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New Ostrowski-Type Inequalities for Generalized Convex Functions via Conformable Fractional Derivatives

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Abstract: We have discovered new Ostrowski-type inequalities applicable to functions whose modulus of the conformable fractional derivative is relatively convex. This advancement constitutes a generalization of earlier results established for convex functions by other authors.

Keywords: Conformable fractional derivative, Ostrowski inequality, relative s-convex functions.

1 Introduction

Ostrowski's inequality has been a well-known result in classical mathematical literature since 1938, when A. Ostrowski in [14] established an upper bound for the approximation of the integral average of a function by its pointwise value over a given interval.

More precisely, let $\mathcal{F} : I \subset [0, +\infty) \to \mathbb{R}$ be a function differentiable in the interior I° of the interval I. Assume that the derivative $\mathcal{F}'$ belongs to the space of integrable functions over the interval $[u, w]$, with $u, w \in I$ and $u < w$. If $|\mathcal{F}'(x)| \leq M$ holds, then the following inequality is valid:

$$\left|\mathcal{F}(x) - \frac{1}{w-u}\int_u^w \mathcal{F}(u)\,du\right| \leq \frac{M}{w-u}\left[\frac{(x-u)^2 + (w-x)^2}{2}\right]$$

In recent years, various generalizations of Ostrowski's inequality have emerged for different classes of functions, such as functions of bounded variation, Lipschitz functions, monotone functions, absolutely continuous functions, convex functions, s-convexas and h-convexas. Additionally, extensions have been developed for n-veces differentiable mappings, with error estimates associated with certain specific mean values, as well as applications in numerical quadrature rules. For a broader perspective on these developments and generalizations, consulting recent works in the literature is recommended (see, [1], [2], [3], [4], [5], [6], [7], [9], [12], [19], [21], [22]).

In [17], in 2019, Ostrowski-type inequalities were introduced for functions whose modulus of the derivative is relatively convex.

On the other hand, in 2014, the conformable fractional derivative was introduced in [10]. Works related to this fractional derivative, as well as other fractional derivatives, can be found in the literature on fractional calculus (see, [20], [15], [16], [18]).

In this work, we present new Ostrowski-type inequalities for functions whose modulus of the conformable fractional derivative is relatively convex. To this end, we will take into account the results from [17] and [10].

2 Preliminaries

In this section, we present the fundamental concepts on which this work is developed. First, we introduce the classical notion of convexity, originally formulated by W. J. Jensen.

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Definition 1(ver [3]). *Let I be an interval in $\mathbb{R}$. A function $\mathcal{F}: I \to \mathbb{R}$ is said to be convex if, for any $x, y \in I$ and any $\tau \in (0,1)$, if the following inequality holds:*

$$\mathcal{F}(\tau x + (1-\tau)y) \leq \tau \mathcal{F}(x) + (1-\tau)\mathcal{F}(y).$$

If the inequality holds in the opposite direction, then $\mathcal{F}$ is said to be concave..

In 1961, W. Orlicz introduced the concept of s-convexity in [13], and later, in 1978, W. Breckner formulated a second version of this definition in [6]. These definitions are presented below.

Definition 2(ver [9]). *Let $0 \leq s \leq 1$. A function $\mathcal{F}: [0,+\infty) \to \mathbb{R}$ is said to be s-convex in the first sense, or s_1-convex, if it satisfies the inequality:*

$$\mathcal{F}(\alpha x + \beta y) \leq \alpha^s \mathcal{F}(x) + \beta^s \mathcal{F}(y),$$

for all $x, y \in [0,+\infty)$ and $\alpha, \beta \in (0,1)$ such that

$$\alpha^s + \beta^s = 1.$$

Similarly, a function $\mathcal{F}$ is said to be s-convex in the second sense, or s_2-convex, if it satisfies the inequality:

$$\mathcal{F}(\alpha x + \beta y) \leq \alpha^s \mathcal{F}(x) + \beta^s \mathcal{F}(y),$$

for all $x, y \in [0,+\infty)$ and $\alpha, \beta \in (0,1)$ such that

$$\alpha + \beta = 1.$$

If in either of the above inequalities the relation is reversed, the function $\mathcal{F}$ is said to be s-concave in the first or second sense, respectively.

The following results are relevant for the development of this work. First, we present a theorem that establishes an Ostrowski-type inequality using a function whose modulus of the derivative is convex.

Theorem 1(ver [3]). *Let $\mathcal{F}: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function in the interior I° of the interval I, and suppose that $\mathcal{F}'$ belongs to the space of integrable functions on $[u,w]$, where $u, w \in I$ with $u < w$. If $\mathcal{F}'$ is convex on $[u,w]$, then the following inequality holds:*

$$\left|\mathcal{F}(x) - \frac{1}{w-u}\int_u^w \mathcal{F}(u)\,du\right|$$
$$\leq \frac{(w-u)}{6}\left[\left(4\left(\frac{w-x}{w-u}\right)^3 - 3\left(\frac{w-x}{w-u}\right)^2 + 1\right)|\mathcal{F}'(u)| + \left(9\left(\frac{w-x}{w-u}\right)^2 - 4\left(\frac{w-x}{w-u}\right)^3 - 6\left(\frac{w-x}{w-u}\right) + 2\right)|\mathcal{F}'(w)|\right].$$

and it is valid for all $x \in [u,w]$.

This inequality is optimal in the sense that the constant $\frac{1}{6}$ cannot be replaced by a smaller value.

Furthermore, using functions whose modulus of the derivative is s-convex in the second sense, M. Alomari et al. [4] established the following result.

Theorem 2([3]). *Let $\mathcal{F}: I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function in the interior I° of the interval I, and suppose that $\mathcal{F}'$ belongs to the space of integrable functions on $[u,w]$, where $u, w \in I$ with $u < w$. If $\mathcal{F}'$ is s-convex in the second sense on $[u,w]$, for some fixed $s \in (0,1]$, and it holds that $|\mathcal{F}'(x)| \leq M$ for all $x \in [u,w]$, then the following inequality holds for each $x \in [u,w]$:*

$$\left|\mathcal{F}(x) - \frac{1}{w-u}\int_u^w \mathcal{F}(u)\,du\right| \leq \frac{M}{w-u}\left[\frac{(x-u)^2 + (w-x)^2}{s+1}\right].$$

Other equally relevant results were established by M. Alomari in [2], who formulated a version of the Ostrowski inequality using functions whose q-th powers of the modulus of their derivative are s-convex in the second sense.

