0.04616	0.01611	0.02103	0.01050	0.01540
		$\hat{\beta}$	0.38744	1.67314	0.12949	0.29669	0.05218	0.11100	0.02129	0.07778
	AD	$\hat{\alpha}$	0.03222	0.10421	0.01719	0.03139	0.00428	0.01620	0.00266	0.01199
		$\hat{\beta}$	0.14972	0.72080	0.05394	0.18091	0.02423	0.08188	0.00291	0.05938
	RTAD	$\hat{\alpha}$	0.06391	0.12827	0.02759	0.03555	0.00952	0.01746	0.00725	0.01290
		$\hat{\beta}$	0.19066	0.77922	0.06587	0.18050	0.02993	0.07982	0.00767	0.05704
	PE	$\hat{\alpha}$	0.07171	0.09929	0.03714	0.03157	0.03091	0.01664	0.02509	0.01241
		$\hat{\beta}$	0.08003	0.48833	0.05648	0.14011	0.04347	0.06818	0.04903	0.05233
Bayesian	SEL	$\hat{\alpha}$	0.23994	0.14394	0.37081	0.16644	0.42601	0.19629	0.43983	0.20553
		$\hat{\beta}$	0.37031	0.62576	0.75368	0.74846	0.92614	0.95706	0.97209	1.02372
	LINEX1	$\hat{\alpha}$	0.25873	0.15601	0.37953	0.17344	0.43091	0.20062	0.44362	0.20896
		$\hat{\beta}$	0.46031	0.77932	0.80627	0.84553	0.95885	1.02375	0.99811	1.07834
	LINEX2	$\hat{\alpha}$	0.22164	0.13309	0.36218	0.15968	0.42114	0.19203	0.43606	0.20214
		$\hat{\beta}$	0.28872	0.51614	0.70348	0.66300	0.89434	0.89475	0.94667	0.97194
	GEL1	$\hat{\alpha}$	0.23098	0.13929	0.36684	0.16341	0.42382	0.19440	0.43815	0.20403
		$\hat{\beta}$	0.34506	0.59744	0.73957	0.72519	0.91761	0.94057	0.96536	1.01016
	GEL2	$\hat{\alpha}$	0.21310	0.13061	0.35888	0.15746	0.41944	0.19065	0.43478	0.20105
		$\hat{\beta}$	0.29530	0.54736	0.71146	0.68031	0.90057	0.90814	0.95191	0.98338

Table 6: Credible Interval for MLE and BE

Case	Estimator	Non-Bayesian ML Lower	Non-Bayesian ML Upper	Bayesian SEL Lower	SEL Upper	LINEX1 Lower	LINEX1 Upper	LINEX2 Lower	LINEX2 Upper	GEL1 Lower	GEL1 Upper	GEL2 Lower	GEL2 Upper
I	$\hat{\alpha}$	1.32797	2.30881	0.98084	1.56334	2.13219	0.56885	1.65330	2.06256	0.40926	1.71108	2.06240	0.35132
	$\hat{\beta}$	1.77568	4.15102	2.37535	2.47911	3.90280	1.42370	2.79658	3.83129	1.03471	2.83984	3.76856	0.92872
II	$\hat{\alpha}$	1.44835	2.55101	1.10267	1.80002	2.44521	0.64519	1.91724	2.38840	0.47116	1.99043	2.39651	0.40609
	$\hat{\beta}$	1.90865	4.50718	2.59853	2.70314	4.29905	1.59591	3.15478	4.34976	1.19498	3.16094	4.21943	1.05849
III	$\hat{\alpha}$	1.59382	2.80101	1.20719	1.82354	2.52653	0.70300	1.98395	2.46296	0.47902	2.01650	2.43137	0.41487
	$\hat{\beta}$	0.84586	2.01186	1.16600	1.08317	1.70779	0.62462	1.19975	1.62624	0.42649	1.21834	1.58176	0.36342
IV	$\hat{\alpha}$	1.55551	2.68555	1.13004	1.86351	2.51974	0.65623	1.96902	2.45166	0.48263	2.03940	2.45609	0.41669
	$\hat{\beta}$	1.94348	4.56812	2.62463	2.77710	4.41421	1.63712	3.22145	4.45179	1.23035	3.26094	4.34059	1.07965

7 Applications

The first data set represents the failure times of the 84 Aircraft Windshield contained in [69] and presented in Table (7) below.

The second data is the COVID-19 death rate for Angola from 14/06/2020 to 20/2/2022. The data is reported in https://data.worldbank.org/indicator/SH.DYN.MORT and presented in Table (8).

The third dataset is the breaking stress of carbon fibers studied by [71] and reported in Table (9).

The last data set contains 30 observations of the March precipitation pattern (in inches) in Minneapolis/St Paul studied by [72] and presented in Table (10).

The summary statistics presented in Table 11 offer valuable insights into the characteristics of four distinct real-world datasets, each exhibiting varying degrees of asymmetry, which is crucial for the proposed Burr III scaled inverse odds ratio-Rayleigh distribution. For the Aircraft Windshield dataset, with a sample size (n) of 85, the interquartile range (IQR)

Table 7: Aircraft Windshield

0.040	1.866	2.385	3.443	0.301	1.876	2.481	3.467	0.309	1.899	2.610	3.478	0.557
1.911	2.625	3.578	0.943	1.912	2.632	3.595	1.070	1.914	2.646	3.699	1.124	1.981
2.661	3.779	1.248	2.010	2.688	3.924	1.281	2.038	2.82	3.00	4.035	1.281	2.085
2.890	4.121	1.303	2.089	2.902	4.167	1.432	2.097	2.934	4.240	1.480	2.135	2.962
4.255	1.505	2.154	2.964	4.278	1.506	2.190	3.000	4.305	1.568	2.194	3.103	4.376
1.615	2.223	3.114	4.449	1.619	2.224	3.117	4.485	1.652	2.229	3.166	4.570	1.652
2.300	3.344	4.602	1.757	2.324	3.376	4.663						