Theorem 3([2]). *Let $\mathcal{F}: I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function in the interior I° of the interval I, and suppose that $\mathcal{F}'$ belongs to the space of integrable functions on $[u,w]$, where $u, w \in I$ with $u < w$. If $|\mathcal{F}'|^q$ is s-convex in the second sense on $[u,w]$ for some $s \in (0,1]$ with $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and if $|\mathcal{F}(x)| \leq M$ for all $x \in [u,w]$, then the following inequality holds:*

$$\left|\mathcal{F}(x) - \frac{1}{w-u}\int_u^w \mathcal{F}(u)\,du\right| \leq \frac{M}{(1+p)^{1/p}}\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(x-u)^2 + (w-x)^2}{w-u}\right].$$

Theorem 4([2]). *Let $\mathcal{F}: I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function in the interior I° of the interval I, and suppose that $\mathcal{F}'$ belongs to the space of integrable functions on $[u,w]$, where $u, w \in I$ with $u < w$. If $|\mathcal{F}'|^q$ is s-convex in the second sense on $[u,w]$ for some fixed $s \in (0,1]$, with $q \geq 1$, and if $|\mathcal{F}'(x)| \leq M$ for all $x \in [u,w]$, then the following inequality is satisfied:*

$$\left|\mathcal{F}(x) - \frac{1}{w-u}\int_u^w \mathcal{F}(u)\,du\right| \leq M\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(x-u)^2 + (w-x)^2}{2(w-u)}\right].$$

In the framework of generalized convexity, a class of functions called relatively convex functions with respect to a given function has emerged.

Notably, A. Noor, K. I. Noor and M. U. Awan introduced the following definitions in [11] .

Definition 3([3]). *Let k_g be a subset of H. It is said that k_g is relatively convex with respect to the function $g : H \to H$ if the following condition holds:*

$$\tau g(v) + (1-\tau)u \in k_g,$$

for all $u, v \in H$, $u, g(v) \in k_g$ $\tau \in [0,1]$.

This notion allows generalizing the classical idea of convexity by incorporating a reference function g, which determines the relative convex structure of the set.

Definition 4([11]). *Let I be an interval in $\mathbb{R}$. A function $\mathcal{F} : k_g \subseteq \mathbb{R} \to \mathbb{R}$ is said to be relatively convex with respect to the function $g : \mathbb{R} \to \mathbb{R}$ if, for all $x, y \in \mathbb{R}$ and for all $\tau \in [0,1]$, the following inequality holds:*

$$\mathcal{F}(\tau g(x) + (1-\tau)y) \leq \tau \mathcal{F}(g(x)) + (1-\tau)\mathcal{F}(y).$$

If the inequality above is satisfied in the opposite sense, then $\mathcal{F}$ is said to be relatively concave with respect to g.

A simple case of constructing these relatively convex sets was presented in [8] within a study of the noise level around railway stations located in urban areas, represented by the set $[0,50] \cup [125,130]$.

When the railway transport system is moved outside the cities, the noise level is reduced to the interval $[0,50]$ The authors define the function $g : \mathbb{R} \to \mathbb{R}$ as:

$$g(x) = \begin{cases} x, & \text{si } x \in [0,50] \\ 0, & \text{otherwise.} \end{cases}$$

This function describes the efforts to maintain the sound level within normal limits. Under this definition, it was concluded that the set $[0,50] \cup [125,130]$ is relatively convex with respect to g.

Definition 5([11]). *A function $\mathcal{F} : K_g \to [0,+\infty)$ is said to be relatively s-convex in the second sense with respect to the function $g : H \to H$, where $s \in (0,1]$ is fixed, if the following inequality holds:*

$$\mathcal{F}(\tau g(x) + (1-\tau)g(y)) \leq \tau^s \mathcal{F}(g(x)) + (1-\tau)^s \mathcal{F}(g(y))$$

for all $x, y \in [0,+\infty)$, $g(x), y \in k_g$ and $\tau \in [0,1]$.

If the inequality above is satisfied in the opposite sense, we say that, $\mathcal{F}$ is relatively s-concave in the second sense.

The following results constitute the foundation of this work and were established in [17] by M. Vivas-Cortez et al.

Lemma 1([17]). *Let $\mathcal{F} : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, where $u, w \in I$ with $u < w$, and $g : \mathbb{R} \to \mathbb{R}$ be a function. If $\mathcal{F}'$ belongs to the space of integrable functions on $[u,w]$, then the following equality holds:*

$$\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz = \frac{(g(x)-u)^2}{w-u}\int_0^1 \tau \mathcal{F}'(\tau g(x) + (1-\tau)u)\,d\tau - \frac{(g(x)-w)^2}{w-u}\int_0^1 \tau \mathcal{F}'(\tau g(x) + (1-\tau)w)\,d\tau.$$

for all $x \in g^{-1}(I)$.

Lemma 2([17]). *Let $\mathcal{F} : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, where $u, w \in I$ with $u < w$, and $g : \mathbb{R} \to \mathbb{R}$ be a function. If $\mathcal{F}'$ belongs to the space of integrable functions on $[u,w]$, then the following equality holds:*

$$\mathcal{F}(x) - \frac{1}{w-g(u)}\int_{g(u)}^w \mathcal{F}(z)\,dz = (g(u)-w)\int_0^1 p(\tau)\mathcal{F}'\left(\tau g(u) + (1-\tau)w\right)\,d\tau,$$

for all $\tau \in [u,w]$.

The following definition is fundamental for the development of this work and was introduced by R. Khalil in [10].

Definition 6([10]). *Let $\mathcal{F} : [0,\infty) \to \mathbb{R}$. The "conformable fractional derivative" of order α is defined as*

$$T_\alpha \mathcal{F}(\tau) = \lim_{\varepsilon \to 0} \frac{\mathcal{F}(\tau + \varepsilon \tau^{1-\alpha}) - \mathcal{F}(\tau)}{\varepsilon},$$

for all $\tau > 0$ and $\alpha \in (0,1)$.

This conformable fractional derivative is also known as the α-derivative.

An important result that follows from this definition is the following:

Theorem 5([10]). *If $\mathcal{F}$ is α-differentiable, then*

$$T_\alpha \mathcal{F}(\tau) = \tau^{1-\alpha}\mathcal{F}'(\tau).$$

3 Main Results

In this section, we present new Ostrowski-type inequalities for functions whose modulus of the α-derivative is relatively convex and relatively s-convex in the second sense.

The following lemma will be necessary

Lemma 3. *Let $\mathcal{F} : I \subseteq \mathbb{R} \to \mathbb{R}$ be a α-differentiable function on the interior I° of I, where $u, w \in I$ with $u < w$, and $g : \mathbb{R} \to \mathbb{R}$ be a function. If $T_\alpha \mathcal{F}$ belongs to the space of integrable functions on $[u, w]$, then the following equality holds:*

$$\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz = \frac{(g(x)-u)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)u)(\tau g(x)+(1-\tau)u)^{\alpha-1}d\tau - \frac{(g(x)-w)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)w)(\tau g(x)+(1-\tau)w)^{\alpha-1}d\tau.$$

*Proof.*We start from the relation given in Lemma 1 of [17] and substitute $T_\alpha(\mathcal{F})(\tau)$ for $\mathcal{F}'$, using the relation given in [10], which states that $\mathcal{F}'(\tau) = \frac{T_\alpha(\mathcal{F})(\tau)}{\tau^{1-\alpha}}$. Thus, we obtain

$$\frac{(g(x)-u)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)u)(\tau g(x)+(1-\tau)u)^{\alpha-1}d\tau$$

and

$$\frac{(g(x)-w)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)w)(\tau g(x)+(1-\tau)w)^{\alpha-1}d\tau$$

From these two expressions, the desired result follows.