Table 8: COVID-19 Mortality Rate

0.0400000	0.0588235	0.0229885	0.1034483	0.0437956	0.0196078	0.0436681	0.0604839
0.0392157	0.0521173	0.0313725	0.0311751	0.0260417	0.0375000	0.0265018	0.0285344
0.0272206	0.0376712	0.0189145	0.0166240	0.0095559	0.0104575	0.0182868	0.0134745
0.0133531	0.0200445	0.0192000	0.0430108	0.0267686	0.0152505	0.0164234	0.0246305
0.0431894	0.0168675	0.0392857	0.0599251	0.0411765	0.0282686	0.0219780	0.0298507
0.0160858	0.0208955	0.0127737	0.0132979	0.0113519	0.0134228	0.0173847	0.0180505
0.0133191	0.0334262	0.0206795	0.0261669	0.0308151	0.0308765	0.0335498	0.0357143
0.0267983	0.0302663	0.0272109	0.0278578	0.0404908	0.0446334	0.0420561	0.0412044
0.0472779	0.0368393	0.0311383	0.0397910	0.0228466	0.0166540	0.0284974	0.0334686
0.0217028	0.0392857	0.0326531	0.0267857	0.0234375	0.0190476	0.0130719	0.0021231
0.0015169	0.0023099	0.0058021	0.0101074	0.0121951	0.0037123	0.0068027	0.0122699
0.0097087							

Table 9: Carbon Fibre Stress

3.7	2.74	2.73	2.5	3.6	3.11	3.27	2.87	1.47	3.11	4.42	2.41	3.19	3.22	1.69
3.28	3.09	1.87	3.15	4.9	3.75	2.43	2.95	2.97	3.39	2.96	2.53	2.67	2.93	3.22
3.39	2.81	4.2	3.33	2.55	3.31	3.31	2.85	2.56	3.56	3.15	2.35	2.55	2.59	2.38
2.81	2.77	2.17	2.83	1.92	1.41	3.68	2.97	1.36	0.98	2.76	4.91	3.68	1.84	1.59
3.19	1.57	0.81	5.56	1.73	1.59	2.0	1.22	1.12	1.71	2.17	1.17	5.08	2.48	1.18
3.51	2.17	1.69	1.25	4.38	1.84	0.39	3.68	2.48	0.85	1.61	2.79	4.7	2.03	1.8
1.57	1.08	2.03	1.61	2.12	1.89	2.88	2.82	2.05	3.65					

Table 10: Precipitation Pattern

0.77	1.74	0.81	1.20	1.95	1.2	0.47	1.43	3.37	2.2	3	3.09	1.51	2.1	0.52
1.62	1.31	0.32	0.59	0.81	2.81	1.87	1.18	1.35	4.75	2.48	0.96	1.89	0.9	2.05

is 1.510, indicating the spread of the central 50% of the data. This dataset shows a mean of 2.385 and a standard deviation (SD) of 1.113. Its skewness (Sk) value of 0.08654 suggests it is nearly symmetric, exhibiting only a slight positive skew. The kurtosis of 2.36543 indicates a platykurtic distribution, meaning its tails are lighter and its peak is flatter than a normal distribution. In contrast, the COVID-19 Mortality Rate dataset, comprising 89 observations, presents a distinct asymmetrical pattern. It has a significantly lower mean of 0.02701 and a standard deviation of 0.01567, reflecting a tighter spread for this particular metric. A notable outlier exists at 0.10345. Crucially, its skewness of 1.44654 demonstrates strong positive skewness, indicating a longer tail to the right, while a high kurtosis of 7.92605 suggests a leptokurtic distribution with a sharper peak and heavier tails compared to a normal distribution, making it a prime candidate for models designed for asymmetric and heavy-tailed data. The Carbon Fibre Stress dataset, the largest with $n = 100$, records a mean of 2.6214 and a standard deviation of 1.01389. This dataset includes an outlier at 5.560. Its skewness of 0.36815 indicates a moderate positive skew, suggesting some asymmetry. The kurtosis value of 3.10494 implies it is slightly

Table 11: Summary Statistics

Statistics	Aircraft Windshield	COVID-19 Mortality Rate	Carbon Fibre Stress	Precipitation Pattern
n	85	89	100	30
Q_1	1.86600	0.01642	1.84000	0.91500
Q_3	3.37600	0.03684	3.22000	2.08750
IQR	1.51000	0.02042	1.38000	1.17250
Outlier	-	0.10345	5.56000	4.75000
Mean	2.38500	0.02701	2.62140	1.67500
Var	1.23915	0.00025	1.02796	1.00123
SD	1.11317	0.01567	1.01389	1.00062
Range	4.62300	0.10193	5.17000	4.43000
Sk	0.08654	1.44654	0.36815	1.08668
Kurtosis	2.36543	7.92605	3.10494	4.20688

leptokurtic, possessing slightly heavier tails than a normal distribution. Finally, the Precipitation Pattern dataset, with $n = 30$, has a mean of 1.675 and a standard deviation of 1.00062. This dataset also contains an outlier at 4.750. Similar to the COVID-19 mortality data, it exhibits strong positive skewness, with an Sk of 1.08668, confirming its asymmetric nature. The kurtosis of 4.20688 further supports a leptokurtic shape, indicating a more pronounced peak and fatter tails. These varying degrees of skewness and kurtosis across the datasets underscore the need for flexible distributions capable of accurately capturing such diverse real-world data characteristics.