Theorem 6. *Let $\mathcal{F} : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a α-differentiable function on the interior I° of I, such that $T_\alpha \mathcal{F}$ belongs to the space of integrable functions, where $u, w \in I$ with $u < w$.*

If $|T_\alpha(\mathcal{F})|$ is relatively convex with respect to the function $g : \mathbb{R} \to \mathbb{R}$ on $[u, w]$ and $|T_\alpha(\mathcal{F})(x)| \le M$, then the following inequality holds:

$$\left|\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \le \frac{M}{w-u}\max\left\{u^{\alpha-1}, w^{\alpha-1}, g(x)^{\alpha-1}\right\}\left[\frac{(g(x)-u)^2+(g(x)-w)^2)}{2}\right]$$

for all $x \in g^{-1}(I)$:

*Proof.*We start from Lemma 3, and it follows that:

$$\left|\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \le \left|\frac{(g(x)-u)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)u)(\tau g(x)+(1-\tau)u)^{\alpha-1}d\tau\right| + \left|\frac{(g(x)-w)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)w)(\tau g(x)+(1-\tau)w)^{\alpha-1}d\tau\right| = |A| + |B|$$

Since $|T_\alpha(\mathcal{F})|)$ is relatively convex with respect to g and $|T_\alpha(\mathcal{F})(x)| \le M$, it follows that:

$$\begin{aligned}|A| &\le \frac{(g(x)-u)^2}{w-u}\int_0^1 \tau(\tau|T_\alpha(\mathcal{F})(g(x)| + (1-\tau)|T_\alpha(\mathcal{F})g(u)|)|(\tau g(x)+(1-\tau)u)|^{\alpha-1}d\tau \\ &\le \frac{(g(x)-u)^2}{w-u}\int_0^1 \tau(\tau M + (1-\tau)M)\max\left\{u^{\alpha-1}, g(x)^{\alpha-1}\right\}d\tau \\ &= \frac{M}{w-u}\max\left\{u^{\alpha-1}, g(x)^{\alpha-1}\right\}(g(x)-u)^2\int_0^1 \tau\,d\tau \\ &= \frac{M}{w-u}\max\left\{u^{\alpha-1}, g(x)^{\alpha-1}\right\}\frac{(g(x)-u)^2}{2}\end{aligned}$$

In a similar way, it follows that:

$$|B| \le \frac{M}{w-u}\max\left\{w^{\alpha-1}, g(x)^{\alpha-1}\right\}\frac{(g(x)-w)^2}{2}$$

From the last two inequalities, the result is obtained.

*Remark.*If $\alpha = 1$ the result coincides with Theorem 5 from [17], and if $g(x) = x$, the classical Ostrowski inequality is obtained.

*Example 1.*For the function $F(x) = e^x$, whose modulus $|T_\alpha F(x)| = |T_\alpha e^x| = |x^{1-\alpha}e^x|$ is relatively convex with respect to the function $g(x) = x^2$, we consider the case $\alpha = \frac{1}{2}$ on the interval $[1, 2]$, for which the inequality from Theorem 6 holds.

Since $|2^{1/2}e^2| \approx 10.45$, we can choose $M = 10.5$, and noting that

$$\max_{x\in[1,2]}\left\{1^{-1/2},\ 2^{-1/2},\ (x^2)^{-1/2}\right\} = 1,$$

we obtain the following inequality:

$$\left|e^{x^2} - \frac{e^2-e^1}{1}\right| \le 10.5\left(x^4 - 3x^2 + \frac{5}{2}\right).$$

This inequality is illustrated in the following figure.

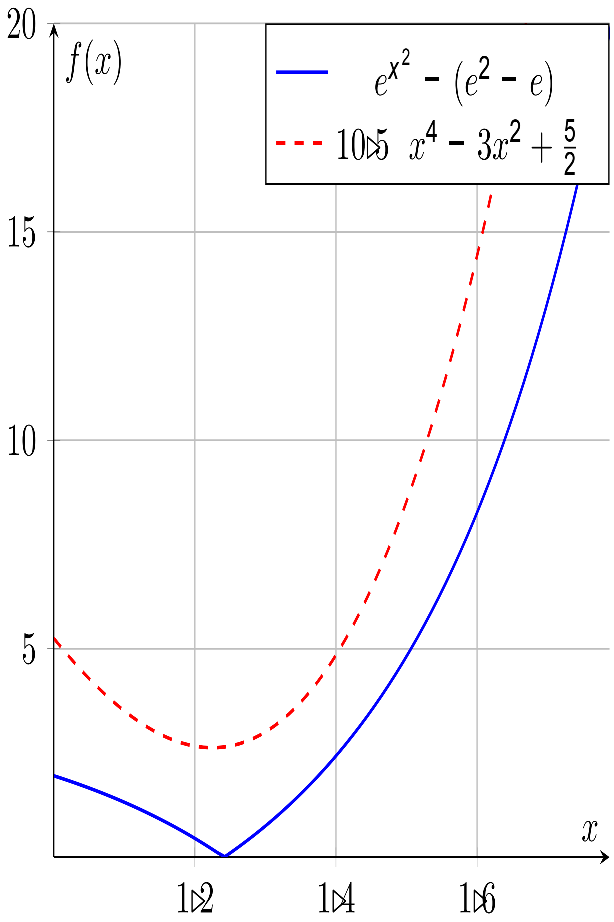

Figure 1: Graph of $\left|e^{x^2} - \frac{e^2-e}{1}\right| \le 10.5\left(x^4 - 3x^2 + \frac{5}{2}\right)$.

Lemma 4. *Let $\mathcal{F} : I \subseteq \mathbb{R} \to \mathbb{R}$ be a α-differentiable function on the interior I° of I, where $u, w \in I$ with $u < w$, and $g : \mathbb{R} \to \mathbb{R}$ be a function. If $T_\alpha \mathcal{F}$ belongs to the space of integrable functions on $[u, w]$ for all $\tau \in [0, 1]$:*

$$\mathcal{F}(x) - \frac{1}{w - g(u)} \int_{g(u)}^{w} \mathcal{F}(z)\, dz = (g(u) - w) \int_0^1 p(\tau) T_\alpha(\mathcal{F})(\tau g(u) + (1-\tau)w) (\tau g(u) + (1-\tau)w)^{\alpha-1}\, d\tau,$$

where $p(\tau)$ is defined as follows:

$$p(\tau) = \begin{cases} \tau, & \tau \in \left[0, \frac{w-x}{w-g(u)}\right] \\ \tau - 1, & \tau \in \left(\frac{w-x}{w-g(u)}, 1\right] \end{cases}$$

for all $x \in [u, w]$.

Proof. Starting from Lemma 2 given in [17], we substitute $T_\alpha(\mathcal{F})(\tau)$ by $\mathcal{F}'(t)$, so that $\mathcal{F}'(\tau) = \frac{T_\alpha(\mathcal{F})(\tau)}{\tau^{1-\alpha}}$, and we obtain:

$$\int_0^{\frac{w-x}{w-g(u)}} \tau \mathcal{F}'(\tau g(u) + (1-\tau)w)\, d\tau = \int_{0)}^{\frac{w-x}{w-g(u)}} \tau T_\alpha(\mathcal{F})(\tau g(u) + (1-\tau)w)(\tau g(u) + (1-\tau)w)^{\alpha-1}\, d\tau,$$

and

$$\int_{\frac{w-x}{w-g(u)}}^{1} \tau \mathcal{F}'(\tau g(u) + (1-\tau)w)\, d\tau = \int_{\frac{w-x}{w-g(u)}}^{1} \tau T_\alpha(\mathcal{F})(\tau g(u) + (1-\tau)w)(\tau g(u) + (1-\tau)w)^{\alpha-1}\, d\tau.$$

From these two expressions, the result is obtained.