It is most desirable to compare new distributions to some standard distributions. This, ideally, justifies the creation of the new distribution when the new outperforms the existing. Based on the above assumptions, we compare the proposed KM-DUS-W distribution with the parent distribution, namely the Weibull distribution proposed by [1], and Gumbel distribution by [73], log-normal (Lnorm) distribution by [75], new generalized logistic-x transformed exponential (NGLXT-E) distribution by [74], and Burr Type XII (BurrXII) distribution by [70].

Table 12: Estimation Accuracy, Goodness of Fit and MLEs

Data	Distribution	LL	AIC	CAIC	BIC	HQIC	W	A	KS	P-value	$\text{MLE}_{\text{Shape}}$	$\text{MLE}_{\text{Scale}}$
Aircraft Windshield	KMDUSW	-131.280	266.555	266.701	271.440	268.520	0.056	0.571	0.051	0.9788	2.3310(0.2065)	0.0856(0.0230)
	Weibull	-131.290	266.577	266.723	271.462	268.542	0.058	0.586	0.053	0.9685	2.8681(0.1358)	2.3937(0.2102)
	Gumbel	-133.150	270.298	270.444	275.183	272.263	0.667	0.661	0.076	0.7114	2.0155(0.1198)	1.0428(0.0833)
	Lnorm	-155.660	315.318	315.465	320.204	217.283	0.577	3.866	0.155	0.0329	0.7926(0.0742)	0.6836(0.0524)
	NGLXT-E	-129.070	262.139	262.285	267.024	264.104	0.065	0.536	0.066	0.8485	0.5561(0.0505)	0.2364(0.0106)
	BurrXII	-171.680	347.361	347.507	352.246	349.326	0.917	5.696	0.276	4.842×10^{-06}	3.1792(0.4489)	0.3503(0.0590)
COVID-19 Mortality Rate	KMDUSW	252.290	-500.577	-500.438	-495.600	-498.571	0.048	0.439	0.066	0.8322	1.7574(0.1411)	4.4388(2.8446)
	Weibull	251.940	-499.877	-499.737	-494.899	-497.870	0.053	0.477	0.068	0.8049	0.0303(0.0019)	1.7916(0.1422)
	Gumbel	252.620	-501.241	-501.101	-496.264	-499.235	0.047	0.344	0.070	0.7739	0.0201(0.0014)	0.0122(0.0010)
	Lnorm	242.420	-480.832	-480.692	-475.855	-478.826	0.318	2.186	0.117	0.1727	-3.8097(0.0760)	0.7167(0.0537)
	NGLXT-E	246.600	-489.194	-489.055	-484.217	-487.188	0.106	0.859	0.107	0.2560	0.8179(0.0670)	21.2002(1.3180)
	BurrXII	251.960	-499.928	-499.788	-494.950	-497.922	0.053	0.476	0.068	0.8071	1.7935(0.1413)	529.3311(249.1726)
Carbon Fibre Stress	KMDUSW	-141.360	286.725	286.848	291.935	288.833	0.060	0.394	0.058	0.8901	2.7360(0.2121)	0.0521(0.0144)
	Weibull	-141.530	287.059	287.182	292.269	289.167	0.062	0.416	0.061	0.8574	2.9438(0.1111)	2.7928(0.2141)
	Gumbel	-144.210	292.416	292.540	297.626	294.525	0.184	0.939	0.096	0.3219	2.1328(0.0962)	0.9091(0.0681)
	Lnorm	-148.420	300.840	300.963	306.050	302.948	0.277	1.483	0.118	0.1254	0.8774(0.0444)	0.4439(0.0314)
	NGLXT-E	-143.210	290.418	290.542	295.629	292.527	0.075	0.607	0.076	0.6126	0.4956(0.0390)	0.2303(0.0085)
	BurrXII	-189.480	382.967	383.091	388.177	385.076	0.873	4.902	0.272	7.286×10^{-07}	5.9391(1.2795)	0.1874(0.0438)
Precipitation Pattern	KMDUSW	-38.620	81.232	81.677	84.035	82.129	0.021	0.163	0.064	0.9997	1.7676(0.2465)	0.3253(0.0908)
	Weibull	-38.640	81.287	81.731	84.089	82.183	0.022	0.169	0.069	0.9988	1.8923(0.2020)	1.8090(0.2491)
	Gumbel	-38.690	81.384	81.828	84.186	82.281	0.017	0.131	0.067	0.9993	1.2353(0.1408)	0.7337(0.1079)
	Lnorm	-38.480	80.951	81.395	83.753	81.848	0.030	0.198	0.091	0.9641	0.3374(0.1137)	0.6227(0.0804)
	NGLXT-E	-40.050	84.091	84.541	86.899	84.993	0.051	0.368	0.111	0.8511	0.7770(0.1129)	0.3431(0.0353)
	BurrXII	-40.260	84.516	84.961	87.318	85.413	0.070	0.430	0.138	0.6182	3.2558(0.6456)	0.5770(0.1372)

Table (12) presents a comparative study of estimation accuracy, goodness-of-fit statistics, and maximum likelihood estimates (MLEs) for six distributions under comparison—KMDUSW, Weibull, Gumbel, Lognormal (Lnorm), NGLXT-E, and BurrXII—on four data sets: Aircraft Windshield, COVID-19 Mortality Rate, Carbon Fibre Stress, and Precipitation Pattern. For every dataset, the log-likelihood (LL), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC) values are given, and then the Cramér–von Mises (W), Anderson–Darling (A), and Kolmogorov–Smirnov (KS) statistics along with their respective p-values. The MLEs of shape and scale parameters with their respective standard errors in parentheses are given in the last two columns. Statistically, in most data sets, the KMDUSW distribution fares better as reflected by greater (or lesser) LL values and smaller AIC, BIC, CAIC, and HQIC values compared to all other models, which suggests that the distribution achieves a superior balance of goodness of fit and model simplicity. For example, from the Aircraft Windshield data, KMDUSW has the lowest AIC (266.555) and highest p-value (0.9788) for the KS test, which indicates high accordance with the empirical data. The same is the case with the Carbon Fibre Stress and Precipitation Pattern datasets, wherein KMDUSW also achieves good GOF indices and KS p-values of more than 0.89 and 0.99, respectively. BurrXII does not fare well on any of the datasets uniformly, as reflected in its high AIC and BIC values and extremely low p-values, often below 0.001, showing lack of fit. On the other hand, though Gumbel and Lognormal occasionally give moderate fit (as in a few KS statistics), their model selection measures overall are worse than for KMDUSW. NGLXT-E is fine in certain cases but is generally outperformed by KMDUSW in estimation accuracy along with quality of fit. MLEs under various distributions vary in scale, with the BurrXII distribution yielding exceedingly large scale parameters and standard errors in certain cases (e.g., COVID-19 Mortality Rate), which indicates instability or overdispersion. The KMDUSW model, by contrast, has sensibly behaved and relatively small standard errors parameter estimates, further indicating its stability and suitability for modeling the given datasets.

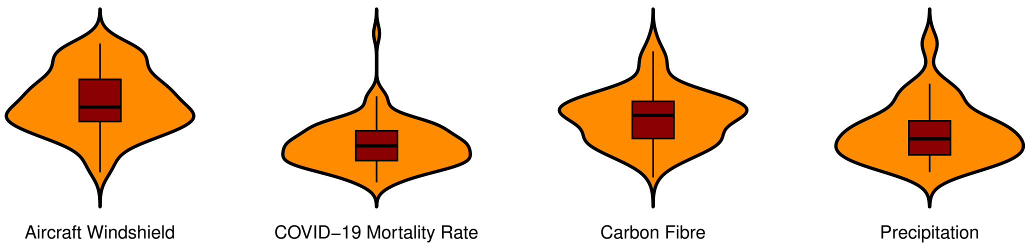


Fig. 4: Boxplots superimposed on Violin plots

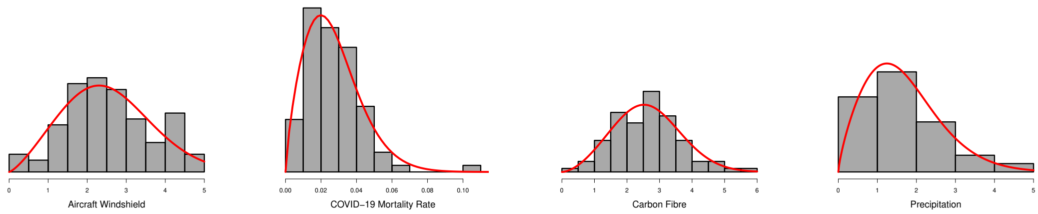


Fig. 5: Density plots superimposed on Histogram

Figure (4) is the boxplots superimposed on the violin plots indicating outliers in the data sets except the Aircraft Windshield data. Figures (5), (6), (7), (9) and (10) represent the density plots superimposed on the histograms, empirical CDFs superimposed on the KM-DUS-W CDFs, empirical survival functions superimposed on the KM-DUS-W survival functions, PP and QQ plots. These plots depict the extent of fit of the proposed KM-DUS-W distributions to the data sets. Figure (8), which is the Total Time on Test (TTT) Plot is a non-parametric plot that is crucial in modeling of survival and reliability events. For the TTT plots, the observed failure times is ordered as:

$$t_{(1)} < t_{(2)} < \ldots < t_{(n)}$$

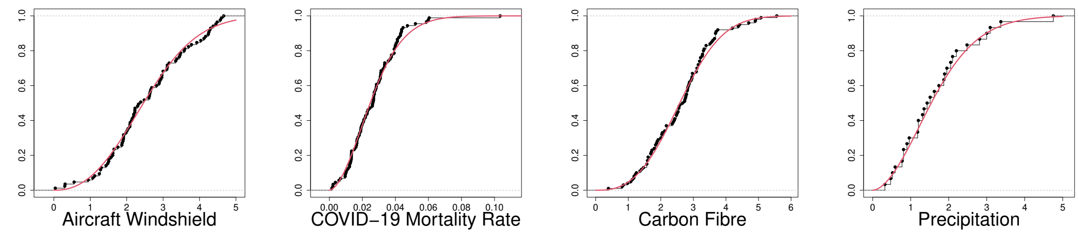


Fig. 6: Empirical CDF with superimposed KM-DUS-W CDF

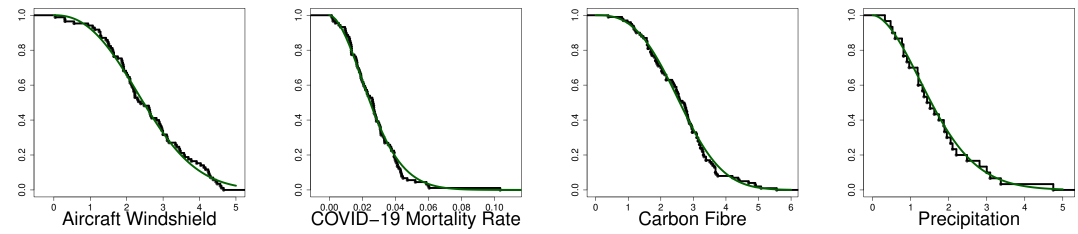


Fig. 7: Empirical S(x) with superimposed KM-DUS-W Survival Function

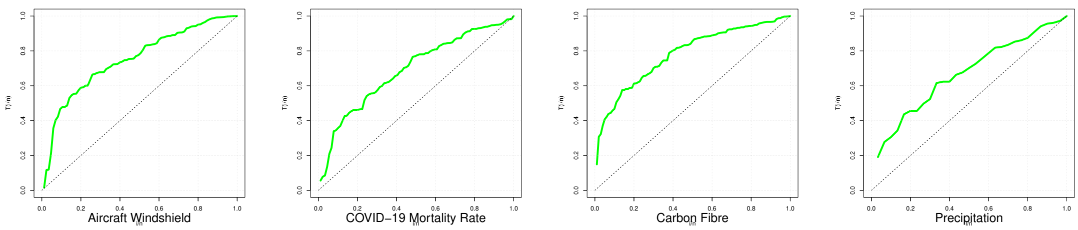


Fig. 8: TTT plots

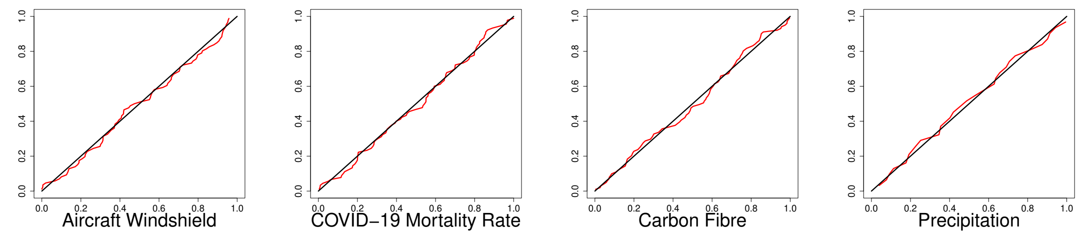


Fig. 9: PP plots

The scaled TTT plot is constructed by plotting the points:

$$\left(\frac{i}{n}, \frac{\sum_{j=1}^{i} t_{(j)} + (n-i)t_{(i)}}{\sum_{j=1}^{n} t_{(j)}} \right), \quad \text{for } i = 1, 2, \ldots, n$$

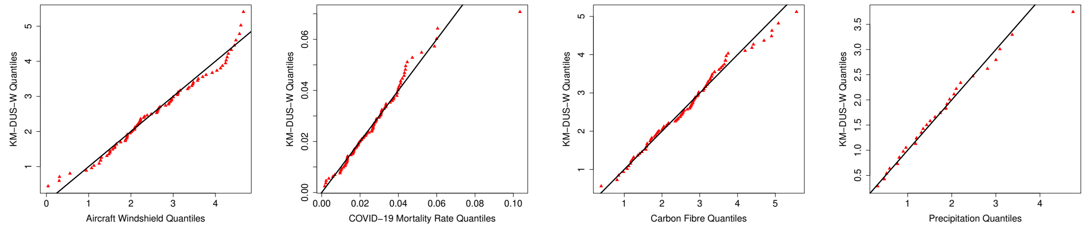


Fig. 10: QQ plots

where $\frac{i}{n}$ represents the proportion of failures, and the vertical axis represents the proportion of the total test time. The TTT plots for the data sets indicate increasing hazard rates.

8 Conclusion

In this study, a new family of lifetime distributions called the Kavya-Manoharan Dinesh-Umesh-Sanjay (KM-DUS) family was constructed by compounding the KM and DUS distribution generators without introducing any additional parameters, thus ensuring model parsimony. A special sub-model of the family, the KM-DUS-Weibull (KMDUSW) distribution, was derived and its flexibility was demonstrated with analytical forms of its CDF, PDF, and hazard rate function. The applicability of the new distribution was examined using four real datasets—Aircraft Windshield, COVID-19 Mortality Rate, Carbon Fibre Stress, and Precipitation Pattern. Descriptive statistics revealed that the datasets manifested heterogeneous features, including various levels of skewness, kurtosis, and outliers. Comparative goodness-of-fit analyses based on log-likelihood values and a set of information criteria (AIC, BIC, CAIC, HQIC), as well as classical GOF statistics (Cramér–von Mises, Anderson–Darling, and Kolmogorov–Smirnov tests), indicated that the KMDUSW distribution performed better than standard models such as Weibull, Gumbel, Lognormal, BurrXII, and NGLXT-E for most of the datasets. Notably, KMDUSW achieved the lowest AIC values and highest KS p-values in datasets like Aircraft Windshield, Carbon Fibre Stress, and Precipitation Pattern, reflecting close conformity to empirical data. In addition, the KMDUSW model demonstrated estimation stability through reasonable parameter sizes and tiny standard errors in MLEs, unlike the BurrXII distribution that tended to give inflated estimates. Overall, the KM-DUS family—and specifically the KMDUSW model—stands out as a robust and flexible addition to statistical modeling of actual data, with increased fit and interpretability without additional model complexity.

References

[1] Weibull, Waloddi. ”A statistical distribution function of wide applicability.” *J. Appl. Mech.*, 1951.
[2] Kavya, P and Manoharan, M. ”Some parsimonious models for lifetimes and applications.” *Journal of Statistical Computation and Simulation*, 91(18):3693–3708, 2021.
[3] Brooks, Stephen. ”Markov chain Monte Carlo method and its application.” *Journal of the royal statistical society: series D (the Statistician)*, 47(1):69–100, 1998.
[4] Van Ravenzwaaij, Don and Cassey, Pete and Brown, Scott D. ”A simple introduction to Markov Chain Monte–Carlo sampling.” *Psychonomic bulletin & review*, 25(1):143–154, 2018.
[5] Cheng, R and Amin, N. ”Maximum product of spacings estimation with application to the lognormal distribution (Mathematical Report 79-1).” *Cardiff: University of Wales IST*, 1979.
[6] Chipepa, Fastel and Oluyede, B and Makubate, Boikanyo and others. ”A new generalized family of odd Lindley-G distributions with application.” *International Journal of Statistics and Probability*, 8(6):1–22, 2019.
[7] Eugene, Nicholas and Lee, Carl and Famoye, Felix. ”Beta-normal distribution and its applications.” *Communications in Statistics-Theory and methods*, 31(4):497–512, 2002.
[8] Jones, MC. ”Families of distributions arising from distributions of order statistics.” *Test*, 13:1–43, 2004.
[9] Ramos, Manoel Wallace A and Marinho, Pedro Rafael D and Cordeiro, Gauss M and da Silva, Ronaldo V and Hamedani, Gholamhossein. ”The Kumaraswamy-G Poisson family of distributions.” *Journal of Statistical Theory and Applications*, 2015.
[10] Chesneau, Christophe and Jamal, Farrukh. ”The sine Kumaraswamy-G family of distributions.” *Journal of Mathematical Extension*, 15, 2020.
[11] Arshad, Rana Muhammad Imran and Tahir, Muhammad Hussain and Chesneau, Christophe and Jamal, Farrukh. ”The Gamma Kumaraswamy-G family of distributions: theory, inference and applications.” *Statistics in Transition new series*, 21(5):17–40, 2020.