Theorem 7. *Let $\mathcal{F} : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a α-differentiable function on the interior I° of I, such that $T_\alpha \mathcal{F}$ belongs to the space of integrable functions, where $u, w \in I$ with $u < w$. If $|T_\alpha(\mathcal{F})|$ is relatively convex with respect to the function $g : \mathbb{R} \to \mathbb{R}$ on $[u, w]$, then the following inequality holds:*

$$\left|\mathcal{F}(x) - \frac{1}{w-g(u)} \int_{g(u)}^{w} \mathcal{F}(z)\, dz\right| \le \frac{(g(u)-w)}{6} \max\{g(u)^{\alpha-1}, w^{\alpha-1}\} \left[\left(4\left(\frac{w-x}{w-g(u)}\right)^3 - 3\left(\frac{w-x}{w-g(u)}\right)^2 + 1\right) |T_\alpha(\mathcal{F})(g(u))| + \left(9\left(\frac{w-x}{w-g(u)}\right)^2 - 4\left(\frac{w-x}{w-g(u)}\right)^3 - 6\left(\frac{w-x}{w-g(u)}\right) + 2\right) |T_\alpha \mathcal{F}(w)|\right]$$

for each $x \in [u, w]$. The constant $\frac{1}{6}$ cannot be replaced by a smaller value.

Proof. Using Lemma 4, the triangle inequality, and the fact that $|T_\alpha \mathcal{F}|$ is relatively convex with respect to the function $g : [u, w] \to \mathbb{R}$, the following holds:

$$\left|\mathcal{F}(x) - \frac{1}{w-g(u)} \int_{g(u)}^{w} \mathcal{F}(z)\, dz\right| \le \left|(g(u) - w) \int_0^{\frac{w-x}{w-g(u)}} \tau T_\alpha(\mathcal{F})(\tau g(u) + (1-\tau)w) (\tau g(u) + (1-\tau)w)^{\alpha-1}\, d\tau\right|$$

$$
\begin{aligned}
&+\left|(g(u)-w)\int_{\frac{w-x}{w-g(u)}}^{1}(1-\tau)T_\alpha(\mathcal{F})(\tau g(u)+(1-\tau)w)\right.\\
&\qquad\left.(\tau g(u)+(1-\tau)w)^{\alpha-1}\,d\tau\right|\\
&\le (g(u)-w)\int_0^{\frac{w-x}{w-g(u)}}\tau(\tau|T_\alpha(\mathcal{F})g(u)|\\
&\quad+(1-\tau)|T_\alpha(\mathcal{F})w|)|\tau g(u)+(1-\tau)w|^{\alpha-1}\,d\tau\\
&\quad+(g(u)-w)\int_{\frac{w-x}{w-g(u)}}^{1}(1-\tau)(\tau|T_\alpha(\mathcal{F})g(u)|\\
&\quad+(1-\tau)|T_\alpha(\mathcal{F})(w)|)|\tau g(u)+(1-\tau)w|^{\alpha-1}\,d\tau\\
&\le (g(u)-w)\max\{g(u)^{\alpha-1},w^{\alpha-1}\}\left[|T_\alpha(\mathcal{F})g(u)|\right.\\
&\qquad\left.\int_0^{\frac{w-x}{w-g(u)}}\tau^2\,d\tau+|T_\alpha(\mathcal{F})(w)|\int_0^{\frac{w-x}{w-g(u)}}\tau-\tau^2\,d\tau\right]\\
&+(g(u)-w)\max\{g(u)^{\alpha-1},w^{\alpha-1}\}\left[|T_\alpha(\mathcal{F})g(u)|\right.\\
&\qquad\left.\int_{\frac{w-x}{w-g(u)}}^{1}\tau-\tau^2\,d\tau+|T_\alpha(\mathcal{F})(w)|\int_{\frac{w-x}{w-g(u)}}^{1}(1-\tau)^2\,d\tau\right]\\
&=(g(u)-w)\max\{g(u)^{\alpha-1},w^{\alpha-1}\}\left[|T_\alpha(\mathcal{F})g(u)|\right.\\
&\qquad\frac{1}{3}\left(\frac{w-x}{w-g(u)}\right)^3+|T_\alpha(\mathcal{F})w|\left(\frac{1}{2}\left(\frac{w-x}{w-g(u)}\right)^2\right.\\
&\qquad\left.\left.-\frac{1}{3}\left(\frac{w-x}{w-g(u)}\right)^3\right)\right]\\
&+(g(u)-w)\max\{g(u)^{\alpha-1},w^{\alpha-1}\}\left[|T_\alpha(\mathcal{F})g(u)|\right.\\
&\qquad\left(\frac{1}{2}-\frac{1}{3}-\frac{1}{2}\left(\frac{w-x}{w-g(u)}\right)^2+\frac{1}{3}\left(\frac{w-x}{w-g(u)}\right)^3\right)\\
&\quad+|T_\alpha(\mathcal{F})(w)|\left(1-1+\frac{1}{3}-\left(\frac{w-x}{w-g(u)}\right)+\left(\frac{w-x}{w-g(u)}\right)^2\right.\\
&\qquad\left.\left.-\frac{1}{3}\frac{w-x}{w-g(u)}\right)^3\right]\\
&=\frac{g(u)-w}{6}\max\{g(u)^{\alpha-1},w^{\alpha-1}\}\left[|T_\alpha(\mathcal{F})g(u)|\right.\\
&\quad\left(4\left(\frac{w-x}{w-g(u)}\right)^3-3\left(\frac{w-x}{w-g(u)}\right)^2+1\right)+|T_\alpha(\mathcal{F})(w)|\\
&\left.\left(9\left(\frac{w-x}{w-g(u)}\right)^2-4\left(\frac{w-x}{w-g(u)}\right)^3-6\left(\frac{w-x}{w-g(u)}\right)+2\right)\right]
\end{aligned}
$$

*Remark.*If $\alpha = 1$ in this Theorem 7, we obtain Theorem 6 from [17] and if $g(u) = u$, we recover the result of Theorem 1.

*Example 2.*For the function $F(x) = e^x$, whose modulus is given by

$$|T_\alpha F(x)| = |T_\alpha e^x| = |x^{1-\alpha}e^x|,$$

we observe that it is relatively convex with respect to the function $g(x) = x^3$, with $\alpha = 0.4$, on the interval $[2,4]$. Then, the inequality from Theorem 7 holds.

Let us verify this, noting that:

$$\max_{x\in[2,4]}\left\{(x^3)^{-0.6},\ 2^{-0.6},\ 4^{-0.6}\right\} = 0.44.$$

Therefore, the following inequality is obtained:

$$
\begin{aligned}
&|e^x - 731.59|\\
&\le 0.293\left[\frac{10254.9}{16}x^3-\frac{71031.9}{16}x^2+\frac{401155.2}{16}x+\frac{208}{16}\right].
\end{aligned}
$$

This inequality is illustrated in the following figure.