[12] Ahmed, Mohamed Ali and Mahmoud, Mahmoud Riad and ElSherbini, Elsayed Ahmed. "The new Kumaraswamy Kumaraswamy family of generalized distributions with application." *Pakistan Journal of Statistics and Operation Research*, pages 159–180, 2015.
[13] Chakraborty, Subrata and Handique, Laba and Jamal, Farrukh. "The Kumaraswamy Poisson-G family of distribution: its properties and applications." *Annals of Data Science*, 9(2):229–247, 2022.
[14] Jamal, Farrukh and Arslan Nasir, Muhammad and Ozel, Gamze and Elgarhy, M and Mamode Khan, Naushad. "Generalized inverted Kumaraswamy generated family of distributions: theory and applications." *Journal of Applied Statistics*, 46(16):2927–2944, 2019.
[15] Tahir, MH and Cordeiro, Gauss M and Alzaatreh, Ayman and Mansoor, M and Zubair, M. "The logistic-X family of distributions and its applications." *Communications in statistics-Theory and methods*, 45(24):7326–7349, 2016.
[16] Elbatal, Ibrahim and Alotaibi, Naif and Almetwally, Ehab M and Alyami, Salem A and Elgarhy, Mohammed. "On odd perks-G class of distributions: properties, regression model, discretization, Bayesian and non-Bayesian estimation, and applications." *Symmetry*, 14(5):883, 2022.
[17] Cordeiro, Gauss M and De Castro, Mário. "A new family of generalized distributions." *Journal of statistical computation and simulation*, 81(7):883–898, 2011.
[18] Anzagra, Lea and Sarpong, Solomon and Nasiru, Suleman. "Chen-G class of distributions." *Cogent Mathematics & Statistics*, 7(1):1721401, 2020.
[19] Rezaei, Sadegh and Sadr, Behnam Bahrami and Alizadeh, Morad and Nadarajah, Saralees. "Topp–Leone generated family of distributions: Properties and applications." *Communications in Statistics-Theory and Methods*, 46(6):2893–2909, 2017.
[20] Kumar, Dinesh and Singh, Umesh and Singh, Sanjay Kumar. "A method of proposing new distribution and its application to Bladder cancer patients data." *J. Stat. Appl. Pro. Lett*, 2(3):235–245, 2015.
[21] Cordeiro, Gauss M and Alizadeh, Morad and Diniz Marinho, Pedro Rafael. "The type I half-logistic family of distributions." *Journal of Statistical Computation and Simulation*, 86(4):707–728, 2016.
[22] Soliman, Amal Hassan and Elgarhy, Mohammed Abd Ellattif and Shakil, Mohammed. "Type II half logistic family of distributions with applications." *Pakistan Journal of Statistics and Operation Research*, pages 245–264, 2017.
[23] Alyami, S. A., Elbatal, I., Alotaibi, N., Almetwally, E. M., and Elgarhy, M. "Modeling to factor productivity of the United Kingdom food chain: Using a new lifetime-generated family of distributions." *Sustainability*, 14:8942, 2022.
[24] Almarashi, M., Jamal, F., Chesneau, C., and Elgarhy, M. "A new truncated Muth generated family of distributions with applications." *Complexity*, vol. 2021, Article ID 1211526, 14 pages.
[25] Al-Moisheer, A. S., Elbatal, I., Almutiry, W., and Elgarhy, M. "Odd inverse power generalized Weibull generated family of distributions: Properties and applications." *Mathematical Problems in Engineering*, vol. 2021, Article ID 5082192.
[26] Alotaibi, N., Elbatal, I., Shrahili, M., Al-Moisheer, A. S., Elgarhy, M., and Almetwally, E. M. "Statistical inference for the Kavya-Manoharan Kumaraswamy model under ranked set sampling with applications." *Symmetry*, 15:587, 2023.
[27] Alghamdi, S. M., Shrahili, M., Hassan, A. S., Elbatal, I., Amin, E. A., and Elgarhy, M. "Analysis of milk production and failure data: Using unit exponentiated half logistic power series class of distributions." *Symmetry*, 15:714, 2023.
[28] Elbatal, I., Diab, L. S., Ben Ghorbal, A., Yousof, H. M., Elgarhy, M., and Ali, E. I. A. "A new losses (revenues) probability model with entropy analysis, applications and case studies for value-at-risk modeling and mean of order-p analysis." *AIMS Mathematics*, 9(3):7169–7211, 2024.
[29] Oluyede, Broderick and Chipepa, Fastel and Wanduku, Divine. "Exponentiated Half Logistic-Power Generalized Weibull-G Family of Distributions: Model, Properties and Applications." *Eurasian Bulletin of Mathematics*, 3(3):134–161, 2021.
[30] Bello, Olalekan Akanji and Doguwa, Sani Ibrahim and Yahaya, Abubakar and Jibril, Haruna Mohammed. "A Type I Half Logistic Exponentiated-G Family of Distributions: Properties and Application." *Communication in Physical Sciences*, 7(3), 2021.
[31] Musekwa, Regent Retrospect and Gabaitiri, Lesego and Makubate, Boikanyo. "A new technique to generate families of continuous distributions." *Revista Colombiana de Estadística*, 47(2):329–354, 2024.
[32] Shaw, William T and Buckley, Ian RC. "The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map." *arXiv preprint arXiv:0901.0434*, 2009.
[33] Loving, Luigi. "Research around the income curve." *Annals of Pure and Applied Mathematics*, 2:123–159, 1925.
[34] Gomes-Silva, Frank S and Percontini, Ana and de Brito, Edleide and Ramos, Manoel W and Venâncio, Ronaldo and Cordeiro, Gauss Moutinho. "The odd Lindley-G family of distributions." *Austrian journal of statistics*, 46(1):65–87, 2017.
[35] Hassan, Amal S and Nassr, Said G. "Power Lindley-G family of distributions." *Annals of Data Science*, 6:189–210, 2019.
[36] Yousof, Haitham M and Altun, Emrah and Ramires, Thiago G and Alizadeh, Morad and Rasekhi, Mahdi. "A new family of distributions with properties, regression models and applications." *Journal of Statistics and Management Systems*, 21(1):163–188, 2018.
[37] Cordeiro, Gauss Moutinho and Alizadeh, Morad and Ozel, Gamze and Hosseini, Bistoon and Ortega, Edwin Moises Marcos and Altun, Emrah. "The generalized odd log-logistic family of distributions: properties, regression models and applications." *Journal of statistical computation and simulation*, 87(5):908–932, 2017.
[38] Yousof, HM and Rasekhi, Mahdi and Altun, Emrah and Alizadeh, Morad. "The extended odd Fréchet family of distributions: properties, applications and regression modeling." *International Journal of Applied Mathematics and Statistics*, 30(1):1–30, 2018.
[39] Alizadeh, Morad and Lak, Fazlollah and Rasekhi, Mahdi and Ramires, Thiago G and Yousof, Haitham M and Altun, Emrah. "The odd log-logistic Topp–Leone G family of distributions: heteroscedastic regression models and applications." *Computational Statistics*, 33:1217–1244, 2018.
[40] Zhao, Name. "The Type-I Heavy-Tailed Exponential (TI-HTE) Distribution." *Journal Name*, xx(yy):zz–ww, 2020.

[41] Tahir, MH and Hussain, M Adnan and Cordeiro, Gauss M. ”A new flexible generalized family for constructing many families of distributions.” *Journal of Applied Statistics*, 49(7):1615–1635, 2022.
[42] Odhah, Omalsad Hamood and Alshanbari, Huda M and Ahmad, Zubair and Rao, Gadde Srinivasa. ”A weighted cosine-G family of distributions: properties and illustration using time-to-event data.” *Axioms*, 12(9):849, 2023.
[43] Muhammad, Mustapha and Bantan, Rashad AR and Liu, Lixia and Chesneau, Christophe and Tahir, Muhammad H and Jamal, Farrukh and Elgarhy, Mohammed. ”A new extended cosine—G distributions for lifetime studies.” *Mathematics*, 9(21):2758, 2021.
[44] M. A., M., Al-Kandari, N. M., and Mohammad, R. Z. ”Fundamental properties of the characteristic function using the compound Poisson distribution as the sum of the gamma model.” *Modern Journal of Statistics*, 1(1):49–57, 2025.
[45] Hammad, A. T., Hafez, E. H., Shahzad, U., Yıldırım, E., Almetwally, E. M., and Kibria, B. M. G. ”New modified Liu estimators to handle the multicollinearity in the beta regression model: Simulation and applications.” *Modern Journal of Statistics*, 1(1):58–79, 2025.
[46] Gupta, Ramesh C and Kundu, Debasis. ”Modeling failure time data by the generalized exponential distribution.” *Communications in Statistics-Theory and Methods*, 28(4):899–914, 1999.
[47] Elbatal, Ibrahim and Aryal, Ganesh R. ”Exponentiated modified Weibull extension distribution.” *International Journal of Statistics and Probability*, 2(3):1–16, 2013.
[48] Noori, N. A., Khaleel, M. A., Khalaf, S. A., and Dutta, S. ”Analytical modeling of expansion for odd Lomax generalized exponential distribution in framework of neutrosophic logic: A theoretical and applied on neutrosophic data.” *Innovation in Statistics and Probability*, 1(1):47–59, 2025.
[49] Tahir, Muhammad Hanif and Cordeiro, Gauss M. ”The Marshall-Olkin extended Weibull distribution.” *Communications in Statistics-Theory and Methods*, 44(18):3971–3987, 2015.
[50] Almalki, Saleh J and Yuan, John. ”Modifications of Weibull and exponential distributions: A review.” *Reliability Engineering & System Safety*, 124:32–55, 2014.
[51] Taha, K. M. and Bakouch, H. S. ”A novel generalized class of distributions: Properties and applications.” *Austrian Journal of Statistics*, 50(2):1–20, 2021.
[52] Bantan, R. A. R., Jamal, F., Chesneau, C., and Elgarhy, M. ”Theory and Applications of the Unit Gamma/Gompertz Distribution.” *Mathematics*, 9:1850, 2021.
[53] Alyami, S. A., Elbatal, I., Alotaibi, N., Almetwally, E. M., Okasha, H. M., and Elgarhy, M. ”Topp–Leone Modified Weibull Model: Theory and Applications to Medical and Engineering Data.” *Applied Sciences*, 12:10431, 2022.
[54] Abouelmagd, T. H. M., SaeedAl-mualim, Elgarhy, M., Afify, Z. A., and Ahmad, M. ”Properties of the Four-Parameter Weibull Distribution and its Applications.” *Pakistan Journal of Statistics*, 33(6):449–466, 2017.
[55] Al-Marzouki, S., Jamal, F., Chesneau, Ch., and Elgarhy, M. ”Type II Topp Leone Power Lomax Distribution with Applications.” *Mathematics*, 8(1):1–26, 2020.
[56] Alkarni, S. H., Afify, A. Z., Elbatal, I., and Elgarhy, M. ”The Extended Inverse Weibull Distribution: Properties and Applications.” *Complexity*, Article ID 3297693, 11 pages, 2020.
[57] Almarashi, A. M., Badr, M. M., Elgarhy, M., Jamal, F., and Chesneau, Ch. ”Statistical Inference of the Half-Logistic Inverse Rayleigh Distribution.” *Entropy*, 22:1–23, 2020.
[58] Husain, Q. N., Qaddoori, A. S., Noori, N. A., Abdullah, K. N., Suleiman, A. A., and Balogun, O. S. ”New expansion of Chen distribution according to the neutrosophic logic using the Gompertz family.” *Innovation in Statistics and Probability*, 1(1):60–75, 2025.
[59] El Gazar, A. M., Ramadan, D. A., ElGarhy, M., and El-Desouky, B. S. ”Estimation of parameters for inverse power Ailamujia and truncated inverse power Ailamujia distributions based on progressive type-II censoring scheme.” *Innovation in Statistics and Probability*, 1(1):76–87, 2025.
[60] Orji, G. O., Etaga, H. O., Almetwally, E. M., Igbokwe, C. P., Aguwa, O. C., and Obulezi, O. J. ”A new odd reparameterized exponential transformed-X family of distributions with applications to public health data.” *Innovation in Statistics and Probability*, 1(1):88–118, 2025.
[61] Al-Shomrani, Ali and Arif, Osama and Shawky, A and Hanif, Saman and Shahbaz, Muhammad Qaiser. ”Toppâ€“Leone Family of Distributions: Some Properties and Application.” *Pakistan Journal of Statistics and Operation Research*, pages 443–451, 2016.
[62] Rudin, Walter. *Principles of Mathematical Analysis*. 1976.
[63] Onyekwere, C. K., Aguwa, O. C., and Obulezi, O. J. ”An updated Lindley distribution: Properties, estimation, acceptance sampling, actuarial risk assessment and applications.” *Innovation in Statistics and Probability*, 1(1):1–27, 2025.
[64] Saboor, A., Jamal, F., Shafq, S., and Mumtaz, R. ”On the versatility of the unit logistic exponential distribution: Capturing bathtub, upside-down bathtub, and monotonic hazard rates.” *Innovation in Statistics and Probability*, 1(1):28–46, 2025.
[65] Gemeay, A. M., Moakofi, T., Balogun, O. S., Ozkan, E., and Hossain, M. M. ”Analyzing real data by a new heavy-tailed statistical model.” *Modern Journal of Statistics*, 1(1):1–24, 2025.
[66] Sapkota, L. P., Kumar, V., Tekle, G., Alrweili, H., Mustafa, M. S., and Yusuf, M. ”Fitting real data sets by a new version of Gompertz distribution.” *Modern Journal of Statistics*, 1(1):25–48, 2025.
[67] Noori, N. A., Abdullah, K. N., and Khaleel, M. A. ”Development and applications of a new hybrid Weibull-inverse Weibull distribution.” *Modern Journal of Statistics*, 1(1):80–103, 2025.
[68] Fisher, R.A. ”Theory of statistical estimation.” *Mathematical Proceedings of the Cambridge Philosophical Society*, 22(5):700–725, 1925.
[69] Murthy, DN Prabhakar and Xie, Min and Jiang, Renyan. *Weibull models*. 2004.