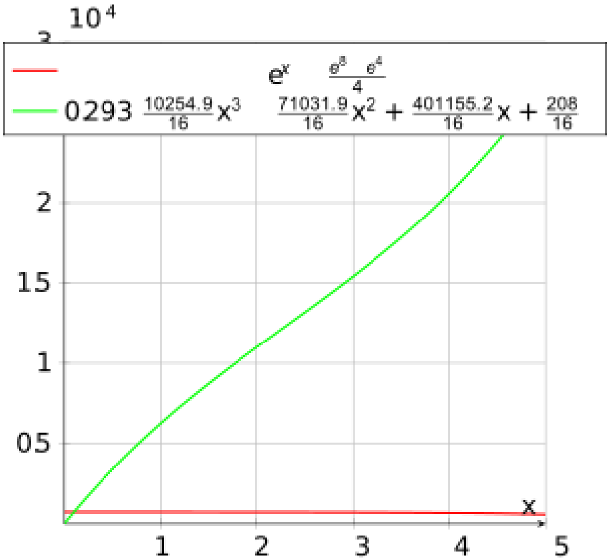

Figure 2: Graphs of the functions $\left|e^x - \frac{e^8-e^4}{4}\right|$ and $0.293\left(\frac{10254.9}{16}x^3-\frac{71031.9}{16}x^2+\frac{401155.2}{16}x+\frac{208}{16}\right)$.

The following result pertains to functions whose modulus of the α-derivative is relatively s-convex in the second sense.

Theorem 8. *Let $\mathcal{F} : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be an α-differentiable function on the interior I° of I such that $T_\alpha(\mathcal{F})$ belongs to the space of integrable functions, where $u, w \in I$ with $u < w$. If $|T_\alpha(F)|$ is relatively s-convex with respect to the function $g : \mathbb{R} \to \mathbb{R}$ for some fixed $s \in (0,1]$, and $|T_\alpha(F)(x)| \le M$ for all $x \in [u,w]$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\mathcal{F}(g(x))-\frac{1}{w-u}\int_u^w\mathcal{F}(z)\,dz\right|\le\\
&\frac{M}{w-u}\max\left\{g(x)^{\alpha-1},u^{\alpha-1},w^{\alpha-1}\right\}\left[\frac{(g(x)-u)^2+(g(x)-w)^2}{s+1}\right]
\end{aligned}
$$

for all $x \in [v,w]$.

*Proof.*By Lemma 3, the triangular inequality, and the fact that $|T_\alpha \mathcal{F}|$ is relatively s-convex with respect to the function $g : [u,w] \to \mathbb{R}$ and that $|T_\alpha(F)(x)| \le M$ for all $x \in [u,w]$, it follows that:

$$\left|\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \le \left|\frac{(g(x)-u)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})(\tau g(x)(1-\tau)u)(\tau g(x)+(1-\tau)u)^{\alpha-1}\,d\tau\right| + \left|\frac{(g(x)-w)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)w)(\tau g(x)+(1-\tau)w)^{\alpha-1}\,d\tau\right|$$
$$\le \frac{(g(x)-u)^2}{w-u}\int_0^1 \tau(\tau^s)|T_\alpha(\mathcal{F}(g(x))| + (1-t)^s |T_\alpha(\mathcal{F})(u)|)|\tau g(x)+(1-\tau)u|^{\alpha-1}\,d\tau + \frac{(g(x)-w)^2}{w-u}\int_0^1 \tau(\tau^s)|T_\alpha(\mathcal{F}(g(x))| + (1-t)^s|T_\alpha(\mathcal{F})(w)|)|\tau g(x)+(1-\tau)w|^{\alpha-1}\,d\tau$$
$$\le \frac{(g(x)-u)^2}{w-u} M \max\{g(x)^{\alpha-1}, u^{\alpha-1}\}\int_0^1 \tau(\tau^s+(1-\tau)^s)\,d\tau + \frac{(g(x)-w)^2}{w-u} M \max\{g(x)^{\alpha-1}, w^{\alpha-1}\}\int_0^1 \tau(\tau^s+(1-\tau)^s)\,d\tau$$
$$\frac{(g(x)-u)^2+(g(x)-w)^2}{w-u} M \max\{g(x)^{\alpha-1}, u^{\alpha-1}, w^{\alpha-1}\}\left[\frac{1}{s+2}+\frac{1}{(s+1)(s+2)}\right]$$
$$= \frac{M}{w-u}\max\{g(x)^{\alpha-1}, u^{\alpha-1}, w^{\alpha-1}\}\left[\frac{(g(x)-u)^2+(g(x)-w)^2}{s+1}\right]$$

*Example 3.*For the function $F(x) = x^2$, whose modulus $|T_\alpha F(w)| = |T_\alpha x^2| = |x^{1-\alpha}2x|$ is relatively s-convex with respect to the function $g(x) = \sin(x)$, it holds that for $\alpha = 0.7$ on the interval $[1,4]$, the inequality from Theorem 8 is satisfied.

Since $|2x^{1.3}| \approx 12.13$, we can choose $M = 12.2$, and noting that

$$\max_{x\in[1,4]}\{1^{-0.3}, 4^{-0.3}, \sin(x)^{-0.3}\} = 1.05,$$

we obtain the following inequality:

$$\left|(\sin(x))^2 - 7\right| \le 4.27\left((\sin(x))^2 - 5\sin(x) + \frac{17}{2}\right).$$

This inequality is illustrated in the following figure.

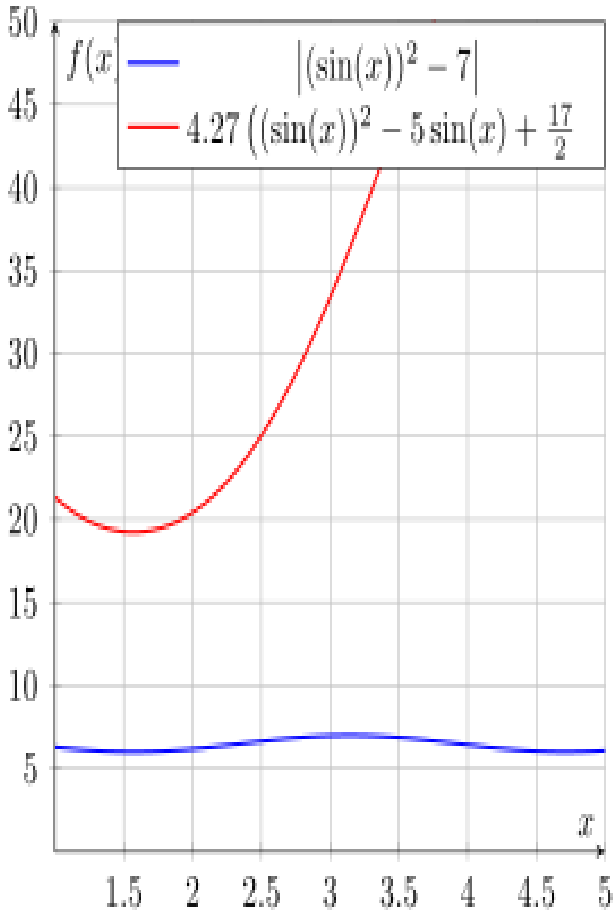

Figure 3: Graphs of the functions $\left|(\sin(x))^2 - 7\right|$ and $4.27\left((\sin(x))^2 - 5\sin(x) + \frac{17}{2}\right)$.

*Remark.*If in Theorem 8 we set $\alpha = 1$, we obtain Theorem 7 de [17], and if $g(x) = x$, we recover Theorem 2.

Theorem 9. *Let $\mathcal{F} : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be an α-differentiable function on the interior I° of I such that $T_\alpha(\mathcal{F})$ belongs to the space of integrable functions, where $u, w \in I$ with $u < w$. If $|T_\alpha(F)|$ is relatively s-convex with respect to the function $g : \mathbb{R} \to \mathbb{R}$ for some fixed $s \in (0,1]$ and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ with $|T_\alpha(f)|(x) \le M$ for all $x \in [u,w]$, then the following inequality holds:*

$$\left|\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \le \left[\frac{(g(x)-u)^2+(g(x)-w)^2}{w-u}\right]\frac{M}{(p+1)^{1/q}}\left(\frac{2}{s+1}\right)^{1/q}\max\{g(x)^{\alpha-1}, u^{\alpha-1}, w^{\alpha-1}\}$$

Proof. By Lemma 3 and using Hölder's inequality, we obtain:

$$\left|\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \le$$

$$\left|\frac{(g(x)-v)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)u)(\tau g(x)+ (1-\tau)u)^{\alpha-1}\,d\tau\right| + \left|\frac{(g(x)-w)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})(\tau g(x)+ (1-\tau)u)(\tau g(x)+(1-\tau)u)^{\alpha-1}\,d\tau\right|$$

$$\le \frac{(g(x)-u)^2}{w-u}\int_0^1 \tau|T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)u)||(\tau g(x)+ (1-\tau)u)^{\alpha-1}|\,d\tau + \frac{(g(x)-w)^2}{w-u}\int_0^1 |T_\alpha(\mathcal{F})(\tau g(x)+ (1-\tau)u)||(\tau g(x)+(1-\tau)u)^{\alpha-1}|\,d\tau$$

$$\le \frac{(g(x)-u)^2}{w-u}\left(\int_0^1 \tau^p\,d\tau\right)^{1/p}\left(\int_0^1 |T_\alpha(\mathcal{F})|(\tau g(x)+ (1-\tau)u)|^q \max\{g(x)^{\alpha-1},u^{\alpha-1}\}^q\,d\tau\right)^{1/q} + \frac{(g(x)-w)^2}{w-u} \left(\int_0^1 \tau^p\,d\tau\right)^{1/p}\left(\int_0^1 |T_\alpha(\mathcal{F})|(\tau g(x)+(1-\tau)w)|^q \max\{g(x)^{\alpha-1},w^{\alpha-1}\}^q\,d\tau\right)^{1/q}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Since $|T_\alpha(\mathcal{F})|^q$ is relatively s-convex in the second sense with respect to the function g and $|T_\alpha(\mathcal{F})(x) \le M|$, we have

$$\int_0^1 |T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)u)|^q\,d\tau \le \int_0^1 (\tau^s|T_\alpha(\mathcal{F})(g(x))|^q + (1-\tau)^s|T_\alpha(\mathcal{F})(u)|)\,d\tau \le M^q\int_0^1 (\tau^s+(1-\tau)^s\,d\tau = \frac{2M^q}{s+1}$$

and

$$\int_0^1 |T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)w)|^q\,d\tau \le \frac{2M^q}{s+1},$$

Thus, we obtain:

$$\left|\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \le \left[\frac{(g(x)-u)^2+(g(x)-w)^2}{w-u}\right]\left(\frac{1}{p+1}\right)^{1/p} \max\{g(x)^{\alpha-1},u^{\alpha-1},w^{\alpha-1}\}\left(\frac{2M^q}{s+1}\right)^{1/q}$$

$$= \left[\frac{(g(x)-u)^2+(g(x)-w)^2}{w-u}\right]\frac{M}{(p+1)^{1/q}}\left(\frac{2}{s+1}\right)^{1/q} \max\{g(x)^{\alpha-1},u^{\alpha-1},w^{\alpha-1}\}$$

Remark. If $\alpha=1$ in this Theorem 9, we obtain Theorem 8 from [17] and if $g(x)=x$, we recover the result of Theorem 3.

Example 4. For the function $F(x)=x^{1/4}$, whose modulus

$$|T_\alpha F(x)| = |T_\alpha x^{1/4}| = \left|\frac{1}{4}x^{1-\alpha}x^{-3/4}\right|$$

is relatively s-convex with respect to the function $g(x)=x^{1/7}$, it holds that for $\alpha=0.3$, $s=0.3$, and $q=3$ on the interval $[2,7]$, the inequality from Theorem 9 is satisfied.

Since

$$\left|\frac{1}{4}x^{-0.05}\right| \approx 0.2411,$$

we may choose $M=0.25$, and observing that

$$\max_{x\in\{2,7\}}\left\{(x^{1/7})^{-0.7},\ 2^{-0.7},\ 7^{-0.7}\right\} = 0.94,$$

we obtain the following inequality:

$$\left|x^{1/28}-1.445\right| \le 0.124\left(\frac{2}{7}x^{2/7}-\frac{18}{7}x^{1/7}+\frac{53}{7}\right).$$

This inequality is illustrated in the following figure.

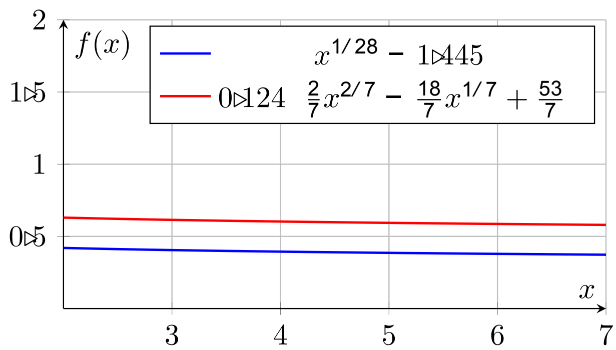

Figure 4: Graphs of the functions $\left|x^{1/28}-1.445\right|$ and $0.124\left(\frac{2}{7}x^{2/7}-\frac{18}{7}x^{1/7}+\frac{53}{7}\right)$.

Theorem 10. *Let $\mathcal{F} : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be an α-differentiable function on the interior I° of I such that $T_\alpha(\mathcal{F})$ belongs to the space of integrable functions, where $u, w \in I$ with $u < w$. If $|T_\alpha(\mathcal{F})|^q$ is relatively s-convex with respect to the function $g : \mathbb{R} \to \mathbb{R}$ for some fixed $s \in (0,1]$, $q > 1$, and $|T_\alpha(\mathcal{F})(x)| \le M$ for all $x \in [u, w]$, then the following inequality holds:*

$$\left|\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \le M\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(g(x)-u)^2 + (g(x)-w)^2}{2(w-u)}\right] \max\{g(x)^{\alpha-1}, u^{\alpha-1}, w^{\alpha-1}\}$$

For each $x \in [u, w]$.