[70] Burr, Irving W. "Cumulative Frequency Functions." *The Annals of Mathematical Statistics*, 13(2):215–232, 1942.
[71] Nichols, Michele D and Padgett, William J. "A bootstrap control chart for Weibull percentiles." *Quality and reliability engineering international*, 22(2):141–151, 2006.
[72] Hinkley, David. "On quick choice of power transformation." *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 26(1):67–69, 1977.
[73] Gumbel, Emil Julius. *Statistics of extremes*. 1958.
[74] Obulezi, Okechukwu J and Obiora-Ilouno, Happiness O and Osuji, George A and Kayid, Mohamed and Balogun, Oluwafemi Samson. "A new family of generalized distributions based on logistic-x transformation: sub-model, properties and useful applications." *Research in Statistics*, 3(1):2477232, 2025.
[75] Galton, Francis. "XII. The geometric mean, in vital and social statistics." *Proceedings of the Royal Society of London*, 29(196-199):365–367, 1879.
[76] Oramulu, Dorathy O and Alsadat, Najwan and Kumar, Anoop and Bahloul, Mahmoud Mohamed and Obulezi, Okechukwu J. "Sine generalized family of distributions: Properties, estimation, simulations and applications." *Alexandria Engineering Journal*, 109:532–552, 2024.
[77] Ekemezie, Divine-Favour N and Anyiam, Kizito E and Kayid, Mohammed and Balogun, Oluwafemi Samson and Obulezi, Okechukwu J. "DUS Topp–Leone-G Family of Distributions: Baseline Extension, Properties, Estimation, Simulation and Useful Applications." *Entropy*, 26(11):973, 2024.

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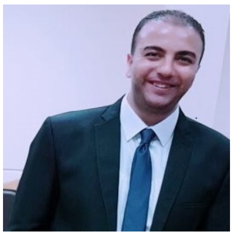

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