Proof. Suppose $q > 1$. By Lemma 3, and using the power mean inequality, we obtain:

$$\left|\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \le \left|\frac{(g(x)-u)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})\,(\tau g(x) + (1-\tau)u)(\tau g(x) + (1-\tau)u)^{\alpha-1} d\tau\right| + \left|\frac{(g(x)-w)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})\,(\tau g(x) + (1-\tau)w)(\tau g(x) + (1-\tau)w)^{\alpha-1} d\tau\right|$$

$$\le \frac{(g(x)-v)^2}{w-u}\left(\int_0^1 \tau\,d\tau\right)^{1-1/q}\left(\int_0^1 \tau(|T_\alpha(\mathcal{F})(\tau g(x) + (1-\tau)u)||\tau g(x) + (1-\tau)u|^{\alpha-1})^q)\,d\tau\right)^{1/q}$$
$$+ \frac{(g(x)-w)^2}{w-u}\left(\int_0^1 \tau\,d\tau\right)^{1-1/q}\left(\int_0^1 \tau(|T_\alpha(\mathcal{F})(\tau g(x) + (1-\tau)w)||\tau g(x) + (1-\tau)w|^{\alpha-1})^q)\,d\tau\right)^{1/q}$$

Since $|T_\alpha(\mathcal{F})|^q$ is relatively s-convex in the second sense with respect to the function g and $|T_\alpha(\mathcal{F})(x) \le M$, we have

$$\int_0^1 \tau(|T_\alpha(\mathcal{F})(\tau g(x) + (1-\tau)u)| |(\tau g(x) + (1-\tau)u)^{\alpha-1}|^q)\,d\tau$$
$$\le \max\{g(x)^{\alpha-1}, u^{\alpha-1}\}^q \int_0^1 \tau^{s+1}|T_\alpha(\mathcal{F})(g(x))|^q + \tau(1-t)^s|T_\alpha(\mathcal{F})(u)|)|^q\,d\tau$$
$$= \max\{g(x)^{\alpha-1}, u^{\alpha-1}\}M^q\frac{1}{s+1}$$

and

$$\int_0^1 \tau(|T_\alpha(\mathcal{F})(\tau g(x) + (1-\tau)w)| |(\tau g(x) + (1-\tau)w)^{\alpha-1}|^q)\,d\tau$$
$$\le \max\{g(x)^{\alpha-1}, w^{\alpha-1}\}^q M^q\frac{1}{s+1}$$

Thus, we obtain:

$$\left|\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \le M\left(\frac{1}{2}\right)^q\left(\frac{1}{s+1}\right)^q \max\{g(x)^{\alpha-1}, u^{\alpha-1}, w^{\alpha-1}\}\left[\frac{(g(x)-u)^2 + (g(x)-w)^2}{w-u}\right]$$

Remark. If in Theorem 10 we set $\alpha = 1$, we obtain Theorem 9 de [17], and if $g(x) = x$, we recover Theorem 4.

Example 5. For the function $F(x) = x^{1/2}$, whose modulus $|T_\alpha F(x)|^q = |T_\alpha x^{1/2}|^2 = \left|\frac{1}{2}x^{1-\alpha}x^{-1/2}\right|^2$ is relatively s-convex with respect to the function $g(x) = x^{1/3}$, it holds that for $\alpha = 0.6$ and $s = 0.4$ on the interval $[2,4]$, the inequality of Theorem 10 is satisfied.

Since $\left|\frac{1}{2}x^{-0.1}\right| \approx 0.466$, we may choose $M = 0.5$, and noting that

$$\max_{x\in[2,4]}\left\{2^{-0.4},\ 4^{-0.4},\ \left(x^{1/3}\right)^{-0.4}\right\} = \max_{x\in[2,4]}\{0.76,\ 0.57,\ 0.91\} = 0.91$$

we obtain the following inequality:

$$\left|x^{1/6} - 1.724\right| \le 0.27\left(x^{2/3} - 6x^{1/3} + 10\right).$$

This inequality is illustrated in the following figure.

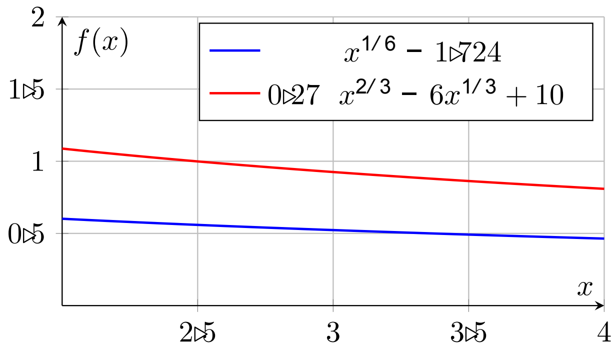

Figure 5: Graphs of the functions $\left|x^{1/6} - 1.724\right|$ and $0.27\left(x^{2/3} - 6x^{1/3} + 10\right)$.

The following result is known as the Hermite-Hadamard inequality for relatively s-convex functions in the second sense,is not an original contribution of this work. Its proof can be found in [11] and is essential for the proof of the final theorem.

Theorem 11. *Let $\mathcal{F} : K_g \subseteq \mathbb{R} \to \mathbb{R}$ a relatively s-convex function in the second sense. Then, the following inequality holds:*

$$2^{s-1}\mathcal{F}\left(\frac{u+g(w)}{2}\right) \leq \frac{1}{g(w)-u}\int_u^{g(w)} \mathcal{F}(z)\,dz \leq \frac{\mathcal{F}(u)+\mathcal{F}(g(w))}{s+1}$$

If $\mathcal{F}$ is relatively s-concave in the second sense, then the inequalities are reversed.

Theorem 12. *Let $\mathcal{F} : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be an α-differentiable function on the interior I° of I such that $T_\alpha(\mathcal{F})$ belongs to the space of integrable functions, where $u, w \in I$ with $u < w$. If $|T_\alpha(\mathcal{F})|^q$ is relatively s-**concave** with respect to the function $g : \mathbb{R} \to \mathbb{R}$ for some fixed $s \in (0,1]$, $q \geq 1$, and $\frac{1}{p}+\frac{1}{q}=1$, and if $|T_\alpha(\mathcal{F})(x)|)| \leq M$ for all $x \in [u,w]$ then the following inequality holds:*

$$\left|\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \leq \frac{2^{\frac{s-1}{q}}\max\{g(x)^{\alpha-1},u^{\alpha-1},w^{\alpha-1}\}}{(1+p)^{1/p}(w-u)}\left[(g(x)-u)^2\left|T_\alpha(\mathcal{F})\left(\frac{u+g(x)}{2}\right)\right| + (g(x)-w)^2\left|T_\alpha(\mathcal{F})\left(\frac{w+g(x)}{2}\right)\right|\right]$$

*Proof.*Let $q > 1$, by Lemma 3 and using Hölder's inequality, we have:

$$\left|\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \leq \left|\frac{(g(x)-u)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)u)(\tau g(x)+(1-\tau)u)^{\alpha-1}\,d\tau\right| + \left|\frac{(g(x)-w)^2}{w-u}\int_0^1 \tau T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)w)(\tau g(x)+(1-\tau)w)^{\alpha-1}\,d\tau\right|$$

$$\leq \frac{(g(x)-u)^2}{w-u}\left(\int_0^1 \tau^p\,d\tau\left(\int_0^1 |T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)u||\tau g(x)+(1-\tau)u|^{\alpha-1})^q\,d\tau\right)^{1/q}$$

Since $|T_\alpha(\mathcal{F})|^q$ is relatively s-concave in the second sense, we obtain by Theorem 11:

$$\int_0^1 |\tau T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)u)|^q\,d\tau \leq 2^{s-1}\left|T_\alpha(\mathcal{F})\left(\frac{g(x)+u}{2}\right)\right|^9$$

and

$$\int_0^1 |\tau T_\alpha(\mathcal{F})(\tau g(x)+(1-\tau)w)|^q\,d\tau \leq 2^{s-1}\left|T_\alpha(\mathcal{F})\left(\frac{g(x)+w}{2}\right)\right|^9$$

From which we obtain:

$$\left|\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \leq \frac{2^{\frac{s-1}{q}}\max\{g(x)^{\alpha-1},u^{\alpha-1},w^{\alpha-1}\}}{(1+p)^{1/p}(w-u)}\left[(g(x)-u)^2\left|T_\alpha(\mathcal{F})\left(\frac{u-g(x)}{2}\right)\right| + (g(x)-w)^2\left|T_\alpha(\mathcal{F})\left(\frac{w-g(x)}{2}\right)\right|\right]$$

*Example 6.*For the function $F(x) = \sqrt{10-x}$, whose modulus

$$|T_\alpha F(x)|^2 = \left|T_\alpha\sqrt{10-x}\right|^2 = \left|x^{1-\alpha}\cdot\frac{1}{2\sqrt{10-x}}\right|^2$$

is relatively s-concave with respect to the function $g(x) = x^2$, it holds that for $\alpha = 0.4$, $s = 0.2$, and $p = 2$, on the interval $[2.5, 3]$, the inequality of Theorem 12 is satisfied.

Since

$$\max_{x\in[2.5,3]}\left\{(x^2)^{-0.6},\ 2.5^{-0.6},\ 3^{-0.6}\right\} = 0.58,$$

we obtain the following inequality:

$$\left|\sqrt{10-x^2}-2.69\right| \leq 0.26\left[(x^2-2.5)^2\cdot\left|\frac{\left(\frac{2.5+x^2}{2}\right)^{1.2}}{4\left(10-\frac{2.5+x^2}{2}\right)}\right| + (x^2-3)^2\cdot\left|\frac{\left(\frac{3+x^2}{2}\right)^{1.2}}{4\left(10-\frac{3+x^2}{2}\right)}\right|\right]$$

This inequality is illustrated in the following figure.

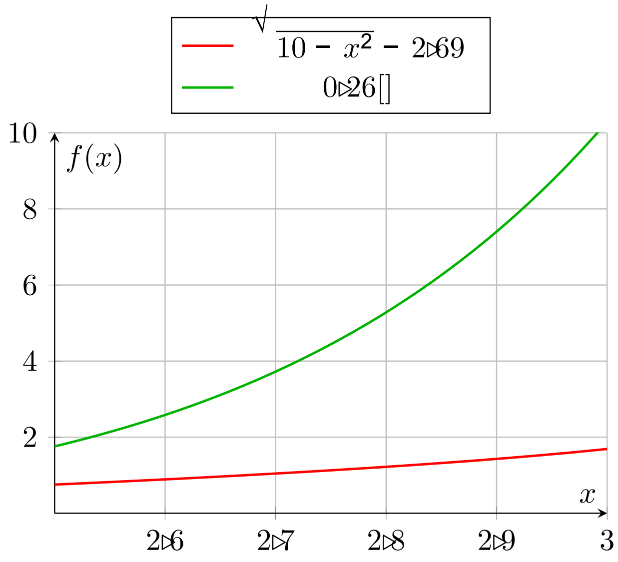

Figure 6: Graphs of the functions $\left|\sqrt{10-x^2}-2.69\right|$ and its upper bound.

4 Implications derived.

Corollary 1. *Let $\mathcal{F}: I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be an α-differentiable function on the interior I° of I such that $T_\alpha(\mathcal{F})$ belongs to the space of integrable functions, where $u, w \in I$ with $u < w$. If $|T_\alpha(F)|$ is relatively s-convex with respect to the function $g: \mathbb{R} \to \mathbb{R}$ for some fixed $s \in (0,1]$ and $p, q > 1$ satisfy $\frac{1}{p}+\frac{1}{q}=1$ with $|T_\alpha(f)|(x) \leq M$ for all $x \in [u,w]$, then the following inequality holds:*

$$\left|\mathcal{F}(g(x)) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \leq \frac{M}{(1+p)^{1/p}} \max\{g(x)^{\alpha-1}, u^{\alpha-1}, w^{\alpha-1}\} \left[\frac{(g(x)-u)^2+(g(x)-w)^2}{w-u}\right]$$

for all $x \in [u,w]$.

*Proof.*If in Theorem 9, we set $s = 1$, the result is obtained.

Corollary 2. *In Theorem 10, if we choose the function $g(x) = \frac{u+w}{2}$, then we have:*

$$\left|\mathcal{F}\left(\frac{u+w}{2}\right) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \leq \left(\frac{M(w-u)}{4}\right)\left(\frac{2}{s+1}\right)^{\frac{1}{q}} \max\{g(x)^{\alpha-1}, u^{\alpha-1}, w^{\alpha-1}\}$$

with $q \geq 1$, where $s \in (0,1]$ and $|T_\alpha(\mathcal{F})|^q$ is relatively s- convex in the second sense with respect to the function $g: \mathbb{R} \to \mathbb{R}$.

Corollary 3. *If in Theorem 12 we choose $s = 1$ and $g(x) = \frac{u+w}{2}$, then we have:*

$$\left|\mathcal{F}\left(\frac{u+w}{2}\right) - \frac{1}{w-u}\int_u^w \mathcal{F}(z)\,dz\right| \leq \frac{(w-u)\max\{g(x)^{\alpha-1}, u^{\alpha-1}, w^{\alpha-1}\}}{4(1+p)^{1/p}} \left[\left|T_\alpha(\mathcal{F})\left(\frac{3u+w}{4}\right)\right| + \left|T_\alpha(\mathcal{F})\left(\frac{3w+u}{4}\right)\right|\right]$$

5 Conclusions

We trust that the concepts and techniques developed in this paper provide a solid foundation for interested readers to further explore and investigate new applications of these recently introduced functions. We believe that their potential impact can extend across various fields of pure and applied sciences, thereby promoting the advancement of knowledge and the development of future research in these areas.

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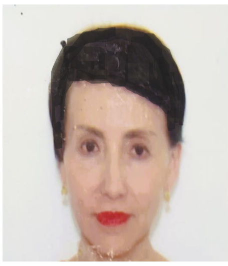

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